NOTES ON DIFFERENTIAL GEOMETRY FOR THE 2015 MEXICO LASS

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CHAPTER 1

Introduction

This is a course on various aspects of differential geometry, designed for young, to be, string theorists. Unfortunately, to be a string theorist it seems one needs to know a good bit of most branches of mathematics. It is impossible to cover such a wide field (even if I did know the material). Our vista is somewhat narrower. I will introduce the basic definitions of topological spaces, differential manifolds (real and complex), Kähler manifolds and then special holonomy manifolds.

Along the way I will introduce tools from algebraic topology, differential topology, differential geometry and algebraic geometry without proof (or even justification in some instances).

One of my aims is to pass on the idea that on some given space one may introduce interesting structures. Most importantly, though those structure carry the same name they may be inequivalent. For example, speces may carry different topologies, the same topological space may carry many differentiable structures or none at all. A differentiable manifold may allow for many inequivalent metrics. Also a differentiable manifold may have a complex structure, or many complex structures or none at all of these and so on.

This course has been heavily influenced[1] by the following four sources. In the beginning there was Eguchi, Glikey and Hanson [**3**] from which generations of physicists have learn't the mathematical ideas and techniques relevant to their research and not just in string theory. Many still keep a copy under their pillow. I first came across Calabi-Yau manifolds in a magnificent course by Phillip Candelas [**2**]. This is a wonderful introduction to complex manifolds, Kähler manifolds and 3 dimensional Calabi-Yau manifolds. Then there is the string theory bible, namely volume II of the book by Green, Scwharz and Witten [**4**]. And for all things special holonomy the book by Joyce [**5**].

[1] A polite way of saying that I have seriously plagerised from these.

CHAPTER 2

Topology

This is as basic as it gets. We are presented with a set

2.1. The Definitions

DEFINITION 2.1.1. A topological space $(X, \mathcal{T})$ is a non-empty set X and a collection of subsets of X, $\mathcal{T}$ such that:

(1) $X \in \mathcal{T}$
(2) $\emptyset \in \mathcal{T}$
(3) If $U_1, \ldots, U_n \in \mathcal{T}$ then $U_1 \cap \cdots \cap U_n \in \mathcal{T}$
(4) Let I be some indexing set, if for each $i \in I$, $U_i \in \mathcal{T}$ then $\cup_{i\in I} U_i \in \mathcal{T}$

DEFINITION 2.1.2. The sets $U_i \in \mathcal{T}$ are the open sets of the topological space $(X, \mathcal{T})$.

There is always the roughest topology that one can put on any set namely $\mathcal{T} = (X, \emptyset)$ which is known as the indiscreet topology. Likewise one also has the finest topology that can be put on a set namely the discrete topology where $\mathcal{T}$ contains all subsets of X.

It is common to abbreviate the topological space to 'X' which I will also do, however, you should be aware that the set X can carry different topologies as we have just seen.

DEFINITION 2.1.3. The standard topology of the real line $\mathbb{R}$ is the one where a set $U \subseteq \mathbb{R}$ is open if for every $x \in U$ there are real numbers a and b such that x is contained in the open interval (a, b) and that interval is contained in U, $x \in (a, b) \subseteq U$.

This is the 'open-ball' topology of the real line. One defines the open-ball topology for $\mathbb{R}^n$ analogously.

DEFINITION 2.1.4. A topological space is Hausdorff if for each pair of distinct points a, b in X there are open subsets U, $V \subset X$ with $a \in U$, $b \in V$ and such that $U \cap V = \emptyset$.

A topological space with the indiscrete topology is not Hausdorff.

We also need the concept of compactness.

DEFINITION 2.1.5. An open covering of a topological space X is a collection of open sets $\{U_i\}_{i\in I}$ such that $\cup_{i\in I} U_i = X$ and this has a finite sub-covering if a finite number of the U_i's can be chosen which still cover X. That is for $K \subseteq I$ and finite $\cup_{i\in K} U_i = X$.

DEFINITION 2.1.6. A topological space X is compact if every open covering has a finite sub-covering.

DEFINITION 2.1.7. A topological space X is second countable if there exists some countable (possibly infinite) collection $\mathcal{U} = \{U_i\}_{i \in I}$ of open subsets such that any open set of X can be expressed as the union of some sub family of $\mathcal{U}$.

CHAPTER 3

Real Manifolds

Giving topological spaces a bit more structure allows one to get a handle on differentiability of functions on those spaces or even of maps between such spaces. But this is an extra structure above that of a topological space and so it can happen, and does happen, that the same topological space can have inequivalent differentiable structures on it. Milnor [**7**] showed that the seven sphere posses several distinct differentiable (smooth) structures. Other, rather more exotic, examples include the topological 4-manifold $\mathbb{R}^4$ which admits uncountably distinct smooth structures. The converse also occurs, there are examples of topological manifolds that admit no smooth structure at all [**6**].

3.1. Charts and Atlases

Manifolds should be thought of as spaces, of a fixed dimension n say, made of rubber so that you can stretch them as much as you like without tearing them. Locally open subsets of such a space 'look like' open subsets of $\mathbb{R}^n$. More concretely for X a toplogical space we require that we have 1-1 (injective) maps between open subsets $\phi_i(U_i) \to V_i$ where U_i is an open subset of X and V_i an open subset of $\mathbb{R}^n$ (with the standard topology). The pair (ϕ_i, U_i) is called a chart and the collection of all (ϕ_i, U_i) $i \in I$ is called an atlas (yes these names do come from good old fashioned pirating days on the high seas).

Definition 3.1.1. A C^r $n-$dimensional manifold X is a Hausdorff topological space together with an atlas (ϕ_i, U_i) where the U_i are some collection of open subsets of X and the ϕ_i are injective maps to open subsets of $\mathbb{R}^n$ satisfying the following three properties:

(1) $\cup_{i\in I} U_i = X$
(2) For $U_i \cap U_j \neq \emptyset$ then $\phi_{ij} = \phi_i \circ \phi_j^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is C^r for all $i,\, j \in I$.
(3) The atlas is maximal, that is, if (ϕ, U) is a chart such that $\phi \circ \phi_i^{-1}$ and $\phi_i \circ \phi^{-1}$ are C^r for all $i \in I$ then (ϕ, U) is part of the atlas.

We will consider those manifolds which are smooth that is from now on all our manifolds will be C^∞.

Example 3.1.1. The n-dimensional sphere S^n in $\mathbb{R}^{n+1}$, $S^n \subset \mathbb{R}^{n+1}$ defined by the equation $\sum_{i=1}^{n+1} (x^i)^2 = 1$ is a smooth manifold. One may define the charts by stereographic projection, namely

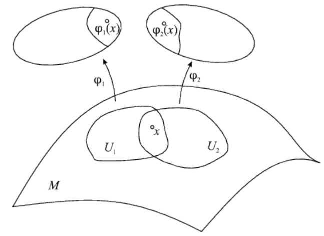


FIGURE 1. Here we see how the same point is mapped under two different charts. The point is the same just its 'coordinates' have changed. Requiring that ϕ_{12} be smooth says that we may change from one set of coordinates to the other smoothly. Diagram thanks to .

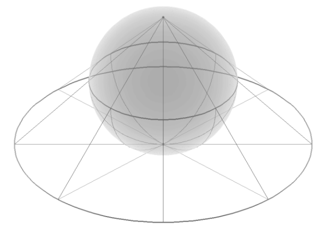


FIGURE 2. Stereographic projection from the 'north-pole' $x^{n+1} = 1$ to $\mathbb{R}^n$ at $x^{n+1} = -1$. Excluding the north-pole we have the map $x^i \longrightarrow x_1^i = x^i/(1-x^{n+1})$. A second patch can be obtained by stereographic projection from the 'south-pole' which is the map $x^i \longrightarrow x_2^i = x^i/(1+x^{n+1})$. Diagram thanks to Wikipedia.

The transition function is $x_2^i = (\phi_{21} \circ x_1)^i =$

EXAMPLE 3.1.2. The prime examples of manifolds for string theorists, Riemann surfaces of genus 0, 1 and g.

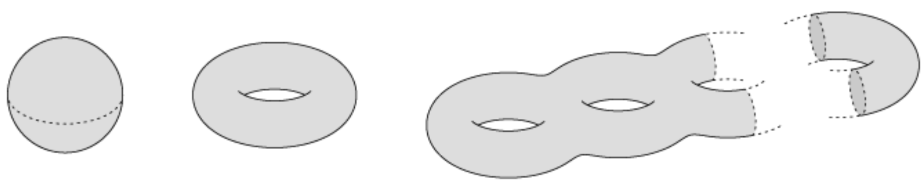


FIGURE 3. Three Riemann surfaces, a sphere of genus zero, a torus of genus one and a general genus g surface. Diagram thanks to .

DEFINITION 3.1.2. A manifold is orientable if there exists an atlas (ϕ_i, U_i) such that for every non empty intersection $U_i \cap U_j$ the Jacobian at every point is positive

$$\det\left(\frac{\partial\phi_i(x)}{\partial\phi_j(x)}\right) > 0, \quad x \in U_i \cap U_j$$

The way to think of the derivative is to consider ϕ_i on the intersection as $\phi_{ij}(\phi_j(x))$.

It is not at all true that every manifold is orientable. The real projective spaces $\mathbb{RP}^n$ are defined by identifying lines in $\mathbb{R}^{n+1}\backslash\{0\}$, $(x^1, \ldots, x^{n+1}) \simeq \lambda(x^1, \ldots, x^{n+1})$ with $\lambda \in \mathbb{R}\backslash\{0\}$. The $\mathbb{RP}^n$ are orientable for n odd and non-orientable when n is even.

EXAMPLE 3.1.3. The real projective space $\mathbb{RP}^2 = S^2/\mathbb{Z}_2$ the identification is by antipodal points.

3.2. Tangent Spaces

Suppose that C is a curve in some space X so that $C : \mathbb{R} \to X$. In local coordinates at $x \in U_i \subset X$ we have that the parameterised curve through x would be $\phi_i^\mu(x(t))$ which we write as $x^\mu(t)$ (more correctly as $x_i^\mu(t)$ so that we keep track of the patch- but this is just too much notation). The tangent vector in local coordinates is $\dot{x}^\mu(t)$ but what is it 'on' X itself?

In looking for an intrinsic description of vectors and vector fields mathematicians have come up with the following viewpoint. Let's fix on $\mathbb{R}^n$ and note that a vector v at a point $p \in \mathbb{R}^n$ can be defined by its action on functions. So for $f : \mathbb{R}^n \to \mathbb{R}$ we define

$$v(f) = \sum_{\mu=1}^{n} v^\mu \partial_\mu f\big|_p \quad \in \mathbb{R}$$

From this point of view a vector is a map of a function to a number. Furthermore, if we change coordinates v^μ and ∂_μ change inversely so that nothing changes. This is intrinsic. In the case of the tangent to the curve that we started with $v^\mu = \dot{x}^\mu(t)$.

We may use the same definition for functions on an $n-$ dimensional manifold X, so let $f : X \to \mathbb{R}$. In any open subset U_i we have a chart (ϕ_i, U_i) and $\widehat{f}_i = f \circ \phi_i^{-1}$ is a map from the image of the open set in $\mathbb{R}^n$ to $\mathbb{R}$. We define the action of a vector (field) at the point $p \in X$ by

$$\begin{aligned} v(f)(p) &= \sum_{\mu=1}^{n} v_i^\mu(\phi_i(p)) \, \frac{\partial f \circ \phi_i^{-1}(\phi_i(p))}{\partial \phi_i^\mu(p)} \\ &= \sum_{\mu=1}^{n} v_i^\mu(x_i) \, \frac{\partial \widehat{f}_i(x_i)}{\partial x_i^\mu} \end{aligned}$$

How do these transform if we change charts or say on overlaps so that $p \in U_i \cap U_j$? The answer is that if we are dealing with intrinsic objects then they do not depend on the choice of chart. Let's see how this works for functions first. On the intersection $U_i \cap U_j$ we also have local coordinates $\phi_j(p)$ and the local function

$$\widehat{f}_j(x_j) = f(p) = \widehat{f}_i(x_i)$$

which is how a function transforms under a general coordinate transformation. The point here is that the function f is a given map and does not care how we wish to view it locally.

Turning our attention to the vector we have

Definition 3.2.1. A vector field v on an n-dimensional manifold X is given by

$$v = \sum_{\mu=1}^{n} v_i^\mu(x_i) \, \frac{\partial}{\partial x_i^\mu}$$

on each open subset $U_i \subset X$ and does not depend on the chart.

To ensure that this is a good definition we require on the overlap $U_i \cap U_j$ that

$$v = \sum_{\mu=1}^{n} v_i^\mu(x_i) \, \frac{\partial}{\partial x_i^\mu} = \sum_{\mu=1}^{n} v_j^\mu(x_j) \, \frac{\partial}{\partial x_j^\mu}$$

The relationship between the partial derivatives is given by the chain rule and so the way the components of the vector transform is

$$v_i^\mu = \frac{\partial x_i^\mu}{\partial x_j^\nu} \, v_j^\nu$$

which is the transformation that you recognize from special and general relativity (and I have moved over to the Einstein summation convention). Once more this is the transformation rule that says that

$$v|_{U_i} = v|_{U_j}\,, \quad \text{on} \quad U_i \cap U_j$$

so that the vector field v is globally defined and does not depend on the chart.

Definition 3.2.2. The space of tangent vectors at a point $x \in X$ is called the tangent space at x and is denoted by $T_x X$.

Definition 3.2.3. The tangent bundle TX of X is the collection of the tangent spaces at all points of X

$$TM = \bigcup_{x \in X} T_x X$$

There is a projection map $\pi : TX \longrightarrow X$, which sends the 'fibre' $T_x X$ to x.

In the definition the fibre is the vector space which sits above the point at which it is defined. A vector field is now naturally thought of as a section of the tangent bundle,

Definition 3.2.4. A section, v of TX is a map $v : X \longrightarrow TX$, such that $\pi \circ v = Id$, where Id is the identity map of X to X.

Notice that the condition $\pi \circ v = Id$ guarantees that $v(x) \in T_x X$ as it should be for a vector field.

