OUTLINE(1)

1) Density operators acting on $\mathcal{H}$, partial trace, purification, distances

2) Measure in the set of pure quantum states

2′) Algorithms for producing random pure states

- **a)** Hurwitz parametrization
- **b)** Rescaling random gaussian variables

3) Measures in the set of mixed quantum states

- **a)** Measures related to metric: the Hilbert–Schmidt measure
- **b)** Measures induced by partial tracing of composite systems

3′) An algorithm for producing density matrices HS–distributed

4) Pure state entanglement & entanglement monotones

5) Mean values: averaging entropy…

- **a)** …on the Hilbert–Schmidt measure
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OUTLINE(2)

6) Determinants of reduced $\boldsymbol{\rho_A}$ and their $\boldsymbol{N}^{\text{th}}$ roots

- **a)** Determinant (D)
- **b)** G–concurrence (G)

7) Average moments of G–concurrence

8) HS–Probability distribution $P_N^{(\beta)}(G)$

8′) Asymptotic behavior of $P_N^{\mathbb{C}}(G)$ for $D \to 0$

8″) Asymptotic behavior of $P_N^{\mathbb{R}}(G)$ for $D \to 0$

8‴) Asymptotic behavior of $P_N^{(\beta)}(G)$ for $D \to (1/N)^N$

9) Asymptotic behavior for $P_N^{(\beta)}(G)$ at large N

10) Induced distributions $P_{N\ell_1,N\ell_2}^{(\beta)}(G)$ at large N

11) Concentration of measure and Levy's Lemma

- Outlook & Perspective

1. Density operators acting on $\mathcal{H}$

Define the set $\mathcal{M}_N$ of all density operators ρ which act in an N–dimensional Hilbert space $\mathcal{H}$ and satisfy:

- :: Hermiticity ($\rho^\dagger = \rho$);
- :: normalization ($\mathrm{Tr}\,\rho = 1$);
- :: positivity ($\rho \geqslant 0$);

Pure states

Are those ρ such that $\mathrm{Tr}\,\rho^2 = 1$, or equivalently such that $\rho = |\Psi\rangle\langle\Psi|$ ·

Partial trace

Once that a bipartite Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^N \otimes \mathbb{C}^K$ is endowed with a NK–dimensional *product basis* $|m,\mu\rangle := |m\rangle_A \otimes |\mu\rangle_B$, then each linear operator $D : \mathcal{H} \to \mathcal{H}$ can be expressed in a matrix form

$$D_{\substack{m\mu\\ n\nu}} := \langle m,\mu|\ D\ |n,\nu\rangle = {}_A\langle m| \otimes_B \langle\mu|\ D\ |n\rangle_A \otimes |\nu\rangle_B$$

and *partial tracing over B* means

$$\mathop{\mathrm{Tr}}_B [D] = \sum_{\mu=1}^{K} {}_B\langle\mu|\ D\ |\mu\rangle_B =: D_A \text{ and similar expression for } D_B\ .$$

In the basis of $\mathcal{H}_A = \mathbb{C}^N$ and $\mathcal{H}_B = \mathbb{C}^K$, D_A and D_B read

$$(D_A)_{mn} = \sum_{\mu=1}^{K} D_{\substack{m\mu\\ n\mu}} \qquad \text{and} \qquad (D_B)_{\mu\nu} = \sum_{m=1}^{N} D_{\substack{m\mu\\ m\nu}} \quad .$$

Purification.

Every mixed state ρ_A acting on a finite N–dimensional Hilbert space $\mathcal{H}_A$ can be viewed as the reduced state of some pure state living in an extended bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ ·

Theorem [Purification]. *Let ρ_A be a density matrix acting on a Hilbert space $\mathcal{H}_A$ of finite dimension N, then there exist a Hilbert space $\mathcal{H}_B$ and a pure state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that the partial trace of $|\Psi\rangle\langle\Psi|$ with respect to $\mathcal{H}_B$*

$$\operatorname*{Tr}_B [|\Psi\rangle\langle\Psi|] = \rho_A \quad .$$

Proof:

A density matrix is by definition positive semidefinite. So ρ_A has square root factorization $\rho_A = MM^\dagger$.

Let $\mathcal{H}_B$ be another copy of the N–dimensional Hilbert space and consider the pure state $|\Psi\rangle$, living on the bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, represented in a product basis as

$$|\Psi\rangle = \sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}\, |i\rangle_A \otimes |j\rangle_B \quad ,$$

Direct calculation gives

$$\operatorname*{Tr}_B [|\Psi\rangle\langle\Psi|] = \sum_{i,\ell=1}^{N} \left(MM^\dagger\right)_{i\ell} |\, i\,\rangle_A \otimes {}_A\langle \ell\,| = \rho_A \qquad \square$$

Since square root decompositions of a positive semidefinite matrix are not unique, neither are purifications.

Density matrices capture partial information about a system. We can get partiality because we are seeing a subsystem of a larger system.

A useful distance.

For any $\rho, \sigma \in \mathcal{M}_N \subset \mathbb{R}^{N^2-1}$ one defines the so–called

Hilbert–Schmidt distance $\quad D_{\mathrm{HS}}(\rho,\sigma) = \sqrt{\mathrm{Tr}\left[(\rho-\sigma)^2\right]} \quad .$

Example: Mixed states of a two level system: $N = 2$

$$\rho_{\vec{\xi}} = \frac{\mathbb{I}_2}{2} + \frac{\vec{\xi} \cdot \vec{\sigma}}{2} \quad , \quad \vec{\sigma} := \left(\sigma_x \, , \, \sigma_y \, , \, \sigma_z \right)$$ Pauli matrices

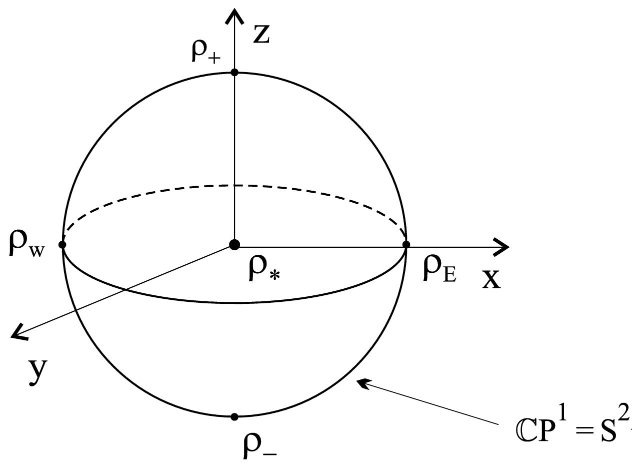

Figure 1: Space of mixed state of a qubit:
the HS–metric induces the flat geometry of a Bloch ball $\mathbf{B}^3 \subset \mathbb{R}^3$.

$$\mathrm{Tr}\, \rho^2 \leqslant 1 \quad \Longrightarrow \quad R_2 = \max \left| \vec{\xi} \right| = 1$$

The HS–distance between any two density operator...

$$D_{\mathrm{HS}}\left(\rho_{\vec{\xi}_1}, \rho_{\vec{\xi}_2}\right) = \sqrt{\mathrm{Tr}\left[\left(\rho_{\vec{\xi}_1} - \rho_{\vec{\xi}_2}\right)^2\right]} = D_{\mathrm{E}}(\vec{\xi}_1, \vec{\xi}_2) \quad ,$$

...proves to be equal to the Euclidean distance between the two coherence vectors $\vec{\xi}_1$ and $\vec{\xi}_2$.

