Divergent series, Borel Summation, 3-D Navier-Stokes

Saleh Tanveer
(Ohio State University)

Collaborators: Ovidiu Costin, Guo Luo

Research supported in part by
- Institute for Math Sciences (IC), EPSRC & NSF.
Navier-Stokes existence—background

- Global Existence of smooth 3-D Navier-Stokes solution is an important open problem.
- Deviation from linear stress-strain relation or incompressibility is potentially important if N-S solutions are singular.
- Usual numerical calculations do not address this issue because errors are not controlled, rigorously.
- Globally smooth solutions known only when Reynolds number small.
- Generally, smooth solutions for smooth data on $[0, T]$ known to exist, for $T$ scaling inversely with initial data/forcing.
- Global weak solutions known since Leray, but not known whether they are unique. For unforced problem in $\mathbb{T}^3$, such a solution becomes smooth again for $t > T_c$, $T_c$ depends on IC.
Borel Summation—background and main idea

- Borel summation generates, under suitable conditions, a one-one correspondence between series and functions that preserve algebraic operations (Ecalle, Costin,..).

- Borel sum can involve large or small variable(s)/ parameter(s).

- Formal expansion for $t << 1$: $v(x, t) = v_0(x) + \sum_{m=1}^{\infty} t^m v_m(x)$ generally divergent for the initial value problem $v_t = \mathcal{N}[v]$, $v(x, 0) = v_0$, $\mathcal{N}$ being some differential operator.

- Borel Sum of this series gives actual solution, which transcends restriction $t << 1$

- For Navier-Stokes, the Borel sum is given by

$$v(x, t) = v_0(x) + \int_0^\infty U(x, p) e^{-p/t} dp$$

Equation for $U$ obtained by inverse-Laplace transforming N-S.
Incompressible 3-D Navier-Stokes in Fourier-Space

Consider 3-D N-S in infinite geometry or periodic box. Similar results expected for finite domain with no-slip BC using eigenfunctions of Stokes operator as basis. In Fourier-Space

\[ \hat{v}_t + \nu |k|^2 \hat{v} = -ik_j P_k [\hat{v}_j \hat{v}] + \hat{f}(k) \]

\[ P_k = \left( I - \frac{k(k \cdot)}{|k|^2} \right) \quad , \quad \hat{v}(k, 0) = \hat{v}_0(k) \]

where \( P_k \) is the Hodge projection in Fourier space, \( \hat{f}(k) \) is the Fourier-Transform of forcing \( f(x) \), assumed divergence free and \( t \)-independent. Subscript \( j \) denotes the \( j \)-th component of a vector. \( k \in \mathbb{R}^3 \) or \( \mathbb{Z}^3 \). Einstein convention for repeated index followed. \( * \) denotes Fourier convolution.

Decompose \( \hat{v} = \hat{v}_0 + \hat{u}(k, t) \), inverse-Laplace Transform in \( 1/t \) and invert the differential operator on the left side
Integral equation associated with Navier-Stokes

We obtain:

\[
\hat{U}(k, p) = \int_0^p \mathcal{K}_j(p, p'; k) \hat{H}_j(k, p') dp' + \hat{U}^{(0)}(k, p) \equiv \mathcal{N} [\hat{U}] (k, p)
\]

(1)

\[
\mathcal{K}_j(p, p'; k) = \frac{ik_j \pi}{z} \{z'Y_1(z')J_1(z) - z'Y_1(z)J_1(z')\}
\]

\[
z = 2|k|\sqrt{\nu p} , \ z' = 2|k|\sqrt{\nu p'} , \ \hat{H}_j = P_k \left\{ \hat{v}_{0,j} \right\} + \hat{U}_j \hat{\hat{v}}_0 + \hat{U}_j \hat{\hat{U}}
\]

\[
\hat{U}^{(0)}(k, p) = 2\frac{J_1(z)}{z} \hat{v}_1(k) , \ P_k = \left( I - \frac{k(k\cdot)}{|k|^2} \right)
\]

\[
\hat{v}_1(k) = (-\nu |k|^2 \hat{v}_0 - ik_j \mathcal{P}_k [\hat{v}_{0,j} \hat{\hat{v}}_0]) + \hat{f}(k),
\]

