QED : The Short Course

Contents

1 Introduction

These notes are not really a complete set. I have presumed that you have seen quantum electrodynamics (QED) before, in one form or another. This means that I can cover many different things in a rather brief and concise manner, as a way of reminding you of the subject. These notes are, therefore, rather telegraphic! However, more will be said and explained in the lectures. And also outside of the lectures!

The subjects that I have chosen to talk about are, in a sense, basic ideas or techniques that will be used or presumed in later courses. Here, I have tried to emphasize the main points as well as to show you how to calculate. As well as that I thought that the subject would be a bit dry if we did not calculate something testable. Indeed, at the very end I will make a comparison of the magnetic moment of the electron, as predicted by QED, and the experimental value.

In any case everything that I have to say is covered in most text books on field theory. You will find a small list of books that might be useful for your studies at the end of these notes.

If you do not understand something please do not hesitate to ask me or anyone else here. As this is a school the whole point is to participate!

2 The Action and its Symmetries

The Dirac action is

$$S = \int d^4x \left[-\frac{1}{4} F_{0\ \mu\nu} F_0^{\mu\nu} + \overline{\psi}_0 \left(i\partial\!\!\!/ + e_0 A\!\!\!/_0 - m_0 \right) \psi_0 \right], \tag{2.1}$$

where e_0 is the (bare) coupling constant, m_0 is the bare mass and the fields are also "bare". As usual a "slash" means $B\!\!\!/ = \gamma^\mu B_\mu$.

Conventions

My metric is $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. The gamma matrices γ_μ are 4×4 matrices, and they satisfy

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}. \tag{2.2}$$

There are many representations for these but I will not pick on one. One thing we want is that under hermitian conjugation

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0. \tag{2.3}$$

The adjoint spinor $\overline{\psi}_0$ is given by

$$\overline{\psi}_0 = \psi_0^\dagger \gamma_0. \tag{2.4}$$

Do you remember why you need γ_0 in this equation? If not, here is a hint: what happens when you look at Lorentz transformations for the adjoint spinor? If you did remember the reason, then here is another question: what should you do if there are more time directions?

The field strength $F_{0\ \mu\nu} = \partial_\mu A_{0\ \nu} - \partial_\nu A_{0\ \mu}$.

Equations of Motion and Gauge Symmetry

The equations of motion are

$$\partial_\mu F_0^{\mu\nu} = -e_0 \overline{\psi}_0 \gamma^\nu \psi_0 \tag{2.5}$$

$$\left(i\partial\!\!\!/ - m_0 \right) \psi_0 = -e_0 A\!\!\!/_0 \psi_0 \tag{2.6}$$

$$\overline{\psi}_0 \left(i \overleftarrow{\partial\!\!\!/} + m_0 \right) = e_0 \overline{\psi}_0 A\!\!\!/_0 \tag{2.7}$$

By differentiating (2.5) with ∂_μ and using anti-symmetry of $F^{\mu\nu}$ one finds that the current

$$j_0^\mu = e_0 \overline{\psi}_0 \gamma_\mu \psi_0, \tag{2.8}$$

should be conserved

$$\partial_\mu j_0^\mu = 0. \tag{2.9}$$

One can show that this is indeed the case by making use of the other two equations of motion (2.6,2.7). It is gauge invariance that guarantees the compatibility of the equations.

The gauge field A_μ contains redundant, unphysical information. The action enjoys the following symmetry

$$\begin{aligned} A_{0\,\mu}(x) \to A'_{0\,\mu}(x) &= A_{0\,\mu}(x) + \frac{1}{e_0}\partial_\mu \lambda(x) \\ \psi_0(x) \to \psi_0(x)' &= \mathrm{e}^{i\lambda(x)}\psi_0(x) \\ \overline{\psi}_0(x) \to \overline{\psi}_0(x)' &= \overline{\psi}_0(x)\mathrm{e}^{-i\lambda(x)} \end{aligned} \tag{2.10}$$

These are called local gauge transformations, since the parameter λ is an arbitrary function (of x).

In fact the current (2.8) is the Noether current associated with the **constant** gauge transformations i.e. those λ which do not depend on the space time position. (Noethers theorem says that you get a conserved current if you have a **global** continuous symmetry).

Notice that it is the fact that the gauge field has an inhomogeneous transformation that ensures that the action is gauge invariant. For each different gauge field in a theory there will be exactly one associated gauge parameter $\lambda(x)$. (In the next course you will see how this works out for non-Abelian groups.)

Covariant Derivatives

In general one may want to construct a gauge invariant action made up of many different types of fields, spinors, scalars, ... We want the gauge invariance with respect to transformations associated with one gauge field[1], say A_μ, (there may be other gauge fields hanging around but we do not care about those at the moment). The way to achieve invariance is to make use of covariant derivatives. Let ϕ_α be a complex 'anything' (scalar, spinor, ...). Define the covariant derivative, ∇_μ on ϕ_α by

$$\nabla_\mu \phi_\alpha := \partial_\mu \phi_\alpha - ieq_\alpha A_\mu \phi_\alpha. \tag{2.11}$$

Gauge transformations are

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \lambda(x), \tag{2.12}$$

$$\phi_\alpha(x) \to \phi_\alpha(x)' = \mathrm{e}^{iq_\alpha \lambda(x)}\phi_\alpha(x). \tag{2.13}$$

[1] For these classical considerations I do not need to talk about "bare" objects so I will drop the zero subscript in this part.

The covariant derivative transforms just like ϕ_α does

$$\nabla_\mu \phi_\alpha(x) \to (\nabla_\mu \phi_\alpha(x))' = \mathrm{e}^{iq_\alpha \lambda(x)} \nabla_\mu \phi_\alpha(x). \tag{2.14}$$

One very important property of the covariant derivative comes from taking the Bianchi identity (which looks just like the Jacobi identity in Lie algebra theory). This comes about as follows. First consider the commutator

$$\left[\nabla_\mu, \nabla_\nu\right] \phi_\alpha(x) = -iq_\alpha e F_{\mu\nu} \phi_\alpha(x). \tag{2.15}$$

Since

$$[\nabla_\mu, [\nabla_\nu, \nabla_\rho]] + [\nabla_\nu, [\nabla_\rho, \nabla_\mu]] + [\nabla_\rho, [\nabla_\mu, \nabla_\nu]] = 0, \tag{2.16}$$

one obtains

$$\nabla_\mu F_{\nu\rho} + \nabla_\nu F_{\rho\mu} + \nabla_\rho F_{\mu\nu} = 0, \tag{2.17}$$

which in 4-dimensions is equivalent to

$$\nabla_\nu \tilde{F}^{\nu\mu} = 0, \tag{2.18}$$

where

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}. \tag{2.19}$$

The Bianchi identity is not a dynamical statement - since we have not made use of the equations of motion - rather it is a kinematical statement. I could also have replaced the covariant derivatives in (2.17) with partial derivatives since the photon carries no charge, but when you come to non-Abelian Yang-Mills theories this is the form that the identity takes.

Now with the covariant derivative in hand one can construct gauge invariant actions easily, for example one could take

$$\int d^4x \ \left(\nabla_\mu \phi_\alpha\right)^\dagger \left(\nabla^\mu \phi_\alpha\right). \tag{2.20}$$

Needless to say, you should **check** all of these things!

Gauge Fixing

The photon has less physical degrees of freedom than the number of components of the gauge field A_μ. Gauge invariance is the redundancy in this description of the physics. We undo this "over counting" by gauge fixing[2]. I will choose a standard covariant gauge fixing namely I will add to the action the following term

$$\int d^4x \ -\frac{1}{2\alpha} (\partial^\mu A_{0\,\mu})^2. \tag{2.21}$$

[2]There is also a mathematical problem, and that is since we have gauge invariance there is no propagator for the gauge field. Put another way, there is no unique solution to the equation (2.5).

Now the total action is no longer gauge invariant since (2.21) transforms, under an infinitesimal gauge transformation, to

$$\int d^4x \; -\frac{1}{\alpha}(\partial^\mu A_{0\,\mu})\partial_\mu\partial^\mu\lambda. \tag{2.22}$$

In fact the Maxwell equation (2.5) takes on a very simple form when $\alpha = 1$,

$$\partial_\mu\partial^\mu A_{0\,\nu} \;=\; -e_0\overline{\psi}_0\gamma_\nu\psi_0. \tag{2.23}$$

Feynman Rules

So much for the classical theory, now its time to go quantum. One can read off the Feynman rules from the action (2.1) and (2.21), (a 0 superscript here means tree level)

- The photon propagator is

$$\begin{aligned} D^{(0)}_{\mu\nu}(p) &= \int d^4x\, \mathrm{e}^{ip.(x-y)} \langle 0|\, T\,(A_{0\,\mu}(x)A_{0\,\nu}(y))\,|0\rangle \\ &= \left(\eta_{\mu\nu} - (1-\alpha)\frac{p_\mu p_\nu}{p^2}\right)\frac{i}{p^2+i\epsilon} \end{aligned} \tag{2.24}$$

- The electron propagator is

$$\begin{aligned} S^{(0)}_F(p)_\alpha{}^\beta &= \int d^4x\, \mathrm{e}^{ip.(x-y)} \langle 0|\, T\left(\psi_{0\,\alpha}(x)\overline{\psi}{}^\beta_0(y)\right)|0\rangle \\ &= \left(\frac{i}{\not p - m_0 + i\epsilon}\right)_\alpha^{\;\beta} \end{aligned} \tag{2.25}$$

  Notice that there is a direction running from y to x that is from $\overline{\psi}_0$ to ψ_0. It is easiest to keep track of this by keeping tabs on the up and down labels.

- The vertex is

$$ie_0\,(\gamma_\mu)_\alpha^{\;\beta} \tag{2.26}$$

  (times a momentum conservation delta function).

- For every closed fermionic loop multiply by a factor of -1.

- Integrate over every independent (read loop) momentum p with the measure

$$\int \frac{d^4p}{(2\pi)^4}. \tag{2.27}$$

- Multiply the diagram by a symmetry factor.

- Sum over all possible topologies of diagrams.