The same holds true for arbitrary N, provided that $\rho_{\vec{\xi}} = \dfrac{\mathbb{I}_N}{N} + \dfrac{\vec{\xi} \cdot \vec{\lambda}}{N}$

- $\vec{\lambda}$ are the $\left(N^2 - 1\right)$ generators of $SU(N)$ fulfilling $\mathrm{Tr}\left(\lambda_i \lambda_j\right) = N\delta_{ij}$;
- the coherence vector span a subset of the ball of radius $\sqrt{N-1}$;

$\Longrightarrow$ we still have a flat Euclidean geometry for the $\vec{\xi}$-space.

V. C., H.–J. Sommers and K. Życzkowski,
Subnormalized states and trace–nonincreasing maps,
J. Math. Phys. **48(5)**, 052110 (2007)

2. Measure in the set of pure quantum states

A unique, unitary invariant measure exist (Fubini–Study measure),

$$P_{\mathrm{FS}}\left(|\Psi\rangle\right) = P_{\mathrm{FS}}\left(U\,|\Psi\rangle\right), \qquad |\Psi\rangle \in \mathcal{H}_N \; ;$$

2′. Algorithms for producing random pure states.

a) Hurwitz parametrization. You do need:

- $(N-1)$ polar angle φ_k, uniformly distributed in $[0, 2\pi)$
- $(N-1)$ azimuthal angle ϑ_k, distributed in $[0, \pi/2]$ according to

$$P\left(\vartheta\right) = k \sin\left(2\vartheta_k\right)\left(\sin\vartheta_k\right)^{2k-2} \quad , \quad k \in \{1, 2, \ldots, N-1\}$$

(In practice: set $\vartheta_k = \arcsin\left(\xi_k^{1/2k}\right)$, with ξ_k uniformly distributed in $[0, 1]$)

b) Rescaling random gaussian variables

- **generate** N Gaussian random $\mathbb{C}$ (or $\mathbb{R}$) –numbers $\{c_i\}_{i=1}^N$

$$P\left(c_1, c_2, \ldots, c_N\right) \propto \exp\left(-\sum_{i=1}^{N} |c_i|^2\right)$$

- **rescale** such numbers $\{x_i\}_{i=1}^N$ dividing by their ℓ_2–norm

$$y_i = \frac{c_i}{\sqrt{\sum_{i=1}^{N} |c_i|^2}}$$

- You got the N coordinates $\{y_i\}_{i=1}^N$ of a random pure state $|\Psi\rangle \in \mathcal{H}_N = \mathbb{C}^N$ (or $\mathbb{R}^N$) !!!

3. Measures in the set of mixed quantum states

Unitary invariance $P(\rho) = P\left(U\rho U^\dagger\right)$ does not distinguish a single measure. This property is characteristic of all product measures

$$\mu = P\left(\vec{\lambda}\right) \times \nu_{\mathrm{H}} ,$$

where ν_{H} is the Haar measure of $U(N)$ (or $O(N)$), the vector $\vec{\lambda}$ represent the spectrum of ρ, while $P\left(\vec{\lambda}\right)$ is any distribution defined in the eigenvalues simplex Δ_N.

Examples

a) Hilbert–Schmidt measure ::

$$P_{\mathrm{HS}}^{(\beta)}(\lambda_1, \lambda_2, \ldots, \lambda_N) = C_N^{(\beta)}\, \delta\left(1 - \sum_{i=1}^{N} \lambda_i\right) \prod_{j<k}^{N} \left|\lambda_j - \lambda_k\right|^{\beta} . \qquad (1)$$

enjoy to following property: every ball (in sense of a given metric) of a fixed radius has the same measure.

In formula (1), β is the repulsion exponent widely used in RMT.

$$\beta = \begin{cases} 2 & \text{for} \quad \mathrm{GUE} \\ 1 & \text{for} \quad \mathrm{GOE} \end{cases} .$$

b) Measures induced by partial tracing of composite systems ::

In order to generate a random density matrix of size N:

- **Construct** a composite Hilbert space $\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_K$;
- **Generate** generate a random pure state $|\Psi\rangle \in \mathcal{H}$ according to the natural, FS–measure on $\mathbb{C}P^{NK-1}$ (or $\mathbb{R}P^{NK-1}$);
- **Obtain** a mixed quantum state by partial tracing, $\rho_N = \mathrm{Tr}_K (|\Psi\rangle\langle\Psi|)$.

The probability distribution induced in this way into the simplex of eigenvalues reads

$$P^{(\beta)}_{N,K}(\vec{\lambda}) = C^{(\beta)}_{N,K}\, \delta\left(1 - \sum_{i=1}^{N} \lambda_i\right) \prod_i \lambda_i^{[\beta(K-N)+\beta-2]/2} \prod_{j<k}^{N} \left|\lambda_j - \lambda_k\right|^{\beta}$$

Observe that the induced distribution above can be recasted into the Hilbert–Schmidt distribution, here rewritten,

$$P^{(\beta)}_{\mathsf{HS}}(\lambda_1, \lambda_2, \ldots, \lambda_N) = C^{(\beta)}_N\, \delta\left(1 - \sum_{i=1}^{N} \lambda_i\right) \prod_{j<k}^{N} \left|\lambda_j - \lambda_k\right|^{\beta} \quad ,$$

provided that we choose $K = N - 1 + 2/\beta$, that is

$$K = \begin{cases} N & \text{for complex } \rho \qquad (\beta = 2 \text{ or abbr. } \mathbb{C}) \\ N+1 & \text{for real } \rho \qquad (\beta = 1 \text{ or abbr. } \mathbb{R}) \end{cases}$$

3′. Algorithm for producing density matrices HS–distributed.

