Lectures at the International School on Quantum Gravity: La Plata

Contents

1 Introduction

The subject matter for this school is quantum gravity, which may mean one of many things. In any case one thing we are sure of is that whatever formulation one is interested in that formulation will require a number of mathematical tools. I have been asked to provide a quick introduction to some of those tools and some of the ideas that might be required.

I should apologise in advance. It is impossible, in 3 lectures, to cover the required ground in a comprehensive or even in a completely comprehensible way. I am sorry for this. Also I hope that my presentation does not turn the student off the subject matter.

The first lecture is a lightning course on differential forms some cohomology and some homology theory. The second is about Kaluza-Klein reduction, that is how to start with a theory in more than 4 dimensions and end up with one in 4 dimensions. The forms and the cohomology groups will re-appear here. The third lecture is about special choices that one may wish to make on the type of spaces that make up the extra dimensions that appeared in the second lecture.

2 Differential Forms and Hodge Theory

Let X be a manifold of dimension n. On such a space we can consider antisymmetric tensors of rank p where $0 \leq p \leq n$. There is a nice way to write such tensors which allows us to avoid using too many labels. The idea is to use differentials dx^{μ} and to define

Definition 2.1 The wedge product of differentials is

$$dx^\mu \wedge dx^\nu = \frac{1}{2}\left(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu\right) = -dx^\nu \wedge dx^\mu$$

Because of the antisymmetric wedge product it is sometimes useful to treat the differentials as if they are Grassman variables and to drop the wedge $\wedge$ and write

$$dx^\mu dx^\nu = -dx^\nu dx^\mu$$

instead. However, you should be careful when doing this as there is also the tendency, in the physics literature, to write

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

and here one certainly does not mean the wedge product. Rather, one means the tensor product in this expression and, as the metric $g_{\mu\nu}$ is symmetric, on really means the symmetric tensor product.

The way one can represent a rank p antisymmetric tensor is as a p 'form' (p is referred to as the degree of the form),

$$\alpha_p = \frac{1}{p!}\alpha_{\mu_1,\dots,\mu_p}\, dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}, \tag{2.1}$$

so that, for example a zero form is a scalar

$$\alpha_0 = \phi, \tag{2.2}$$

while a one form is a vector

$$A = A_\mu\, dx^\mu, \tag{2.3}$$

and a two-form is an antisymmetric tensor of rank 2

$$B = \frac{1}{2}B_{\mu\nu}\, dx^\mu \wedge dx^\nu, \tag{2.4}$$

and so on.

Notice that one can wedge a form α_p with a form β_q to get a form λ_{p+q}. Degrees are additive since we are only counting the number of dx's. We have

$$\lambda_{p+q} = \alpha_p \wedge \beta_q = (-1)^{pq}\beta_q \wedge \alpha_p, \tag{2.5}$$

the last equality following from antisymmetry of the dx's.

One, of the many, nice things about the form notation (actually the tensor product notation) is how tensors (and in particular forms) transform under general coordinate transformations. If in the x co-ordinates a tensor is $T(x)_{\mu_1\dots\mu_n}$ and in the x' coordinates it is $T'(x')_{\mu_1\dots\mu_n}$ then we have

$$T = T(x)_{\mu_1\dots\mu_n}\, dx^{\mu_1} \otimes \dots \otimes dx^{\mu_n} = T'(x')_{\mu_1\dots\mu_n}\, dx'^{\mu_1} \otimes \dots \otimes dx'^{\mu_n}$$

and this is about the best possible definition of a tensor.

The space of p-forms is refereed to as $\Omega^p(X)$. One is usually more explicit than this. Everything we have done works if the forms are real or if they are complex and one makes this clear by writing $\Omega^p(X,\mathbb{R})$ or $\Omega^p(X,\mathbb{C})$ as the case may be.

Exterior Derivative

There is a natural differentiation operator, d, that one can now introduce

Definition 2.2 The exterior derivative acting on forms is

$$d = dx^\mu \partial_\mu,$$

where it is understood that the dx^μ part just acts by wedge multiplication.

So acting on zero forms this is,

$$d\phi = \partial_\mu \phi \, dx^\mu, \tag{2.6}$$

while on one forms we find

$$dA = dA_\mu dx^\mu = \partial_\nu A_\mu \, dx^\nu \wedge dx^\mu = \frac{1}{2} F_{\nu\mu} \, dx^\nu \wedge dx^\mu, \tag{2.7}$$

which is nice for physicists because they recognize the field strength of a gauge field entering and finally

$$dB = \frac{1}{2}\partial_\rho B_{\mu\nu} \, dx^\rho \wedge dx^\mu \wedge dx^\nu = \frac{1}{3!}\left(\partial_\rho B_{\mu\nu} + \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu}\right) dx^\rho \wedge dx^\mu \wedge dx^\nu. \tag{2.8}$$

Acting with d raises the degree by one and a way of saying this is to write

$$d : \Omega^p(X) \to \Omega^{p+1}(X). \tag{2.9}$$

On products of forms one has

$$d\left(\alpha_p \wedge \beta_q\right) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q. \tag{2.10}$$

Since d is anticommuting it also has the useful property that

$$d^2 = 0. \tag{2.11}$$

Stokes Theorem

Another nice aspect of differential forms is that they can be integrated. For example suppose we wish to integrate over a two manifold then traditionally we would like to make use of a measure[1] that we write as $d^2x = dx \wedge dy$. In terms of forms, in two dimensions

$$\begin{aligned} B &= \frac{1}{2} B_{\mu\nu}\, dx^{\mu} \wedge dx^{\nu} \\ &= B_{xy}\, dx \wedge dy \end{aligned} \tag{2.12}$$

and so it is natural to integrate B. The important point is that no metric is required.

Stokes theorem tells you that if you integrate a form which is a "total" derivative (exact) on a manifold with boundary then integral is only on the boundary

Theorem 2.3 (Stoke's Theorem)

$$\int_X d\alpha_{n-1} = \int_{\partial X} \alpha_{n-1}$$

where ∂X stands for the boundary of X (one has to be careful with 'signs' here because the boundary components may have different orientations).

The Hodge Star and an Inner Product on the Space of Forms

So far there has been no metric. In these notes I use the terms Riemannian metric and Riemannian manifold in particular way. For a metric to be Riemannian it must locally have a Euclidean signature (not a Lorentzian one) and then a Riemannian manifold is one which comes equipped with a Riemannian metric. In physics one usually is not so strict but rather qualifies the description by saying g is a Riemannian metric with Euclidean signature. You have been warned!

If the manifold M comes equipped with a Riemannian metric g then there is an operation, known as the Hodge star.