S Matrix Rules

To pass from Feynman diagrams to S-Matrix elements there is a simple algorithm. One does the following, in the order given,

- Amputate all external legs, that is remove all external propagators, (or equivalently, multiply each external leg by the inverse propagator).
- Put all the external momenta on shell.

Then, on the fermion legs that you have cut off, for particles with momentum p and spin s, you multiply by

- $u_\alpha(p,s)$ for an incoming electron.
- $\overline{u}^\alpha(p,s)$ for an outgoing electron.
- $\overline{v}^\alpha(p,s)$ for an incoming positron.
- $v_\alpha(p,s)$ for an outgoing positron.

On the amputated photon legs, for photons with momentum k and helicity λ, you multiply by

- $\epsilon_\mu(k,\lambda)$ for an incoming photon.
- $\epsilon^*_\mu(k,\lambda)$ for an outgoing photon.

Ward Identities

These tell you how **non** gauge invariant your expectation values are. Please note the negative. You have all seen derivations before, so let me briefly derive two Ward (-Takahashi-Green) identities. I will use path integral language but as always you **ought** to rederive these in the operator approach As insisted on by Marco Moriconi). The idea is that if we perform a gauge transformation (2.10) then in the path integral this is thought of as a change of variables[3] so for any theory based on a set of fields Φ we have

$$\begin{aligned}\int D\Phi\, \mathrm{e}^{iS(\Phi)} G(\Phi) &= \int D\Phi\, \mathrm{e}^{iS(\Phi+\delta\Phi)} G(\Phi+\delta\Phi) \\ &= \int D\Phi\, \mathrm{e}^{iS(\Phi)} G(\Phi) \\ &\quad + \int D\Phi\, \mathrm{e}^{iS(\Phi)} \left(iG(\Phi)\delta\Phi \frac{\delta S(\Phi)}{\delta\Phi} + \delta\Phi \frac{\delta G(\Phi)}{\delta\Phi}\right), \end{aligned} \qquad (2.28)$$

[3]One may also derive the Dyson-Schwinger equations in this way.

from which we conclude that

$$\int D\Phi\, \mathrm{e}^{iS(\Phi)} \left(iG(\Phi)\delta\Phi \frac{\delta S(\Phi)}{\delta\Phi} + \delta\Phi \frac{\delta G(\Phi)}{\delta\Phi} \right) = 0. \tag{2.29}$$

I presumed that the change of variables Jacobian is unity!

Apply this procedure to

$$\int DA_{0\,\mu} D\psi_0 D\overline{\psi}_0, \mathrm{e}^{iS(A_{0\,\mu}, \psi_0, \overline{\psi}_0)} A_{0\,\mu}(x) \tag{2.30}$$

where the variation of the fields is by gauge transformations (2.10) and the action is the sum of (2.1) and (2.21) to find

$$\langle \frac{i}{\alpha} A_{0\,\mu}(x) \partial_\rho \partial^\rho \partial^\nu A_{0\,\nu}(y) \rangle = \partial_\mu \delta^4(x-y) \tag{2.31}$$

I made use of (2.22).

On the other hand we can do the same thing to

$$\int DA_{0\,\mu} D\psi_0 D\overline{\psi}_0, \mathrm{e}^{iS(A_{0\,\mu}, \psi_0, \overline{\psi}_0)} \psi_{0\,\alpha}(x) \overline{\psi}_0^\beta(y) \tag{2.32}$$

to obtain

$$\frac{1}{\alpha} \Box_z \langle \psi_{0\,\alpha}(x) \partial^\mu A_{0\,\mu}(z) \overline{\psi}_0^\beta(y) \rangle = \langle \psi_{0\,\alpha}(x) \overline{\psi}_0^\beta(y) \rangle \left(\delta(x-z) - \delta(y-z) \right). \tag{2.33}$$

If we Fourier transform these equation we find

$$k^\mu \langle A_{0\,\mu}(k) A_{0\,\nu}(q) \rangle = -\alpha \frac{k_\nu}{k^2 + i\epsilon} \delta^4(k+q), \tag{2.34}$$

and

$$k^\mu \langle \psi_{0\,\alpha}(p) A_{0\,\mu}(k) \overline{\psi}_0^\beta(q) \rangle = \langle \psi_{0\,\alpha}(p) \overline{\psi}_0^\beta(q+k) \rangle - \langle \psi_{0\,\alpha}(p+k) \overline{\psi}_0^\beta(q) \rangle \tag{2.35}$$

When passing to momentum space we usually put in the explicit delta function so that

$$\begin{aligned} \langle A_{0\,\mu}(k) A_{0\,\nu}(q) \rangle &= D_{\mu\nu}(k) \delta^4(k+q) && (2.36) \\ \langle \psi_{0\,\alpha}(p) \overline{\psi}_0^\beta(q) \rangle &= S_\alpha{}^\beta(p) \delta^4(p+q) && (2.37) \\ \langle \psi_{0\,\alpha}(p) A_{0\,\mu}(k) \overline{\psi}_0^\beta(q) \rangle &= (G_{0\,\mu})_\alpha{}^\beta(p,k) \delta^4(p+q+k). && (2.38) \end{aligned}$$

So in momentum space the Ward Identities (finally) take on the simple form

$$\begin{aligned} k^\mu D_{0\,\mu\nu}(k) &= -\alpha \frac{k_\nu}{k^2 + i\epsilon}, && (2.39) \\ k^\mu (G_\mu)_\alpha{}^\beta(p,k) &= \frac{\alpha}{k^2 + i\epsilon} \left(S_0{}_\alpha^\beta(p+k) - S_0{}_\alpha^\beta(p) \right). && (2.40) \end{aligned}$$

Ward Identities at Tree Level

This is an exercise!

The interesting thing about (2.39) is that it tells us that the longitudinal component of the two point function for the photon is always a tree level object. And not only is it tree level, but it is completely a "gauge" fixing object. This is nice since, after all, the physical photon has components transverse to the line of flight. Can we check this to be sure it is true beyond tree level? Well, the at one loop there is only one Feynman diagram for the two point function and the corresponding mathematical expression is (I have cut the external legs off)[4]

$$-1e_0^2 \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\left(\gamma_\mu \frac{1}{\not{p} - m_0 + i\epsilon} \gamma_\nu \frac{1}{\not{p} + \not{k} - m_0 + i\epsilon}\right). \tag{2.41}$$

But we are in a bind! The integral diverges, since for large momenta it goes like

$$\int d^4p/p^2 \sim p^2 \,|^\infty \tag{2.42}$$

A similar story holds for the one and higher loop contributions to the electron propagator and to the three point vertex as well. But, and this is important, infinities to the Delbruck scattering diagram do not start until second loop order.

There is no sense to checking anything with integrals that diverge. In order to get something we will have to "regularise" the integrals. And that is what we will do further on.

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3 Renormalization in a Nutshell

Beyond tree-level Feynman diagrams diverge. And, as I just argued, it makes no sense to talk about these ill defined objects. For the moment, however, presume that we have somehow made sense of all the integrals that appear in the theory. That is, that we have regularized them in some way.

This discussion of infinities has been in terms of the bare fields and bare parameters. The physical fields and physical parameters should describe a well behaved finite field theory. Renormalization is the method which enables us to pass from the bare (yet regularized) theory to the finite theory. The assumption (which can be proved, but I will not do so) is that all we have to do to achieve a well defined theory is to relate the bare variables to the physical (or almost physical) ones by multiplicative factors called renormalization constants.

[4] I have put a factor of -1 in front of this expression. The minus sign comes from the fact that we have a fermion loop. The factor 1 is the symmetry factor and it arises as follows. One gets a factor of $1/2$ from expanding the interaction part of the action out to second order. Then there is another factor of 2 since there are two ways to join the external photon legs to the vertices. There is only one way to join the fermions up, so $2 \times 1/2 = 1$.

Let

$$A_{0\,\mu} = \sqrt{Z_3}\,A_\mu, \quad \psi_0 = \sqrt{Z_2}\,\psi_0 \quad \overline{\psi}_0 = \sqrt{Z_2}\,\overline{\psi}, \tag{3.1}$$

and

$$e_0 = \frac{Z_1}{\sqrt{Z_3}Z_2}\,e, \quad m_0 = \frac{Z_m}{Z_2}\,m, \tag{3.2}$$

be the relationship between the bare quantities and the renormalized ones. The action now takes the form

$$S = \int d^4x \left[-\frac{Z_3}{4} F_{\mu\nu}F^{\mu\nu} + Z_2\,\overline{\psi} i \partial\!\!\!/ \psi + Z_1 e \overline{\psi}\, A\!\!\!/ \psi - Z_m\,\overline{\psi}\, m \psi \right], \tag{3.3}$$

and at the level of Greens functions we have

$$\begin{aligned} &\langle 0 | \mathrm{T}\left(A_\mu \ldots \overline{\psi} \ldots \psi \ldots \right) |0\rangle (e, m) \\ &= Z_3^{-\frac{N_\gamma}{2}} Z_2^{-\frac{N_f}{2}} \langle 0 | \mathrm{T}\left(A_{0\,\mu} \ldots \overline{\psi}_0 \ldots \psi_0 \ldots \right) |0\rangle (e_0, m_0). \end{aligned} \tag{3.4}$$

To complete the specification of the renormalized theory we need to fix the four unknown constants, Z_1, Z_2, Z_3 and Z_m. We can do this in various ways but certainly the most intuitive is to demand that the renormalized theory yields the physical parameters, that is that the renormalized coupling constant e is actually the physical coupling constant and that m is the physical mass of the electron. We will make these choices presently, but for now leave the precise identification open.

There is one more renormalization that we need to make in general and that has to do with the gauge fixing parameter. I will not bother with it here, but it is an important ingredient in non-Abelian theories.

Power Counting and Renormalizability

By saying that the theory is renormalizable simply by the multiplicative renormalization above, (3.1) and (3.2) I am implying that you do not need to "especially" correct for infinities in Greens functions from terms that are, for example, of the Pauli-type

$$\frac{e}{m}\overline{\psi}(x)\sigma_{\mu\nu}\psi(x)F^{\mu\nu}. \tag{3.5}$$

To see why we should be pleased that such a term is not needed, requires some formal machinary, mostly based on dimension counting, that I do not wish to get into here. Rather, let me explain the situation in words and in some generality, so we are not, for the moment, restricted to QED.