\*, denotes Fourier Convolution, * denotes Laplace convolution, while \* denotes Fourier followed by Laplace convolution. \(J_1\) and \(Y_1\) are the usual Bessel functions.
Introduce norm $\| \cdot \|_{\mu, \beta}$ and $\| \cdot \|$ for $\mu > 3$, $\beta \geq 0$ so that

$$
\| \hat{w} \|_{\mu, \beta} = \sup_{k \in \mathbb{R}^3} e^{\beta|k|} (1 + |k|)^\mu |\hat{w}(k)|
$$

$$
\| \hat{U} \| = \sup_{p > 0} e^{-\alpha p} (1 + p^2) \| \hat{U}(\cdot, p) \|_{\mu, \beta}
$$

Lemma 1: If $\| \hat{v}_0 \|_{\mu + 2, \beta}$ and $\| \hat{f} \|_{\mu, \beta}$ are finite, then an upper bound for $\alpha$ can be found in terms of $\hat{v}_0$ and $\hat{f}$ so that the integral equation (1) has a unique solution for $p \in \mathbb{R}^+$ for which $\| \hat{U} \| < \infty$.

Theorem 1: Under same conditions as in Lemma 1, the 3-D Navier-Stokes has a unique solution for $\text{Re} \frac{1}{t} > \alpha$. Furthermore, $\hat{v}(\cdot, t)$ is analytic for $\text{Re} \frac{1}{t} > \alpha$ and $\| \hat{v}(\cdot, t) \|_{\mu + 2, \beta} < \infty$ for $t \in [0, \alpha^{-1})$.

Theorem 2 deals with Borel Summability and the nature of the asymptotic expansion $\hat{v} \sim \hat{v}_0 + t\hat{v}_1$ and will not be discussed.
Remarks on Theorem 1

Remark 1: Local existence results in Theorem 1 already known through classical methods. However, in the present formulation, global existence problem can be cast into a question of asymptotics of a known solution to integral equation. A sub-exponential growth as $p \to \infty$ gives global existence.

Remark 2: Errors in Numerical solutions rigorously controlled, unlike usual N-S calculations. Discretization in $p$ and Galerkin approximation in $k$ results in:

$$\hat{U}_\delta(k, m\delta) = \delta \sum_{m' = 0}^{m} K_{m,m'} \mathcal{P}_N \mathcal{H}_\delta(k, m'\delta) + \hat{U}^{(0)}(k, m\delta)$$

$$\equiv \mathcal{N}_\delta \left[ \hat{U}_\delta \right] \quad \text{for} \quad k_j = -N, \ldots, N, \quad j = 1, 2, 3$$

$\mathcal{P}_N$ is the Galerkin Projection into $N$ -Fourier modes. $\mathcal{N}_\delta$ has properties similar to $\mathcal{N}$. The continuous solution $\hat{U}$ satisfies $\hat{U} = \mathcal{N}_\delta \left[ \hat{U} \right] + E$, where $E$ is the truncation error. Thus, $\hat{U} - \hat{U}_\delta$ can be estimated using same tools as in Theorem 1.
Numerical Solutions to integral equation

We choose the Kida initial conditions and forcing

\[ v_0(x) = (v_1(x_1, x_2, x_3, 0), v_2(x_1, x_2, x_3, 0), v_3(x_1, x_2, x_3, 0)) \]

\[ v_1(x_1, x_2, x_3, 0) = v_2(x_3, x_1, x_2, 0) = v_3(x_2, x_3, x_1, 0) \]

\[ v_1(x_1, x_2, x_3, 0) = \sin x_3 (\cos 3x_2 \cos x_3 - \cos x_2 \cos 3x_3) \]

\[ f_1(x_1, x_2, x_3) = \frac{1}{5} v_1(x_1, x_2, x_3, 0) \]

High Degree of Symmetry makes computationally less expensive

Corresponding Euler problem believed to blow up in finite time;

so good candidate to study viscous effects

In the plots, "constant forcing" corresponds to \( f = (f_1, f_2, f_3) \) as above, while zero forcing refers to \( f = 0 \). Recall sub-exponential growth in \( p \) corresponds to global N-S solution.
Numerical solution to integral equation-plot-1

\[ \|\hat{U}(., p)\|_{4,0} \text{ vs. } p \text{ for } \nu = 1, \text{ constant forcing.} \]
Numerical solution to integral equation-plot-2

\[ \| \hat{U}(., p) \|_{4,0} \text{ vs. } p \text{ for } \nu = 1, \text{ no forcing} \]
Numerical solution to integral equation-plot-3

\[ \| \hat{U}(\cdot, p) \|_{4,0} \text{ vs. } p \text{ for } \nu = 0.16, \text{ constant forcing} \]
Numerical solution to integral equation-plot-4

\[ \| \hat{U}(., p) \|_{4,0} \text{ vs. } p \text{ for } \nu = 0.1, \text{ constant forcing} \]
Numerical solution to integral equation-plot-5