Definition 2.4 Given a Riemannian metric g on X there is a Hodge star $*$ action on the basis of forms given by

$$*\left(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\right) = \frac{\sqrt{g}}{(n-p)!} \epsilon^{\mu_1,\ldots,\mu_p}{}_{\mu_{p+1},\ldots,\mu_n}\, dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_n}$$

where

$$\epsilon_{\mu_1,\ldots,\mu_n} = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \\ 0 & \text{when two indices agree} \end{cases} .$$

For a p-form α_p we have

$$\begin{aligned} \alpha_p &= \frac{1}{p!} \alpha_{\mu_1,\ldots,\mu_p}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} \\ *\alpha &= \frac{\sqrt{g}}{(n-p)!\, p!} \epsilon^{\mu_1,\ldots,\mu_p}{}_{\mu_{p+1},\ldots,\mu_n} \alpha_{\mu_1,\ldots,\mu_p}\, dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_n}. \end{aligned} \tag{2.13}$$

[1] We would also add $\sqrt{g}$ where g is the metric, but we will come to that later.

Theorem 2.5 The Hodge star maps is a map

$$* : \Omega^p(X) \to \Omega^{n-p}(X)$$

so that

$$*^2 : \Omega^p(X) \to \Omega^p(X)$$

and in fact $*^2 = \pm 1$.

To prove this one just acts twice with the Hodge star and checks.

Definition 2.6 The Riemannian volume form dV is defined to be $dV = *1$,

$$dV = \frac{1}{n!}\sqrt{g}\,\epsilon_{\mu_1 \ldots \mu_n}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n} = \sqrt{g}\, d^n x$$

One may invert this relation to find

$$dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n} = \sqrt{g}\, d^n x = \bar{\epsilon}^{\nu_1,\ldots,\nu_n} d^n x,$$

where $\bar{\epsilon}$ is defined to be

$$\bar{\epsilon}^{\nu_1,\ldots,\nu_n} = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \\ 0 & \text{when two indices agree} \end{cases}$$

Definition 2.7 The metric inner product is

$$(\alpha, \beta) = \int_M \alpha \wedge *\beta,$$

and explicitly this is

$$(\alpha, \beta) = \frac{1}{p!}\int_M \sqrt{g}\,\alpha_{\mu_1,\ldots,\mu_p}\,\beta^{\mu_1,\ldots,\mu_p}\, d^n x.$$

We need to show that the explicit form is indeed correct. One can easily determine the integrand,

$$\begin{aligned} \alpha * \beta &= \frac{\sqrt{g}}{(n-p)!p!p!}\alpha_{\nu_1,\ldots,\nu_p}\beta_{\mu_1,\ldots,\mu_p}\epsilon^{\mu_1,\ldots,\mu_p}{}_{\mu_{p+1},\ldots,\mu_n} dx^{\nu_1}\ldots dx^{\nu_p} dx^{\mu_{p+1}}\ldots dx^{\mu_n} \\ &= \frac{\sqrt{g}}{(n-p)!p!p!}\alpha_{\nu_1,\ldots,\nu_p}\beta_{\mu_1,\ldots,\mu_p}\epsilon^{\mu_1,\ldots,\mu_p}{}_{\mu_{p+1},\ldots,\mu_n}\bar{\epsilon}^{\nu_1,\ldots,\nu_p\,\mu_{p+1},\ldots,\mu_n} d^n x. \end{aligned}$$

Note that

$$\epsilon^{\mu_1,\ldots,\mu_p}{}_{\mu_{p+1},\ldots,\mu_n}\bar{\epsilon}^{\nu_1,\ldots,\nu_p\,\mu_{p+1},\ldots,\mu_n} = (n-p)!\,(g^{\mu_1\nu_1}\ldots g^{\mu_p\nu_p} \pm \text{permutations})$$

the $(n-p)!$ comes from the possible permutations of the $(n-p)$ indices that we are summing over. The total number of permutations in the brackets on the right hand side is $p!$. You can see this by noting that μ_1 can be paired with any of the p, ν_i, while then for μ_2 we have a choice of $p-1$, ν_i and so on. We have then

$$\alpha * \beta = \frac{\sqrt{g}}{p!} \alpha_{\mu_1,\ldots,\mu_p} \, \beta^{\mu_1,\ldots,\mu_p} \, d^n x,$$

and so

$$(\alpha, \beta) = \frac{1}{p!} \int_M \sqrt{g} \, \alpha_{\mu_1,\ldots,\mu_p} \, \beta^{\mu_1,\ldots,\mu_p} \, d^n x.$$

■

Note also that there is a simple relationship between ϵ and $\overline{\epsilon}$,

$$\overline{\epsilon}^{\mu_1,\ldots,\mu_n} = \det g \, g^{\mu_1\nu_1} \ldots g^{\mu_n\nu_n} \, \epsilon_{\nu_1,\ldots,\nu_n}.$$

Theorem 2.8 The inner product is positive semi definite, that is

$$(\alpha, \alpha) \in \mathbb{R}_+$$

and equals zero iff $\alpha = 0$.

This one can see directly from the explicit form of the inner product since everything in sight is real and positive semi-definite.

Remark 2.9 If we had allowed for complex differential forms the appropriate definition of the inner product would include complex conjugation of the first differential form.

Remark 2.10 If we had allowed for a metric with a Lorentzian signature we could not conclude that the inner product is positive semi-definite.

Adjoint of the Exterior Derivative

For this section suppose that X is a compact and closed manifold - so that it has no boundary. Then we can define an adjoint to the exterior derivative. It is the adjoint with respect to the inner product that we defined above and requires us to integrate by parts (which is why we want X to have no boundary).

Definition 2.11 The adjoint δ of d with respect to the Riemannian inner product is

$$(d\alpha, \beta) = (\alpha, \delta\beta).$$

Up to a sign one easily sees that

$$\delta = \pm * d * . \tag{2.14}$$

Note that for $d\alpha$ and β to have the same degree in the inner product on the left hand side of (2.11) α must have degree one less than β but on the righthand side this must be the same as the degree of $\delta\beta$. Thus it is clear that δ lowers the degree by one, so one writes

$$\delta : \Omega^p(X) \to \Omega^{p-1}(X).$$

The degree of zero forms cannot be lowered so

$$\delta\phi(x) = 0,$$

however the degree of one forms can be lowered

$$\delta A = \delta A_\mu \, dx^\mu = \nabla^\mu \, A_\mu$$

and this is also nice for physicists because we recognize the covariant divergence of a vector (which is what we would use for gauge fixing on a curved manifold). Notice that the metric enters, this is unavoidable since it entered in the very definition of δ.