Suppose the mass dimension of some correlation function is d. This means that just from the point of view of dimensions it goes like m^d where m is a typical mass in the theory. Also suppose that the coupling constant in the theory, λ, has mass dimension a, so we can swap it with m^a say. Now we consider the correlation function at the N'th order in λ. It will go like

$$\lambda^N I \tag{3.6}$$

where I is the whole bunch of momentum integrals that we need to perform. What is the mass dimension of I? Well we see that it must be $d - Na$, that is $I \sim m^{d-Na}$.

On the other hand, the integral I is really a product of integrals over products of propagators, the propagators give us powers of momenta in numerators, so symbolically we have

$$I \sim \int \frac{(d^D p)^L}{(p^2 - m^2)^P} \sim m^{LD-2P}. \tag{3.7}$$

If I has positive mass dimensions, $LD - 2P > 0$, then clearly integrals of the type (3.7) diverge and the bigger the difference is the more divergent they are! But we worked out that the mass dimension of I, is m^{d-Na}, so if $d - Na > 0$, then we will have a divergent integral. The problem is that if $a < 0$, then the difference is indeed always positive and what is worse as we go to higher orders in perturbation theory, meaning as we increase N, the difference increases, so the divergences are getting worse and worse. Such a theory is said to be non-renormalizable.

Suppose instead that $a > 0$ then, apparently, the situation improves as we increase N. I say apparantly as the discussion has been a little rough, but in any case the feel of the problem is vastly different to that when $a > 0$, and theories with $a < 0$ are said to be super-renormalizable.

The inbetween theories, with $a = 0$, are the renormalizable theories. QED is an example of such a theory. So to answer the question of why we should be happy not have a Pauli-type term to deal with, note that the coupling constant is e/m, which means that it corresponds to a non-renormalizable interaction!

Gauge Symmetry in the Renormalized Theory

Notice that (3.3) is just a rewriting of the original action and so also enjoys the following symmetry

$$\begin{aligned} A_\mu(x) \to A'_\mu(x) &= A_\mu(x) + \partial_\mu \lambda(x) \\ \psi(x) \to \psi(x)' &= \mathrm{e}^{ie\frac{Z_1}{Z_2}\lambda(x)}\psi(x) \\ \overline{\psi}(x) \to \overline{\psi}(x)' &= \overline{\psi}(x)\mathrm{e}^{-ie\frac{Z_1}{Z_2}\lambda(x)}. \end{aligned} \tag{3.8}$$

These local gauge transformations are supposed now to be finite (we are working with the finite theory) and this implies that the ratio $\frac{Z_1}{Z_2}$ is finite. Since there is a gauge symmetry there are associated Ward identities. The most significant for us is

$$k^\mu \Gamma_\mu(p, p-k) = ie\frac{Z_1}{Z_2}\left(S^{-1}(p) - S^{-1}(p-k)\right) \tag{3.9}$$

and, if the Greens functions and the coupling constants are finite objects, this again implies that the ratio of Z_1 to Z_2 is finite.

Renormalization Conditions

As already mentioned we need to impose some conditions on the theory in order to specify the renormalization constants. Indeed we will let nature dictate the choices. So we want e to be the physical charge and m to be the physical mass of the electron. What do these conditions mean?

We begin with the mass. In field theory the physical mass of a fundamental particle (which is represented by a field) is the the pole of the propagator or, equivalently, the zero of the inverse propagator. Write for the complete inverse electron propagator

$$S^{-1}(p, m) = -i\left(\not{p} - m - \Sigma(p, m)\right), \tag{3.10}$$

so that our first condition is that[5]

$$\Sigma(p, m)\big|_{\not{p}=m} = 0. \tag{3.11}$$

This helps to determine Z_m. The next condition is that the residue of the pole be unity

$$\frac{d}{d\not{p}}\Sigma(p, m)\bigg|_{\not{p}=m} = 0, \tag{3.12}$$

which now allows us to fix Z_2.

In the case of the photon (as long as we are in four dimensions) the complete propagator takes the form

$$D_{\mu\nu}(k^2) = -i\frac{(\eta_{\mu\nu} - k_\mu k_\nu/k^2)}{k^2\left(1 - \hat{\Pi}(k^2)\right)} - i\alpha\frac{k_\mu k_\nu/k^2}{k^2 + i\epsilon} \tag{3.13}$$

which always has a pole at $k^2 = 0$. I have imposed the Ward identity, at the renormalized level, to write the propagator in this way. We need to fix the residue so that at $k^2 = 0$, the residue of the physical part is unity, this means that we take

$$\hat{\Pi}(0) = 0, \tag{3.14}$$

[5]OK. The equation that follows is nonsense as it stands. You can never let $\not{p} = m$ just like that. Take the trace of both sides to see that this is not possible. To make sense of it multiply Σ against $u(p, m)$ where $(\not{p} - m)u(p, m) = 0$. Mostly you will never see the equations with the spinors in there, so I have also opted not to do it, but don't forget that they are there.

which fixes Z_3.

Finally, Z_1 is determined by requiring that

$$\Gamma_\mu(p,p) = e\gamma_\mu. \tag{3.15}$$

Renormalization Conditions Imply $Z_1 = Z_2$

With the choices that we have made the Ward Identity (3.8) implies that $Z_1 = Z_2$. Lets see how this comes about. First differentiate the Ward identity with respect to k^μ to obtain

$$\Gamma_\mu(p,p-k) + k^\nu \frac{\partial}{\partial k^\mu}\Gamma_\nu(p,p-k) = -ie\frac{Z_1}{Z_2}\frac{\partial}{\partial k^\mu}S^{-1}(p-k), \tag{3.16}$$

which at $k=0$ is the "differential" Ward identity

$$\Gamma_\mu(p,p) = ie\frac{Z_1}{Z_2}\frac{\partial}{\partial p^\mu}S^{-1}(p). \tag{3.17}$$

Evaluate this expression at $\not{p} = m$, to obtain

$$\Gamma_\mu(p,p)\big|_{\not{p}=m} = ie\frac{Z_1}{Z_2}\frac{\partial}{\partial p^\mu}\, S^{-1}(p)\Big|_{\not{p}=m} \tag{3.18}$$

In order to make a little sense of this write,

$$\Sigma(p) = C(p^2,m^2)\not{p} - D(p^2,m^2)m \tag{3.19}$$

and note that the conditions (3.11) and (3.12), imply that

$$C(m^2,m^2) - D(m^2,m^2) = 0, \tag{3.20}$$

and

$$C(m^2,m^2) + 2m^2\frac{d}{dp^2}\Big(C(p^2,m^2) - D(p^2,m^2)\Big)\Big|_{p^2=m^2} = 0. \tag{3.21}$$

Now,

$$\frac{\partial}{\partial p^\mu}\, S^{-1}(p)\Big|_{\not{p}=m} = -i\gamma_\mu + i\frac{\partial}{\partial p^\mu}\, \Sigma(p)|_{\not{p}=m}\,, \tag{3.22}$$

and (this is now an exercise) one has that

$$\frac{\partial}{\partial p^\mu}\, \Sigma(p)|_{\not{p}=m} = 0, \tag{3.23}$$

to obtain

$$\frac{\partial}{\partial p^\mu}\, S^{-1}(p)\Big|_{\not{p}=m} = -i\gamma_\mu. \tag{3.24}$$

Putting all the pieces together, and making use of (3.15) we find that (3.18) amounts to

$$e\gamma_\mu = e\frac{Z_1}{Z_2}\gamma_\mu, \tag{3.25}$$

from which we obtain

$$Z_1 = Z_2. \tag{3.26}$$

On Shell

Let

$$\bar{\Gamma}_\mu(p,p') = \bar{u}(p')\Gamma_\mu(p,p')u(p). \tag{3.27}$$

Then the Ward identity becomes

$$k^\mu\bar{\Gamma}_\mu(p,p') = \bar{u}(p')\left[S^{-1}(p') - S^{-1}(p)\right]u(p). \tag{3.28}$$

It is quite useful also to consider this Ward identity for on shell electrons, $p^2 = m^2$ and $p'^2 = m^2$. Using the general form of the inverse electron propagator

$$S(p)^{-1} = \not{p} - m - \Sigma(p), \tag{3.29}$$

and the normalization condition $\Sigma(\not{p} = m) = 0$ in (3.28) we find

$$k^\mu\bar{\Gamma}_\mu(p,p) = 0. \tag{3.30}$$

This means that the 'physical' current, $\bar{\Gamma}_\mu(p,p')$ is conserved, even if the photon is "off"-shell.

The Renormalized Coupling e

Since, with the prescription that we have chosen, $Z_1 = Z_2$, we learn that

$$e = \sqrt{Z_3}\, e_0. \tag{3.31}$$

This tells us that the difference between the bare charge and the physical charge rests entirely in the renormalization of the gauge field. This is a particularly satisfying result since it tells us that we do not need to know how the electron is being dressed (corrected).

4 Dimensional Regularization

We saw in the last two lectures that the Feynman diagrams involving loops lead to divergent integrals. One way handle such integrals is simply to cut them off, giving them finite upper bounds. This way of regularizing infinite integrals is not very convenient in dealing with theories which have local symmetries. This inconvenience comes about because such a procedure does not respect gauge invariance. Ideally we would prefer to adopt a regularization scheme which respects all of the symmetries of the classical action. In general this is not possible. However, there are regularization schemes which preserve the local symmetries of the classical action. Dimensional regularization is the most convenient of them.

The local gauge symmetry is not dimension dependent. One can formulate gauge theories in any number of space time dimensions. However, the degree of divergence of a loop integral strongly depends on the space-time dimension. The main idea of dimensional regularization is to evaluate our integrals in a space-time in which they converge. The result will turn out to be an analytic function of the space-time dimension D with possible poles at integer values of D for some divergent integrals.

One of the aims in this lecture is to explain these ideas. Another thing that I think is important to mention, but which I shall not go into is that dimensional regularization is not just a regularization of diagrams. Rather, dimensional regularization can be viewed as a prescription that defines the complete theory. One then adopts the view that space-time **is** D dimensional. Relativistic invariance is guaranteed since one uses a D-dimensional Lorentz invariant action to define the theory. Gauge invariance is guaranteed, at least for QED, since that D-dimensional action is invariant (subtleties arise in the standard model because of the presence of γ_5).