The pure state of the bipartite system $\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_K$ can be represented in a product basis as

$$|\psi\rangle = \sum_{i=1}^{N} \sum_{j=1}^{K} A_{ij} \, |i\rangle \otimes |j\rangle \, .$$

- **generate** NK Gaussian random $\mathbb{C}$ (or $\mathbb{R}$) –numbers $\{A_{ij}\}$
  $\Longrightarrow$ pick a $N \times K$ matrix A, with K suitably chosen, from the Ginibre ensemble of non–Hermitian, complex (non–symmetric, real) matrices;
- **rescale** such numbers $\{A_{ij}\}$ dividing by their ℓ_2–norm
  $\Longrightarrow$ divide A for $\sqrt{\mathrm{Tr}\, AA^\dagger}$;

. . . now $|\psi\rangle$ is FS–distributed in $\mathcal{H}_N \otimes \mathcal{H}_K$. . .

- **obtain** a mixed state by partial tracing, $\rho_N = \mathrm{Tr}_K \left(|\Psi\rangle\langle\Psi|\right)$
  $\Longrightarrow \rho_N = AA^\dagger / \, \mathrm{Tr}\, AA^\dagger$;

Random mixed states are produced by generating normalized Wishart matrices $AA^\dagger / \, \mathrm{Tr}\, AA^\dagger$, with A belonging to the Ginibre ensemble.

$$K = \begin{cases} 1 & \text{for } \rho_N = |\Psi\rangle\langle\Psi| \text{ pure quantum state FS–distributed} \\ N & \text{for } \rho_N \text{ complex mixed quantum state HS–distributed} \\ N+1 & \text{for } \rho_N \text{ real mixed quantum state HS–distributed} \end{cases}$$

Example: $N = 2$

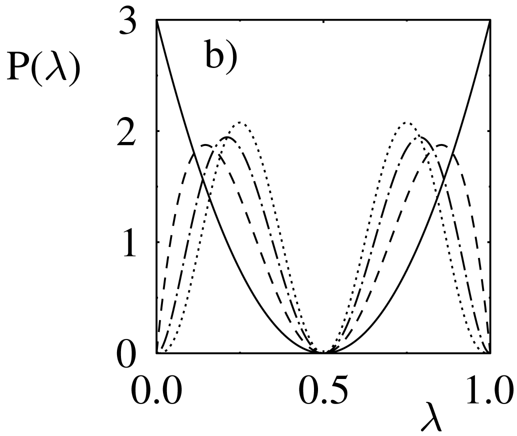

Figure 2: Measures $P_{2,K}$ induced by partial tracing; solid line represents HS–measure i.e. $K = 2$, dashed line $K = 3$, dash-dotted line $K = 4$, dotted line $K = 5$.

Example: $N = 3$

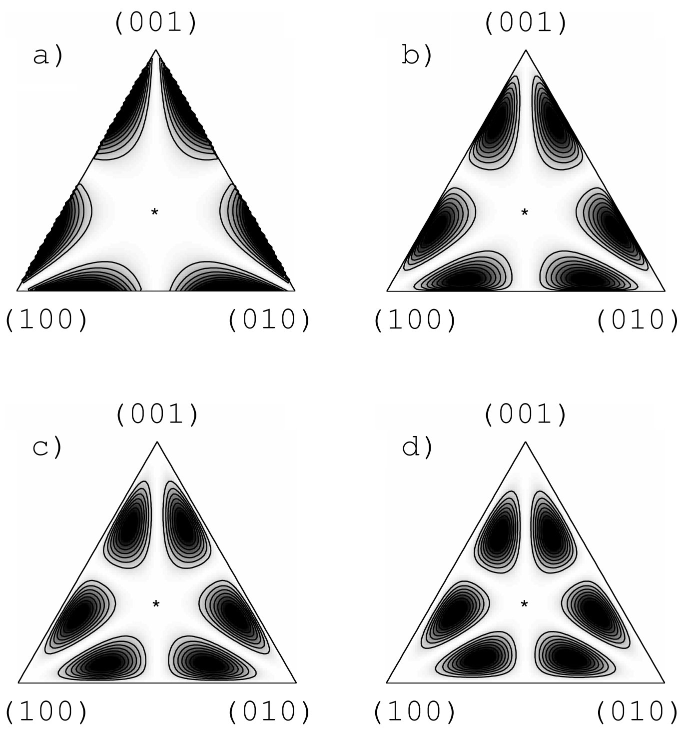

Figure 3: **(a)** HS–measure equal to $P^{(2)}_{3,3}$. Other measures induced by partial tracing: **(b)** $P^{(2)}_{3,4}$, **(c)** $P^{(2)}_{3,5}$ and **(d)** $P^{(2)}_{3,6}$.

4. Pure state entanglement

A fundamental difference between classical and quantum systems is:

we put systems together by **tensor product** *in quantum mechanics rather than by cartesian product.*

Consider two noninteracting systems A and B, with respective Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. The Hilbert space of the **composite system** is the tensor product $\mathcal{H}_A \otimes \mathcal{H}_B$.

If the first system is in state $|\psi\rangle_A$ and the second in state $|\phi\rangle_B$, the state of the composite system is $|\psi\rangle_A \otimes |\phi\rangle_B$, or $|\psi\rangle_A \, |\phi\rangle_B$, for short.

States of the composite system which can be represented in this form are called **separable states**, or **product states**.

Not all states are product states. Fix a basis $\{|i\rangle_A\}$ for $\mathcal{H}_A$ and a basis $\{|j\rangle_B\}$ for $\mathcal{H}_B$. The most general state in the bipartite overall system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^N \otimes \mathbb{C}^K$ is represented in a product basis as

$$|\Psi\rangle = \sum_{i=1}^{N} \sum_{j=1}^{K} A_{ij} \, |i\rangle_A \otimes |j\rangle_B \quad ,$$

and admit a **Schmidt decomposition**

$$|\Psi\rangle = \sum_{k=1}^{N} \sqrt{\lambda_k} \, |e(k)\rangle_A \otimes |f(k)\rangle_B \quad .$$

$\Longrightarrow$ a separable state $|\Psi\rangle$ is characterized by a vector of Schmidt coefficients with only **one** non–vanishing entry.

Schmidt coefficients $\{\lambda_k\}$ coincide with the spectrum of the eigenvalues of the **reduced density matrix** $\rho_A = \mathrm{Tr}_B\,(|\Psi\rangle\,\langle\Psi|)$.

$\Longrightarrow$ a state $|\Psi\rangle$ is separable **iff** its reduced density matrix is a **pure state**.

If a state is not separable, it is called an **entangled state**.

$\Longrightarrow$ a state is **entangled iff** its reduced density matrix is a **mixed state**.

For example, given two basis vectors $\{|0\rangle_A, |1\rangle_A\}$ of $\mathcal{H}_A$ and two basis vectors $\{|0\rangle_B, |1\rangle_B\}$ of $\mathcal{H}_B$, the following is an entangled state:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B\Big) \quad . \tag{2}$$

If the composite system is in this state, it is impossible to attribute to either system A or system B a definite pure state. Instead, their states are superposed with one another. In this sense, the systems are "entangled".