3.3. Cotangent Spaces

Given a vector space V one gets for free a dual vector space V^* of linear maps from V to $\mathbb{R}$ or to $\mathbb{C}$ depending on the context. In physics vector field components are contravariant while their duals are covariant. Having identified the vector fields as contravariant we now want to determine the covariant objects.

Definition 3.3.1. The cotangent space $T_x^* X$ at $x \in X$ is the dual vector space to the tangent space $T_x X$.

Definition 3.3.2. The cotangent bundle $T^* X$ of X is the collection of the cotangent spaces at all points of X

$$T^* M = \bigcup_{x \in X} T_x^* X$$

There is a projection map $\pi : T^* X \longrightarrow X$, which sends the 'fibre' $T_x^* X$ to x.

I use the same symbol π for the projection map though it acts on different spaces. I do this as the obvious alternatives π_* and π^* are taken.

Given a basis $\{e_i\}$ for a vector space V there is a natural dual basis $\{w^i\}$ for V^* given by $w^j(e_i) = \delta_i^j$. For the vector fields the basis is d/dx^μ so the designation of the basis for the cotangent bundle is dx^μ (noting that $dx^\mu/dx^\nu = \delta_\nu^\mu$).

Definition 3.3.3. A section, w of T^*X is a map $w : X \longrightarrow TX$, such that $\pi \circ w = Id$.

In each open subset $U_i \subset X$ we have

$$w = \sum_{\mu=1}^{n} w_{i\,\mu}(x_i)\, dx_i^\mu \tag{3.3.1}$$

and on the overlap $U_i \cap U_j$

$$w_{i\,\mu}(x_i) = \sum_{\nu=1}^{n} \frac{\partial x_j^\nu}{\partial x_i^\mu}\, w_{j\,\nu}(x_j)$$

which guarantees that the section is globally well defined.

3.4. Tensor Products and Tensors

Tensors of various covariant and covariant degrees are sections of tensors products of the tangent and cotangent bundles of X.

Definition 3.4.1. A tensor τ of degree $(r,\, s)$ is a section of the tensor product

$$\overbrace{TX \otimes \cdots \otimes TX}^{\text{r factors}} \otimes \underbrace{T^*X \otimes \cdots \otimes T^*X}_{\text{s factors}}$$

in local coordinates in $U_i \subset X$

$$\tau = \sum_{\substack{\mu_1\cdots\mu_r \\ \nu_1\ldots\nu_s}} \tau^{\mu_1\ldots\mu_r}_{\nu_1\ldots\nu_s}(x_i)\, \frac{\partial}{\partial x_i^{\mu_1}} \otimes \cdots \otimes \frac{\partial}{\partial x_i^{\mu_r}} \otimes dx_i^{\nu_1} \otimes \cdots \otimes dx_i^{\nu_s}$$

These tensors are reducible.

3.5. Differential Forms

Let X be a manifold of dimension n. On such a space we can consider antisymmetric tensors of rank p where $0 \le p \le n$. There is a nice way to write such tensors which allows us to avoid using too many labels. The idea is to use a Grassmann odd (i.e. anticommuting) object dx^μ with the properties that products (called wedge products) are antisymmetric

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu = \frac{1}{2}\left(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu\right) \tag{3.5.1}$$

though I will sometimes (without warning) drop the wedge $\wedge$ and write

$$dx^\mu dx^\nu = -dx^\nu \, dx^\mu, \tag{3.5.2}$$

instead. The way one can represent a rank p antisymmetric tensor is as a p 'form' (p is referred to as the degree of the form),

$$\alpha_p = \frac{1}{p!}\, \alpha_{\mu_1,\dots,\mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \tag{3.5.3}$$

where

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} = \frac{1}{p!} \sum_{\sigma \in \text{permutations}} (-1)^{|\sigma|}\, dx^{\mu_{\sigma(1)}} \otimes \cdots \otimes dx^{\mu_{\sigma(p)}} \tag{3.5.4}$$

For example a zero form is a scalar

$$\alpha_0 = \phi, \tag{3.5.5}$$

while a one form is a vector

$$\alpha_1 = A_\mu \, dx^\mu, \tag{3.5.6}$$

and a two-form is an antisymmetric tensor of rank 2

$$\alpha_2 = \frac{1}{2} B_{\mu\nu}\, dx^\mu \wedge dx^\nu, \tag{3.5.7}$$

and so on.

Definition 3.5.1. The space of smooth p-forms is referred to as $\Omega^p(X, \mathbb{R})$ if the coefficients $\alpha_{\mu_1,\dots,\mu_p}$ are real or $\Omega^p(X, \mathbb{C})$ if the coefficients are complex and we say the form is real valued in the first case or complex valued in the second.

As we can add p-forms together and they are still p-forms and also multiply p-forms either by real or complex numbers we deduce that $\Omega^p(X, \mathbb{R})$ is an infinite dimensional vector space over the reals while $\Omega^p(X, \mathbb{C})$ is an infinite dimensional vector space over $\mathbb{C}$. You should check that these spaces indeed satisfy all of the conditions for being a vector space.

Definition 3.5.2. Differential forms have a natural degree associated with them, namely the degree of a $p-$form is p.

There is another operation on the spaces of forms one can wedge a form α_p with a form β_q to get a form λ_{p+q}. Formally we would write that there is a map

$$\Omega^p(X, \mathbb{R}) \otimes \Omega^q(X, \mathbb{R}) \xrightarrow{\wedge} \Omega^{p+q}(X, \mathbb{R}) \tag{3.5.8}$$

and likewise for complex forms. Degrees are additive since we are only counting the number of dx's. We have

$$\lambda_{p+q} = \alpha_p \wedge \beta_q = (-1)^{pq}\, \beta_q \wedge \alpha_p, \tag{3.5.9}$$

the last equality following from antisymmetry of the dx's.

The structures that we have just described almost form an algebra.

DEFINITION 3.5.3. An algebra over a field K is vector space V with a bi-linear product

$$V \otimes_K V \to V$$

In our case both $\mathbb{R}$ and $\mathbb{C}$ are fields so all is well. But as we saw the bi-linear map (3.5.8) mixes our vector spaces so we are not quite there. However, we can form another vector space out of the ones that we have

DEFINITION 3.5.4.

$$\Omega^*(X, \mathbb{R}) = \Omega^0(X, \mathbb{R}) \oplus \Omega^1(X, \mathbb{R}) \oplus \cdots \oplus \Omega^{n-1}(X, \mathbb{R}) \oplus \Omega^n(X, \mathbb{R})$$

(likewise for complex valued forms)

and now

$$\Omega^*(X, \mathbb{R}) \otimes \Omega^*(X, \mathbb{R}) \xrightarrow{\wedge} \Omega^*(X, \mathbb{R}) \tag{3.5.10}$$

Furthermore, there is a natural grading of this algebra given by whether the degree is odd or even and this is reflected in (3.5.9).

DEFINITION 3.5.5. The graded algebra $\Omega^*(X, \mathbb{R})$ is called Cartan's exterior algebra or just the exterior algebra of forms.

3.6. Exterior Derivative

There is a natural differentiation operator, d, that one can now introduce called the exterior derivative and it is defined as

$$d = \sum_\mu dx^\mu \partial_\mu, \tag{3.6.1}$$

where it is understood that the dx^μ part just acts by wedge multiplication. This definition makes sense as both dx^μ and ∂_μ change by the chain rule but inversely as we change patches so that on the overlap

$$\sum_\mu dx_i^\mu \left.\frac{\partial}{\partial x_i^\mu}\right|_{U_i \cap U_j} = \sum_\mu dx_j^\mu \left.\frac{\partial}{\partial x_j^\mu}\right|_{U_i \cap U_j}$$

By the definition acting on a zero form α_0 with d we have,

$$d\alpha_0 = d\phi = \partial_\mu \phi \, dx^\mu, \tag{3.6.2}$$

while on one forms we find

$$d\alpha_1 = dA_\mu \, dx^\mu = \partial_\nu A_\mu \, dx^\nu \wedge dx^\mu = \frac{1}{2} F_{\nu\mu} \, dx^\nu \wedge dx^\mu, \tag{3.6.3}$$

which is nice for physicists because they recognize the field strength of a gauge field entering and finally

$$\begin{aligned} d\alpha_2 &= \frac{1}{2} \partial_\rho B_{\mu\nu} \, dx^\rho \wedge dx^\mu \wedge dx^\nu \\ &= \frac{1}{3!} \left(\partial_\rho B_{\mu\nu} + \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu}\right) dx^\rho \wedge dx^\mu \wedge dx^\nu. \end{aligned} \tag{3.6.4}$$

Acting with d raises the degree by one and a way of saying this is to write

$$d : \Omega^p(X) \to \Omega^{p+1}(X). \tag{3.6.5}$$

putting all forms together we get yet another action on the exterior algebra

$$d : \Omega^*(X) \to \Omega^*(X). \tag{3.6.6}$$

On products of forms one has

$$d\left(\alpha_p \wedge \beta_q\right) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q. \tag{3.6.7}$$

Since d is anticommuting it also has the useful property that

$$d^2 = 0. \tag{3.6.8}$$

as one may easily check. One says that d is nilpotent.

3.7. Stokes Theorem

Another nice aspect of differential forms is that they can be integrated. For example suppose we wish to integrate over a two manifold then traditionally we would like to make use of a measure[1] that we write as $d^2x = dxdy$. In terms of forms, in two dimensions

$$\begin{aligned} \alpha_2 &= \frac{1}{2} B_{\mu\nu}\, dx^\mu \wedge dx^\nu \\ &= B_{xy}\, dx \wedge dy \end{aligned} \tag{3.7.1}$$

and so it is natural to integrate α_2! The important point is that no metric is required.

Stokes theorem tells you that if you integrate a form which is a "total" derivative (exact) on a manifold with boundary then integral is only on the boundary

$$\int_X d\alpha_{n-1} = \int_{\partial X} \alpha_{n-1} \tag{3.7.2}$$

where ∂X stands for the boundary of X (one has to be careful with 'signs' here because the boundary components may have different orientations).

[1] We would also add $\sqrt{g}$ where g is the metric, but we will come to that later.

CHAPTER 4

Riemannian Geometry

So far there has been no metric on any of our spaces. It happens to be a fact that one can always find a Riemannian metric on any differential manifold with Euclidean signature. Usually there are many inequivalent metrics that can be put on the manifold X, so when talking about a Riemannian manifold we need to include the metric that we are referring to, so we write (X, g) for a Riemannian manifold. Not every smooth manifold admits a metric whose signature is Lorentzian, for example, only the 2-torus of all the compact closed Riemann surfaces admits a metric with Lorentzian signature. The necessary and sufficient condition for the existence of a Lorentzian metric on a smooth and compact manifold is that it have globally defined non-zero vector fields which is equivalent to the Euler character vanishing.

Unless otherwise stated the signature is always taken to be Euclidean.

4.1. The Hodge Star and an Inner Product on the Space of Forms

If the manifold X comes equipped with a Riemannian metric g then there is an operation, known as the Hodge star. I will give two equivalent definitions.

DEFINITION 4.1.1. The Riemannian volume measure dV_g of the Riemannian manifold (X, g) is given by

$$dV_g = \sqrt{g}\, d^n x$$

DEFINITION 4.1.2. Let $\alpha,\, \beta \in \Omega^p(X, \mathbb{C})$ where (X, g) is a Riemannian manifold and set the point wise inner product to be

$$(\alpha,\, \beta)_x = \sum \left(\alpha_{\mu_1 \dots \mu_p} \overline{\beta_{\nu_1 \dots \nu_p}}\, g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p}\right)(x)$$

EXERCISE 4.1.1. Show that for $\alpha \in \Omega^p(X, \mathbb{C})$ that $(\alpha,\, \alpha)_x \in \mathbb{R}$ while for $\alpha \in \Omega^p(X, \mathbb{R})$ that $(\alpha,\, \alpha)_x \in \mathbb{R}_+$.

DEFINITION 4.1.3. The inner product of $\alpha,\, \beta \in \Omega^p(X, \mathbb{C})$ where (X, g) is a Riemannian manifold is

$$(\alpha,\, \beta) = \int_X (\alpha,\, \beta)_x\, dV_g$$

DEFINITION 4.1.4 (Hodge Star I). The Hodge star is a map $* : \Omega^p(X, \mathbb{C}) \longrightarrow \Omega^{n-p}(X, \mathbb{C})$ defined so that for any $\beta \in \Omega^p(X, \mathbb{C})$ then $\alpha \wedge (*\beta) = (\alpha,\, \beta)_x\, dV_g\ \forall \alpha \in \Omega^p(X, \mathbb{C})$.

PROPOSITION 4.1.1. The Hodge star operator acting on $\beta \in \Omega^p(X, \mathbb{C})$ on an n-dimensional Riemannian manifold (X, g) satisfies

$$*(*\beta) = (-1)^{p(n-p)}\beta$$

and consequently, as $*$ is invertible, it provides an isomorphism

$$\Omega^p(X,\, \mathbb{C}) \simeq \Omega^{n-p}(X,\, \mathbb{C})$$

EXERCISE 4.1.2. Show that $*1 = dV_g$.

DEFINITION 4.1.5 (Hodge Star II). Given a Riemannian metric g on X there is a Hodge star $*$ action on the basis of forms given by

$$*\left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}\right) = \frac{\sqrt{g}}{(n-p)!}\epsilon^{\mu_1,\ldots,\mu_p}{}_{\mu_{p+1},\ldots,\mu_n}\, dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_n}$$

where

$$\epsilon_{\mu_1,\ldots,\mu_n} = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \\ 0 & \text{when two indices agree} \end{cases}.$$

For a p-form α_p we have

$$\begin{aligned} \alpha_p &= \frac{1}{p!}\alpha_{\mu_1,\ldots,\mu_p}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \\ *\alpha &= \frac{\sqrt{g}}{(n-p)!\,p!}\epsilon^{\mu_1,\ldots,\mu_p}{}_{\mu_{p+1},\ldots,\mu_n}\overline{\alpha_{\mu_1,\ldots,\mu_p}}\, dx^{\mu_{p+1}} \wedge \cdots \wedge dx^{\mu_n}. \end{aligned} \tag{4.1.1}$$

EXERCISE 4.1.3. Prove Proposition 4.1.1.