Dimensionful Couplings

The mass dimension of the coupling constant (charge) in four dimensions is zero. However as we change dimensions the mass dimension of the charge varies. In fact it goes like

$$(\mu^2)^{\frac{4-D}{2}} \tag{4.1}$$

where μ^2 is some arbitrary mass squared scale parameter.

Performing Integrals

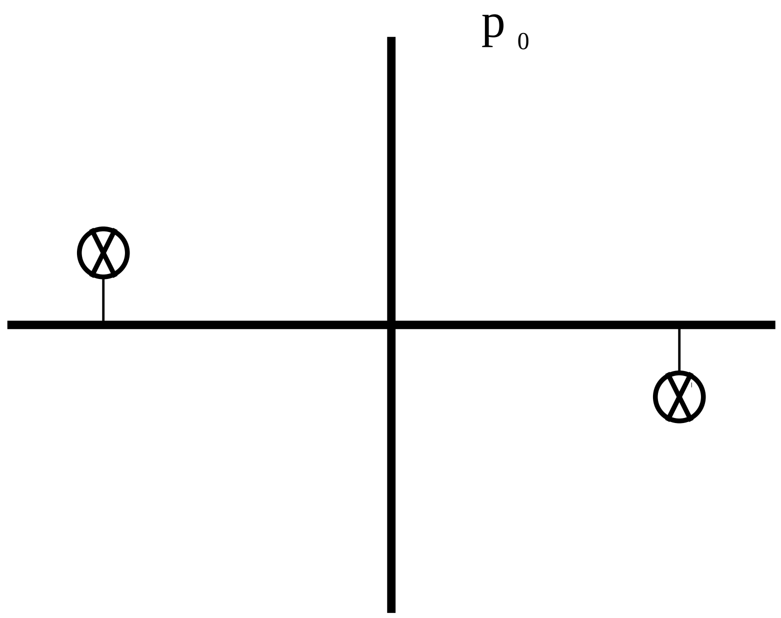
The type of integrals that we are going to encounter will be of the form

$$I(r,s,D) = \int \frac{d^D p}{(2\pi)^D} \frac{(p^2)^r}{(p^2 - M^2 + i\epsilon)^s}. \tag{4.2}$$

The $i\epsilon$ prescription tell us how to avoid poles in the complex p_0 plane. The p_0 integration can be performed with the help of Cauchy's residue theorem. The poles in the complex p_0-plane are located at

$$p_0 = \pm\sqrt{\underline{p}^2 + M^2} \mp i\epsilon. \tag{4.3}$$

The square root is manifestly real when $\underline{p}^2 + M^2 \geq 0$ and the singularities of the p_0 integration are located as shown in the figure below.



The p_0 Integration Plane

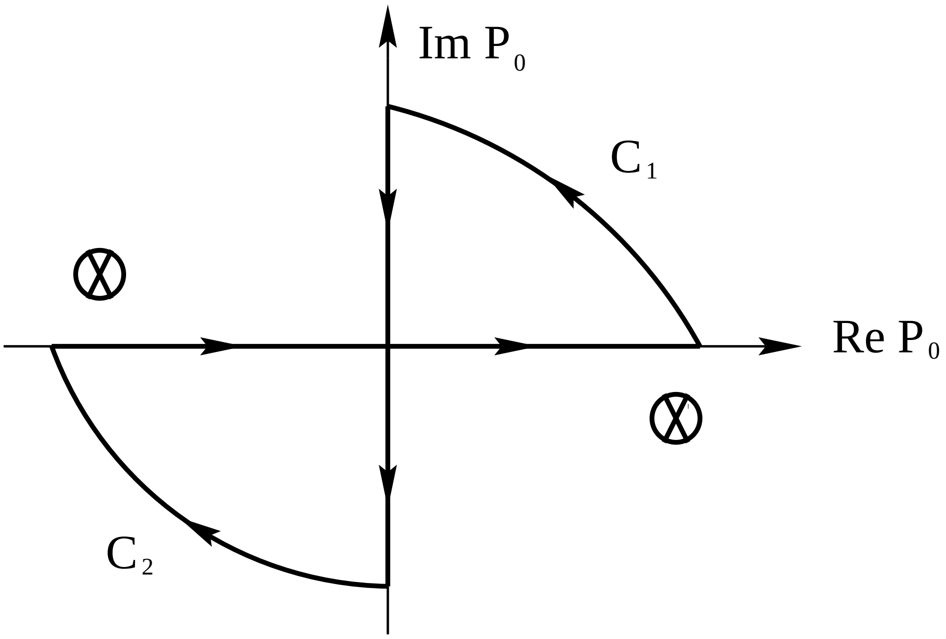
Consider the following identity

$$\begin{aligned} 0 &= \int_C \frac{dp_0}{(2\pi)} f(p_0, \underline{p}) \\ &= \left(\int_{-\infty}^{\infty} + \int_{C_1} + \int_{i\infty}^{-i\infty} + \int_{C_2}\right) \frac{dp_0}{(2\pi)} f(p_0, \underline{p}) \end{aligned} \tag{4.4}$$

where the contour C is chosen as indicated below, and therefore it does not enclose any of the poles of the integrands.

We shall eventually let the radii of C_1 and C_2 go to infinity. In this limit, providing the integrand decays fast enough at infinity, one obtains

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{dp_0}{(2\pi)} f(p_0, \underline{p}) \\ &= \int_{-i\infty}^{+i\infty} \frac{dp_0}{(2\pi)} f(p_0, \underline{p}) \\ &= i \int_{-\infty}^{+\infty} \frac{dp_D}{(2\pi)} f(ip_D, \underline{p}). \end{aligned} \tag{4.5}$$



The Contour Used In The Integration

where in the last step a change of integration variable has been performed

$$p_0 = ip_D, \tag{4.6}$$

(that is we Euclideanised the integral).

Our task is to calculate integrals of the form (4.2), which we do by rotating the contour, as we have just done, so

$$I(r,s,D) = i(-1)^{(r+s)} \int \frac{d^D p_E}{(2\pi)^D} \frac{(p_E^2)^r}{(p_E^2 + M^2)^s} \tag{4.7}$$

where $p_E^2 = p_1^2 + \cdots + p_{D-1}^2 + p_D^2$ is the length square of a D-dimensional Euclidean vector. Note that we also dropped the $i\epsilon$, because it is no longer needed.

Note also that the integrand of the D-dimensional integral on the right hand-side of (4.7) has rotational symmetry. The angular integrations can be done. We will be left with a radial integral to perform

$$I(r,s,D)(M^2) = i(-1)^{(r+s)} \frac{\mathrm{Vol}(\mathrm{S_{D-1}})}{(2\pi)^{\mathrm{D}}} \int_0^\infty dt\, \frac{t^{D-1+2r}}{(t^2 + M^2)^s} \tag{4.8}$$

where $\mathrm{Vol}(\mathrm{S_{D-1}})$ represents the surface area of a unit sphere in D-dimensional Euclidean space R^D. It is given by

$$\mathrm{Vol}(\mathrm{S_{D-1}}) = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \tag{4.9}$$

The Γ-function entering in this equation is defined by

$$\Gamma(\varepsilon) = \int_0^\infty dt\, t^{\varepsilon-1} e^{-t} \qquad \mathrm{Re}\ \varepsilon > 0. \tag{4.10}$$

The most important property of this function is that it can be analytically continued to the entire complex ε- plane. It has poles at $\varepsilon = 0$ as well as at negative integer values of ε.

The radial integral in (4.8) can be performed with the help of a general formula

$$\int_0^\infty \frac{u^\alpha \; du}{(u+a)^\beta} = a^{\alpha+1-\beta} \frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha-1)}{\Gamma(\beta)}. \tag{4.11}$$

After a little algebra we find that

$$\begin{aligned} &I(r,s,D)(M^2) \\ &= \frac{i(-1)^{(r+s)}}{(4\pi)^{D/2}}.\left(M^2\right)^{D/2+r-s}.\frac{\Gamma(r+D/2)\Gamma(s-r-D/2)}{\Gamma(D/2)\Gamma(s)}. \end{aligned} \tag{4.12}$$

Feynman Parameterization

To proceed we also need to introduce a simple formula of Feynman

$$\frac{1}{ab} = \int_0^1 dx \; \frac{1}{[ax+(1-x)b]^2}. \tag{4.13}$$

The proof of this is not hard, here is one way to do it: note that

$$\frac{1}{[ax+(1-x)b]^2} = \frac{1}{b-a}\frac{\partial}{\partial x}\frac{1}{[ax+(1-x)b]} \tag{4.14}$$

substituting this into the right hand side of (4.13) one can perform the integral to obtain

$$\frac{1}{b-a}.\left.\frac{1}{[ax+(1-x)b]}\right|_0^1 = \frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{ab} \tag{4.15}$$

which indeed is the left hand side of (4.13).

5 1-loop Correction to the Photon 2-Point Function

This one of the simplest contexts in which we can introduce the method of dimensional regularization. In a previous lecture we saw that it is given by

$$-1e_0^2 \int \frac{d^4p}{(2\pi)^4} \; \mathrm{Tr}\left(\gamma_\mu \frac{1}{\not p - m_0 + i\epsilon}\gamma_\nu \frac{1}{\not p + \not k - m_0 + i\epsilon}\right). \tag{5.1}$$

In 4-space-time dimensions this integral diverges (at least appears to) quadratically in the momenta. It is, however, convergent if the space-time dimension is smaller than two. We thus consider a D-dimensional integral

$$\Pi^D_{\mu\nu}(k) = -e_0^2 \int \frac{d^Dp}{(2\pi)^D} \; \mathrm{Tr}\left(\gamma_\mu \frac{1}{\not p - m_0 + i\epsilon}\gamma_\nu \frac{1}{\not p + \not k - m_0 + i\epsilon}\right). \tag{5.2}$$

where now k^2 stands for

$$k^2 = k_0^2 - k_1^2 - \cdots - k_{D-1}^2.$$

We evaluate (5.2) for those values of D for which the integral makes sense. The result will then be analytically continued in D. The quadratic singularity[6] at $D = 4$ will manifest itself as a simple pole.