> <u>Obs:</u> Here, a local measurement causes a state reduction of the entire system state $|\Psi\rangle$, and therefore changes the probabilities for potential future measurements on either subsystems.

For example, the density matrix of A for the entangled state (2) is

$$\rho_A = \frac{1}{2}\Big(|0\rangle_A\langle 0|_A + |1\rangle_A\langle 1|_A\Big) = \frac{1}{2}\,\mathbb{1}_A \quad .$$

This demonstrates that, as expected, the reduced density matrix for an entangled pure ensemble is a mixed ensemble. Also not surprisingly, the density matrix of A for the pure product state $|\psi\rangle_A \otimes |\phi\rangle_B$ is

$$\rho_A = |\psi\rangle_A\langle\psi|_A \quad .$$

Entanglement Monotones:

DEF: Quantity not–increasing under the actions (even combined) of local operation, that is when the only admitted actions on the overall system are of the kind

$$\sum_{ijk\cdots} \cdots \left(A_{ijk} \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes B_{ij}\right)\left(A_i \otimes \mathbb{1}\right) \rho \left(A_i^\dagger \otimes \mathbb{1}\right)\left(\mathbb{1} \otimes B_{ij}^\dagger\right)\left(A_{ijk}^\dagger \otimes \mathbb{1}\right)\cdots$$

$$\sum_i A_i^\dagger A_i = \mathbb{1} \quad , \quad \sum_j B_{ij}^\dagger B_{ij} = \mathbb{1}\ \forall i \quad , \quad \sum_k A_{ijk}^\dagger A_{ijk} = \mathbb{1}\ \forall ij \quad \ldots$$

5. Mean values: averaging entropy

To characterize, to what extend a given state ρ is mixed, one may use the Von Neumann entropy

$$S(\rho) = -\operatorname{Tr} \rho \ln \rho$$

with $S(\rho) = 0$ for any pure state and $S(\rho) = \ln N$ for the maximally mixed state. We have computed the following averages over

a) Hilbert–Schmidt measure

$$\langle S(\rho)\rangle_{\mathrm{HS}} = \ln N - \frac{1}{2} + O\left(\frac{\ln N}{N}\right) \quad . \qquad (3)$$

b) Induced measure [Page, 1993]

$$\langle S(\rho)\rangle_{N,K} = \sum_{n=K+1}^{KN} \frac{1}{n} - \frac{N-1}{2K}$$
$$= \psi(NK+1) - \psi(K+1) - \frac{1}{2}\frac{N}{K} + \frac{1}{2K} \quad .$$

One can observe that $\langle S(\rho)\rangle_{N,N} = \langle S(\rho)\rangle_{\mathrm{HS}}$, within the accuracy of the asymptotic approximation in (3).

6. Determinants of reduced ρ_A and their N^{th} roots

Consider pure states $|\Psi\rangle$ on the bipartite system

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^N \otimes \mathbb{C}^K$$

and their reduced density matrices

$$\rho_A = \mathop{\mathrm{Tr}}_B \left(|\Psi\rangle\langle\Psi|\right) \ ,$$

characterized by their spectrum of eigenvalues $\vec{\lambda}$. We now focus on:

a) the determinant *D* of the reduced density matrix ρ_A

$D := \det \rho_A$, [M. Sinołęcka, K. Życzkowski and M. Kuś, 2002]

given by the product of all eigenvalues $D = \prod_{i=1}^{N} \lambda_i$, and

b) its rescaled N^{th} root $G := ND^{\frac{1}{N}}$, called ***G*–concurrence**, and given by the (rescaled) geometric mean of all $\vec{\lambda}$

$$G = N \left(\prod_{i=1}^{N} \lambda_i \right)^{\frac{1}{N}} \quad . \qquad \text{[G. Gour, 2005]}$$

Both G and D are important **entanglement monotones**.

We aim to compute mean values and to describe probability distributions for the determinant and G–concurrence of random density matrices generated with respect to the flat, Euclidean, HS–measure.

7. Average moments of G–concurrence

The moments of the G–concurrence on the HS–probability distribution $P_N^{(\beta)}(G)$ are given by

$$\begin{cases} \langle G_{\mathbb{C}}^M \rangle_N = N^M \dfrac{\Gamma(N^2)}{\Gamma(M+N^2)} \prod_{j=1}^{N} \dfrac{\Gamma\left(\frac{M}{N}+j\right)}{\Gamma(j)} \\ \langle G_{\mathbb{R}}^M \rangle_N = N^M \dfrac{\Gamma\left(\frac{N^2+N}{2}\right)}{\Gamma\left(M+\frac{N^2+N}{2}\right)} \prod_{j=1}^{N} \dfrac{\Gamma\left(\frac{M}{N}+\frac{j+1}{2}\right)}{\Gamma\left(\frac{j+1}{2}\right)} \end{cases} ; \quad (4)$$

similar expression have been found for $\langle D_{(\beta)}^M \rangle_N$.