The Laplacian on Forms

Definition 2.12 The Laplacian Δ on forms of any degree is defined to be

$$\Delta = d\delta + \delta d,$$

and it does not change the degree of the form it acts on since d raises the degree by one while δ lowers the degree by one.

One immediate consequence of the form of the Laplacian is that on compact closed Riemannian X the spectrum is real and positive semi-definite. This means that the eigenvalues of Δ are real and zero or positive. The proof of this is exactly the same as for the Hamiltonian of the simple harmonic oscillator in quantum mechanics, so we will stop to prove it.

Theorem 2.13 Let X be a compact closed Riemannian manifold then the spectrum of Δ is positive semi-definite.

Proof: Let ω be an eigenvalue $\Delta\,\omega = \lambda\omega$ and ω non-zero then

$$\begin{aligned} \lambda &= (\omega, \Delta\omega)/(\omega,\omega) \\ &= (d\omega, d\omega)/(\omega,\omega) + (\delta\omega,\delta\omega)/(\omega,\omega) \end{aligned}$$

Now every term in the second line is real so λ must be real and by Theorem 2.8 the inner product is positive semi-definite so that each term is also positive semi-definite. ■

If you work it out you can see that the Laplacian takes the form

$$\Delta = -g^{\mu\nu}\,\nabla_\mu \nabla_\nu + \dots$$

where the ellipses indicate terms with lower numbers of derivatives. You should notice two things. The first is the sign, it is correct so that the spectrum is positive semi-definite on a Riemannian manifold. The second is that if we scale the metric by $g \to tg$ the first term on the right hand side scales by t^{-1}. This is true for the other terms too a result which is quite simple and which will be useful later on so we prove it.

Theorem 2.14 Let $*$ be the Hodge star of the Riemannian metric g and Δ the associated Laplacian. Set $*_t$ to be the Hodge star of the Riemannian metric tg, $t \in (0, \infty)$ and Δ_t the associated Laplacian, then

$$\Delta_t = t^{-1}\,\Delta$$

Proof: From the definition of the Hodge star (2.4) we see that

$$*_t\,\alpha_p = t^{n/2-p}\; *\alpha_p$$

Now, from (2.14)

$$\delta_t \alpha_p = \pm *_t d *_t \alpha_p = t^{-1} \delta\alpha_p$$

and this is enough to prove the result since the scaling does not depend on the degree of the form. ∎

This means that if λ is an eigenvalue of Δ then $t^{-1}\lambda$ is an eigenvalue of Δ_t (and the eigenvalues of the two Laplacians are one to one). This means that we can change the eigenvalues at will (by varying the metric in this way), unless the eigenvalue is zero.

Infact we have a stronger result that we will not prove, namely

Theorem 2.15 Let X be a compact, closed, oriented, Riemannian manifold. The eigenvalues of the Laplacian on p-forms has a discrete spectrum $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$.

Which means that by scaling the metric we can make the first non-zero eigenvalue λ_1 (and hence all non-zero eigenvalues) as large as we like. This argument does not apply to the zero eigenvalue.

Definition 2.16 A form ω is said to be harmonic if it satisfies

$$\Delta\,\omega = 0$$

Theorem 2.17 A harmonic form ω on a compact closed Riemannian manifold satisfies

$$d\omega = 0, \quad \delta\omega = 0$$

and is said to be closed and co-closed.

Proof: By the proof of Theorem 2.13 we have

$$0 = (d\omega, d\omega) + (\delta\omega, \delta\omega)$$

where each term is positive semi-definite by Theorem 2.8 and to be zero each term must be zero so again by Theorem 2.8 we deduce that ω is closed, $d\omega = 0$ and co-closed $\delta\omega = 0$. ■

Harmonic forms play an important role in differential geometry as we will see shortly.

3 Homology and Cohomology Groups

In this section I review, very briefly and rather heuristically, some of the constructs that we will be using later on. As some of the basic invariants of a smooth manifold are the homology groups (and equivalently over $\mathbb{R}$ the cohomology groups) we start there. A fine review of this material, aimed at physicists, is [1]. From now on X is a smooth, compact, oriented and connected manifold.

Homology

Given a set of $p-$dimensional oriented sub-manifolds N_i of X one can form a $p-$chain.

Definition 3.1 A $p-$chain a_p is

$$\sum_i c_i N_i \,,$$

where the coefficients c_i determine the type of chain one has. If the $c_i \in \mathbb{R}(\mathbb{C})$ the $p-$chain is said to be a real (complex) $p-$chain. For $c_i \in \mathbb{Z}(\mathbb{Z}_2)$, and the chain is called an integer ($\mathbb{Z}_2$) chain and so on.

Definition 3.2 ∂ is the operation that gives the oriented boundary of the manifold it acts on.

We can give some simple examples of this operation. For example, the sphere has no boundary so $\partial S^2 = \emptyset$, while the cylinder, $I \times S^1$, has two circles for a boundary, $\partial(I \times S^1) = \{0\} \times S^1 \oplus \{1\} \times -S^1$. I have written $\oplus$ to indicate that we will be 'adding' sub-manifolds to form chains when one should have used the union symbol $\cup$. The minus sign in front of the second S^1 is to indicate that it has opposite relative orientation. One defines the boundary of a $p-$chain to be a $(p-1)-$chain by

$$\partial a_p = \sum_i c_i \partial N_i .$$

Notice that $\partial^2 = 0$, as the boundary of a boundary is empty.

Definition 3.3 A $p-$cycle is a $p-$chain without boundary, i.e. if $\partial a_p = 0$ then a_p is a $p-$cycle.

Definition 3.4 Let $Z_p = \{a_p : \partial a_p = 0\}$ be the set of $p-$cycles and let $B_p = \{\partial a_{p+1}\}$ be the set of $p-$boundaries of $(p+1)-$chains. The $p-$th simplicial homology group of X is defined by

$$H_p = Z_p/B_p.$$

Cohomology

Definition 3.5 Let Z^p be $\{\omega_p : d\omega_p = 0\}$ the set of closed $p-$forms and let B^p be $\{\omega_p : \omega_p = d\omega_{p-1}\}$ the set of exact $p-$forms. The $p-$th De Rham cohomology group is defined by

$$\mathrm{H}^p(X, \mathbb{R}) = Z^p/B^p.$$

De Rham's Theorems

Definition 3.6 The inner product of a $p-$cycle, $a_p \in H_p(X, \mathbb{R})$ and a closed $p-$form, $\omega_p \in \mathrm{H}^p(X, \mathbb{R})$ is

$$\pi(a_p, \omega_p) = \int_{a_p} \omega_p.$$

Notice that this does not depend on the representatives used for, by Stokes theorem,

$$\begin{aligned}
\pi(a_p + \partial a_{p+1}, \omega_p) &= \int_{a_p + \partial a_{p+1}} \omega_p = \int_{a_p} \omega_p + \int_{a_{p+1}} d\omega_p = \pi(a_p, \omega_p) \\
\pi(a_p, \omega_p + d\omega_{p-1}) &= = \int_{a_p} \omega_p + \int_{\partial a_p} \omega_{p-1} = \pi(a_p, \omega_p).
\end{aligned}$$