Checking the Ward Identity

Now that we have a regularized, meaning well defined, integral we can see if the Ward Identity (2.39) holds. In any case for us it means that $k^\mu \Pi^D_{\mu\nu}(k) = 0$. We get

$$k^\mu \Pi^D_{\mu\nu}(k) = -e_0^2 \int \frac{d^D p}{(2\pi)^D} \ \mathrm{Tr} \left(\not{k} \frac{1}{\not{p} - m_0 + i\epsilon} \gamma_\nu \frac{1}{\not{p} + \not{k} - m_0 + i\epsilon} \right), \tag{5.3}$$

now use the fact that $\not{k} = (\not{p} + \not{k} - m_0 + i\epsilon) - (\not{p} - m_0 + i\epsilon)$ to obtain

$$\begin{aligned} k^\mu \Pi^D_{\mu\nu}(k) &= -e_0^2 \int \frac{d^D p}{(2\pi)^D} \ \mathrm{Tr} \left(\gamma_\nu \frac{1}{\not{p} + \not{k} - m_0 + i\epsilon} - \gamma_\nu \frac{1}{\not{p} - m_0 + i\epsilon} \right) \\ &= 0, \end{aligned} \tag{5.4}$$

by shifting integration variables in the first integral in the second line[7]. Great, the Ward identity holds, and we did not have to perform an integral to show it.

But you also get something back. We now know that

$$\Pi^D_{\mu\nu}(k) = \left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi^D(k^2), \tag{5.5}$$

for some $\Pi^D(k^2)$. In fact, by contracting (5.5) with the metric we get

$$\Pi^D(k^2) = \eta^{\mu\nu} \Pi^D_{\mu\nu}(k)/(D-1), \tag{5.6}$$

a fact that we we will use below.

Here is How it Works

Use the 'Feynman parameterization' (4.13) to rewrite $\Pi^D_{\mu\nu}$ in a more convenient form, namely,

$$\begin{aligned} \Pi^D_{\mu\nu}(k) &= -e_0^2 \int_0^1 dx \int \frac{d^D p}{(2\pi)^D} \ \mathrm{Tr} \ (\gamma_\mu(\not{p} + (1-x)\not{k} - m_0)\gamma_\nu(\not{p} - x\not{k} - m_0)) \\ &\quad \cdot \frac{1}{(p^2 + x(1-x)k^2 - m_0^2 + i\epsilon)^2}. \end{aligned} \tag{5.7}$$

In obtaining this formula we also shifted the integration variable p to $p - xk$. Now use the γ_μ rules,

[6] We will see that the situation is slightly better than this.

[7] We are allowed to make this shift because everything in sight is now well defined. If we had not regularized then we would not be able to trust these formal manipulations.

γ Identities

Using the basic anticommutation rules of the γ_μ matrices one can derive the following identities

$$\begin{aligned}
\gamma_\lambda\gamma_\mu\gamma^\lambda &= (2-D)\gamma_\mu, &(5.8)\\
\gamma_\lambda\gamma_\mu\gamma_\nu\gamma^\lambda &= (D-4)\gamma_\mu\gamma_\nu + 4\delta_{\mu\nu}, &(5.9)\\
\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho\gamma^\lambda &= (6-D)\gamma_\mu\gamma_\nu\gamma_\rho - 4\left(\gamma_\mu\delta_{\nu\rho} - \gamma_\nu\delta_{\mu\rho} + \gamma_\rho\delta_{\mu\nu}\right), \\
&= -2\gamma_\rho\gamma_\nu\gamma_\mu + (4-D)\gamma_\mu\gamma_\nu\gamma_\rho, &(5.10)
\end{aligned}$$

and so on.

This is How it Works Ctd.

to write the integral for $\Pi^D(k^2)$ instead

$$(D-1)\Pi^D(k) = -e_0^2\int_0^1 dx \int \frac{d^Dp}{(2\pi)^D}\ \mathrm{Tr}\,(\mathbb{I})\,\frac{(2-D)[p^2 - x(1-x)k^2] + Dm^2}{(p^2 + x(1-x)k^2 - m_0^2 + i\epsilon)^2}. \quad (5.11)$$

The poles are in the imaginary p_0 plane are at

$$p_0 = \pm\sqrt{-x(1-x)k^2 + \underline{p}^2 + m_0^2}\ \mp i\epsilon. \quad (5.12)$$

For k space-like, i.e. $k^2 < 0$, or light-like $k^2 = 0$, the square root is manifestly real and the singularities of the p_0 integration are located as shown in the figure (4). In this case we can easily perform the p-integrations. The value of $\Pi^D_{\mu\nu}(k^2)$ for non space-like or null-like k 's can then be obtained by analytic continuation in k^2.

Write the numerator in the integrand of (5.11) as

$$\begin{aligned}
&(2-D)[p^2 - x(1-x)k^2] + Dm^2 \\
&\quad = (2-D)[p^2 + x(1-x)k^2 - m_0^2] + 2m^2 - 2(2-D)x(1-x)k^2, \quad (5.13)
\end{aligned}$$

so that

$$\begin{aligned}
(D-1)\Pi^D(k) &= -ie_0^2\int_0^1 dx\ \Big((2-D)I(0,1)(M^2) \\
&\qquad + [2m^2 - 2(2-D)x(1-x)k^2]I(0,2)(M^2)\Big), \quad (5.14)
\end{aligned}$$

where

$$M^2 = m_0^2 - x(1-x)k^2. \quad (5.15)$$

Now use the "Wick rotated" formulae for the integrals, to obtain,

$$\Pi^D(k) = -\frac{2ie_0^2}{(4\pi)^{D/2}}\,\mathrm{Tr}\,(\mathbb{I})\int_0^1 dx\,\Gamma(2-D/2)\frac{x(1-x)k^2}{(M^2)^{2-D/2}}. \quad (5.16)$$

The Lurking Infinity!

So we have seen clearly that Π^D is a meromorphic function of D with well defined singularities residing at the positions of the poles of the Γ functions. This is where that infinity we saw, right at the very beginning, has been moved to.

The pole structure of $\Gamma(\epsilon)$ follows immediately from the basic property

$$\Gamma(\epsilon+1) = \epsilon\,\Gamma(\epsilon) \tag{5.17}$$

which follows from (4.10) by an integration by parts. From this relation we see that

$$\Gamma(\epsilon)|_{\epsilon\to 0} \to \frac{1}{\epsilon} + \ldots\quad . \tag{5.18}$$

The ellipses ... in this equation indicate a constant plus terms of first order and higher in ϵ, i.e,

$$\Gamma(\epsilon)|_{\epsilon\to 0} \to \frac{1}{\epsilon} - \gamma + \emptyset(\epsilon) \tag{5.19}$$

where $\gamma = 0.5772...$ is the Euler number.

Define ϵ by

$$\epsilon = 4 - D. \tag{5.20}$$

And clearly the expression (5.16) diverges as

$$\Gamma(\epsilon/2) = \frac{2}{\epsilon} - \gamma +\ 0(\epsilon). \tag{5.21}$$

Renormalization

The physical renormalization condition is that the residue of the pole of the photon propagator should be unity. That is

$$\hat{\Pi}(0) = 0, \tag{5.22}$$

so we need the relationship between Π^D and $\hat{\Pi}$. This is straightforward to work out

$$\hat{\Pi}(k^2) = -i\frac{\Pi^D(k^2)}{k^2} + (1 - Z_3) + \ldots . \tag{5.23}$$

The renormalization condition then tells us that

$$Z_3 = 1 - \frac{2e_0^2}{(4\pi)^{D/2}}\,\mathrm{Tr}\,(\mathbb{I}) \int_0^1 dx\ \Gamma(2 - D/2)\,\frac{x(1-x)}{(m^2)^{2-D/2}}. \tag{5.24}$$

The renormalization constant has an infinite part,

$$-\frac{2\alpha_0}{3\pi}\frac{1}{\epsilon}, \tag{5.25}$$

where $\alpha_0 = e_0^2/4\pi$, as well as a non-trivial finite piece at one loop order.

To the order that we are working we conclude that

$$\hat{\Pi}(k^2) = -\frac{2e^2}{(4\pi)^{D/2}}\,\mathrm{Tr}\,(\mathbb{I})\int_0^1 dx\;\Gamma(2-D/2)\,x(1-x)\;\left[\frac{1}{(M^2)^{2-D/2}} - \frac{1}{(m^2)^{2-D/2}}\right] \tag{5.26}$$

Now expand,

$$\begin{aligned}\Gamma(\epsilon/2).\left[\frac{1}{(M^2)^{2-D/2}} - \frac{1}{(m^2)^{2-D/2}}\right] &= \Gamma(\epsilon/2).\frac{\epsilon}{2}.\ln\left(\frac{m^2}{M^2}\right)\\ &= \ln\left(\frac{m^2}{M^2}\right) + 0\,(\epsilon)\,.\end{aligned} \tag{5.27}$$

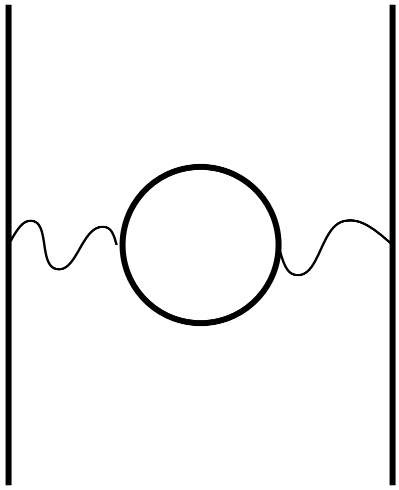
Putting this back into the definition of $\hat{\Pi}(k^2)$, we find that

$$\begin{aligned}\hat{\Pi}(k^2) &= -\frac{2e^2}{(4\pi)^{D/2}}\,\mathrm{Tr}\,(\mathbb{I})\int_0^1 dx\,x(1-x)\ln\left(\frac{m^2}{M^2}\right) + 0\,(\epsilon)\\ &= -\frac{2\alpha}{\pi}\int_0^1 dx\,x(1-x)\ln\left(\frac{m^2}{M^2}\right) + 0\,(\epsilon)\,.\end{aligned} \tag{5.28}$$

Notice that I did not have to tell you what $\mathrm{Tr}\,(\mathbb{I})$ is away from 4-dimensions and that also we would have recovered the same answer even if we had used $(4\pi)^2$ rather than $(4\pi)^{D/2}$ in the denominator. So from this point of view the renormalized quantities are rather robust (at one loop) and do not really depend too much on the details of the continuation in dimension.