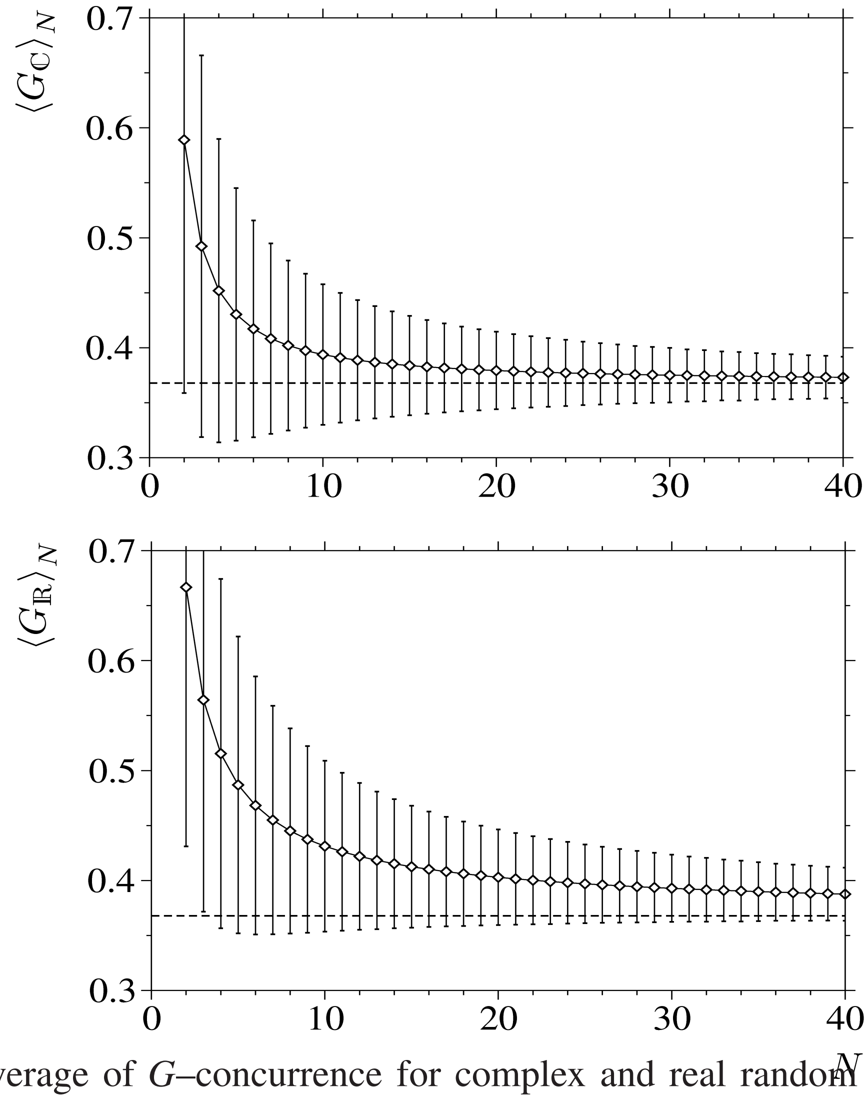

Figure 4: Average of G–concurrence for complex and real random pure states of a $N \times (N+2-\beta)$ system distributed accordingly to the FS measure. The average is computed by means of equation (4); error bars represent the variance of $P_N^{(\beta)}(G)$. Dashed line represent the asymptote $G_\star = 1/\mathrm{e}$.

8. HS–Probability distribution $P_N^{(\beta)}(G)$

When $N = 2$, the HS–Probability distribution $P_2^{(\beta)}(G)$ is given by

$$\begin{cases} P_2^{\mathbb{C}}(G) &= 3\, G\, \sqrt{1 - G^2} \\ P_2^{\mathbb{R}}(G) &= 2\, G \end{cases} \quad , \qquad G \in [0, 1] \;\cdot \tag{5}$$

For $N > 2$ we construct the HS–distribution from all moments $\langle G_{(\beta)}^M \rangle_N$. The distribution $P_N^{(\beta)}(G)$ is given by an inverse Laplace transformation, consisting in the following integral (see Figure 5):

$$P_N^{(\beta)}(G) = \int_{-i\infty}^{+i\infty} \frac{\mathrm{d}M}{2\pi i}\, G^{-(1+M)}\, \langle G_{(\beta)}^M \rangle_N \tag{6}$$

The same relation holds between $P_N^{(\beta)}(D)$ and $\langle D_{(\beta)}^M \rangle_N$.

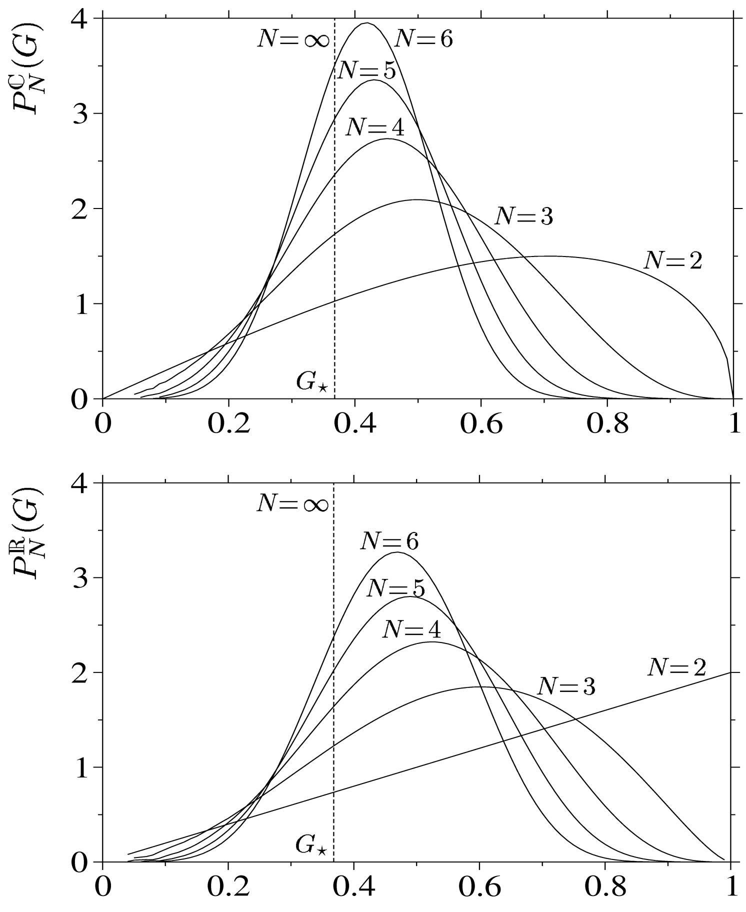

Figure 5: G–concurrence's distributions $P_N^{(\beta)}(G)$ are compared for different N. The distributions are obtained by performing numerically the inverse Laplace transformation of equation (6). Dashed vertical line is centered in $G_\star = 1/\mathrm{e}$.

8′. Asymptotic behavior of $P_N^{\mathbb{C}}(G)$ for $D \to 0$.

In this part of the spectrum, the probability $P_N^{\mathbb{C}}(D)$ can be expanded in a power series with some logarithmic corrections, as follows:

$$P_N^{\mathbb{C}}(D) \simeq Z_N^{\mathbb{C}} + X_N^{\mathbb{C}} \cdot D \log D + \widetilde{X}_N^{\mathbb{C}} \cdot D + O\left(D^2 (\log D)^2\right) \qquad (7)$$