DEFINITION 4.1.6. The Riemannian volume form dV_g is defined to be $dV_g = *1$,

$$dV_g = \frac{1}{n!}\sqrt{g}\,\epsilon_{\mu_1\ldots\mu_n}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} = \sqrt{g}\, d^n x$$

One may invert this relation to find

$$dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} = \sqrt{g}\, d^n x = \bar{\epsilon}^{\nu_1,\ldots,\nu_n} d^n x,$$

where $\bar{\epsilon}$ is defined to be

$$\bar{\epsilon}^{\nu_1,\ldots,\nu_n} = \begin{cases} 1 & \text{even permutation} \\ -1 & \text{odd permutation} \\ 0 & \text{when two indices agree} \end{cases}$$

DEFINITION 4.1.7. The metric inner product is

$$(\alpha, \beta) = \int_M \alpha \wedge *\overline{\beta},$$

and explicitly this is

$$(\alpha, \beta) = \frac{1}{p!}\int_M \sqrt{g}\,\alpha_{\mu_1,\ldots,\mu_p}\,\overline{\beta^{\mu_1,\ldots,\mu_p}}\, d^n x.$$

We need to show that the explicit form is indeed correct. One can easily determine the integrand,

$$\begin{aligned}\alpha * \beta &= \frac{\sqrt{g}}{(n-p)!p!p!}\alpha_{\nu_1,\dots,\nu_p}\overline{\beta_{\mu_1,\dots,\mu_p}}\epsilon^{\mu_1,\dots,\mu_p}{}_{\mu_{p+1},\dots,\mu_n} dx^{\nu_1}\dots dx^{\nu_p}dx^{\mu_{p+1}}\dots dx^{\mu_n}\\ &= \frac{\sqrt{g}}{(n-p)!p!p!}\alpha_{\nu_1,\dots,\nu_p}\overline{\beta_{\mu_1,\dots,\mu_p}}\epsilon^{\mu_1,\dots,\mu_p}{}_{\mu_{p+1},\dots,\mu_n}\overline{\epsilon}^{\nu_1,\dots,\nu_p\,\mu_{p+1},\dots,\mu_n} d^n x.\end{aligned}$$

Note that

$$\epsilon^{\mu_1,\dots,\mu_p}{}_{\mu_{p+1},\dots,\mu_n}\overline{\epsilon}^{\nu_1,\dots,\nu_p\,\mu_{p+1},\dots,\mu_n} = (n-p)!\,(g^{\mu_1\nu_1}\dots g^{\mu_p\nu_p} \pm \text{permutations})$$

the $(n-p)!$ comes from the possible permutations of the $(n-p)$ indices that we are summing over. The total number of permutations in the brackets on the right hand side is $p!$. You can see this by noting that μ_1 can be paired with any of the p, ν_i, while then for μ_2 we have a choice of $p-1$, ν_i and so on. We have then

$$\alpha * \beta = \frac{\sqrt{g}}{p!}\alpha_{\mu_1,\dots,\mu_p}\,\overline{\beta^{\mu_1,\dots,\mu_p}}\,d^n x,$$

and so

$$(\alpha,\beta) = \frac{1}{p!}\int_M \sqrt{g}\,\alpha_{\mu_1,\dots,\mu_p}\,\overline{\beta^{\mu_1,\dots,\mu_p}}\,d^n x.$$

□

Note also that there is a simple relationship between ϵ and $\overline{\epsilon}$,

$$\overline{\epsilon}^{\mu_1,\dots,\mu_n} = \det g\, g^{\mu_1\nu_1}\dots g^{\mu_n\nu_n}\,\epsilon_{\nu_1,\dots,\nu_n}.$$

THEOREM 4.1.2. The inner product for $\alpha \in \Omega^p(X,\mathbb{R})$ is positive semi definite, that is

$$(\alpha,\alpha) \in \mathbb{R}_+$$

and equals zero iff $\alpha = 0$.

PROOF. This one can see directly from the explicit form of the inner product since everything in sight is real and positive semi-definite. □

REMARK 4.1.1. If we had allowed for a metric with a Lorentzian signature we could not conclude that the inner product is positive semi-definite.

4.2. Adjoint of the Exterior Derivative

For this section suppose that X is a compact and closed manifold - so that it has no boundary, $\partial X = \emptyset$. Then we can define an adjoint to the exterior derivative. It is the adjoint with respect to the inner product that we defined above and requires us to integrate by parts (which is why we want X to have no boundary).

DEFINITION 4.2.1. The adjoint δ of d with respect to the Riemannian inner product is

$$(d\alpha,\beta) = (\alpha,\delta\beta).$$

Up to the determination of the sign one easily sees that

$$\delta = (-1)^{np+n+1} * d * . \tag{4.2.1}$$

Clearly as d is nilpotent so too is δ,

$$\delta^2 = 0$$

Note that for $d\alpha$ and β to have the same degree in the inner product on the left hand side of (4.2.1) α must have degree one less than β but on the right hand side this must be the same as the degree of $\delta\beta$. Thus it is clear that δ lowers the degree by one, so one writes

$$\delta : \Omega^p(X, \cdot) \to \Omega^{p-1}(X \cdot).$$

and this extends to the exterior algebra of forms

$$\delta : \Omega^*(X \cdot) \to \Omega^*(X \cdot).$$

The degree of zero forms cannot be lowered so

$$\delta\phi(x) = 0,$$

however the degree of one forms can be lowered

$$\delta A = \delta A_\mu \, dx^\mu = \nabla^\mu A_\mu$$

and this is also nice for physicists because we recognize the covariant divergence of a vector (which is what we would use for gauge fixing on a curved manifold).

Notice that the metric enters, this is unavoidable since it entered in the very definition of δ. In particular this means that if you put two different metrics on the same topological space you will get two different δ's.

4.2.1. The Laplacian on Forms

Definition 4.2.2. The Laplacian Δ on forms of any degree is defined to be

$$\Delta = d\delta + \delta d,$$

and it does not change the degree of the form it acts on since d raises the degree by one while δ lowers the degree by one.

One immediate consequence of the form of the Laplacian is that on compact closed Riemannian X the spectrum is real and positive semi-definite. This means that the eigenvalues of Δ are real and zero or positive. The proof of this is exactly the same as for the Hamiltonian of the simple harmonic oscillator in quantum mechanics, so we will stop to prove it.

Theorem 4.2.1. Let X be a compact closed Riemannian manifold then the spectrum of Δ is positive semi-definite.

PROOF. Let ω be an eigenvalue $\Delta\,\omega = \lambda\omega$ and ω non-zero then

$$\begin{aligned}\lambda &= (\omega, \Delta\omega)/(\omega,\omega) \\ &= (d\omega, d\omega)/(\omega,\omega) + (\delta\omega,\delta\omega)/(\omega,\omega)\end{aligned}$$

Now every term in the second line is real so λ must be real and by Theorem 4.1.2 the inner product is positive semi-definite so that each term is also positive semi-definite. □

EXERCISE 4.2.1. Consider S^1 with the metric $g = d\theta \otimes d\theta$ where θ is the angular coordinate $[0, 2\pi)$. What are the eigenfunctions and eigenvalues of the Laplacian acting on functions?

If you work it out you can see that the Laplacian takes the form

$$\Delta = -g^{\mu\nu}\,\nabla_\mu\nabla_\nu + \dots$$

where the ellipses indicate terms with lower numbers of derivatives. You should notice two things. The first is the sign, it is correct so that the spectrum is positive semi-definite on a Riemannian manifold. That the sign is correct you can see by asking what would you expect on flat $\mathbb{R}^n$? Acting with Δ on a plane wave, $\exp(ip.x)$, we would get $p^2 \exp(ip.x)$ so that the eigenvalue is $p^2 \geq 0$. The second is that if we scale the metric by $g \to tg$ the first term on the right hand side scales by t^{-1}. This is true for the other terms too a result which is quite simple and which will be useful later on so we prove it.

THEOREM 4.2.2. Let $*$ be the Hodge star of the Riemannian metric g and Δ the associated Laplacian. Set $*_t$ to be the Hodge star of the Riemannian metric tg, $t \in (0,\infty)$ and Δ_t the associated Laplacian, then

$$\Delta_t = t^{-1}\,\Delta$$

PROOF. From the definition of the Hodge star (4.1.5) we see that

$$*_t\,\alpha_p = t^{n/2-p}\; * \alpha_p$$

Now, from (4.2.1)

$$\delta_t\alpha_p = \pm *_t d *_t \alpha_p = t^{-1}\delta\alpha_p$$

and this is enough to prove the result since the scaling does not depend on the degree of the form. □

This means that if λ is an eigenvalue of Δ then $t^{-1}\lambda$ is an eigenvalue of Δ_t (and the eigenvalues of the two Laplacians are one to one). This means that we can change the eigenvalues at will (by varying the metric in this way), unless the eigenvalue is zero.

Infact there is a more interesting result that we will not prove, namely

THEOREM 4.2.3. Let X be a compact, closed, oriented, Riemannian manifold. The eigenvalues of the Laplacian on p-forms has a discrete spectrum $0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots$.

Which means that by scaling the metric we can make the first non-zero eigenvalue λ_1 (and hence all non-zero eigenvalues) as large as we like. This argument does not apply to the zero eigenvalue. This result is of particular interest to us as we can follow the Riemannian volume while we scale.

PROPOSITION 4.2.4. The Riemannian volume of tg is related to that of g by

$$\mathrm{Vol}_{tg}(X) = \int_X *_t\, 1 = t^{n/2} \int_X * 1 = t^{n/2}\mathrm{Vol}_g(X)$$

As $t \downarrow 0$ the non-zero eigenvalues are becoming large but the volume is becoming small.

DEFINITION 4.2.3. A form ω is said to be harmonic if it satisfies

$$\Delta\, \omega = 0$$

THEOREM 4.2.5. A harmonic form ω on a compact closed Riemannian manifold satisfies

$$d\omega = 0, \quad \delta\omega = 0$$

and is said to be closed and co-closed.

PROOF. By the proof of Theorem 4.2.1 we have

$$0 = (d\omega, d\omega) + (\delta\omega, \delta\omega)$$

where each term is positive semi-definite by Theorem 4.1.2 and to be zero each term must be zero so again by Theorem 4.1.2 we deduce that ω is closed, $d\omega = 0$ and co-closed $\delta\omega = 0$. □

Harmonic forms, are unaffected by any scalings of the metric that we perform and play an important role in differential geometry as we will see shortly.

CHAPTER 5

Betti Numbers, Homology, Cohomology and Hodge Theory

Perhaps the most basic invariants of a topological manifold X are its Betti numbers $b_i(X)$. Our aim here is to give the definitions of these numbers but then to show how they may be determined by considering X as a topological manifold (homology), or through knowledge of the action of the exterior derivative the smooth structure that X has as a differentiable manifold (cohomology) and finally by considering zero eigen-forms of the Laplacian on thinking of X as (X, g) a Riemannian manifold (Hodge theory). One does a bit better in that three points of view allow us to find isomorphisms between certain groups whose dimensions are given by the Betti numbers.

5.1. Homology

Given a set of $p-$dimensional oriented submanifolds N_i of X one can form a $p-$chain.

Definition 5.1.1. A $p-$chain a_p is formal sum

$$\sum_i c_i N_i \,, \tag{5.1.1}$$

where the coefficients c_i determine the type of chain one has. If the $c_i \in \mathbb{R}(\mathbb{C})$ the $p-$chain is said to be a real (complex) $p-$chain. For $c_i \in \mathbb{Z}(\mathbb{Z}_2)$, and the chain is called an integer ($\mathbb{Z}_2$) chain and so on.

∂ is the operation that gives the oriented boundary of the manifold it acts on. For example, the sphere has no boundary so $\partial S^2 = \emptyset$, while the cylinder, $I \times S^1$, has two circles for a boundary, $\partial(I \times S^1) = \{0\} \times S^1 \oplus \{1\} \times -S^1$. I have written $\oplus$ to indicate that we will be 'adding' submanifolds to form chains when one should have used the union symbol $\cup$. The minus sign in front of the second S^1 is to indicate that it has opposite relative orientation. One defines the boundary of a $p-$chain to be a $(p-1)-$chain by

$$\partial a_p = \sum_i c_i \partial N_i . \tag{5.1.2}$$

Notice that $\partial^2 = 0$, as the boundary of a boundary is empty.

Definition 5.1.2. A $p-$cycle is a $p-$chain without boundary, i.e. if $\partial a_p = \emptyset$ then a_p is a $p-$cycle.

DEFINITION 5.1.3. Let $Z_p = \{a_p : \partial a_p = \emptyset\}$ be the set of $p-$cycles and let $B_p = \{\partial a_{p+1}\}$ be the set of $p-$boundaries of $(p+1)-$chains. The $p-$th simplicial homology group of X is defined by

$$\mathrm{H}_p(X,\,\cdot) = \mathrm{Z}_p(X,\,\cdot)/\mathrm{B}_p(X,\,\cdot)$$

The equivalence relation in the definition is the following: if $a_p \in \mathrm{Z}_p$, that is $\partial a_p = \emptyset$, then

$$a_p' = a_p + \partial c_{p+1}$$

for any c_{p+1} is identified with a_p in H_p.

The simplicial homology groups only depend on the fact that X is a topological space and not on its smooth structure. It is also a fact that the dimensions of these groups are finite (and for $p > n$ they are empty).

DEFINITION 5.1.4. The p-th Betti number $b_p(X)$ is defined to be

$$b_p(X) = \dim_{\mathbb{R}} \mathrm{H}_p(X,\,\mathbb{R})$$

The Betti numbers are 'topological invariants' that is they only depend on the topology of the manifold X and not on any other structure that it may have (for example if X is a Riemannian manifold they do not depend on the metric one has on X).