\( \hat{U}(k, p) \) vs. \( p \) for \( k = (1, 1, 17) \), \( \nu = 0.1 \), no forcing.
\[ \log \| \hat{U}(\cdot, p) \|_{4,0} \text{ vs. } \log p \text{ for } \nu = 0.001, \text{ constant forcing} \]
Issues raised by numerical computations

Numerical solutions to integral equation available on finite interval \([0, p_0]\), yet N-S solution requires \([0, \infty)\) interval since
\[ \hat{v}(k, t) = \hat{v}_0 + \int_0^\infty e^{-p/t} \hat{U}(k, p) dp \]

Actually, the integral over \(\int_0^{p_0}\) gives an approximate N-S solution, with errors that can be bounded for a time interval \([0, T]\), if computed solution to integral equation eventually decreases with \(p\) on a sufficiently large interval \([0, p_0]\).

Further, a non-increasing \(\hat{U}\) over a sufficiently large interval \([0, p_0]\) gives smaller bounds on growth rate \(\alpha\) as \(p \to \infty\). Therefore, in such cases smooth NS solution exists over a long interval \([0, \alpha^{-1})\).

Recall for unforced problem in \(\mathbb{T}^3\), even weak solution to NS becomes smooth for \(t > T_c\), with \(T_c\) estimated from initial data. Hence global existence follows under some conditions.
Extending Navier-Stokes interval of existence

For $\alpha_0 \geq 0$, define

$$\epsilon = \nu^{-1/2} p_0^{-1/2}, \quad a = \|\hat{v}_0\|_{\mu,\beta}, \quad c = \int_{p_0}^{\infty} \|\hat{U}^{(0)}(., p)\|_{\mu,\beta} e^{-\alpha_0 p} dp$$

$$\epsilon_1 = \nu^{-1/2} p_0^{-1/2} \left( 2 \int_0^{p_0} e^{-\alpha_0 s} \|\hat{U}(., s)\|_{\mu,\beta} ds + \|\hat{v}_0\|_{\mu,\beta} \right)$$

$$b = \frac{e^{-\alpha_0 p_0}}{\sqrt{\nu p_0 \alpha}} \int_0^{p_0} \|\hat{U}^* \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{\mu,\beta} ds$$

**Theorem 3:** A smooth solution to 3-D Navier-Stokes equation exists in the $\|\cdot\|_{\mu,\beta}$ space on the interval $[0, \alpha^{-1})$, when $\alpha \geq \alpha_0$ is chosen to satisfy

$$\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4b\epsilon - \epsilon_1^2}$$

**Remark:** If $p_0$ is chosen large enough, $\epsilon, \epsilon_1$ is small when computed solution in $[0, p_0]$ decays with $q$. Then $\alpha$ can be chosen rather small.
Relation of Optimal $\alpha$ to Navier-Stokes singularities

$$\hat{U}(k, p) = \frac{1}{2\pi i} \int_{c_0 - i\infty}^{c_0 + i\infty} e^{p/t} [\hat{v}(k, t) - \hat{v}_0(k)] \, d \left[ \frac{1}{t} \right]$$

Rightmost singularity(ies) of NS solution $\hat{v}(k, t)$ in the $1/t$ plane determines optimal $\alpha$. $\gamma$ gives dominant oscillation frequency.
Laplace-transform and accelerated representation

To get rid of the effect of complex singularity, it is prudent to seek a more general Laplace-transform involves

\[ \hat{v}(k, t) = \hat{v}_0(k) + \int_0^\infty e^{-q/t^n} \hat{U}(k, q) dq \]

We have arguments to show for at least the unforced problem, if there are complex singularities \( t_s \) in the right-half plane, but not on the real axis, then a nonzero lower bound for \( |\arg t_s| \) exists. Then, for sufficiently large \( n \), no singularities in the \( \tau = t^{-n} \) plane in the right-half plane. Hence, \( \hat{U}(k, q) \) will not grow with \( q \) \( \hat{U}(k, q) \) satisfies an integral equation similar to the one satisfied by \( \hat{U}(k, p) \) and Theorems similar to Theorem 1 follow. In the context of ODEs, change of variable \( p \rightarrow q \) is called acceleration (Ecalle)
Extending Navier-Stokes interval of existence