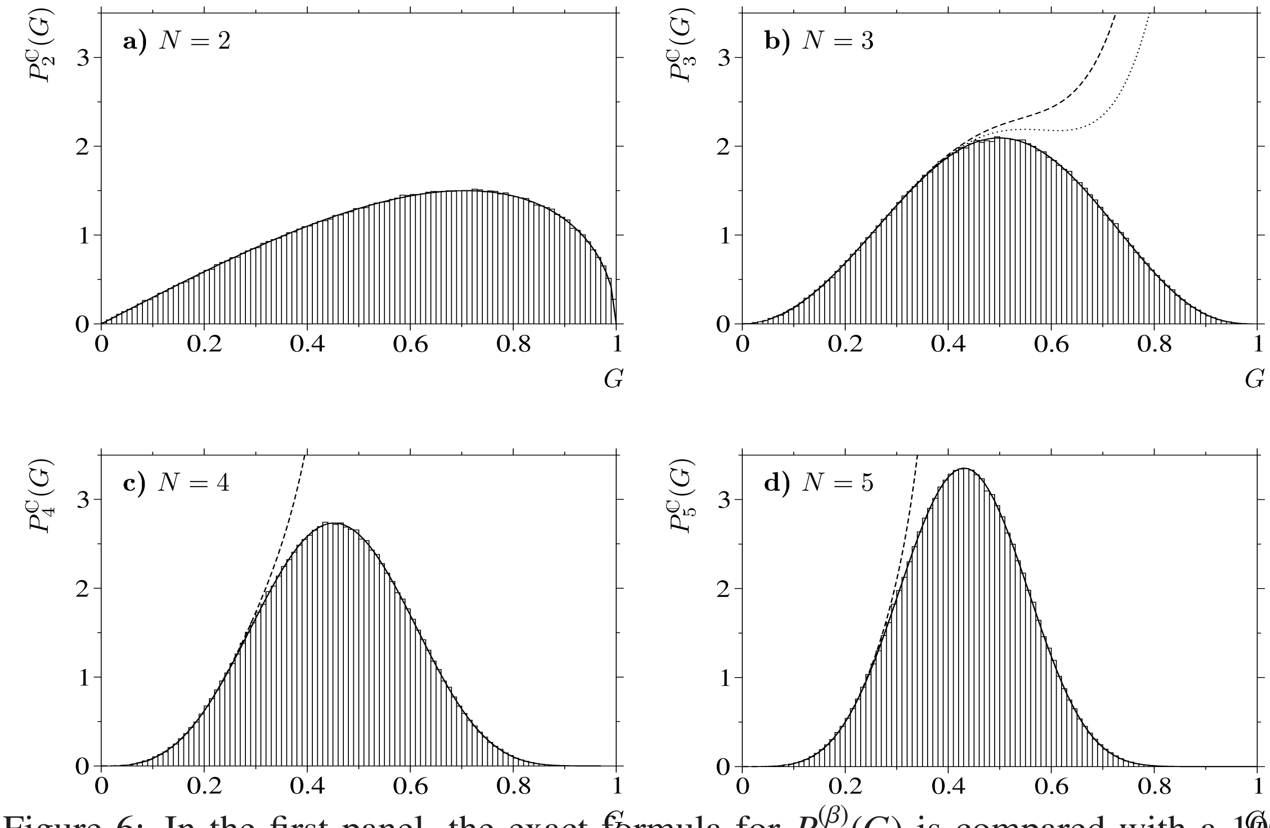

Figure 6: In the first panel, the exact formula for $P_2^{(\beta)}(G)$ is compared with a 100 bins histogram of 10^6 G–concurrence of 2×2 complex density matrices distributed accordingly to the HS measure. The other panels shows histograms (for different N) together with the distribution of G–concurrence obtained by inverse Laplace transforming (plotted in solid lines). The left asymptote given by eq. (7) is also plotted in dashed line for comparison.

$$Z_N^{\mathbb{C}} = \frac{\Gamma(N^2)}{\Gamma(N^2 - N)\,\Gamma(N)} \quad , \quad X_N^{\mathbb{C}} = \frac{\Gamma(N^2)}{\Gamma(N^2 - 2N)\,\Gamma(N)\,\Gamma(N-1)} \quad ,$$

$$\widetilde{X}_N^{\mathbb{C}} = X_N^{\mathbb{C}} \cdot \Big[N + N\psi(N^2 - 2N) - 4 - 2\psi(1) - (N-2)\psi(N-2)\Big] \quad .$$

NOTE $P_N^{(\beta)}(G)\,\mathrm{d}G = P_N^{(\beta)}(D)\,\mathrm{d}D$ $\Longrightarrow$ $G \cdot P_N^{(\beta)}(G) = N \cdot D \cdot P_N^{(\beta)}(D)$

From now on formulae and figures will be given indifferently for both G and D distribution, being clear their mutual relation.

8″. Asymptotic behavior of $P_N^{\mathbb{R}}(G)$ for $D \to 0$.

The expansion of probability $P_N^{\mathbb{R}}(D)$, corresponding real ρ_A of small determinant, is still a power series (plus logarithmic corrections) but the exponents are now semi–integer:

$$P_N^{\mathbb{R}}(D) \simeq Z_N^{\mathbb{R}} + Y_N^{\mathbb{R}} \cdot D^{\frac{1}{2}} + X_N^{\mathbb{R}} \cdot D \log D + \widetilde{X}_N^{\mathbb{R}} \cdot D + \\ + W_N^{\mathbb{R}} \cdot D^{\frac{3}{2}} \log D + \widetilde{W}_N^{\mathbb{R}} \cdot D^{\frac{3}{2}} + \mathcal{O}\left(D^2 (\log D)^2\right) \tag{8}$$

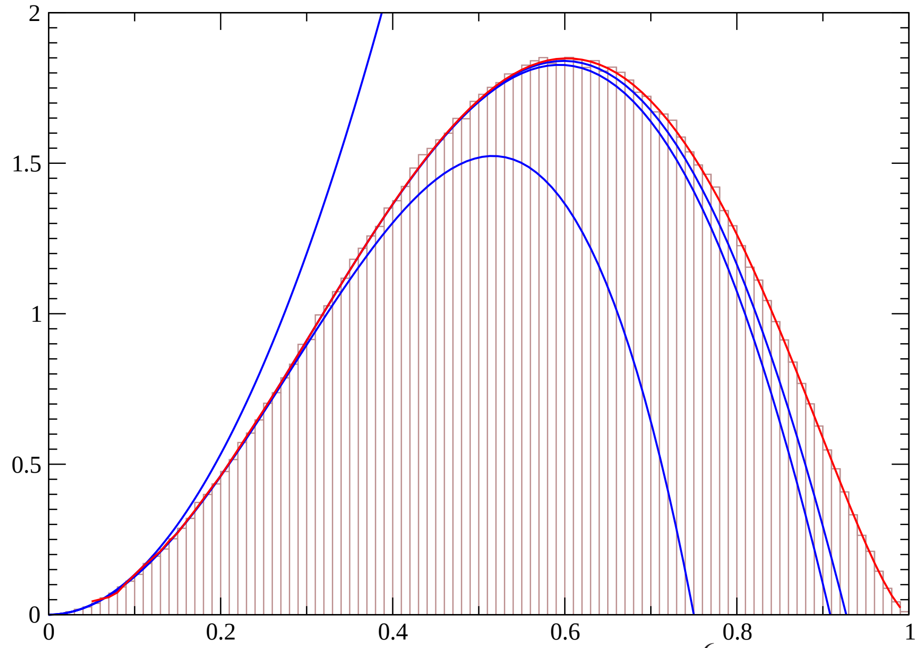

Figure 7: In this plot we shows a 100 bins histogram of 10^6 G–concurrence of a 3×3 real density matrix distributed accordingly to the HS measure. The true distribution of G–concurrence is represented by red line. Blue lines represent the contribution to eq. (8), correspondent to $Z_N^{\mathbb{R}}$, $Y_N^{\mathbb{R}}$, $X_N^{\mathbb{R}}$ and $\widetilde{X}_N^{\mathbb{R}}$, added one by one.