5.2. Cohomology

DEFINITION 5.2.1. Let Z^p be $\{\omega_p : d\omega_p = 0\}$ the set of closed $p-$forms and let B^p be $\{\omega_p : \omega_p = d\omega_{p-1}\}$ the set of exact $p-$forms. The $p-$th De Rham cohomology group is defined by

$$\mathrm{H}^p(X,\,\cdot) = \mathrm{Z}^p(X,\,\cdot)/\mathrm{B}^p(X,\,\cdot).$$

The equivalence relation means that if ω_p and ω_p' are in $\mathrm{Z}^p(X,\,\cdot)$ but are related by

$$\omega_p' = \omega_p + dc_{p-1}$$

for any $p-1$ form c_{p-1} then they represent the same element in $\mathrm{H}^p(X,\,\cdot)$.

5.3. De Rham's Theorems

Notice that the cohomology groups require X to be a manifold (and not just a topological space) since we need the concept of a smooth form. Nevertheless, the theorems of De Rham show that the homology and cohomology groups are intimately related as we will now see.

DEFINITION 5.3.1. The inner product of a $p-$cycle, $a_p \in H_p$ and a closed $p-$form, $\omega_p \in H^p$ is

$$\pi(a_p, \omega_p) = \int_{a_p} \omega_p. \tag{5.3.1}$$

Notice that this does not depend on the representatives used for, by Stokes theorem,

$$\begin{aligned} \pi(a_p + \partial a_{p+1}, \omega_p) &= \int_{a_p+\partial a_{p+1}} \omega_p = \int_{a_p} \omega_p + \int_{a_{p+1}} d\omega_p = \pi(a_p, \omega_p) \\ \pi(a_p, \omega_p + d\omega_{p-1}) &= \int_{a_p} \omega_p + \int_{\partial a_p} \omega_{p-1} = \pi(a_p, \omega_p). \end{aligned}$$

One may thus think of the **period** π as a mapping

$$\pi : \mathrm{H}_p(X,\, \mathbb{R}) \otimes \mathrm{H}^p(X,\, \mathbb{R}) \to \mathbb{R} \tag{5.3.2}$$

For X compact and closed De Rham has established two important theorems. Let $\{a_i\}$, $i = 1, \ldots b_p(X)$, be a set of independent $p-$cycles forming a basis $\mathrm{H}_p(X,\, \mathbb{R})$, where the $p-$th Betti number $b_p(X) = \dim_{\mathbb{R}} \mathrm{H}_p(X,\, \mathbb{R})$.

THEOREM 5.3.1. Given any set of periods ν_i, $i = 1 \ldots b_p$, there exists a closed $p-$form ω for which

$$\nu_i = \pi(a_i, \omega) = \int_{a_i} \omega.$$

THEOREM 5.3.2. If all the periods of a $p-$form ω vanish,

$$\pi(a_i, \omega) = \int_{a_i} \omega = 0 \tag{5.3.3}$$

then ω is an exact form.

Putting the previous two theorems together, we have that if $\{\omega_i\}$ is a basis for $\mathrm{H}^p(X,\, \mathbb{R})$ then the period matrix

$$\pi_{ij} = \pi(a_i, \omega_j) \tag{5.3.4}$$

is invertible. This implies

COROLLARY 5.3.1. The homology and cohomology groups, $\mathrm{H}_p(X,\, \mathbb{R})$ and $\mathrm{H}^p(X,\, \mathbb{R})$, are dual to each other with respect to the inner product π and so are naturally isomorphic. In particular the Betti numbers are

$$b_p(X) = \dim_{\mathbb{R}} \mathrm{H}_p(X,\, \mathbb{R}) = \dim_{\mathbb{R}} \mathrm{H}^p(X,\, \mathbb{R})$$

5.4. Poincaré Duality

If X is compact, orientable and closed of dimension n then $\mathrm{H}^n(X, \mathbb{R}) = \mathbb{R}$, as, up to a total differential, any $\omega_n \in \mathrm{H}^n(X, \mathbb{R})$ is proportional to the volume form. One aspect of Poincaré duality is the

THEOREM 5.4.1. $\mathrm{H}^p(X,\mathbb{R})$ is dual to $\mathrm{H}^{n-p}(X,\mathbb{R})$ with respect to the inner product

$$(\omega_p,\, \omega_{n-p}) = \int_X \omega_p \wedge \omega_{n-p}. \tag{5.4.1}$$

This implies that $\mathrm{H}^p(X,\,\mathbb{R})$ and $\mathrm{H}^{n-p}(X,\,\mathbb{R})$ are isomorphic as vector spaces, so that, in particular,

COROLLARY 5.4.1. The Betti numbers of a compact closed manifold X satisfy

$$b_p(X) = b_{n-p}(X)$$

We need the following statement

THEOREM 5.4.2. Given any $p-$cycle a_p there exists an $(n-p)-$form α, called the Poincaré dual of a_p, such that for all closed $p-$forms ω

$$\int_{a_p} \omega = \int_X \alpha \wedge \omega. \tag{5.4.2}$$

5.5. The Künneth Formula

This is a formula that relates the cohomology groups of a product space $X_1 \times X_2$ to the cohomology groups of each factor. The formula is

$$\mathrm{H}^p(X_1 \times X_2, \mathbb{R}) = \sum_{q=0}^{p} \mathrm{H}^q(X_1,\, \mathbb{R}) \otimes \mathrm{H}^{p-q}(X_2,\, \mathbb{R}). \tag{5.5.1}$$

In particular this implies that $b_p(X_1 \times X_2) = \sum_{q+r=p} b_q(X_1)\, b_r(X_2)$.

Notation: I will denote a basis for $\mathrm{H}_1(X,\mathbb{R})$ by γ_i and the dual basis for $\mathrm{H}^1(X,\mathbb{R})$ by $[\gamma_i]$. For $\mathrm{H}_2(X,\mathbb{R})$ I denote the basis by Σ_i (as a mnemonic for a two dimensional manifold) and the corresponding basis for $\mathrm{H}^2(X,\mathbb{R})$ by $[\Sigma_i]$.

5.6. Hodge Theory

The De Rham theorems are very powerful and very general. Rather than working with a cohomology class $[\omega_p]$ of a $p-$form ω_p it would be nice to be able to choose a canonical representative of the class. This is what Hodge theory gives us and along the way also gives us a different characterization of the cohomology groups.

Up till now, we have not needed a metric on the manifold X. The introduction of a metric, though not needed in the general framework, leads to something new. Let X be equipped with a metric g. Define the Hodge star $*$ operator as in (4.1.5), $* : \Omega^p \to \Omega^{(n-p)}$, by

$$*\omega_{\mu_1,\dots,\mu_p} dx^{\mu_1} \dots dx^{\mu_p} = \frac{\sqrt{g}}{(n-p)!} \omega_{\mu_1,\dots,\mu_p} \epsilon^{\mu_1 \dots \mu_p}{}_{\mu_{p+1} \dots \mu_n} dx^{\mu_{p+1}} \dots dx^{\mu_n}. \tag{5.6.1}$$

The symbol $\epsilon_{\mu_1 \ldots \mu_n}$ is 0 if two labels are repeated and $\pm$ for even or odd permutations respectively. All labels are raised and lowered with respect to the metric.

THEOREM 5.6.1. For a compact manifold without boundary, any p-form can be uniquely decomposed as a sum of an exact, a co-exact and a harmonic form,

$$\omega_p = d\alpha_{(p-1)} + \delta\beta_{(p+1)} + \gamma_p$$

this is referred to as the Hodge decomposition.

The harmonic form γ_p is our representative for $[\omega_p]$. This corresponds to choosing $d * \omega_p = 0$ as the representative. From the point of view of gauge theory, this amounts to the usual Landau gauge (extended to higher dimensional forms).

I will not prove the complete theorem but, rather, establish that the decomposition is an orthogonal one with respect to the metric inner product. So the aim is to show that

$$(d\alpha_{(p-1)}, \delta\beta_{(p+1)}) = 0, \quad (d\alpha_{(p-1)}, \gamma_p) = 0, \quad (\delta\beta_{(p+1)}, \gamma_p) = 0 \tag{5.6.2}$$

The first of these follows from the fact that we can, by Definition 4.2.1, move the adjoint exterior derivative from $\delta\beta_{(p+1)}$ to $d\alpha_{(p-1)}$ but then we get $d^2\alpha_{(p-1)} = 0$. Likewise, in the second and third expressions we may move either d or δ to act on γ_p but then, by Theorem 4.2.5, these are both zero.

DEFINITION 5.6.1. The space of harmonic p-forms is denoted here by

$$\mathrm{H}^p_\Delta(X, \cdot) = \{\omega_p \in \Omega^p(X, \cdot) | \, \Delta\omega_p = 0\}$$

We note that these spaces are finite dimensional. Indeed for X compact closed we have an isomorphism

$$\mathrm{H}^p_\Delta(X, \mathbb{R}) \simeq \mathrm{H}^p(X, \mathbb{R}) \tag{5.6.3}$$

which you can see as follows: Let $d\omega_p = 0$ then by the Hodge decomposition $\omega_p = d\alpha_{(p-1)} + \delta\beta_{(p+1)} + \gamma_p$ we have that $d\delta\beta_{(p+1)} = 0$, so that $(\beta_{(p+1)}, d\delta\beta_{(p+1)}) = 0 = (\delta\beta_{(p+1)}, \delta\beta_{(p+1)})$ or that $\delta\beta_{(p+1)} = 0$, whence $\omega_p = d\alpha_{(p-1)} + \gamma_p$ so that the harmonic form γ_p is a representative for the cohomology class of ω_p. Conversely if ω_p is harmonic then it must be γ_p since the decomposition is orthogonal.

This gives us yet another another way of determining the Betti numbers of X, namely as

$$b_p(X) = \dim_{\mathbb{R}} \mathrm{H}^p_\Delta(X, \mathbb{R})$$

PROPOSITION 5.6.2. The Hodge star is an isomorphism between the Hodge groups of a connected compact closed manifold X of dimension n

$$\mathrm{H}^p_\Delta(X, \mathbb{R}) \simeq \mathrm{H}^{n-p}_\Delta(X, \mathbb{R})$$

with corresponding isomorphisms with and between $\mathrm{H}^p(X, \mathbb{R})$ and $\mathrm{H}_p(X, \mathbb{R})$. In particular the Betti numbers satisfy,

$$b_p(X) = b_{n-p}(X)$$

CHAPTER 6

Kaluza Klein Reduction

We are in a position to apply the math that we have picked up in previous chapters. The idea here is that space-time is actually a lot bigger than we first thought and that "all" of the physics that we see is actually a consequence of gravitational interactions in the bigger space. In the case of string theory this will not be quite true as there will be other fields also to consider in the higher dimensional space-time X. Physics, for present purposes, is taken to mean the action, since it determines both the classical physics (at the level of equations of motion) and the quantum physics (at the level of the path integral).

6.1. Product Manifolds

Let our space time be denoted by M and have dimension d, this could Minkowski space-time or a Friedman-Walker type universe or $\ldots$. The relationship between what we see, M and the actual space time, X of dimension D is

$$X = M \times K, \qquad \dim_{\mathbb{R}} X = \dim_{\mathbb{R}} M + \dim_{\mathbb{R}} K, \tag{6.1.1}$$

where K is taken to be some compact, closed and 'small' space. Small here means that its volume is much less than that of M. For various reasons, some of which will be explained below, we would in the normal course of things not notice that K is there. Nevertheless, the existence of such a K allows us to deduce various things about possible physics on M.

The space M could be considered to have to have a metric with Euclidean signature, however, for the present consider it to have a Euclidean signature. The internal space K will be given a metric with Euclidean signature. This means that we can apply all the machinery we developed about differential forms to K.

Denote the local coordinates on X by

$$X^M = (x^\mu, y^i) \tag{6.1.2}$$

where the x^μ are coordinates on M and y^i are coordinates on K. For example if $\dim_{\mathbb{R}} X = 10$ and $\dim_{\mathbb{R}} M = 4$, we would have $M = 1, \ldots, 10$, $\mu = 0, \ldots, 3$ and $i = 5, \ldots, 10$.

The Lorentz groups are, in general,

Euclidean Signature	Lorentz Signature	Manifold
$SO(D)$	$SO(D-1,1)$	X
$SO(d)$	$SO(d-1,1)$	M
$SO(D-d)$	$SO(D-d)$	K

but for a product manifold of the type that we are considering (6.1.1) the Lorentz group is $SO(d-1,1) \times SO(D-d) \subset SO(D-1,1)$.

Our problem is that, if physics really comes from the bigger space M_D then all our fields will depend on the X^M coordinates. Let $\Phi(X)$ be some object transforming under $SO(1, D-1)$ in some representation, for example, $\Phi = g_{MN}$ or A_M or ψ_α or $\ldots$. Then there are two related questions to answer,

- How does Φ decompose under $SO(1, d-1) \times SO(D-d)$?
- What does this mean for physics on M?

The aim in this chapter is to answer these questions and in particular to apply the answers to the type IIA and IIB string theories.

6.2. Massless on X but Massive on M?

We concluded in the discussion above that the non-zero modes of the higher dimensional theory on X are massive from the point of view of the lower dimensional theory on M. Lets see how this comes about.