Effective Charge

At one loop, of all the diagrams that can arise for the usual scattering of electrons the only one which is relevant for questions of what the physical effective charge will be is the following one. (The others are there to ensure that the renormalization goes through correctly, as well as to produce other physical effects, of course!)



The Effective Charge

At tree level the photon propagator times the vertices goes like

$$\frac{\alpha}{k^2} \tag{5.29}$$

while the corresponding object for small k^2 arising from including the above diagram is

$$\frac{\alpha_{\text{eff}}(k^2)}{k^2} \tag{5.30}$$

where

$$\alpha_{\text{eff}}(k^2) = \frac{\alpha}{1 - \hat{\Pi}(k^2)}. \tag{5.31}$$

6 The Three Point Function at One Loop

By making use of the Feynman rules we can determine the three point function Γ_μ up to any order in perturbation theory. That is in theory, in practice, to actually compute the loop corrections is quite a daunting task. Even so, brave souls have calculated certain quantities up to some high loop order. Here, we will content ourselves with a one loop calculation of the basic three point vertex. This will give us two pieces of information. Firstly, the finite part will allow us to determine the magnetic moment of a spin 1/2 particle up to this order. Secondly, the infinite part will require us to recall the ideas once more of renormalization.

Definition Of Charge

Before running off to calculate let us ask firstly what we mean by charge. Charge is something that can be measured in an experiment, (Coulombs law for example). In field theory charge is the expectation value of the charge operator, Q, and that operator, is given to us by Noethers theorem

$$Q = \int d^3x \, J_0. \tag{6.1}$$

Conservation of the current translates into a statement that we saw before,

$$k^\mu \bar{u}(p')\Gamma_\mu(p',p)u(p)\big|_{\not{p}=\not{p}'=m} = 0, \tag{6.2}$$

and the charge is then given in terms of the zero component of

$$\bar{\Gamma}_\mu(p,p') = \bar{u}(p')\Gamma_\mu(p',p)u(p), \tag{6.3}$$

with the electrons on shell and with zero incoming photon momentum ($p' = p$ and $\not{p} = m$). Actually our choice of renormalization condition was designed so that this would be the case.

In the following we will calculate the three point vertex in a particular limit. The limit will be where the photon momentum (on the external photon leg) is very small. That is, the photon is producing a soft or weak electromagnetic field. This simplification will allow us to perform the calculations in a straightforward manner. Fortunately, one does not lose much predictive power this way. As an example, we will calculate the one loop anomalous magnetic moment of the electron. Also, as far as renormalization is concerned, this is exactly the right limit in any case. To make life even simpler, we will not work with $\Gamma_\mu(p,p')$ but rather with $\bar{\Gamma}_\mu$. As far as S-matrix elements are concerned this is what we want anyway, so there is no loss in doing this. For the purposes of renormalization this is still fine, since we only want to fix one renormalization constant with this calculation namely Z_1. I will point out the simplifications as we go along.

The Renormalized Vertex

In perturbation theory we will have (after renormalizing) a physical charge e expansion of the 3-point vertex form

$$\Gamma_\mu(p',p) = e\gamma_\mu + \sum_{n=1}^{\infty} e^{2n+1}\Gamma_\mu^n(p',p), \tag{6.4}$$

where the $\Gamma_\mu^n(p',p)$ are matrix functions of the momenta and mass. Recall that the relationship between the renormalized and non-renormalized 3-vertex is

$$\begin{aligned} \Gamma_\mu(p,p') &= \sqrt{Z_3} Z_2 \Gamma_{0\,\mu}(p,p') \\ &= Z_1 e \left(\gamma_\mu + e_0^2 \Gamma^1_{0\,\mu}(p',p) + \dots\right). \end{aligned} \tag{6.5}$$

Up to one loop order we will have that $Z_1 = 1 + e^2 Z_1^1 + \dots$, using this and the renormalization condition in (6.5) we obtain

$$\Gamma^1_\mu(p,p') = \Gamma^1_{0\,\mu}(p,p') - \Gamma^1_{0\,\mu}(p,p')\Big|_{\not{p}=\not{p}'=m}, \tag{6.6}$$

and

$$Z_1^1 \gamma_\mu = -e^2\, \Gamma^1_{0\,\mu}(p,p')\Big|_{\not p=m,\not p'=m\, k=0}. \tag{6.7}$$

A Basis of Matrices Using γ

Any 4×4 matrix can be expanded in a basis of γ matrices. There are 16 components to an arbitrary 4×4 matrix so we will need 16 γ and fortunately we have exactly that many that are independent. These are

$$\mathbb{I}, \quad \gamma_5, \quad \gamma_\mu, \quad \gamma_5\gamma_\mu, \text{ and } \sigma_{\mu\nu}. \tag{6.8}$$

To be independent means that one can uniquely expand any matrix as

$$M = a_0\mathbb{I} + a_5\gamma_5 + a^\mu\gamma_\mu + a_5^\mu\gamma_5\gamma_\mu + a^{\mu\nu}\sigma_{\mu\nu}, \tag{6.9}$$

for some coefficients a. To get back the coefficients just take the trace of M with the corresponding matrix, e.g.

$$a^\mu = \frac{1}{4}\,\mathrm{Tr}\,(\gamma^\mu M)\,. \tag{6.10}$$

Instead of $\gamma_5\gamma_\mu$ (which is a pseudo-vector) one may alternatively use $\gamma_{[\mu}\gamma_\nu\gamma_{\lambda]}$ with the braces [] indicating total antisymmetry.

We can also expand out the $\Gamma^n_\mu(p',p)$ in a basis of γ matrices, however, since QED is parity preserving the $\Gamma^n_\mu(p',p)$ will not depend on γ_5. This means that $\Gamma^n_\mu(p',p)$ will be a function of the matrices $\mathbb{I}$, γ_μ, $\gamma_{[\mu}\gamma_\nu\gamma_{\lambda]}$ and $\sigma_{\mu\nu}$ as well as on the momenta (needed to get the Lorentz labels to come out correctly) and on the mass. We can simplify life further by considering instead

$$\bar\Gamma^n_\mu(p,p') = \bar u(p')\Gamma^n_\mu(p',p)u(p), \tag{6.11}$$

with the electrons on shell. In this case the most general form is

$$\bar\Gamma^n_\mu(k) = \bar u(p')\left[\gamma_\mu\, f^n(k^2) + \frac{1}{2m}\sigma_{\mu\nu}\,k^\nu\, g^n(k^2)\right]u(p). \tag{6.12}$$

Notice that one possible term with the three $\gamma_{[\mu}\gamma_\nu\gamma_{\lambda]}$ matrices has not been explicitly included. The reason for this is that two of the matrices must be contracted with momenta and by using the equations of motion and the standard γ_μ matrix commutation relations one can write such a term as a combination of terms that appear in (6.12). I have smuggled something else in here (while you were not looking). There could also have been a term of the form $\bar u(p')(p+p')_\mu u(p)$ on the right hand side of (6.12). It is certainly allowed since it is consistent with the Ward identity (3.30). Well what happend is that I used the following result, to rewrite such a term as a combination of those already present on the right hand side of (6.12).

Gordons Identity

$$\bar{u}(p')\gamma_\mu u(p) = \frac{1}{2m}\bar{u}(p')\left[(2p_\mu + k_\mu) + i\sigma_{\mu\nu}k^\nu\right]u(p), \tag{6.13}$$

which can be obtained as follows. Since $(\not{p} - m)u(p) = 0$ we have that $\bar{u}(p')\gamma_\mu(\not{p} - m)u(p) = 0$. Likewise, since $\bar{u}(p')(\not{p}' - m) = 0$ we find $\bar{u}(p')(\not{p}' - m)\gamma_\mu u(p) = 0$. Adding the two derived equations together yields (6.13).

Now back to our task, we want to get a handle on the renormalized vertex at the next order of perturbation theory. But, as I have already explained we will, instead, concentrate on the matrix element vertex $\bar{\Gamma}^1_\mu(p, p')$. Renormalization now means that

$$\bar{\Gamma}^1_\mu(p, p) = 0. \tag{6.14}$$

Another way of saying this is that the form factor $f^n(k^2)$ is normalized so that $f^0(0) = 1$, and whenever $n > 0$ $f^n(0) = 0$.

Γ_μ up to One Loop

At one loop $\Gamma_{0\,\mu}$ is, in the Feynman gauge ($\alpha = 1$),

$$\Gamma_{0\,\mu}(p', p) = e_0\gamma_\mu + e_0^3\Gamma^1_{0\,\mu}(p', p), \tag{6.15}$$

where

$$\Gamma^1_{0\,\mu}(p', p) = -i\int \frac{d^4q}{(2\pi)^4}\gamma_\lambda \frac{(\not{p}' - \not{q}) + m}{(p' - q)^2 - m^2 + i\epsilon}\gamma_\mu \frac{(\not{p} - \not{q}) + m}{(p - q)^2 - m^2 + i\epsilon}\gamma^\lambda \frac{1}{q^2 + i\epsilon}. \tag{6.16}$$

Notice also that the integrals which come from terms in the numerator that are proportional to m or to m^2 are finite. Also I have introduced the physical mass into these equations. Logically, this is perhaps not the best way to do things, however, at this order we are not making an error here (since the physical mass will differ from the renormalized mass by a term proportional to e^2).