Coefficients reads $\quad Z_N^{\mathbb{R}} = \frac{2^{N-1}\,\Gamma\left(\frac{N^2+N}{2}\right)}{\Gamma\left(\frac{N^2-N}{2}\right)\Gamma(N)}\,, \qquad Y_N^{\mathbb{R}} = -\sqrt{\pi}\frac{2^{N-1}\,\Gamma\left(\frac{N^2+N}{2}\right)}{\Gamma\left(\frac{N^2-2N}{2}\right)\Gamma\left(\frac{N+1}{2}\right)\Gamma(N-1)}$

and, for $N > 3$, we have†

$$\begin{cases} X_N^{\mathbb{R}} &= -\frac{2^{2N-3}\,\Gamma\left(\frac{N^2+N}{2}\right)}{\Gamma\left(\frac{N^2-3N}{2}\right)\Gamma(N)\,\Gamma(N-2)} \\ \widetilde{X}_N^{\mathbb{R}} &= X_N^{\mathbb{R}}\Big\{N + N\psi\left(\frac{N^2-3N}{2}\right) - 8 - \frac{3}{2}\,\psi\left(\frac{1}{2}\right) - 2\,\psi(1) + \\ & \qquad -\frac{N-3}{2}\,\psi\left(\frac{N-3}{2}\right) - \frac{N-4}{2}\,\psi\left(\frac{N-4}{2}\right)\Big\} \end{cases} .$$

†For $N = 3$ separate calculations yields $\widetilde{X}_3^{\mathbb{R}} = 12 \cdot 5!$ and $X_3^{\mathbb{R}} = 0$.

8‴. Asymptotic behavior of $P_N^{(\beta)}(G)$ for $D \to (1/N)^N$.

Stirling approximation $\Longrightarrow$

$$\begin{cases} P_N^{\mathbb{C}}(D) \simeq A_N^{\mathbb{C}} \cdot \dfrac{(-\log D - N\log N)^{(N^2-3)/2}}{D\,\left[(N^2-3)/2\right]!} \\[2ex] P_N^{\mathbb{R}}(D) \simeq A_N^{\mathbb{R}} \cdot \dfrac{(-\log D - N\log N)^{(N^2+N-6)/4}}{D\,\left[(N^2+N-6)/4\right]!} \end{cases}$$

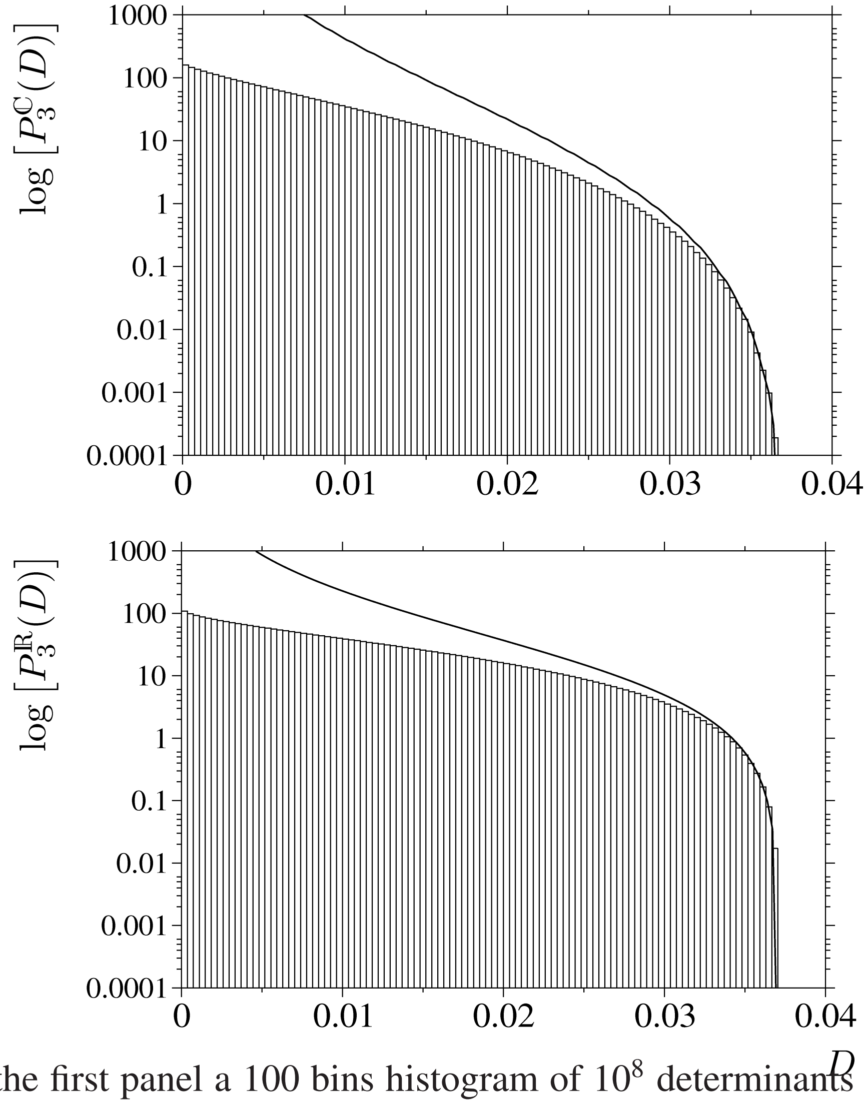

Figure 8: In the first panel a 100 bins histogram of 10^8 determinants of 3×3 complex density matrices distributed accordingly to the HS measure is compared with the right asymptote given by equation (9) (plotted in solid line). Same analysis is depicted in the second panel, but for 3×3 real density matrices.

$$-\log D - N\log N \simeq \simeq 1 - D\,N^N = 1 - G^N \quad \Longrightarrow \quad \begin{cases} P_N^{\mathbb{C}}(G) \simeq \widetilde{A}_N^{\mathbb{C}} \cdot \dfrac{(1-G^N)^{(N^2-3)/2}}{G} \\[2ex] P_N^{\mathbb{R}}(G) \simeq \widetilde{A}_N^{\mathbb{R}} \cdot \dfrac{(1-G^N)^{(N^2+N-6)/4}}{G} \end{cases}$$

9. Asymptotic behavior for $P_N^{(\beta)}(G)$ at large N

When the system becomes eventually large, we found the general result

$$G(M) := \lim_{N\to\infty} \left\langle G_{(\beta)}^M \right\rangle_N = \mathrm{e}^{-M} \tag{9}$$

that holds for both real and complex density matrices HS–distributed.