Let Φ be a massless scalar on $X = M \times K$. Being a scalar of $SO(D-1,1)$ means that Φ is also a scalar of both $SO(D-2,1)$ and $SO(1)$. Fix a diagonal metric on the product space,

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu \otimes dx^\nu + R^2 d\theta \otimes d\theta.$$

This means that the radius of the circle is R as the Riemannian volume (circumference) of the circle is, as we saw before,

$$\mathrm{Vol}(S^1) = \int_0^{2\pi} d\theta\, \sqrt{g} = 2\pi R.$$

The Laplacian Δ_X on the product space is,

$$\begin{aligned} \Delta_X &= \Delta_M + \Delta_{S^1} \\ &= -g^{\mu\nu}\nabla_\mu \nabla_\nu - \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} \\ &= \Delta_M - \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2}. \end{aligned} \tag{6.2.1}$$

As Φ is massless on X it satisfies

$$\Delta_X\, \Phi(x, \theta) = 0,$$

or

$$\Delta_M \Phi(x,\theta) = -\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}\Phi(x,\theta),$$

which in terms of a Fourier series for $\Phi(x,\theta) = \sum_n \Phi^{(n)}(x)e^{in\theta}$ reads

$$\Delta_M \Phi^{(n)}(x) = \frac{n^2}{R^2}\Phi^{(n)}(x). \tag{6.2.2}$$

This last equation (6.2.2) is the Klein-Gordon equation for a massive scalar field. This tells us that from the point of view of M the $\Phi^{(n)}(x)$ are massive, except for $\Phi^{(0)}$, with mass

$$M_n = \frac{|n|}{R}. \tag{6.2.3}$$

It is important to notice that the Laplacian on the internal space $K = S^1$ acts essentially as a mass squared operator for each mode

$$\Delta_{S^1}\Phi(x,\theta) = \sum_n \frac{n^2}{R^2}\,\Phi^{(n)}(x)\,\mathrm{e}^{in\theta} \tag{6.2.4}$$

Lets take stock of the situation. Up to this point we have an infinite tower of scalars on M. Apart from the zero-mode fields they are all massive. We also want K to be small, which means that we want R, which tells us the size of the circle, to be small. But as $R \to 0$ the massive fields are becoming very massive indeed!

We now have to turn our attention to what this means for physics on M. From the point of view of M it takes more and more energy to create such super heavy particles in interactions so that they effectively decouple. Only the massless particles remain.

6.3. General Kaluza-Klein

In this section we will generalise the previous discussion and consider a general situation with some internal compact closed space K. As before we do not see the “internal” space K so it had better be small. A natural question that arises now is what is the field content of our theory given the product structure and where the internal manifold is suitably small?

The answer to that question of course depends on the theory at hand. We have to be a bit more precise about what we are doing. When we say that space-time has the form $M \times K$ what we are suggesting is that there exists a metric which satisfies Einsteins equation’s (augmented with corrections depending on the theory) which can be put on such a space. Now these equations are highly non-linear and very difficult to solve in full generality. The attitude that I will adopt at this point is that we have somehow solved them and that thus we have determined our “background” $M \times K$.

Having fixed the background we will now concentrate on the equations of motion so that we can determine the masses of the fields as seen from the viewpoint of M. The field equations that we consider are those for p-forms in the background gravitational field. These may well not be the only equations of motion that you come across but, at least in string theory, they are equations that we will have to deal with.

Antisymmetric Tensor Zero-Modes

The equation of motion for a rank p antisymmetric tensor B_p in D dimensions is

$$\Delta_D B_p = 0 \tag{6.3.1}$$

where Δ_D is the D- dimensional Laplacian,

$$\Delta_D = \delta d + d\delta \tag{6.3.2}$$

We also impose the gauge fixing condition

$$*d * B_p = 0. \tag{6.3.3}$$

In components (6.3.1) reads

$$-g^{MP}\nabla^M\nabla^P B_{N_1,\ldots,N_p} - R_{M[N_1}B^M{}_{N_2\ldots N_p]} - \frac{p(p-1)}{2}R_{PQ[N_1N_2}B^{PQ}{}_{N_3\ldots N_p]} = 0,$$

while the gauge fixing condition is

$$\nabla^{N_1}B_{N_1,N_2,\ldots,N_p} = 0. \tag{6.3.4}$$

Here is an explicit example for a rank one antisymmetric tensor field (a complicated way of saying a vector). Maxwells equations on a curved manifold are

$$*d * dA = g_{MN}\nabla^M\nabla^N A_R - \nabla^M\nabla_R A_M = 0, \tag{6.3.5}$$

while the gauge fixing condition is

$$*d * A = \nabla^M A_M = 0. \tag{6.3.6}$$

Adding (6.3.6) and, the derivative of, (6.3.6) gives us back (6.3.1). This argument works for all antisymmetric tensors (regardless of the rank).

The nice thing about a product manifold is that you have a product metric for which all the covariant derivatives split

$$\nabla^M = \left(\nabla^\mu, \nabla^i\right), \tag{6.3.7}$$

consequently the equations of motion take the form

$$\left(\Delta_M + \Delta_K\right) B_p = 0, \tag{6.3.8}$$

where Δ_M and Δ_K are the Laplacians on M and on K respectively. Notice that (6.2.1) is a special case of (6.3.8).

Given an equation of this kind we can ‘separate variables’ and expand the p-form in a basis of eigenfunctions of Δ_K. The coefficients of this expansion will depend on M (and be forms there). For example if we consider a 0-form (function) and let $\{\omega_i^{(0)}(y)\}$ be an orthogonal basis of eigenfunctions for Δ_K with eigenvalue λ_i we can write

$$B_0(x,y) = \sum_i g_i(x)\,\omega_i^{(0)}(y)$$

and the equation of motion becomes

$$\sum_i \left(\Delta_M g_i(x)\, \omega_i^{(0)}(y) + g_i(x)\lambda_i \omega_i^{(0)}(y)\right) = 0$$

or, as the functions $\omega_i^{(0)}$ are orthogonal on K, we have

$$\Delta_M g_i(x) + \lambda_i\, g_i(x) = 0$$

Hence (6.3.8) tells us that, from the d dimensional point of view, i.e. from the point of view of M, the Laplacian Δ_K on the internal space K behaves like a mass squared operator, just as it did when K was taken to be a circle (6.2.4).

Now here comes the "crunch" (literally), if K is very small then the non-zero eigenvalues of Δ_K will be very large. You saw an example of this for the case of toroidal compactification-actually we do not need anything so fancy, we just remember that for functions on a circle of radius R the momenta go like $2i\pi n/R$, so that the Laplacian has eigenvalues n^2/R^2. When one makes the circle small $R \downarrow 0$, the non-zero eigenvalues go scooting off to infinity. Fields with such large eigenvalues are super-heavy so that for low energy purposes we will not see nor need them[1]. In general we have our scaling theorem that says if we scale the metric by t, that is $g \to tg$ or the volume $dV \to t^{n/2} dV$ $(n = D - d)$, the eigenvalues of the Laplacian Δ_K get scaled by t^{-1} so, as we decrease the volume of K by letting $t \to 0$, the non-zero eigenvalues are sent to infinity.

After scaling the metric on K the equation of motion becomes

$$\left(\Delta_M + t^{-1}\Delta_K\right) B_p = 0$$

That is what we will do, we will take the volume of K to be very small, $t \downarrow 0$, so that the fields cannot be arbitrary functions of y^i but rather they must be harmonic forms. Given that only harmonic forms of K enter we solve the infinity problem we were running into before since there are only a finite number of harmonic forms on K. So we demand

$$\Delta_K B_p = 0. \tag{6.3.9}$$

This is a very strong condition on the fields since there are relatively few, linearly independent, harmonic forms on a compact manifold. In fact (6.3.9) implies that B_p is covariantly constant. The argument is the same as that of the previous chapter, though we apply it to a form B_p which is the sum of forms of various degrees on K. As an example a one form on X $B_1(x,y) = B_\mu(x,y)dx^\mu + B_i(x,y)dy^i$ which is the sum of a 1-form on M (but a zero form on K) and a 1-form on K (but a zero form on M).

We have

[1]Notice that this is not quite like the way the massive states of string theory are eliminated. There the masses go like M_{planck}. Here, on the otherhand, we decide how big or small K is -unfortunately there seems to be no mechanism which dictates a value, and indeed the size becomes a moduli parameter of the theory.

PROPOSITION 6.3.1. Let the differentiable manifold X be a product ofdifferentiable manifolds $X = X_1 \times X_2$ with X_2 compact closed and $B_{(q,p-q)}$ the component of $B_p \in \Omega^p(X, \mathbb{C})$ that has degree q as a form on X_1 and degree $p-q$ as a form on X_2. Take $\{\omega^i_{p-q}\}$ for i some indexing set to be a basis of the infinite dimensional vector space $\Omega^{p-q}(X_2, \mathbb{C})$. Then,

$$B_{(q,p-q)}(x, y) = \sum_i B^i_q(x) \wedge \omega^i_{p-q}, \quad B^q_i \in \Omega^q(X_1, \mathbb{C})\ \forall i$$

In any case from (6.3.9) we have

$$\begin{aligned} 0 &= \int_K B_p *_K \Delta_K B_p \\ &= \int_K (d_K B_p *_K d_K B_p + \delta_K B_p *_K \delta_K B_p) \end{aligned} \tag{6.3.10}$$

where the subscript K on the Hodege star, the exterior derivative and its adjoint remind us that they live on K only in these equations. The integrand is an absolute square (as only forms of the same degree on K will be paired) and so for the integral to vanish one therefore finds,

$$d_K B_p = \delta_K B_p = 0. \tag{6.3.11}$$

By Proposition (6.3.1) we see that we are only interested in those forms $B^{(q,p-q)}$ where the ω^i_{p-q} are harmonic. We will denote by $\{\gamma^A_p\}$ a basis for harmonic p-forms for $0 \le p \le n$ (though the indexing set changes as we change p).

The solutions to our equations are therefore

$$B_p(X) = \sum_{q=0}^{p} \sum_{A=1}^{b_q(K)} B^A_{p-q}(M) \wedge \gamma^A_q(K) \tag{6.3.12}$$

EXERCISE 6.3.1. How do you understand this formula if $p > \dim_{\mathbb{R}} M$?

The fields on space-time are just the $B^A_{p-q}(M)$. For any p form on X we therefore get an array of fields on M, namely

Degree on M	p-form	$p-1$-form	$p-2$ -form	...	1-form	0-form
Number	1	$b_1(K)$	$b_2(K)$	...	$b_{p-1}(K)$	$b_p(K)$

Lets summarize this situation: The topology of the internal manifold really dictates the content of the fields that one sees in space-time.

EXERCISE 6.3.2. Write down the low energy field content on M for the bosonic fields (not the metric yet) that appear in the IIA and IIB theories where $X_{10} = M \times K$ and K is compact closed and $0 \le \dim_{\mathbb{R}} K \le 10$.

CHAPTER 7

Special Holonomy Manifolds

A rather important development in the study of differentiable manifolds (and bundles over them) was the introduction of the notion of holonomy. Given a connection on a bundle the holonomy tells one how an object changes as it is parallel transported around a closed loop. Vector bundles are naturally associated to G principle G bundles and the holonomy is then generically G. The holonomy of an orientable n-dimensional real manifold is generically $SO(n)$ (as it is the holonomy of the tangent bundle which is associated to the frame bundle). However, there may be special connections available on the vector bundle (or in the Riemannian case, special metrics) for which the holonomy is a subgroup of G. $SO(n)$ will act on various associated bundles of the frame bundle on our manifold, say E, and having only a subgroup act implies we get a splitting of that bundle $E = E_1 \oplus \cdots \oplus E_r$ into irreducible representation spaces of the subgroup of $SO(n)$.

Manifolds which admit special holonomy have covariantly constant spinors on them (mathematicians call them parallel spinors). The importance of this to string theory is that the presence of a covariantly constant spinor implies the existence of a supersymmetric ground state.

7.1. Berger's Classification, Wang's Theorem and Spinors

Let $x \in M$ be a point on our manifold and E_x the fibre of E above it. Parallel transport with respect to a connection on E and the Christoffel connection on TM along a closed loop C from x to x in M defines an isometry $E_x \longrightarrow E_x$. Physicists know group elements acting on the fibres as path ordered or Wilson loops (without the trace).

Definition 7.1.1. The holonomy group $\mathrm{Hol}(D)$ of the connection D is the subgroup of $SO(n)$ ($O(n)$ if M is not oriented) generated by the isometries of all closed loops C.

Suppose we have a covariantly constant section s of E then it detects the holonomy group as follows. Since s is covariantly constant then along a curve $C(t)$ from x_0 to x

$$C^M D_M s = 0 \tag{7.1.1}$$

which is easily solved to give us

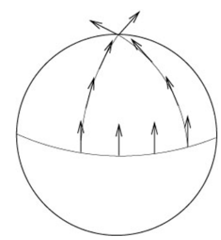
$$\begin{aligned} s(x) &= P \exp \int_C A \,.\, s(x_0) \\ &= U(x,\, x_0)\,.\, s(x_0) \end{aligned} \tag{7.1.2}$$

Now for an arbitrary closed loop starting and ending at x we find that

$$s(x) = U(x,\, x)\,.\,s(x) \equiv U(x)\,.\,s(x) \tag{7.1.3}$$

$$\mathrm{Hol}(D) = \{U(x) \in SO(n) |\, U(x).s(x) = s(x)\} \subseteq SO(n) \tag{7.1.4}$$

Figure 1. Parallel transport of a vector, or a section of a bundle in general, yields a vector which depends on the path taken.



The main theorem in the field which applies to the holonomy of the tangent bundle.

Theorem 7.1.1 (Berger [**1**]). Let M be an oriented connected, simply connected Riemannian manifold which is neither symmetric nor locally a product manifold, then its holonomy can only be the SO group or one of the following subgroups

$\mathrm{Hol}(\nabla)$	Name	Ricci Flat
$SO(n)$		
$U(n) \subset SO(2n)$	Kähler manifold	
$SU(n) \subset SO(2n)$	Calabi-Yau manifold	Yes
$Sp(n) \subset SO(4n)$	Hyper Kähler manifold	Yes
$G_2 \subset SO(7)$	G_2 manifold	Yes
$\mathrm{Spin}(7) \subset SO(8)$	Spin(7) manifold	Yes

It turns out that studying covariantly constant spinors allows for a more straightforward analysis. There are two reason we are interested in these. Firstly, given a covariantly constant spinor one gets for free covariantly constant forms, as we will see. Secondly, spinors are space time scalars and so technically easier to deal with.