One may thus think of the **period** π as a mapping

$$\pi : H_p(X, \mathbb{R}) \otimes \mathrm{H}^p(X, \mathbb{R}) \to \mathbb{R}$$

For X compact and closed De Rham has established two important theorems. Let $\{a_i\}$, $i = 1, \ldots b_p$, be a set of independent $p-$cycles forming a basis $H_p(X, \mathbb{R})$, where the $p-$th Betti number $b_p(X) = \dim_{\mathbb{R}} \mathrm{H}_p(X, \mathbb{R})$. The Betti numbers are all finite $b_p(X) < \infty$ (as you can see by inspection) and are topological invariants of X.

Theorem 3.7 Given any set of periods ν_i, $i = 1 \ldots b_p$, there exists a closed $p-$form ω for which

$$\nu_i = \pi(a_i, \omega) = \int_{a_i} \omega.$$

Theorem 3.8 If all the periods of a $p-$form ω vanish,

$$\pi(a_i, \omega) = \int_{a_i} \omega = 0$$

then ω is an exact form.

Putting the two preceding theorems together, we have that if $\{\omega_i\}$ is a basis for $\mathrm{H}^p(X, \mathbb{R})$ then the period matrix

$$\pi_{ij} = \pi(a_i, \omega_j)$$

is invertible. This implies that $\mathrm{H}^p(X, \mathbb{R})$ and $H_p(X, \mathbb{R})$ are dual to each other with respect to the inner product π and so are naturally isomorphic. An immediate consequence is that the Betti numbers are then also given as

$$b_p(X) = \dim_{\mathbb{R}} \mathrm{H}^p(X, \mathbb{R})$$

in particular the spaces of closed forms modulo exact forms are finite dimensional and give us topological information about X.

Poincaré Duality

If X is compact, orientable and closed of dimension n then $\mathrm{H}^n(X, \mathbb{R}) = \mathbb{R}$, as, up to a total differential, any $\omega_n \in \mathrm{H}^n(X, \mathbb{R})$ is proportional to the volume form. One aspect of Poincaré duality is the following

Theorem 3.9 $\mathrm{H}^p(X, \mathbb{R})$ is dual to $\mathrm{H}^{n-p}(X, \mathbb{R})$ with respect to the pairing

$$(\omega_p, \omega_{n-p}) = \int_X \omega_p \wedge \omega_{n-p}.$$

This theorem implies that $\mathrm{H}^p(X, \mathbb{R})$ and $\mathrm{H}^{n-p}(X, \mathbb{R})$ are isomorphic as vector spaces, so that, in particular, $b_p(X) = b_{n-p}(X)$.

Remark 3.10 Note that this is not the metric inner product that we defined previously.

We need the following statement

Theorem 3.11 Given any $p-$cycle a_p there exists an $(n-p)-$form α, called the Poincaré dual of a_p, such that for all closed $p-$forms ω

$$\int_{a_p} \omega = \int_X \alpha \wedge \omega.$$

The Künneth Formula

This is a formula that relates the cohomology groups of a product space $X_1 \times X_2$ to the cohomology groups of each factor. The formula is

$$\mathrm{H}^p(X_1 \times X_2, \mathbb{R}) = \sum_{q=0}^{p} \mathrm{H}^q(X_1, \mathbb{R}) \otimes \mathrm{H}^{p-q}(X_2, \mathbb{R}).$$

In particular this implies that $b_p(X_1 \times X_2) = \sum_{q+r=p} b_q(X_1) b_r(X_2)$.

Notation: I will denote a basis for $H_1(X, \mathbb{R})$ by γ_i and the dual basis for $\mathrm{H}^1(X, \mathbb{R})$ by $[\gamma_i]$. For $H_2(X, \mathbb{R})$ I denote the basis by Σ_i (as a mnemonic for a two dimensional manifold) and the corresponding basis for $\mathrm{H}^2(X, \mathbb{R})$ by $[\Sigma_i]$.

4 Hodge Theory

The De Rham theorems are very powerful and very general. Rather than working with a cohomology class $[\omega_p]$ of a $p-$form ω_p it would be nice to be able to choose a canonical representative of the class. This is what Hodge theory gives us and along the way also gives us a different characterisation of the cohomology groups.

Up till now, we have not needed a metric on the manifold X. The introduction of a metric, though not needed in the general framework, leads to something new. Let X be equipped with a metric g. Define the Hodge star $*$ operator as in (2.4), $* : \Omega^p \to \Omega^{(n-p)}$, by

$$*\omega_{\mu_1,\ldots,\mu_p}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p} = \frac{\sqrt{g}}{(n-p)!} \omega_{\mu_1,\ldots,\mu_p} \epsilon^{\mu_1 \ldots \mu_p}{}_{\mu_{p+1} \ldots \mu_n}\, dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_n}.$$

The symbol $\epsilon_{\mu_1 \ldots \mu_n}$ is 0 if two labels are repeated and $\pm$ for even or odd permutations respectively. All labels are raised and lowered with respect to the metric.

Theorem 4.1 For a compact manifold without boundary, any p-form can be uniquely decomposed as a sum of an exact, a co-exact and a harmonic form,

$$\omega_p = d\alpha_{(p-1)} + \delta\beta_{(p+1)} + \gamma_p$$

this is referred to as the Hodge decomposition. Furthermore, the decomposition is orthogonal with respect to the metric inner product.

Part Proof: It is easy to show that the decomposition is orthogonal so we will do that. First we have $(\gamma_p, d\alpha_{(p-1)}) = (\delta\gamma_p, \alpha_{(p-1)}) = 0$ as γ_p is co-closed by Theorem 2.17. Likewise $(\gamma_p, \delta\beta_{(p+1)}) = 0$ as γ_p is closed. Finally $(d\alpha_{(p-1)}, \delta\beta_{(p+1)}) = (d^2\alpha_{(p-1)}, \beta_{(p+1)}) = 0$.
■

Now suppose that ω_p is a closed form then, using the Hodge decomposition Theorem 4.1,

$$d\omega_p = 0 = d\delta\beta_{p+1}$$

which implies that $\delta\beta_{p+1} = 0$ since $0 = (\beta, d\delta\beta) = (\delta\beta_{p+1}, \delta\beta_{p+1})$. Consequently we have

Corollary 4.2 A closed p-form ω_p has the Hodge decomposition

$$\omega_p = d\alpha_{p-1} + \gamma_p$$

with γ_p a harmoic form.