For $\alpha_0 \geq 0$, define

$$
\epsilon_1 = \nu^{-1/2} q_0^{-1 + 1/(2n)} , \quad c = \int_{q_0}^{\infty} \|\hat{U}^{(0)}(., q)\|_{\mu, \beta} e^{-\alpha_0 q} dq
$$

$$
\epsilon_1 = \nu^{-1/2} q_0^{-1 + 1/(2n)} \left( 2 \int_{0}^{q_0} e^{-\alpha_0 s} \|\hat{U}(., s)\|_{\mu, \beta} ds + \|\hat{v}_0\|_{\mu, \beta} \right)
$$

$$
b = \frac{e^{-\alpha_0 q_0}}{\sqrt{\nu q_0^{-1 - 1/(2n)}}} \int_{0}^{q_0} \|\hat{U}^* \hat{U} + \hat{v}_0 \cdot \hat{U}\|_{\mu, \beta} ds
$$

Theorem 4: A smooth solution to 3-D Navier-Stokes equation exists in the $\|\cdot\|_{\mu, \beta}$ space on the interval $[0, \alpha^{-1/n})$, when $\alpha \geq \alpha_0$ is chosen to satisfy

$$
\alpha > \epsilon_1 + 2\epsilon c + \sqrt{(\epsilon_1 + 2\epsilon c)^2 + 4bc} - \epsilon_1
$$

Remark: If $q_0$ is chosen large enough, $\epsilon$, $\epsilon_1$ is small when computed solution in $[0, q_0]$ decays with $q$. Then $\alpha$ can be chosen rather small.
Conclusions

We have shown how Borel summation methods provides an alternate existence theory for N-S equation

With this integral equation (IE) approach, the global existence of NS is implied if known solution to IE has subexponential growth. The solution to integral equation in a finite interval can be computed numerically with errors controlled rigorously

Integral equation in an accelerated variable $q$ expected to show no exponential growth unless there is singularity on the real $t$-axis.

The computation over a finite $[0, q_0]$ interval, gives a better upper bound on growth rate exponent $\alpha$ at $\infty$ and hence ensures a longer existence time $[0, \alpha^{-1/n})$ to 3-D Navier-Stokes.

Unresolved issues include Rigorous control of round-off error and obtaining small enough bounds on truncation error for manageable step size.
Key points in the proof-I

Define norm: \( \| \hat{f}(k, p) \| = \sup_{p \geq 0} e^{-\alpha p} (1 + p^2) \| \hat{f}(\cdot, p) \|_{\mu, \beta} \)

Because of properties

\[
\frac{e^{\alpha p}}{(1 + p^2)} \ast \frac{e^{\alpha p}}{(1 + p^2)} = e^{\alpha p} \int_0^p \frac{ds}{(1 + s^2)[1 + (p - s)^2]} \leq \frac{M_0 e^{\alpha p}}{1 + p^2}
\]

\[
\left[ e^{-\beta |k|} (1 + |k|)^{-\mu} \right] \ast \left[ e^{-\beta |k|} (1 + |k|)^{-\mu} \right] \leq \frac{C_0(\mu)e^{-\beta |k|}}{(1 + |k|)^{-\mu}},
\]

the following algebraic properties follow:

\[
\| [\hat{f}(k, p)] \ast [\hat{g}(k)] \|_{\mu, \beta} \leq C_0 \| \hat{f}(\cdot, p) \|_{\mu, \beta} \| \hat{g} \|_{\mu, \beta}
\]

\[
\| \hat{u} \ast \hat{v} \| \leq M_0 C_0 \| \hat{u} \| \| \hat{v} \| , \quad \| \int_0^p |\hat{u}(k, s)| ds \| \leq C \alpha^{-1} \| \hat{u} \|
\]
Key points in the proof-II

From these relations, it is possible to conclude from the integral equation that if

\[ u(p) \equiv \|\hat{U}(., p)\|_{\mu, \beta}, \quad a = \|\hat{v}_0\|_{\mu, \beta}, \quad u^0(p) = \|\hat{U}^{(0)}(., p)\|_{\mu, \beta}, \]

then

\[ u(p) \leq \frac{C}{\sqrt{\nu p}} \int_0^p [u \ast u + au](s)ds + u^{(0)}(p) \]