Theorem 7.1.2 (Wang [**9**]). Let M be a connected, simply connected Riemannian spin manifold which is neither symmetric nor locally a product manifold, together with a non-zero covariantly constant spinor. Then the Ricci tensor vanishes and there are four possible cases

$\mathrm{Hol}(\nabla)$		# of Covariantly Constant Spinors
$SU(n) \subset SO(2n)$		2
$Sp(n) \subset SO(4n)$		n+1
$G_2 \subset SO(7)$		1
$Spin(7) \subset SO(8)$		1

Let us see how the Ricci flatness arises from having a covariant constant spinor. Let η be a covariantly constant spinor

$$D_M \eta = 0. \tag{7.1.5}$$

The covariant derivative on a spinor

$$D_M\eta^\alpha = \partial_M\eta^\alpha + \frac{1}{4}\omega_M^{AB}\,(\sigma_{AB})^\alpha{}_\beta\,\eta^\beta,$$

where ω_M^{AB} is the spin connection, $(A, B, \dots)$ are local Lorentz labels, $(\alpha, \beta, \dots)$ are spinor labels and, with Γ_A the gamma matrices, $\sigma_{AB} = \frac{1}{2}[\Gamma_A, \Gamma_B]$. The interesting point about (7.1.5) is that it is often not possible to find a non-zero η though according to Theorem 7.1.2 only on very special manifolds can this equation have a non-trivial solution. Consider the commutator

$$[D_M, D_N]\,\eta = 0 = R_{MN}^{AB}\sigma_{AB}\,\eta, \tag{7.1.6}$$

where

$$R_{MN}^{AB} = \partial_M\omega_N^{AB} - \partial_N\omega_M^{AB} + [\omega_M, \omega_N]^{AB}$$

are the components of the Riemann curvature two form, R^{AB},

$$R^{AB} = d\omega^{AB} + \omega^{AC}\omega_C{}^B = \frac{1}{2}R_{MN}^{AB}\,dX^M \wedge dX^N.$$

The Riemann curvature tensor is obtained if we make us of the zehn-bein e_M^A and its inverse e_A^M,

$$R_{MN}^{AB} = R_{MNPQ}\,e^{P\,A}\,e^{Q,B},$$

and the integrability condition (7.1.6) can be expressed as

$$R_{MNPQ}\,\Gamma^{PQ}\,.\,\eta = 0. \tag{7.1.7}$$

Notice that the rank three antisymmetric product of Gamma matrices is,

$$\Gamma^{PQR} = \frac{1}{3}\left(\Gamma^P\Gamma^{QR} + \Gamma^Q\Gamma^{RP} + \Gamma^R\Gamma^{PQ}\right) \tag{7.1.8}$$

and that it enjoys cyclic symmetry in the labels, $\Gamma^{PQR} = \Gamma^{QRP} = \Gamma^{RPQ}$. A little algebra shows that

$$\Gamma^P\Gamma^{QR} = \Gamma^{PQR} - g^{RP}\Gamma^Q + g^{PQ}\Gamma^R. \tag{7.1.9}$$

From this we deduce that

$$\begin{aligned} 0 &= R_{MPQR}\Gamma^P\Gamma^{QR}\eta \\ &= R_{MPQR}\left(\Gamma^{PQR} - g^{RP}\Gamma^Q + g^{PQ}\Gamma^R\right)\eta \end{aligned}$$

however, we may make use of the symmetry property of the Riemann curvature tensor

$$R_{MPQR} + R_{MQRP} + R_{MRPQ} = 0$$

to obtain

$$R_{MN}\Gamma^N\eta = 0, \tag{7.1.10}$$

where the Ricci tensor is

$$R_{MN} = R_{MPNQ}\;g^{PQ}.$$

One can get a quick result from this. Since $R_{MN} = R_{NM}$ we can multiply (7.1.10) by Γ^M to obtain

$$R\eta = 0$$

where the scalar curvature is

$$R = R_{MN} g^{MN}$$

and we learn that to have covariantly constant spinors M must allow for zero Ricci scalar. The stronger results follows on noting that the conjugate spinor will satisfy

$$\overline{\eta}\, R_{MN}\, \Gamma^N = 0, \tag{7.1.11}$$

so that

$$\begin{aligned} 0 &= \overline{\eta}\,.\, R_{MN}\, \Gamma^N \Gamma_L\,.\,\eta \\ &= \overline{\eta}\,.\, R_{MN}\, \{\Gamma^N, \Gamma_L\}\,.\,\eta \\ &= 2R_{ML}\, \overline{\eta}.\eta \end{aligned} \tag{7.1.12}$$

If η is non-zero then its norm $\overline{\eta}.\eta$ will also be nowhere zero, since together (7.1.5 and its conjugate imply $\partial_M(\overline{\eta}.\eta) = 0$) so that we conclude that

$$R_{MN} = 0. \tag{7.1.13}$$

7.2. Preserving Supersymmetry

In the previous lecture we learnt how 10 dimensional fields behave in a background of the form $M_4 \times K$. In doing this we saw that for the low energy theory there are still only a finite number of fields that one must keep track of for four dimensional physics. Our task is now to show that there are spaces of the form $M_4 \times K$ which **are** solutions of the super-gravity equations of motion.

But like most things in string theory when you want to answer one question you are instantly faced by another. The one that pops up this time is, do we want a solution that preserves some supersymmetry or not? One quick answer to this question is you would really like a solution with no supersymmetry since, if it exists in four dimensions, it should certainly be broken at low energy scales. Unfortunately, in string perturbation theory if one has broken the supersymmetry completely the theory goes “haywire” (this is not a technical term). What happens is that it seems to be impossible to stop a cosmological constant from being generated, but this will de-stabilize our favourite vacuum (meaning that the equations with a cosmological constant will have a different minimum). So to avoid that particular embarrassment we will search for solutions that preserve some supersymmetry.

Having decided to look for solutions that preserve some supersymmetry has another good side to it, it makes life easier. Solving the equations of 10 dimensional super-gravity, just like that, appears to be a tall order. However, when one is looking for solutions that preserve some supersymmetry, there is a useful trick that can be used.

Suppose we have a set of fields that are solutions to the equations of motion, then to say that the solution preserves some supersymmetry is to mean that the supersymmetry variations vanish for such a background. The trick is the following-if one finds fields for which the supersymmetry variations, for some supersymmetry parameter, vanish then those fields will satisfy the equations of motion. I will not prove that this trick works but give a heuristic proof and then we will just use it. The reason that this is useful is that the supersymmetry transformations are at most first order in derivatives while the equations of motion are second order. We now have a simpler set of equations to solve.

Consider a global supersymmetric theory with an unbroken supercharge Q. This means that Q is conserved and annihilates the vacuum. Let us denote the vacuum by $|\Omega >$. Then for any operator $\mathcal{O}$, $< \Omega|\{Q, \mathcal{O}\}|\Omega >= 0$. If $\mathcal{O}$ is fermionic then, $\{Q, \mathcal{O}\}$ is the supersymmetry transform of $\mathcal{O}$, $\delta\mathcal{O}$. But at the classical level, there is no difference between $< \Omega|\{Q, \mathcal{O}\}|\Omega >$ and $\delta\mathcal{O}$. Consequently finding a vacuum of unbroken supersymmetry (at the tree level) means finding a supersymmetry transformation such that $\delta\mathcal{O} = 0$. At the tree level we can check the transformations of the elementary fields.

The discussion so far has been for a global supersymmetric theory-what changes when the theory in question has local supersymmetry? The supersymmetry parameter now depends on the co-ordinates X^M. Which means, roughly, that for each space time point there is a supersymmetry parameter so there are an infinite number of conserved supercharges. Now one looks for those Q that generate unbroken supersymmetries.

7.3. The $N = 1$ Supersymmetric Ground State in $D = 10$

We begin with the $N = 1$ supegravity theory in 10 dimensions. The soulutions for the tyoe IIA and type IIB theories then follow.

The solutions that are of interest for us here[1] are those where all fields, except the graviton, are taken to be zero. This means that the supersymmetry transformations of bosonic fields automatically vanish, since they depend on the fermions, so we do not have to worry about those. The only transformation that we need to check is that of the gravitino

$$\delta\psi_M = \frac{1}{\kappa} D_M \eta + 0 \tag{7.3.1}$$

where the terms that have been set to zero are quartic fermionic terms plus those involving other fields. The paremeter of the supergravity transformations η is a Majorana-Weyl spinor.

As we have already discussed a supersymmetric ground state is obtained if one can find a solution to $\delta\psi_M = 0$, that is one tries to solve

For η not to identically vanish the ten dimensional space time must have vanishing scalar curvature. For example, a ten dimensional sphere will simply not do! The original equation (7.1.5) is, of course, more restrictive than the those we have derived from it,

[1]Though by no means are these the only ones of interest as you will see in later courses.

so the scalar curvature vanishing is the "least" requirement to obtain a supersymmetric vacuum.

In order to do an honest job of solving (7.1.5) we really ought to discuss the breaking of the holonomy group that such an equation implies. However, that would take us a little afield so what I will do is just show you that there are solutions of the form $M_4 \times K$-and the K have names!

If we consider the M_4 part of (7.1.5) we find, as that part of the spin connection $\omega_\mu^{AB} = 0$, that

$$\partial_\mu \eta = 0 \tag{7.3.2}$$

that is we will be dealing with a ground-state that has global supersymmetry in four dimensions. So essentially then we are left with the K part of the equation

$$\partial_i \eta^\alpha + \frac{1}{4}\omega_i^{ab} \left(\sigma_{ab}\right)^\alpha{}_\beta \, \eta^\beta = 0 \tag{7.3.3}$$

where $a, b = 1, \ldots, 6$ are internal Lorentz labels for K. In order to understand the meaning of this equation it is useful to decompose the spinor η, and the Gamma matrices in terms of $SO(1,3) \times SO(6)$.

Lets start with the Gamma matrices of $SO(1,9)$. In ten dimensions the gamma matrices Γ^M are 32 by 32 matrices which must satisfy the Clifford algebra

$$\left\{\Gamma^M, \Gamma^N\right\} = 2\eta^{MN} \tag{7.3.4}$$

where the signature of the Minkowski metric is $(-, +, \ldots, +)$.

Definition 7.3.1. The $SO(1,3)$ and $SO(6)$ subgroups are generated by

$$\Gamma^{\mu\nu} = \frac{1}{2}\left[\Gamma^\mu, \Gamma^\nu\right], \quad \Gamma^{ij} = \frac{1}{2}\left[\Gamma^i, \Gamma^j\right] \quad \text{with} \quad \left[\Gamma^{\mu\nu}, \Gamma^{ij}\right] = 0$$

where $\mu = 0, \ldots, 3$ and $i = 4, \ldots, 9$.

The fact that the $SO(1,3)$ and $SO(6)$ groups act independently tells us that the spinor is a spinor of $SO(1,3)$ and $SO(6)$ independently, so that we can label the 10 dimensional spinor representations by those of the 4 and 6 dimensional representations.

Definition 7.3.2. Under the $SO(1,3) \times SO(6)$ group the chirality operators in 4, 6 and 10 dimensions, $\Gamma_{(4)}$, $\Gamma_{(6)}$ and $\Gamma_{(10)}$ respectively are

$$\Gamma_{(4)} = i\Gamma^0 \ldots \Gamma^3, \quad \Gamma_{(6)} = -i\Gamma^4 \ldots \Gamma^9 \quad \text{and} \quad \Gamma_{(10)} = \Gamma^0 \ldots \Gamma^9 = \Gamma_{(4)}.\Gamma_{(6)}$$

One checks with the use of Clifford algebra (7.3.4) that

$$\Gamma_{(4)}^2 = \mathbb{I}, \quad \Gamma_{(6)}^2 = \mathbb{I}, \quad \Gamma_{(10)}^2 = \mathbb{I}$$

which means they all have eigenvalues ± 1. Furthermore, a 10 dimensional spinor is positive chiral iff it is simultaneously positive as a 4 and 6 dimensional spinor or simultaneously negative as a 4 and 6 dimensional spinor. Similarly, a 10 dimensional spinor is positive chiral iff its 4 dimensional chirality is opposite to its 6 dimensional chirality.

We could continue in this generality but it probably helps to have a concrete realisation of all of the above. We can exhibit the decomposition of the spinor representations under $SO(1,3) \times SO(6) \subset SO(1,9)$ by giving a tensor product representation for the Γ^M matrices. In six dimensions the gamma matrices are 8 by 8 matrices while in 4 they are 4 by 4. The decomposition of the $SO(1,9)$ gamma matrices to $SO(1,3) \times SO(6)$ amounts to writing the gamma matrices in ten dimensions as tensor products of gamma matrices in four dimensions with those in six dimensions. To do this requires some notation, let γ^i be the gamma matrices in six dimensions and γ^μ those in four. As one possibility we take the ten dimensional gamma matrices to be

$$\Gamma^i = \gamma_{(5)} \otimes \gamma^i, \quad \Gamma^\mu = \gamma^\mu \otimes \mathbb{I}_6. \tag{7.3.5}$$

where $\gamma_{(5)} = i\gamma^0 \dots \gamma^3$ is the chirality operator of the Clifford algebra in 4 dimensions. My notation here for the tensor product is to consider the matrix on the left first then in each of its entries place the matrix on the right.

Exercise 7.3.1. Show that this choice satisfies the Clifford algebra (7.3.4) and also that we get rather explicitly that the chirality operators split neatly $\Gamma_{(4)} = \gamma_5 \otimes \mathbb{I}_6$ and $\Gamma_{(6)} = \mathbb{I}_4 \otimes \gamma_{(7)}$ where $\gamma_{(7)} = -i\gamma^4 \dots \gamma^9$ is the chirality operator of the gamma algebra in 6 dimensions. Furthermore the Lorentz group action factorises,

$$\frac{1}{2}\left[\Gamma^i, \Gamma^j\right] = \mathbb{I}_4 \otimes \frac{1}{2}\left[\gamma^i, \gamma^j\right] \text{ and } \frac{1}{2}\left[\Gamma^\mu, \Gamma^\nu\right] = \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right] \otimes \mathbb{I}_6$$

Thanks to the exercise we see that we may consider the $SO(1,9)$ 32 component spinor as a tensor product of the 4 dimensional spinor of $SO(1,3)$ and the 8 dimensional spinor of $SO(6)$. So we can write

$$\eta^A \simeq \eta^{r\,\overline{\alpha}}, \quad r = 1, \dots, 4 \;\; \overline{\alpha} = 1, \dots, 8 \;\; A = 1, \dots, 32$$

Infact the tensor product is designed so that $SO(1,3) \times SO(6)$ acts by $SO(1,3)$ on the r label and by $SO(6)$ on the $\overline{\alpha}$ label,

$$\left(\frac{1}{2}\left[\Gamma^i, \Gamma^j\right].\eta\right)^{r\overline{\alpha}} = \frac{1}{2}\left[\gamma^i, \gamma^j\right]^{\overline{\alpha}}{}_{\overline{\beta}}\,\eta^{r\overline{\beta}}, \quad \left(\frac{1}{2}\left[\Gamma^\mu, \Gamma^\nu\right].\eta\right)^{r\overline{\alpha}} = \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right]^r{}_s\,\eta^{s\overline{\alpha}}$$

This decomposition is not preserved by $SO(1,9)$ since we also have the Lorentz rotation

$$\frac{1}{2}\left[\Gamma^\mu, \Gamma^i\right] = \gamma^\mu\gamma_{(5)} \otimes \gamma^i$$

which mixes r and $\overline{\alpha}$.