The harmonic form γ_p is our representative for $[\omega_p]$. This corresponds to choosing $d * \omega_p = 0$ as the representative. From the point of view of gauge theory, this amounts to the usual Landau gauge (extended to higher dimensional forms).

5 Kaluza-Klein Reduction

The idea here is that spacetime is actually a lot bigger than we first thought and that "all" of the physics that we see is actually a consequence of gravitational interactions in the bigger space. In the case of string theory this will not be quite true as there will be other fields also to consider in the higher dimensional space-time M_D. Physics, for present purposes, is taken to mean the action, since it determines both the classical physics (at the level of equations of motion) and the quantum physics (at the level of the path integral).

Let our space time be denoted by M and have dimension d, this could Minkowski space-time or a Friedman-Walker type universe or .... The relationship between what we see, M and the actual space time, M_D of dimension D is

$$M_D = M \times K, \qquad \dim_{\mathbb{R}} M_D = \dim_{\mathbb{R}} M + \dim_{\mathbb{R}} K, \tag{5.1}$$

where K is taken to be some compact and small space. Small here means that its volume is much less than that of M. For various reasons, some of which will be explained below, we would in the normal course of things not notice that K is there. Nevertheless, the existence of such a K allows us to deduce various things about possible physics on M.

Notice that we will consider M to have a metric with Lorentzian signature but the internal space K will be given a metric with Euclidean signature. This means that we can apply all the machinery we developed about differential forms to K.

Denote the local coordinates on M_D by

$$X^M = (x^\mu, y^i) \tag{5.2}$$

where the x^μ are coordinates on M and y^i are coordinates on K. For example if $\dim_{\mathbb{R}} M_D = 10$ and $\dim_{\mathbb{R}} M = 4$, we would have $M = 1, \ldots, 10$, $\mu = 0, \ldots, 3$ and $i = 5, \ldots, 10$.

The Lorentz groups are, in general,

$$\begin{aligned} SO(1, D-1) & \quad \text{for} \quad M_D \\ SO(1, d-1) & \quad \text{for} \quad M \\ SO(D-d) & \quad \text{for} \quad K \end{aligned}$$

but for a product manifold of the type that we are considering (5.1) the Lorentz group is $SO(1, d-1) \times SO(D-d) \subset SO(1, D-1)$.

Our problem is that, if physics really comes from the bigger space M_D then all our fields will depend on the X^M coordinates. Let $\Phi(X)$ be some object transforming under $SO(1, D-1)$ in some representation, for example, $\Phi = g_{MN}$ or A_M or ψ_α or $\ldots$. Then there are two related questions to answer,

a) How does Φ decompose under $SO(1, d-1) \times SO(D-d)$?

b) What does this mean for physics on M?

The aim of this lecture is to answer these questions.

The Internal Space is a Circle; $K = S^1$

This is a "warm up exercise". Actually it is historically the first example of this type of reduction. Kaluza and Klein had the idea that one could get electromagnetism coupled to gravity in 4-dimensions by considering a gravitational theory in 5-dimensions, $M_5 = M_4 \times S^1$. Here we consider an equivalent situation, with the 5-dimensional space-time replaced by a D-dimensional space time M_D and with M being $D-1$ having $D-1$ dimensions.

But before proceeding let us ask what we mean by electromagnetism coupled to gravity? Actually what we mean is that the action on M is the sum of the Einstein action and the covariant Maxwell action. These are supposed to both arise, somehow, just from the Einstein action on M_D. I will not show this, but I will try to convince you that it must be true.

The coordinate y on K is really the angular coordinate and we can denote it by θ. There are various ways of writing the metric on $M \times S^1$ for example as

$$g_{MN} = (\tilde{g}_{\mu\nu}, g_{\mu\theta}, g_{\theta\theta}) = (g_{\mu\nu}, \tilde{A}_\mu, \tilde{\phi}),$$

which in terms the line element squared is

$$ds^2 = g_{\mu\nu}\, dx^\mu \otimes dx^\nu + 2\tilde{A}_\mu\, dx^\mu \otimes d\theta + \tilde{\phi}\, d\theta \otimes d\theta,$$

or as

$$ds^2 = g_{\mu\nu}\, dx^\mu \otimes dx^\nu + (\phi\, d\theta + A_\mu\, dx^\mu)^2,$$

which corresponds to writing the metric as

$$g_{MN} = (g_{\mu\nu} + A_\mu A_\nu, \phi A_\mu, \phi^2),$$

both are general forms for the metric.

The fields that make up the metric all depend on X^M, however, since K is a circle we can Fourier decompose the fields. We have

$$\begin{aligned} g_{\mu\nu}(X) &= \sum_{n=-\infty}^{\infty} g_{\mu\nu}^{(n)}(x) e^{in\theta} \\ A_\mu(X) &= \sum_{n=-\infty}^{\infty} A_\mu^{(n)}(x) e^{in\theta} \\ \phi(X) &= \sum_{n=-\infty}^{\infty} \phi^{(n)}(x) e^{in\theta} \end{aligned} \tag{5.3}$$

So at this point it looks as if we have symmetric tensors, vectors and scalars from the point of view of M. However, before making that assertion we still need to check that they transform as they should.

General Coordinate Transformations

General coordinate transformations on M_D are

$$X^M \rightarrow X'^M = X'^M(X)$$

The metric transforms as

$$G_{MN}(X) = \frac{\partial X'^P}{\partial X^M} \frac{\partial X'^Q}{\partial X^N} G'_{PQ}(X').$$

A subset of these transformations will correspond to general coordinate transformations on M. These are transformations of the form

$$(x^\mu, y^i) \rightarrow (x'^\mu(x), y^i)$$

which come about from simply ignoring the fact that K is there at all. Under such a coordinate transformation one has that

$$\begin{aligned} \phi(x,\theta) &= \phi'(x',\theta) \\ A_\mu(x,\theta) &= \frac{\partial x'^\nu}{\partial x^\mu} A'_\nu(x',\theta) \\ g_{\mu\nu}(x,\theta) &= \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g'_{\rho\sigma}(x',\theta) \end{aligned} \tag{5.4}$$

which implies that each mode in (5.3) transforms, from the point of view of M, as a tensor of the appropriate degree. For example the $A_\mu^{(n)}$ are vectors as, by (5.4),

$$A_\mu^{(n)}(x) = \frac{\partial x'^\nu}{\partial x^\mu} A_\nu'^{(n)}(x'), \tag{5.5}$$

and likewise the $g_{\mu\nu}^{(n)}$ are rank 2 symmetric tensors and the $\phi^{(n)}$ are all scalars.