As before, we will work in the regularized theory by passing to D-dimensions, so that we consider

$$\Gamma^1_{0\,\mu}(p', p) = -i\int \frac{d^Dq}{(2\pi)^D}\gamma_\lambda \frac{(\not{p}' - \not{q}) + m}{(p' - q)^2 - m^2 + i\epsilon}\gamma_\mu \frac{(\not{p} - \not{q}) + m}{(p - q)^2 - m^2 + i\epsilon}\gamma^\lambda \frac{1}{q^2 + i\epsilon}. \tag{6.17}$$

We put this in a more tractable form by using the Feynman parameterization

$$\frac{1}{abc} = 2\int_0^1 dx \int_0^{1-x} dy \, \frac{1}{[a+(b-a)x+(c-a)y]^3}. \tag{6.18}$$

(Try to derive this formula). Let,

$$a = q^2 + i\epsilon \tag{6.19}$$
$$b = (p'-q)^2 + i\epsilon \tag{6.20}$$
$$c = (p'-q)^2 + i\epsilon \tag{6.21}$$

and

$$a+(b-a)x+(c-a)y = q^2 - 2q.(xp'+yp) - r + i\epsilon, \tag{6.22}$$

where

$$r = -xp'^2 - yp^2 + m^2(x+y). \tag{6.23}$$

Consequently (6.17) becomes

$$\Gamma^1_{0\,\mu}(p',p) = -2i\int_0^1 dx \int_0^{1-x} dy \int \frac{d^Dq}{(2\pi)^D}\,\frac{N_\mu(p',p,q)}{[q^2-2q.(xp'+yp)-r+i\epsilon]^3}, \tag{6.24}$$

where

$$N_\mu(p',p,q) = \gamma_\lambda(\not{p}'-\not{q}+m)\gamma_\mu(\not{p}-\not{q}+m)\gamma^\lambda. \tag{6.25}$$

One further change of variables helps matters. Make the substitution $q^\mu \to q^\mu + xp'^\mu + yp^\mu$ in the integral (with unit Jacobian) to obtain,

$$\Gamma^1_{0\,\mu}(p',p) = -2i\int_0^1 dx \int_0^{1-x} dy \int \frac{d^Dq}{(2\pi)^D}\,\frac{N_\mu(p',p,q+t)}{[q^2-t^2-r+i\epsilon]^3}, \tag{6.26}$$

where

$$t^\mu = xp'^\mu + yp^\mu. \tag{6.27}$$

Furthermore we can now drop terms odd in q in the integrand of (6.26) since they will vanish on integrating (this is a consequence of the fact that the integral would change sign under $q \to -q$).

Infinite and Finite Parts

We decompose $\Gamma^1_{0\,\mu}$ as

$$\Gamma^1_{0\,\mu}(p',p) = \Lambda^{(0)}_{0\,\mu}(p',p) + \Lambda^{(2)}_{0\,\mu}(p',p), \tag{6.28}$$

where

$$\Lambda^{(0)}_{0\,\mu}(p',p) = -2i\int_0^1 dx \int_0^{1-x} dy \int \frac{d^Dq}{(2\pi)^D}\,\frac{N^{(0)}_\mu(p',p,t)}{[q^2-t^2-r+i\epsilon]^3}, \tag{6.29}$$

and

$$\Lambda^{(2)}_{0\,\mu}(p',p) = -2i\int_0^1 dx \int_0^{1-x} dy \int \frac{d^Dq}{(2\pi)^D}\,\frac{N^{(2)}_\mu(q)}{[q^2-t^2-r+i\epsilon]^3}, \tag{6.30}$$

with $(N_\mu(p',p,q+t) = N^{(0)}_\mu(p',p,t) + N^{(2)}_\mu(q))$,

$$N^{(0)}_\mu(p',p,t) = \gamma_\lambda(\not{p}' - \not{t} + m)\gamma_\mu(\not{p} - \not{t} + m)\gamma^\lambda, \tag{6.31}$$

and

$$N^{(2)}_\mu(q) = \gamma_\lambda\not{q}\gamma_\mu\not{q}\gamma^\lambda. \tag{6.32}$$

Clearly $\Lambda^{(0)}_{0\,\mu}(p',p)$, as we pass back to 4-dimensions, $D \to 4$, is convergent, while $\Lambda^{(2)}_{0\,\mu}(p',p)$ is logarithmically divergent. This means that we can take $D = 4$ directly in (6.29). Using the rules of dimensional regularization we see that $\Lambda^{(2)}_{0\,\mu}(p',p)$ is proportional γ_μ. This equation tells us that, at least up to order e^3, the renormalization constant Z_1 only corrects for divergences in the vertex which are proportional to γ_μ. Fortunately[8], for us this works out since the only infinity that we find is indeed proportional to γ_μ. However, please be clear, while the divergence is in $\Lambda^{(2)}_{0\,\mu}(p',p)$ this does not mean that the renormalization does not have an effect on $\Lambda^{(0)}_{0\,\mu}(p',p)$. It does.

Matrix Elements

Thats as far as I want to go with this level of generality. Now I will make the approximations mentioned before. To proceed we take the fields to be on shell, that is $p^2 = m^2 = (p')^2$, and we take $k^2 << m^2$. Now $r = 0$ on shell and $t^2 = m^2(x^2+y^2) + 2xyp'.p = m^2(x+y)^2 + 2xy(p'.p - m^2) = m^2(x+y)^2 +)(k^2)$. In this limit we see that $\Lambda^{(2)}_{0\,\mu}(p',p)$ does not depend on k. But this means that when we renormalize one of the things that will happen is that we will simply subtract off this piece to arrive at the renormalized vertex, so we can forget it (though of course it is required if one wants Z_1^1 explicitly). That leaves us with $\Lambda^{(0)}_{0\,\mu}(p',p)$ to take care of.

The integral that we have to perform at one-loop, to determine $\Lambda^{(0)}_{0\,\mu}(p',p)$ follows from an easy application of our dimensional regularization rules,

$$\int \frac{d^4q}{(2\pi)^4}\,\frac{1}{[q^2-t^2-r+i\epsilon]^3} = -\frac{i}{32\pi^2}\,\frac{1}{t^2+r}. \tag{6.33}$$

[8] I say fortunately, but in reality there is a proof that the theory is renormalizable to all orders, so it had to happen in the way it does in the text.

We substitute this in (6.29),

$$\Lambda_\mu^{(0)}(p',p) = -\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, \frac{N_\mu^{(0)}(p',p,t)}{t^2+r}. \tag{6.34}$$

This becomes, with our approximations,

$$\Lambda_\mu^{(0)}(p',p) = -\frac{1}{16\pi^2}\int_0^1 dx \int_0^{1-x} dy\, \frac{N_\mu^{(0)}(p',p,t)}{m^2(x+y)^2}. \tag{6.35}$$

The last simplification that we make is to consider instead

$$\bar\Lambda_\mu^{(0)}(p',p)\bar u(p')\Lambda_\mu^{(0)}(p',p)u(p), \tag{6.36}$$

which simplifies the form of $N_\mu^{(0)}(p',p,t)$. In order to get a nice formula one makes use of

$$(\not p + m)\gamma_\lambda u(p) = 2p_\lambda u(p), \quad \bar u(p')\gamma_\lambda(\not p' + m) = 2p'_\lambda \bar u(p'). \tag{6.37}$$

and the γ_μ matrix rules (with $D = 4$) to find that

$$\begin{aligned} &\bar u(p')N_\mu^{(0)}(p',p,t)u(p) \\ &= 4p.p'\bar u(p')\gamma_\mu u(p) - 2\bar u(p')\not t\gamma_\mu \not t u(p) - 2\bar u(p')\left(\not p\not t\gamma_\mu + \gamma_\mu \not t\not p'\right)u(p). \end{aligned} \tag{6.38}$$

Still working within the weak field approximation, after a little algebra, we find that

$$\begin{aligned} \bar u(p')N_\mu^{(0)}(p',p,t)u(p) &= -4m\bar u(p')\left(p'_\mu[x(x+y)-y] + p_\mu[y(x+y)-x]\right)u(p) \\ &\quad + am^2\bar u(p')\gamma_\mu u(p) + O(k^2), \end{aligned} \tag{6.39}$$

where a is just some constant (what is it?). Renormalization will eliminate the term proportional to a, so we may as well forget that too.

This means that we now have to evaluate

$$\bar\Lambda_\mu^{(0)}(p',p) = \frac{1}{4\pi^2}\int_0^1 dx \int_0^{1-x} dy\, \frac{\bar u(p')\left(p'_\mu[x(x+y)-y] + p_\mu[y(x+y)-x]\right)u(p)}{m(x+y)^2} \tag{6.40}$$

The two integrals that have still to be performed are elementary,

$$\int_0^1 dx \int_0^{1-x} dy\, \frac{y(x+y)-x}{(x+y)^2} = \int_0^1 dx \int_0^{1-x} dy\, \frac{x(x+y)-y}{(x+y)^2} = -\frac{1}{4}. \tag{6.41}$$

Finally we obtain,

$$\begin{aligned} \bar\Lambda_\mu^{(0)}(p',p) &= -\frac{1}{16\pi^2}\frac{\bar u(p')(p'_\mu + p^\mu)u(p)}{m} \\ &= \frac{i}{16\pi^2}\frac{\bar u(p')\sigma_{\mu\nu}k^\nu u(p)}{m} + \ldots. \end{aligned} \tag{6.42}$$

where we have made use of the Gordon identity and the ellipses denote terms that are proportional[9] to $\bar{u}(p')\gamma_\mu u(p)$.

To end this section we see that the renormalized 3-point function up to one loop order is

$$\bar{\Gamma}_\mu(p,p') = e\,\bar{u}(p')\gamma_\mu u(p) + \frac{ie^3}{16\pi^2}\frac{\bar{u}(p')\sigma_{\mu\nu}k^\nu u(p)}{m} + O(k^2). \tag{6.43}$$

Question: How would you establish that this result does not depend on the gauge chosen?

7 Magnetic Moment of a Spin 1/2 Particle

In this section we will be concerned with making use of the previous calculation of $\bar{\Gamma}_\mu$ in order to determine the gyromagnetic moment for the electron. Actually, to the order that we are working the calculation gives the same result for all fermionic fields minimally coupled to the gauge field. However, higher order terms, or terms coming from other interactions that we have not taken into account here may mean that different fermions have different magnetic moments. An important example is that of the muon, it does indeed have a magnetic moment different to that of the electron.