Now we turn to the spinors themeselves. Recall that the spin group associated with SO(6) is spin(6)= SU(4) while the spin group of $SO(1,3)$ is $SL(2,\mathbb{C})$ (for $SO(4)$ it is $SU(2) \times SU(2)$). Recall also that a Dirac spinor in 4 space-time dimensions takes the form

$$\Psi_D^{(4)} = \begin{pmatrix} \psi_\alpha \\ \overline{\chi}^{\dot{\alpha}} \end{pmatrix}$$

where ψ_α is in the 2 of $SL(2,\mathbb{C})$ and $\overline{\chi}^{\dot{\alpha}}$ is in the conjugate $2'$ representation. These can be taken to be the positive and negative chirality Weyl spinors (by choice of γ_5). A Majorana spinor would have

$$\Psi_M^{(4)} = \begin{pmatrix} \psi_\alpha \\ \overline{\psi}^{\dot{\alpha}} \end{pmatrix}$$

That is one takes the complex conjugate of the 2 in the $2'$ to make a Majorana spinor.

The fundamental representation, call its ϵ of $SU(4)$ is the 4 and it too is the positive Weyl spinor while the conjugate representation $\overline{4}$, $\overline{\lambda}$, is the negative Weyl spinor.

In 10 dimensions Weyl spinors are complex and 16 dimensional. Under the decomposition (the formal one or the explicit tensor product that we introduced) we have the positive Weyl spinor

$$(\psi \otimes \epsilon) \oplus \left(\overline{\chi} \otimes \overline{\lambda}\right) \in \mathbf{16}_+ = (\mathbf{2}, \mathbf{4}) \oplus (\mathbf{2}', \overline{\mathbf{4}})$$

The notation is, just as we defined above, a tensor product, so that on the right hand side we have $16 = 2 \times 4 + 2 \times 4$ complex components. We may impose that this spinor also be Majorana which amounts to demanding that the components of the $(\mathbf{2}', \overline{\mathbf{4}})$ are the complex conjugates to the $(\mathbf{2}, \mathbf{4})$, which we write suggestively as

$$(\psi \otimes \epsilon) \oplus \left(\overline{\psi} \otimes \overline{\epsilon}\right) \in \mathbf{16}_{M+} = (\mathbf{2}, \mathbf{4}) \oplus \overline{(\mathbf{2}, \mathbf{4})}$$

in which the $\mathbf{2}'$ is the complex conjugate of the $\mathbf{2}$ and the $\overline{\mathbf{4}}$ is the complex conjugate of the $\mathbf{4}$. The left handed Weyl spinor in 10 dimensions is

$$(\overline{\chi} \otimes \epsilon) \oplus \left(\psi \otimes \overline{\lambda}\right) \in \mathbf{16}_- = (\mathbf{2}', \mathbf{4}) \oplus (\mathbf{2}, \overline{\mathbf{4}})$$

and

$$\left(\overline{\psi} \otimes \epsilon\right) \oplus (\psi \otimes \overline{\epsilon}) \in \mathbf{16}_{M-} = (\mathbf{2}', \mathbf{4}) \oplus \overline{(\mathbf{2}', \mathbf{4})}$$

The 32 real Majorana spinor is then just

$$\mathbf{32}_M = \mathbf{16}_{M+} \oplus \mathbf{16}_{M-}$$

It is time to return to the issue at hand. As the spin connection is zero on M the equation for η becomes

$$\partial_i \eta^{r\overline{\alpha}} + \frac{1}{4}\omega_i^{ab} \left(\sigma_{ab}\right)^{\overline{\alpha}}{}_{\overline{\beta}}\, \eta^{r\overline{\beta}} = 0 \tag{7.3.6}$$

where now

$$\sigma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b] = e_a^i e_b^j \, \Gamma_{ij}. \tag{7.3.7}$$

The four dimensional spinor label plays no role, so we may concentrate on the fact that it is a $\mathbf{4}$ of $SU(4)$. The equation to be satisfied is

$$D_i \epsilon = 0 \tag{7.3.8}$$

where ϵ is now a 4-component spinor on K which we have taken to be the positive chirality spinor. We recall that the covariantly constant spinor satisfies

$$\epsilon = U.\epsilon$$

reducing the the holonomy from $SU(4)$ to some subgroup. If there is only one solution, which we put in the form (by an $SU(4)$ transformation)

$$\epsilon = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \epsilon_1 \end{pmatrix}. \tag{7.3.9}$$

Then the subgroup that preserves (7.3.9) must actually live in $SU(3)$, i.e.

$$U_{SU(4)} = \begin{pmatrix} U_{SU(3)} & 0 \\ 0 & 1 \end{pmatrix} \tag{7.3.10}$$

If there are two solutions then, by an $SU(4)$ transformation followed by an $SU(3)$ transformation, we can put the solution in the form

$$\epsilon = \begin{pmatrix} 0 \\ 0 \\ \epsilon_1 \\ \epsilon_2 \end{pmatrix}. \tag{7.3.11}$$

and the subgroup of $SU(4)$ that preserves this is $SU(2)$, i.e. the matrices

$$U_{SU(4)} = \begin{pmatrix} U_{SU(2)} & 0_{2\times 2} \\ 0_{2\times 2} & \mathbb{I}_{2\times 2} \end{pmatrix} \tag{7.3.12}$$

which clearly preserves the structure of (7.3.11).

The more we reduce the holonomy group the more supersymmetry we preserve. The minimal (non-zero) amount of supersymmetry we can keep is when the holonomy is strictly $SU(3)$, which we will assume for the rest of this discussion.

Now as the reduced holonomy is really SU(3) and not some subgroup thereof there is precisely one covariantly constant spinor ϵ. Its complex conjugate (living in the $\overline{\mathbf{4}}$) will be the unique negative chirality spinor that is covariantly constant. What does all of this mean for the supersymmetry in four dimensions?

We ought to put back the other labels. For us η transforms in the $\mathbf{16}_{M+} = (\mathbf{2},\mathbf{4})\oplus\overline{(\mathbf{2},\mathbf{4})}$, where ϵ (7.3.9) is the part in $\mathbf{4}$ while its complex conjugate is in the $\overline{\mathbf{4}}$, i.e. with a slight abuse of notation $\eta = 2\otimes\epsilon\oplus 2'\otimes\overline{\epsilon}$. As there is only one component of the $\mathbf{4}$ part, we find are left with, in four dimensions, the $\mathbf{2}$ of $SO(1,3)$ and its complex conjugate. That is we are left with 4 real components, so that η is finally a four dimensional Majorana spinor. One Majorana spinors worth of supersymmetry corresponds to $N = 1$ supersymmetry in four dimensions. In fact the spinor we are left with is $\mathbf{2}_+ \oplus \mathbf{2'}_-$, where the signs correspond to the $U(1)$ charges of ϵ and $\overline{\epsilon}$.

For the type IIA theory we also have the $\mathbf{16}_{M-}$. Once more, since there is only one non-zero component of the $\mathbf{4}$, this means that we are left with a four component Majorana spinor which means a second supersymmetry in four dimensions. The second four dimensional Majorana spinor is $\mathbf{2}_- \oplus \mathbf{2'}_+$. One can combine the two spinors together and obtain $\left((2,1)\oplus(1,2)\right)_+$ and its complex conjugate. At the end of the day there is $N = 2$ supersymmetry when one compactifies the type IIA theory on a Calabi-Yau manifold.

Had we decomposed the type IIB spinors $2(2,1,4) \oplus 2(1,2,\overline{4})$ we would have obtained two copies of $(2,1)_+ \oplus (1,2)_-$ which has a different $U(1)$ charge to the type IIA theory.

CHAPTER 8

Complex Manifolds

The idea that one may give some space a complex character, or complex structure is familiar. We know that $\mathbb{R}^{2n}$ can be thought of as the complex manifold $\mathbb{C}^n$ in the standard way. But what of compact manifolds? Can all even dimensional real manifolds be given a complex structure? The answer is an emphatic no. One counter example is S^4 (yet $S^3 \times S^1$ can be given a complex structure). On the other hand the same real manifold may allow for more than one complex structure on it. For string theorists these possible independent complex structures arise in many forms one of which is as the famous moduli space of a Riemann surface.

8.1. Complex Coordinates

We start with the most familiar situation namely on $\mathbb{R}^2$ we take complex coordinates

$$z = x + iy, \quad \overline{z} = x - iy \tag{8.1.1}$$

and call it $\mathbb{C}$. Note that

$$dx \wedge dy = \frac{i}{2} dz \wedge d\overline{z}, \tag{8.1.2}$$

and that

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

while

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \quad \frac{\partial}{\partial y} = i\frac{\partial}{\partial z} - i\frac{\partial}{\partial \overline{z}} \tag{8.1.3}$$

We can repeat this and give a complex 'structure' to $\mathbb{R}^{2n}$ with standard coordinates (x^i, y^i), $i = 1, \ldots, n$ by defining the complex coordinates to be

$$z^i = x^i + \sqrt{-1}y^i \tag{8.1.4}$$

then the real manifold $\mathbb{R}^{2n}$ with these complex coordinates is called $\mathbb{C}^n$.

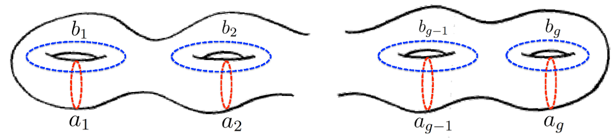
Just as real manifolds are defined as 'smooth' patches of $\mathbb{R}^n$, complex manifolds are defined as 'holomorphic' patches of $\mathbb{C}^n$

Definition 8.1.1. A complex manifold X of complex dimension n, $\dim_{\mathbb{C}} X = n$, is a real (smooth) manifold of dimension $2n$, $\dim_{\mathbb{R}} X = 2n$, with an atlas (ϕ_a, U_a) for which the transition functions $\phi_{ab} = \phi \circ \phi_b^{-1} : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ are holomorphic.

Example 8.1.1. The simplest example of a complex manifold is a point, i.e. $\mathbb{C}^0$.

Example 8.1.2. The first non-trivial examples of compact closed complex manifolds are the Riemann surfaces.

Figure 1. Here, once more we have a genus g Riemann surface



Example 8.1.3. One way to obtain a 2-dimensional torus is to consider the quotient $(\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z})$

Example 8.1.4. The Kodaira-Thurtson surface is an example of a non-trivial complex manifold. This surface S is given by $S = K/H$ with

$$K = \left\{ (z,w) = \begin{pmatrix} 1 & z & w \\ 0 & 1 & \overline{z} \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{C}^2 \right\}$$

while $H = \{(a,b) \in K | a,\, b \in (\mathbb{Z} + \sqrt{-1}\mathbb{Z})\}$ and acts by (multiplication on the right as a matrix) $(a,b) \circ (z,w) = (z+a,\, w + \overline{a}z + b)$.

Consider a manifold that allows for complex co-ordinates (this is not always possible). Then we will have a bunch of complex variables (coordinates) z^i each of which is defined by a pair (x^i, y^i) as in (8.1.1). I use the convention that my complex coordinates carry a label I, so z^I, while their complex conjugates carry a barred label, so $\overline{z}^{\overline{I}}$.

8.2. Projective Spaces and Grassmanians

These are some of the prime examples of complex manifolds. Their importance also comes from the fact that while one might think one can obtain an a nice complex submanifold of $\mathbb{C}^n$ by imposing some set of analytic equations (depending only on z not $\overline{z}$) this is not the case,

Theorem 8.2.1 (someone). The only connected compact analytic submanifold of $\mathbb{C}^n$ is a point.

While that is bad news it turns out that there are many nice submanifolds of the projective spaces.

Definition 8.2.1. The complex projective space $\mathbb{CP}^n$ (or sometimes $\mathbb{P}^n$) is the space of complex lines $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, in $\mathbb{C}^{n+1}\backslash\{0\}$. In equations we have the identification

$$(z^1, \ldots, z^{n+1}) \simeq \lambda\,(z^1, \ldots, z^{n+1}), \quad \lambda \in \mathbb{C}^*$$

One easily constructs an atlas for the projective spaces. The open sets and maps are taken to be

$$U_i = \{z^i \neq 0\}, \quad \phi_i(z^1, \dots, z^{n+1}) = (w_i^1, \dots, w_i^{n+1}) = \frac{1}{z^i}(z^1, \dots, z^{n+1})$$

Note that $\phi_i : \mathbb{CP}^n|_{U_i} \longrightarrow \mathbb{C}^n$ as $\phi_i(\lambda.z) = \phi_i(z)$ and the image is $\mathbb{C}^n$ (not $\mathbb{C}^{n+1}$) since the i-th position in the coordinates is always 1. On overlaps $U_i \cap U_j$ we determine the transition functions by looking at how the coordinates change

$$w_i^k = z^k/z^i = z^k/z^j \,.\, z^j/z^i = w_j^k/w_j^i = \phi_{ij}^k(w_j) \tag{8.2.1}$$

Example 8.2.1. The two sphere S^2 once given a complex structure is $\mathbb{CP}^1$. The coordinates in $\mathbb{C}^2$ are (z^1, z^2) so that, by the discussion above, $\mathbb{CP}^1$ is covered by the two patches $U_1 = \{z^1 \neq 0\}$ and $U_2 = \{z^2 \neq 0\}$ with coordinates $w_1 = z^2/z^1$, $w_2 = z^1/z^2$ and on the overlap $U_1 \cap U_2$ we have that $w_1 = 1/w_2$.