So it has become apparent that we do have symmetric tensors and vector fields as well as scalars on M. So the transformation properties of the fields under the decomposition of the Lorentz group that was claimed at the beginning holds.

Actually we are over blessed since we have an infinite number of each type of field (labelled by an integer n) .... We will have to come to grips with this but first I would remind the reader that having vector fields is not the same as having gauge fields, for that we need gauge invariance. Where is that and what mediates it?

Gauge Symmetry

To answer the question posed above we do not need to keep track of ϕ, so from now on set it to be a constant, $\phi(X) = R$. This sets the volume (= circumference) of the S^1 to be $2\pi R$,

$$\int_{S^1} \sqrt{g_{55}}\, dx^5 = \int_0^{2\pi} R\, d\theta = 2\pi R$$

The only symmetry we have to begin with is general coordinate invariance on M_D. If we want to perform a coordinate transformation that does not change the metric on K, in this case we have fixed $g_{55} = R^2$, then the coordinate transformations will have to have a special form.

Indeed now consider infinitesimal transformations (5.4) of the form

$$(x^\mu, \theta) \to (x^\mu, \theta + R^{-1}\epsilon(x))$$

Under this type of transformation we find that

$$RA_\mu(x,\theta) = RA'_\mu(x,\theta) + \epsilon \frac{\partial}{\partial\theta} A'_\mu(x,\theta) + \partial_\mu \epsilon R,$$

or, to first order in ϵ,

$$\delta A_\mu(x,\theta) = A'_\mu(x,\theta) - A_\mu(x,\theta) = -\frac{\epsilon}{R}\frac{\partial}{\partial\theta} A_\mu(x,\theta) - \partial_\mu \epsilon. \tag{5.6}$$

It is fruitful to reconsider (5.6) in terms of modes, in which case we find

$$\begin{aligned} \delta A_\mu^{(0)}(x) &= -\partial_\mu \epsilon \\ \delta A_\mu^{(n)}(x) &= -i\frac{n}{R}\epsilon A_\mu^{(n)}(x) \qquad n \neq 0. \end{aligned} \tag{5.7}$$

These equations are marvelous! They tell us that we have exactly one gauge field namely $A_\mu^{(0)}$, as it transforms as a gauge field under a $U(1)$ transformation parametrized by $\epsilon(x)$. They also tell us that all the vector fields with non-zero mode number n are charged, with charge $e_n = n/R$ times the basic charge, with respect to this gauge field, since they transform with a phase under the $U(1)$ symmetry. The charge is deduced by looking at the minimal coupling which must be of the form

$$D_\mu A_\nu^{(n)} = (\partial_\mu - \frac{n}{R} A_\mu^{(0)}) A_\nu^{(n)}, \tag{5.8}$$

so that the covariant derivative transforms as

$$D_\mu A_\nu^{(n)} \to \mathrm{e}^{-i\frac{n}{R}\epsilon} D_\mu A_\nu^{(n)}, \tag{5.9}$$

under the transformations (5.7).

The metric transforms as

$$\delta g_{\mu\nu}(x,\theta) = -\frac{\epsilon}{R}\frac{\partial}{\partial\theta} g_{\mu\nu}(x,\theta), \tag{5.10}$$

or in terms of modes

$$\delta g_{\mu\nu}^{(n)}(x) = -i\frac{n}{R}\epsilon\, g_{\mu\nu}^{(n)}(x). \tag{5.11}$$

Hence, only $g_{\mu\nu}^{(0)}(x)$ is not charged under the $U(1)$ symmetry.

There is yet one more thing that we can deduce. The modes with $n \neq 0$, $g_{\mu\nu}^{(n)}$ and $A_\mu^{(n)}$ are massive. The reason for this is that to get the correct degrees of freedom for such fields one should have a gauge invariant action with a gauge symmetry for each independent vector field (or tensor field). But for the charged fields we see from (5.7) and (5.11) there is no such symmetry. For the theory to make sense the only way out is that they be massive, so that such a gauge invariance for each mode is not required (and not present). This allows us to make an important observation. The only massless symmetric tensor is $g_{\mu\nu}^{(0)}(x)$ so it is the metric on M. If there were other symmetric 2-tensors available (if some of the $g_{\mu\nu}^{(n)}(x)$ were massless) then there would be some confusion as to which is “the” metric on M, however, since none of the other modes are massless, we have a unique candidate for the metric and there is no confusion.

Massless on M_D but Massive on M?

We concluded in the discussion above that the non-zero modes of the higher dimensional theory on M_D are massive from the point of view of the lower dimensional theory on M. Lets see how this comes about.

Let Φ be a massless scalar on $M_D = M \times K$ with a diagonal metric,

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu + R^2 d\theta \otimes d\theta.$$

This means that the radius of the circle is R as the Riemannian volume (circumference) of the circle is, as we saw before,

$$\mathrm{Vol}(S^1) = \int_0^{2\pi} d\theta\,\sqrt{g} = 2\pi R.$$

The Laplacian Δ_{M_D} on the product space is,

$$\begin{aligned}
\Delta_{M_D} &= \Delta_M + \Delta_{S^1} \\
&= -g^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} \\
&= \Delta_M - \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}.
\end{aligned} \tag{5.12}$$

As Φ is massless on M_D it satisfies

$$\Delta_{M_D}\Phi(x,\theta) = 0,$$

or

$$\Delta_M\Phi(x,\theta) = -\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\Phi(x,\theta),$$

which in terms of a Fourier series for $\Phi(x,\theta) = \sum_n \Phi^{(n)}(x)e^{in\theta}$ reads

$$\Delta_M\Phi^{(n)}(x) = \frac{n^2}{R^2}\Phi^{(n)}(x). \tag{5.13}$$

This last equation (5.13) is the Klein-Gordon equation for a massive scalar field. This tells us that from the point of view of M the $\Phi^{(n)}(x)$ are massive, except for $\Phi^{(0)}$, with mass

$$M_n = \frac{|n|}{R}. \tag{5.14}$$

It is important to notice that the Laplacian on the internal space $K = S^1$ acts essentially as a mass squared operator for each mode

$$\Delta_{S^1}\Phi(x,\theta) = \sum_n \frac{n^2}{R^2}\,\Phi^{(n)}(x)\,\mathrm{e}^{in\theta} \tag{5.15}$$

Notice that the charge and mass of the fields is correlated

$$M_n^2 = e_n^2. \tag{5.16}$$

Lets take stock of the situation. Up to this point we have an infinite tower of symmetric rank 2 tensor fields of vector fields and of scalars on M. Apart from the zero-mode fields they are all massive and charged. We also want K to be small, which means that we want R, which tells us the size of the circle, to be small. But as $R \to 0$ the massive fields are becoming very massive indeed!