Example ($\beta = 2$)

$$\left\langle G_{\mathbb{C}}^M \right\rangle_N = \left\{ \prod_{k=0}^{M-1} \frac{N}{N^2+k} \right\} \left\{ \left[\frac{M}{N} \Gamma\left(\frac{M}{N}\right) \right]^N \right\} \left\{ \prod_{k=1}^{N-1} \left(1 + \frac{M}{kN}\right)^{N-k} \right\}$$

Equation (9) display

$$\begin{cases} \mu & = \langle G_{(\beta)} \rangle \quad = 1/\mathrm{e} = 0.367\,879\,441\ldots \\ \sigma^2 & = \langle G_{(\beta)}^2 \rangle \quad - \langle G_{(\beta)} \rangle^2 = 0 \end{cases}$$

The limiting distribution, can be earned by performing the Laplace inverse transform, and reads

$$P^{(\beta)}(G) := \lim_{N\to\infty} P_N^{(\beta)}(G) = \delta(G - \mathrm{e}^{-1}) \tag{10}$$

again for both $\beta = 1, 2$.

A similar concentration of reduced density matrices around the maximally mixed state has been recently quantified for bipartite $N \times K$ systems [Hayden et al., 2006].

Concerning the G–concurrence, we have concentration of the distribution around its mean, although in this case the mean is not coinciding with the maximally mixed state (on which $G = 1$).

10. Induced distributions $P^{(\beta)}_{N\ell_1,N\ell_2}(G)$ at large N

The moments of the G–concurrence on the induced probability distribution $P^{(\beta)}_{K_1,K_2}(G)$ are given by

$$\begin{cases} \langle G_{\mathbb{C}}^M \rangle_{K_1,K_2} = K_1^M \dfrac{\Gamma(K_1K_2)}{\Gamma(K_1K_2+M)} \displaystyle\prod_{j=1}^{K_1} \dfrac{\Gamma(K_2 - K_1 + j + M/K_1)}{\Gamma(K_2 - K_1 + j)} \\ \langle G_{\mathbb{R}}^M \rangle_{K_1,K_2} = K_1^M \dfrac{\Gamma\left(\frac{K_1K_2}{2}\right)}{\Gamma\left(\frac{K_1K_2}{2}+M\right)} \displaystyle\prod_{j=1}^{K_1} \dfrac{\Gamma\left(\frac{K_2-K_1+j}{2}+\frac{M}{K_1}\right)}{\Gamma\left(\frac{K_2-K_1+j}{2}\right)} \end{cases} ;$$

We now consider the following high dimensional joint limit: $K_1 = N\ell_1$ and $K_2 = N\ell_2$, with $\ell_1, \ell_2 \in \mathbb{N}^+$, $\ell_2 > \ell_1$ and N eventually large.

In terms of the parameter $q := \ell_2/\ell_1 \in \mathbb{Q}$, we find

$$G(M) := \lim_{N\to\infty} \left\langle G^M_{(\beta)} \right\rangle_{N\ell_1,N\ell_2} = X_q^{-M} \quad , \quad \forall\, \beta \in \{1,2\}$$

with

$$X_q := \frac{1}{\mathrm{e}} \left(\frac{q}{q-1}\right)^{q-1}$$

The limiting distribution $P_{(\ell_1),(\ell_2)}(G)$, can be earned as before and reads

$$P_{(\ell_1),(\ell_2)}(G) := \lim_{N\to\infty} P^{(\beta)}_{N\ell_1,N\ell_2}(G) = \delta(G - X_q)$$

again for both $\beta = 1, 2$.

<u>Obs:</u>

$q \to 1$ (HS–distribution) $\Longrightarrow$ $X_q \to 1/\mathrm{e}$ (previous example)

$q \to \infty$ (large environment) $\Longrightarrow$ $X_q \to 1$ (compl. mixed state)

11. Concentration of measure and Levy's Lemma

Levy's Lemma. *Let $f : \mathbb{S}^k \mapsto \mathbb{R}$ be a function with Lipschitz constant η (with respect to the Euclidean norm) and a point $X \in \mathbb{S}^k$ be chosen uniformly at random. Then*

1. $\Pr\{f(X) - \mathbb{E}f \gtrless \pm\alpha\} \leqslant 2\exp\left(-C_1(k+1)\alpha^2/\eta^2\right)$ *and*
2. $\Pr\{f(X) - m(f) \gtrless \pm\alpha\} \leqslant \exp\left(-C_2(k-1)\alpha^2/\eta^2\right)$

for absolute constant $C_i > 0$ that may be chosen as
$C_1 = (9\pi^3)^{-1}$ *and* $C_2 = (2\pi^2)^{-1}$.
($\mathbb{E}f$ is the mean value of f, $m(f)$ a median for f).

Trivial example

Take $f(x_1, x_2, \ldots, x_{k+1}) = x_1$. Then Levy's Lemma says that the probability of finding a random point on $\mathbb{S}^k$ outside a band of the equator of size 2α approach to 0 as $2\exp\left(-C_1(k+1)\alpha^2\right)$.

$(k \longrightarrow \infty)$ $\xrightarrow[\text{LEMMA}]{\text{LEVY'S}}$ Every equator is "Fat"

Less trivial example

The function f could be a measure, an entangle monotones...

In particular, for $f = S(\rho_A)$, the Von Neumann Entropy, we have

Theorem [Hayden et al., 2006]. *Let $|\psi\rangle\langle\psi|$ be a random pure state FS–distributed on $\mathcal{H} = \mathcal{H}_\mathcal{A} \otimes \mathcal{H}_\mathcal{B} = \mathbb{C}^N \otimes \mathbb{C}^K$, with $K \geqslant N \geqslant 3$. Then*

$$\Pr\left\{S\left[\operatorname*{Tr}_B(|\psi\rangle\langle\psi|)\right] < \ln N - \alpha - \beta\right\} \leqslant \exp\left(-\frac{(NK-1)C_3\alpha^2}{(\ln N)^2}\right),$$

where $\beta = N/K$ and $C_3 = (8\pi^2)^{-1}$.

Outlook & Perspective

Main aim of the project proposed consist in:

- deriving the distribution of eigenvalues induced by differen measures and finding a link to the RMT;
- analyzing ensemble of random matrices related to different measures in the set $\mathcal{M}_N$.

Concerning the G–concurrence:

- finding its mean value over the set of random pure states together with all higher moments;
- describing its complete probability distribution over the set of random pure states:
- showing analytically the effect of concentration of the measure.

(G–concurrence is likely to be the first measure of pure state entanglement for which such a complete analysis is performed).

Moreover, our work may also be considered as a contribution to the Random Matrix Theory:

- we have found the distribution of the determinants of random Wishart matrices $AA^\dagger$, normalized by fixing their trace.