Diracs Formula for the Magnetic Moment

The Dirac equation for an electron in the presence of an electromagnetic field

$$(i\not{D} - m)\psi = 0, \tag{7.1}$$

can be used to determine the magnetic moment of the spin 1/2 particle. We have

$$\begin{aligned}(i\not{D} + m)(i\not{D} - m)\psi &= \left((i\not{D})^2 - m^2\right)\psi \\ &= \left((iD)^2 - e\frac{\sigma^{\mu\nu}}{2}F_{\mu\nu} - m^2\right)\psi = 0\end{aligned} \tag{7.2}$$

You should compare this formula with what you would write down for a spinless particle, ϕ, coupled in a minimal fashion to the electromagnetic field,

$$\left((iD)^2 - m^2\right)\phi = 0. \tag{7.3}$$

The difference between the electrons equation of motion (7.2) and the spinless particles equation of motion (7.3) is the term coupling the spin matrix and the field strength. The spin dependent term is

$$-g\frac{e}{2}\frac{\sigma^{\mu\nu}}{2}F_{\mu\nu} = -g\frac{e}{2}\begin{pmatrix} \sigma.(\mathbf{B}+\mathbf{E}) & 0 \\ 0 & \sigma.(\mathbf{B}-\mathbf{E}) \end{pmatrix}, \tag{7.4}$$

[9] And by now we are used to forgetting these terms, for a good reason.

from which we read off that $g = 2$ for the gyromagnetic ratio.

How do we determine the gyromagnetic ratio directly from the physical current $\bar{\Gamma}^0_\mu$? Actually there is a straightforward way to do this. The electrons spin is felt in the presence of an external electromagnetic field, so we can ask for the scattering of the electron due to such a background. The interaction with a background or external "photon" $A^{\rm ext}_\mu$ is $\overline{\psi}\gamma^\mu\psi A^{\rm ext}_\mu$. The S-matrix element in the external field, at tree level in the quantum fields, is just

$$e\bar{u}(p')\gamma^\mu u(p) A^{\rm ext}_\mu(k) = \bar{\Gamma}^{0\,\mu}(p,p')A^{\rm ext}_\mu(k). \tag{7.5}$$

One can use the Gordon identity to rewrite this expression as

$$\frac{e}{2m}\bar{u}(p')\left((p_\mu + p'_\mu) + i\sigma^\mu{}_\nu k^\nu\right)u(p)A^{\rm ext}_\mu(k). \tag{7.6}$$

The spin coupling term may be written as

$$\frac{e}{4m}\bar{u}(p')\sigma^{\mu\nu}u(p)\,F^{\rm ext}_{\nu\mu}(k), \tag{7.7}$$

where

$$F^{\rm ext}_{\nu\mu}(k) = ik_\nu A^{\rm ext}_\mu(k) - ik_\mu A^{\rm ext}_\nu(k). \tag{7.8}$$

For a static field $F^{\rm ext}_{0\mu}(k) = 0$ and $B_i = \epsilon_{ijk}F^{\rm ext}_{jk}(k)$. The Pauli coupling (7.7) can be thought of as an effective magnetic dipole energy

$$-\mathbf{B}.\mu^0 \tag{7.9}$$

where

$$\mu^0 = \frac{e}{2m}2\bar{u}(p')\frac{\sigma}{2}u(p). \tag{7.10}$$

Since the magnetic dipole moment is

$$2.\frac{e}{2m}\frac{\sigma}{2} \tag{7.11}$$

we conclude that the gyromagnetic ratio is 2.

The One Loop Correction

We have seen that the correct method for extracting the magnetic moment at tree level is to determine the part of

$$\bar{u}(p')\Gamma^{(0)}_\mu(p',p)u(p), \tag{7.12}$$

proportional to $\bar{u}(p')\sigma_{\mu\nu}k^{nu}u(p)$. The one loop contribution is determined in the same way. We repeat the exercise but with (6.43). This time we find that, to one loop order,

$$\mathbf{B}.\mu^0 + \mathbf{B}.\mu^1 = \frac{e}{2m}(2 + \frac{\alpha}{\pi})\bar{u}(p')\frac{\sigma}{2}u(p), \tag{7.13}$$

from which we deduce that the gyromagnetic ratio to this order is

$$g = 2 + \frac{\alpha}{\pi}. \tag{7.14}$$

The Real World

The calculation that we have just performed was first done by Schwinger in 1948 and it has been tested rather stringently in experiments. Let me quote the results. First our calculation gives

$$g - 2 = 2.3228 \times 10^{-3} \tag{7.15}$$

while experiment seems to favour

$$g - 2 = 2.3194 \times 10^{-3}. \tag{7.16}$$

The agreement is quite impressive. Infact, the computations have been done up to three loop order and the agreement becomes startling! To this order one finds

$$g - 2 = 2.3194 \times 10^{-3}, \tag{7.17}$$

and disagreement comes in at the 9-th decimal place.

8 The Infinite Part of Z_1 at 1-Loop

The infinite contribution we saw came from considering the term proportional to $N_\mu^{(2)}(q)$. We derived that (6.30)

$$\Lambda_{0\,\mu}^{(2)}(p', p) = -2i\int_0^1 dx \int_0^{1-x} dy \int \frac{d^Dq}{(2\pi)^D}\,\frac{\gamma_\lambda \not{q}\gamma_\mu \not{q}\gamma^\lambda}{[q^2 - t^2 - r + i\epsilon]^3}. \tag{8.1}$$

We note that we need to extend our integration rules a little to handle the fact that we do not have q^2 in the numerator. The extension is obtained by making use of Lorentz invariance. An integral of the form

$$I_{\mu\nu} = \int \frac{d^Dq}{(2\pi)^D}\, f(q^2)\, q_\mu q_\nu, \tag{8.2}$$

must be proportional to $\eta_{\mu\nu}$ by Lorentz invariance. You can see this as follows, change variables to

$$q_\mu = \Lambda_\mu^{\ \nu} q'_\nu, \tag{8.3}$$

in the integral, where $\Lambda_\mu^{\ \nu}$ is a Lorentz transformation. The measure is $d^4q = d^4q'$ since the determinant of a Lorentz transformation matrix is unity. Also $q^2 = (q')^2$ which is a basic property of Lorentz transformations, so finally

$$\begin{aligned} I_{\mu\nu} &= \int \frac{d^Dq}{(2\pi)^D} f(q^2)\, q_\mu q_\nu \\ &= \int \frac{d^Dq'}{(2\pi)^D} f((q')^2)\, \Lambda_\mu^{\ \rho} q'_\rho \, \Lambda_\nu^{\ \sigma} q'_\sigma \\ &= \Lambda_\mu^{\ \rho} \Lambda_\nu^{\ \sigma} \int \frac{d^Dq}{(2\pi)^D} f(q^2)\, q_\rho q_\sigma \\ &= \Lambda_\mu^{\ \rho} \Lambda_\nu^{\ \sigma}\, I_{\rho\sigma}, \end{aligned} \tag{8.4}$$

where in the third line I simply dropped the prime on the integration variable. Now the only symmetric invariant rank two tensor that we have available is $\eta_{\mu\nu}$, which tells us that $I_{\mu\nu} \sim \eta_{\mu\nu}$ as I claimed before. To find the proportionality constant we contract with the metric and so

$$I_{\mu\nu} = \frac{\eta_{\mu\nu}}{D} \eta^{\rho\sigma} I_{\rho\sigma}. \tag{8.5}$$

Now let

$$f(q^2) = \frac{(q^2)^r}{(q^2 - M^2 + i\epsilon)^s}, \tag{8.6}$$

to find that

$$\int \frac{d^Dq}{(2\pi)^D} \frac{(q^2)^r}{(q^2 - M^2 + i\epsilon)^s} q_\mu q_\nu = \frac{\eta_{\mu\nu}}{D} I(r+1, s, D)(M^2). \tag{8.7}$$

Consequently, with $M^2 = t^2 + r$,

$$\Lambda^{(2)}_{0\,\mu}(p', p) = -2i \int_0^1 dx \int_0^{1-x} dy\, \frac{\gamma_\lambda \gamma_\sigma \gamma_\mu \gamma^\sigma \gamma^\lambda}{D} I(1, 3, D)(M^2). \tag{8.8}$$

We also need to make use of the rule that $\gamma_\sigma \gamma_\mu \gamma^\sigma = (2 - D)\gamma_\mu$ twice to get

$$\Lambda^{(2)}_{0\,\mu}(p', p) = -2i \frac{(2-D)^2}{D} \gamma_\mu \int_0^1 dx \int_0^{1-x} dy\, \frac{i}{(4\pi)^{D/2}} \left(M^2\right)^{D/2-2} \frac{\Gamma(1 + D/2)\Gamma(2 - D/2)}{\Gamma(3)} \tag{8.9}$$

The infinite part sits in $\Gamma(2 - D/2) = 2/\epsilon + \ldots$. To obtain the pole part we may ignore all the other dependence on ϵ which is polynomial since such terms will only contribute a finite correction. For example

$$(M/\mu)^{4-D} = 1 + \epsilon \ln(M/\mu) + \ldots \tag{8.10}$$

and multiplying against $\Gamma(2 - D/2)$ the pole gives us

$$(M/\mu)^{4-D}\Gamma(2 - D/2) = 2/\epsilon + \text{finite terms}. \tag{8.11}$$

So apart from the pole term everywhere else we will set $D = 4$,

$$\begin{aligned}\Lambda^{(2)}_{0\,\mu}(p', p) &= \frac{1}{4\pi^2}\gamma_\mu \int_0^1 dx \int_0^{1-x} dy\, \frac{1}{\epsilon} + \text{finite terms} \\ &= \frac{1}{8\pi^2}\gamma_\mu\, \frac{1}{\epsilon} + \text{finite terms.}\end{aligned} \tag{8.12}$$

Now from (6.7) we deduce that

$$Z_1^1 = -\frac{e^2}{8\pi^2}\frac{1}{\epsilon} + \text{finite terms,} \tag{8.13}$$

or

$$Z_1 = 1 - \frac{e^2}{8\pi^2}\frac{1}{\epsilon} + \text{finite terms.} \tag{8.14}$$

To get this result we have made no approximations, nor did we need to go on shell.

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