We now have to turn our attention to what this means for physics on M. From the point of view of M it takes more and more energy to create such super heavy particles in interactions so that they effectively decouple. Only the massless particles remain.

6 General Kaluza-Klein

In this section we will generalise the previous discussion and consider a general situation with some internal compact closed space K. As before we do not see the "internal" space K so it had better be small. A natural question that arises now is what is the field content of our theory given the product structure and where the internal manifold is suitably small?

The answer to that question of course depends on the theory at hand. We have to be a bit more precise about what we are doing. When we say that space-time has the form $M \times K$ what we are suggesting is that there exists a metric which satisfies Einsteins equation's (augmented with corrections depending on the theory) which can be put on such a space. Now these equations are highly non-linear and very difficult to solve in full generality. The attitude that I will adopt at this point is that we have somehow solved them and that thus we have determined our "background" $M \times K$.

Having fixed the background we will now concentrate on the equations of motion so that we can determine the masses of the fields as seen from the viewpoint of M. The field equations that we consider are those for p-forms in the background gravitational field. These may well not be the only equations of motion that you come across but, at least in string theory, they are equations that we will have to deal with.

Antisymmetric Tensor Zero-Modes

The equation of motion for a rank p antisymmetric tensor B_p in D dimensions is

$$\Delta_D B_p = 0 \tag{6.1}$$

where Δ_D is the D- dimensional Laplacian,

$$\Delta_D = \delta d + d\delta \tag{6.2}$$

We also impose the gauge fixing condition

$$*d * B_p = 0. \tag{6.3}$$

In components (6.1) reads

$$-g^{MP}\nabla^M \nabla^P B_{N_1,\ldots,N_p} + \ldots = 0, \tag{6.4}$$

while the gauge fixing condition is

$$\nabla^{N_1} B_{N_1,N_2,\ldots,N_p} + \ldots = 0. \tag{6.5}$$

Here is an explicit example for a rank one antisymmetric tensor field (a complicated way of saying a vector). Maxwells equations on a curved manifold are

$$*d * dA = g_{MN}\nabla^M \nabla^N A_R - \nabla^M \nabla_R A_M = 0, \tag{6.6}$$

while the gauge fixing condition is

$$*d*A = \nabla^M A_M = 0. \tag{6.7}$$

Adding (6.7) and, the derivative of, (6.6) gives us back (6.1). This argument works for all antisymmetric tensors (regardless of the rank).

The nice thing about a product manifold is that you have a product metric for which all the covariant derivatives split

$$\nabla^M = \left(\nabla^\mu, \nabla^i\right), \tag{6.8}$$

consequently the equations of motion take the form

$$\left(\Delta_M + \Delta_K\right) B_p = 0, \tag{6.9}$$

where Δ_M and Δ_K are the Laplacians on M and on K respectively. Notice that (5.12) is a special case of (6.9).

Given an equation of this kind we can 'seperate variables' and expand the p-form in a basis of eigenfunctions of Δ_K. The coefficients of this expansion will depend on M (and be forms there). For example if we consider a 0-form (function) and let $\{\omega_i^{(0)}(y)\}$ be an orthogonal basis of eigenfunctions for Δ_K with eigenvalue λ_i we can write

$$B_0 = \sum_i g_i(x)\,\omega_i^{(0)}(y)$$

and the equation of motion becomes

$$\sum_i \left(\Delta_M g_i(x)\,\omega_i^{(0)}(y) + g_i(x)\lambda_i\omega_i^{(0)}(y)\right) = 0$$

or, as the functions $\omega_i^{(0)}$ are orthogonal on K, we have

$$\Delta_M g_i(x) + \lambda_i\, g_i(x) = 0$$

Hence (6.9) tells us that, from the d dimensional point of view, i.e. from the point of view of M, the Laplacian Δ_K on the internal space K behaves like a mass squared operator, just as it did when K was taken to be a circle (5.15).

Now here comes the "crunch" (literally), if K is very small then the non-zero eigenvalues of Δ_K will be very large. You saw an example of this for the case of toroidal compactification-actually we do not need anything so fancy, we just remember that for functions on a circle of radius R the momenta go like $2i\pi n/R$, so that the Laplacian has eigenvalues n^2/R^2. When one makes the circle small $R \downarrow 0$, the non-zero eigenvalues go scooting off to infinity. Fields with such large eigenvalues are super-heavy so that for low energy purposes we will not see nor need them[2]. In general we have our scaling

[2]Notice that this is not quite like the way the massive states of string theory are eliminated. There the masses go like M_{planck}. Here, on the otherhand, we decide how big or small K is -unfortunately there seems to be no mechanism which dictates a value, and indeed the size becomes a moduli parameter of the theory.

theorem that says if we scale the metric by t, that is $g \to tg$ or the volume $dV \to t^{n/2} dV$ ($n = D - d$), the eigenvalues of the Laplacian Δ_K get scaled by t^{-1} so, as we decrease the volume of K by letting $t \to 0$, the non-zero eigenvalues are sent to infinity.

After scaling the metric on K the equation of motion becomes

$$\left(\Delta_M + t^{-1}\Delta_K\right) B_p = 0$$

That is what we will do, we will take the volume of K to be very small, $t \downarrow 0$, so that the fields cannot be arbitrary functions of y^i but rather they must be harmonic forms. Given that only harmonic forms of K enter we solve the infinity problem we were running into before since there are only a finite number of harmonic forms on K. So we demand

$$\Delta_K B_p = 0. \tag{6.10}$$

This is a very strong condition on the fields since there are relatively few, linearly independent, harmonic forms on a compact manifold. In fact (6.10) implies that B_p is covariantly constant. From (6.10) we have

$$\begin{aligned} 0 &= \int_K B_p * \Delta_K B_p \\ &= \int_K \left(dB_p * dB_p + \delta B_p * \delta B_p\right) \end{aligned} \tag{6.11}$$

but the integrand is an absolute square (we are in a space of Euclidean signature) so at each point of K it is positive semi-definite[3]. For the integral to vanish one therefore finds,

$$dB_p = \delta B_p = 0. \tag{6.12}$$

Scalars

When $p = 0$ the antisymmetric tensor is a scalar, $\phi(x, y)$, and (6.12) tells us that the scalar is y^i independent,

$$\frac{\partial \phi}{\partial y^i} = 0. \tag{6.13}$$

Consequently, in the low energy theory in four dimensions, the 10-dimensional scalar becomes a true four dimensional scalar.

[3] The d that appears here is the exterior derivative on K only.

Vectors

Recall that a our 10-dimensional vector $A_M = (A_\mu, A_i)$ in this case (6.12) splits into two equations (remember the affinities split so that $\Gamma^M_{i\mu} = 0 = \Gamma^i_{\mu M} = \Gamma^\mu_{iM}$)

$$\begin{aligned}\frac{\partial A_\mu}{\partial y^j} &= 0 &(6.14)\\ \nabla_j A_i - \nabla_i A_j &= 0 &(6.15)\\ \nabla^i A_i &= 0. &(6.16)\end{aligned}$$

The first equality (6.14) tells us that A_μ has no y dependence and so is a single vector on M. The second equation (6.15) tells us that the six dimensional vector A_i is flat, which means that locally it is pure gauge, i.e $A_i = \partial_i \Lambda$ ($A = d\Lambda$) for some Λ will solve this equation. We need to look at this a bit more closely. If Λ is globally defined then plugging $A_i = \partial_i \Lambda$ into (6.16) would tell us that Λ is constant, which in turn would imply that $A_i = 0$. However, we do not need that Λ is globally defined we only need that $\partial_i \Lambda$ is well defined. To make life simple lets pretend that K is $X_5 \times S^1$ for some five dimensional manifold X_5. Let θ be the co-ordinate on the S^1. Now let $\Lambda = \theta$, such a Λ is not globally defined since when $\theta \to \theta + 2\pi$, Λ jumps by 2π, however $\partial_i \Lambda = \delta_i^\theta$ is well defined (it is a constant). The message here is that if K has non-trivial 1-cycles in it (the terminology is circle = 1-cycle) then there can be a non-trivial solution to (6.15,6.16). One denotes the space of non-trivial 1-cycles in K by $H_1(K)$-it is called the first homology group of K and the dimension of this space is denoted by $b_1(K)$, which is called the first betti number of K. There are plenty of manifolds that have no one-cycles, for example spheres do not, and there are plenty of manifolds that do have one-cycles, for example Tori do. Lets write a basis of the solution space as γ^1_a (which are one-forms) where $a = 1, \ldots, b_1(K)$. This means that we can expand those A_i which solve (6.15,6.16) in this basis as

$$A_i(x,y)dy^i = \sum_{a=1}^{b_1(K)} \phi^a(x) \gamma^1_a(y) \tag{6.17}$$

where the ϕ^a are four dimensional scalars.

Lets summarize this situation: the massless (from M's point of view) fields that make up A_M is one vector A_μ and $b_1(K)$ scalars ϕ^a.

Rank 2 Antisymmetric Tensor

In this case the equations become

$$\begin{array}{ll}\partial_i B_{\mu\nu} = 0 & \\ \partial_i B_{j\mu} - \partial_j B_{i\mu} = 0, & \nabla^i B_{i\mu} = 0 \\ \partial_i B_{jk} + \partial_j B_{ki} + \partial_k B_{ij} = 0 & \nabla^i B_{ij} = 0\end{array} \tag{6.18}$$

Now the first equation tells us that $B_{\mu\nu}$ does not depend on y so that from here we get one antisymmetric tensor on M. The second line of (6.18) tells us that, thinking of $B_{i\mu}$ as a $D-d$ dimensional vector with a dummy μ label, for each μ there are again $b_1(K)$ solutions. Once more we expand,

$$B_{i\mu}(x,y)dy^i = \sum_{a=1}^{b_1(K)} B^a_\mu(x)\gamma^1_a \tag{6.19}$$

and so on M we find $b_1(K)$ massless vectors B^a_μ.

The first equation on the last line of (6.18) can be solved by letting $B_{ij} = \partial_i\Sigma_j - \partial_j\Sigma_i$ ($B = d\Sigma$), but the second equation would imply $\Sigma_i = 0$, if Σ_i is globally defined. But as before all we need is that $d\Sigma$ be well defined Σ may jump. This time it is not enough to have just one circle direction because the "curl" $d\Sigma$ requires that Σ have more than one non-zero component. So lets take K, again for simplicity, to have a product form $X_4 \times T^2$, with $y^5 = \theta^1$ and $y^6 = \theta^2$. Notice that both of the equations are satisfied if $\Sigma = (0,0,0,0,-\theta^2,+\theta^1)$ with $B_{56} = 2$. The moral is that if one wants a harmonic two-form then there must be a non-trivial 2-cycle (this time any non-trivial two-manifold) in the manifold in question. The space of such two-cycles is denoted by $H_2(K)$, the second homology group of K and its dimension is the second betti number of K, $b_2(K)$. Let γ^2_A, $A = 1,\ldots,b_2(K)$ be a basis for $H_2(K)$. The upshot this time is that for two-forms which solve the last two equations of (6.18) we may expand

$$B_{ij}(x,y)dy^i \wedge dy^j = \sum_{A=1}^{b_2(K)} \phi^A(x)\gamma^2_A \tag{6.20}$$

so that on M the ϕ^A are $b_2(K)$ scalars.

Rank 3 Antisymmetric Tensor

The story with an antisymmetric tensor tensor of rank three is similar. $C_{\mu\nu\rho}$ is y independent. $C_{\mu\nu i}$ can be expanded as

$$C_{\mu\nu i}(x,y)dy^i = \sum_{a=1}^{b_1(K)} C^a_{\mu\nu}(x)\gamma^1_a, \tag{6.21}$$

which means that we have $b_1(K)$ two forms $C^a_{\mu\nu}$. The field $C_{\mu ij}$ likewise has an expansion as

$$C_{\mu ij}(x,y)dy^i \wedge dy^j = \sum_{A=1}^{b_2(K)} C^A_\mu(x)\gamma^2_A, \tag{6.22}$$

and so there are $b_2(K)$ massless vectors C^A_μ on M. Let $H_3(K)$ be the third homology group of K, and γ^3_r, $r = 1,\ldots,b_3(K)$, a basis for it. Then C_{ijk} has the expansion,

$$C_{ijk}(x,y)dy^i \wedge dy^j \wedge dy^k = \sum_{r=1}^{b_3(K)} C^r(x)\gamma^3_r, \tag{6.23}$$

which means that there are $b_3(K)$ massless scalars $C^r(x)$.

Nonlinear Equations and Zero Modes

Actually as we are taking it for granted that our manifold has the form $M \times K$ means that the we really wish to consider fluctuations of the metric (and other fields) around this configuration. This means that we can linearize the full equations to obtain the linear equation for the quantum fluctuations. But until we know the equations we wish to solve we will not know what to linearize. So that will have to wait till an imaginary next lecture.

References

[1] T. Eguchi, P. Gilkey and A. Hanson, **Gravitation, Gauge theories and Differential Geometry**, Physics Reports (1980) 213-393.