Exercise 8.2.1. From the definition show that $\mathbb{CP}^n = \mathbb{C}^n \cup \dots \cup \mathbb{C}^0$.

Turning back to the question of having analytic submanifolds of $\mathbb{CP}^n$ we quote the most useful result in this area, namely

Theorem 8.2.2 (Chow). Every complex analytic projective variety is algebraic, i.e. it is the common zero set of a finite family of homogeneous holomorphic polynomials.

The word 'projective' in the theorem means that it lies in products of complex projective spaces. One immediately understands that one proposes sets of homogeneous equations to see if we get interesting complex analytic submanifolds.

Definition 8.2.2. The homogeneous coordinates $[z_0, \dots, z_n]$ of $\mathbb{CP}^n$ with $(z_0, \dots, z_n) \in \mathbb{C}^{n+1}/\{0\}$ are defined to be

$$[z_0, \dots, z_n] = \{(\lambda z_0, \dots, \lambda z_n) |\, \lambda \in \mathbb{C}^*\} \in \mathbb{CP}^n$$

Though these are not coordinates in the usual sense, for example they are not unique as $[z_0, \dots, z_n]$ and $[\lambda z_0, \dots, \lambda z_n]$ represent the same point.

Example 8.2.2. The manifold in $\mathbb{CP}^2$ defined by

$$C_P = \left\{[z_0,\, z_1,\, z_2] \in \mathbb{CP}^2 \,|P(z_0, z_1, z_2) = 0\right\}$$

for a generic homogeneous polynomial of degree $d \geq 1$ is a Riemann surface of genus $g = (d-1)(d-2)/2$

Example 8.2.3. A K3 surface can be obtained as quartic hypersurface in $\mathbb{CP}^3$

$$K3 = \left\{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 \,|\, z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\right\}$$

Remark 8.2.1. We will give examples later of Calabi-Yau manifolds that appear as complete intersections in products of complex projective spaces.

EXERCISE 8.2.2. Define a map $f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^2$ by

$$f([w_0, w_1]) = [w_0^2, w_0 w_1, w_1^2]$$

and show that it is well defined. Show that in the $\mathbb{CP}^2$ coordinates $[z_0, z_1, z_2]$ this map implies the homogeneous polynomial equation $z_0 z_2 - z_1^2 = 0$ and determine the space that this defines.

8.3. Complex Structures

We want a slightly more intrinsic understanding of what a complex structure is. Consider a real vector field

$$v = v^x \frac{\partial}{\partial x} + v^y \frac{\partial}{\partial y} = v^z \frac{\partial}{\partial z} + v^{\overline{z}} \frac{\partial}{\partial \overline{z}} \tag{8.3.1}$$

with

$$v^z = v^x + iv^y, \quad v^{\overline{z}} = v^x - iv^y, \quad v^{\overline{z}} = \overline{v^z} \tag{8.3.2}$$

The last equality follows from the fact that we are considering real vector fields. Given that we have complex numbers floating around we may as well consider complex valued vector fields in which case v^z and $v^{\overline{z}}$ are not complex conjugates.

DEFINITION 8.3.1. The complexified tangent bundle of a complex manifold X, is denoted by $TX_{\mathbb{C}} = TX \otimes \mathbb{C}$ and sections of this bundle are complex valued vector fields.

The vector fields (8.3.1) have holomorphic and anti-holomorphic components. These do not mix as we change coordinate charts since the data is holomorphic. Consequently we have an invariant splitting of the complexified tangent bundle as

$$TX_{\mathbb{C}} = T^{(1,0)}X \oplus T^{(0,1)}X \tag{8.3.3}$$

where sections of $TX^{(1,0)}$ have the local form

$$s = \sum_{I=1}^{n} s^I(z, \overline{z}) \frac{\partial}{\partial z^I} \tag{8.3.4}$$

DEFINITION 8.3.2. $T^{(1,0)}X$ is called the holomorphic tangent bundle of the complex manifold X.

We can apply the same logic to sections of the cotangent bundle of X. We consider a section of T^*X

$$w = w_x \, dx + w_y dy = w_z \, dz + w_{\overline{z}} \, d\overline{z} \tag{8.3.5}$$

with

$$w_z = \frac{1}{2}(w_x - iw_y), \quad w_{\overline{z}} = \frac{1}{2}(w_x + iw_y), \quad w_{\overline{z}} = \overline{w_z} \tag{8.3.6}$$

If we allow for complex valued sections we may again drop the condition that $w_{\overline{z}} = \overline{w_z}$.

DEFINITION 8.3.3. The complexified cotangent bundle of a complex manifold X is denoted by $T^*X_{\mathbb{C}} = T^*X \otimes \mathbb{C}$.

Once more we have a natural splitting of this bundle as

$$T^*X_{\mathbb{C}} = T^{*(1,0)}X \oplus T^{*(0,1)}X \tag{8.3.7}$$

where sections of $T^{*(0,1)}X$ are

$$r = \sum_{\overline{I}=1}^{n} r_{\overline{I}}(z,\overline{z})\, d\overline{z}^{\overline{I}} \tag{8.3.8}$$

As you recall a linear map from a vector space to itself, $M : V \longrightarrow V$ is an element of $\mathrm{End}(V) = V \otimes V^*$, which we can write as $M = \sum_{a,b} M^a_{\ b}\, e_a \otimes e^b$. Linear maps are given by matrices and we see the matrix in question is $M^a_{\ b}$. For $v = \sum_a v^a e_a \in V$ we have $M(v) = \sum_{a,b} M^a_{\ b}\, e_a \sum_c v^c e^b(e_c) = \sum_{ab} M^a_{\ b}\, v^b\, e_a$ giving us a new vector with components in the $\{e_a\}$ basis $\sum_b M^a_{\ b}\, v^b$. Now $\mathrm{End}(V) \simeq \mathrm{End}(V^*)$ and therefore M is also understood to act on $w = \sum_a w_a\, e^a \in V^*$ as $M(w) = \sum_{a,b} M^a_{\ b} \sum_c w_c e_a(e^c)\, e^b = \sum_{ab} w_a\, M^a_{\ b}\, e^b$.

Definition 8.3.4. A complex structure is the tensor defined on $U_i \subset X$

$$J = i \sum_I dz_i^I \otimes \frac{\partial}{\partial z_i^I} - i \sum_{\overline{I}} d\overline{z}_i^{\overline{I}} \otimes \frac{\partial}{\partial \overline{z}_i^{\overline{I}}}$$

Theorem 8.3.1. The mixed tensor J enjoys the following properties.

(1) J is well defined.
(2) J is real
(3) $J : T_{\mathbb{C}}X \longrightarrow T_{\mathbb{C}}X$ preserving the decomposition (8.3.3) into holomorphic and anti-holomorphic components.
(4) $J : T^*_{\mathbb{C}}X \longrightarrow T^*_{\mathbb{C}}X$ preserving the decomposition (8.3.7) into holomorphic and anti-holomorphic bundles.
(5) $J \circ J = -Id$, where Id is the appropriate identity map.

Proof. The first condition is obvious as is the second (just complex conjugate or write it in real coordinates). The third and fourth follow as, just by looking at the basis formulae, we have $J \in \mathrm{End}(T^{(1,0)}X) \oplus \mathrm{End}(T^{(0,1)}X)$ as there are no mixing terms in (8.3.4). As a matrix J is diagonal and can be written as

$$J = \begin{pmatrix} iId & 0 \\ 0 & -iId \end{pmatrix}$$

□

The reason J is called the complex structure is apparent, namely if $v \in T^{(1,0)}X$ then $J(v) = iv$.

We can repeat the exercise and decompose all forms into their their holomorphic and anti-holomorphic components. As before, since we have complex numbers floating around we may as well consider forms with complex coefficients. We denote the space of p forms on X with complex coefficients by $\Omega^p(X, \mathbb{C})$. Given a complex manifold we may write forms in terms of the differentials dz and $d\overline{z}$.

Definition 8.3.5. Let $\Omega^{(p,q)}(X,\mathbb{C})$ be the space of forms

$$\omega = \frac{1}{p!\,q!}\sum \omega_{I_1\ldots I_p\,\overline{I}_1\ldots\overline{I}_q}\, dz^{I_1}\wedge\cdots\wedge dz^{I_p}\wedge d\overline{z}^{\overline{I}_1}\wedge\cdots\wedge d\overline{z}^{\overline{I}_q}$$

By simply writing out real differentials dX^M in terms of dz^I and $d\overline{z}^{\overline{I}}$ we arrive at the following relationship.

Proposition 8.3.2. The space of complex p-forms $\Omega^p(X,\,\mathbb{C})$ on X is given by

$$\Omega^p(X,\,\mathbb{C}) = \bigoplus_{r=0}^{p}\Omega^{(r,p-r)}(X,\mathbb{C})$$

Consequently on a complex manifold the $\Omega^{(p,q)}(X,\mathbb{C})$ give 'finer' information than the $\Omega^p(X,\,\mathbb{C})$.

8.4. Almost Complex Structures

Suppose that you do not know that a manifold is complex or not how would you determine that without resorting to creating all the holomorphic patches required? The answer comes from the Newlander-Nirenberg theorem. To state it we need to go back to our mixed tensor J.

Definition 8.4.1. Let X be a real $2n$ dimensional manifold. An almost complex structure on X is a smooth mixed tensor J which satisfies

$$J^P{}_N\, J^N{}_M = -\delta^P_M$$

Theorem 8.4.1 (Newlander-Nirenberg). The almost complex structure J defines a complex structure iff the associated Nijenhuis tensor, N_J

$$N^P{}_{MN}(J) = J^Q{}_M\left(\partial_Q J^P{}_N - \partial_N J^P{}_Q\right) - J^Q{}_N\left(\partial_Q J^P{}_M - \partial_M J^P{}_Q\right)$$

vanishes, $N_J = 0$. Another way of saying this is tha a necessary and sufficient condition for there to exist holomorphic charts, defined by the almost complex structure around each point of X is that the Nijenhuis tensor of J vanishes.

As written the expression for the Nijenhuis tensor does not look very covariant. However,

Exercise 8.4.1. Let g be an arbitrary Riemannian metric on X and ∇ the Levi-Cevita covariant derivative. Show that

$$N^P{}_{MN}(J) = J^Q{}_M\left(\nabla_Q J^P{}_N - \nabla_N J^P{}_Q\right) - J^Q{}_N\left(\nabla_Q J^P{}_M - \nabla_M J^P{}_Q\right)$$

From this we deduce that this is a covariant tensor no matter which metric we pick on X. Having a complex structure or not does not depend on the Riemannian structure.

Proposition 8.4.2. If for some Riemannian structure $\nabla_Q J^P{}_N = 0$ then the Nijenhuis tensor vanishes and the almost complex structure J defines a complex structure.

Proof. This is a corollary to the previous exercise. $\square$

8.5. Hermitian Structures on a Complex Bundle

On any real manifold we know that there are Riemannian metrics. Given the underlying real structure of our complex manifolds we are guaranteed that complex manifolds also has Riemannian metrics. However, one can sharpen this somewhat, tailoring the metrics under consideration to those which are directly compatible with the complex structure on X.

DEFINITION 8.5.1. If V is a complex vector space and $v,\ u,\ w \in V$, $\lambda \in \mathbb{C}$ then an inner product $\langle\ ,\ \rangle$ is said to be Hermitian if

(1) $\langle u, v\rangle = \overline{\langle v, u\rangle}$
(2) $\langle (u+w), v\rangle = \langle u, v\rangle + \langle w, v\rangle$
(3) $\langle \lambda\, u, v\rangle = \lambda\, \langle u, v\rangle$
(4) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ iff $v = 0$.

DEFINITION 8.5.2. A Hermitian metric on a complex manifold X is a metric of the form

$$g = g_{I\overline{I}}\, dz^I \otimes d\overline{z}^{\overline{I}}$$

PROPOSITION 8.5.1. All complex manifolds allow for Hermitian metrics.

PROOF. Let $g(\ ,\ )$ be any Riemannian metric on X then $h(\ ,\ ) = g(\frac{1}{2}(Id - iJ)\,,\ \frac{1}{2}(Id + iJ))$ is a Hermitian metric. $\square$

Note that for a Hermitian metric the components $g_{IJ} = 0$ and $g_{\overline{IJ}} = 0$. Hermitian metrics are interesting as they pair $T^{(1,0)}X$ and $T^{(0,1)}X$.

8.6. Cohomology

On a complex manifold the exterior derivative can be decomposed

$$d = dz^I \partial_I + d\overline{z}^{\overline{I}} \overline{\partial}_{\overline{I}} = \partial + \overline{\partial}. \tag{8.6.1}$$

and we have that

$$\partial : \Omega^{(p,q)}(X, \mathbb{C}) \longrightarrow \Omega^{(p+1,q)}(X, \mathbb{C}), \quad \overline{\partial} : \Omega^{(p,q)}(X, \mathbb{C}) \longrightarrow \Omega^{(p,q+1)}(X, \mathbb{C})$$

From the fact that $d^2 = (\partial + \overline{\partial})^2 = 0$ we have, by counting form degree, that

$$\partial^2 = 0, \quad \{\partial\ ,\ ,\overline{\partial}\} = 0, \quad \overline{\partial}^2 = 0 \tag{8.6.2}$$

Then we can use the splitting of the exterior derivative to "refine" the cohomology groups. We can ask about cohomology related to

DEFINITION 8.6.1. The Dolbeault operator or $\overline{\partial}$- operator on a complex manifold X is defined to be

$$\overline{\partial} = d\overline{z}^{\overline{I}} \overline{\partial}_{\overline{I}}.$$

and it has the property that $\overline{\partial}^2 = 0$.

As an analogue of de Rham's theorem we have

THEOREM 8.6.1 (Dolbeault). Let X be a complex manifold then

$$\mathrm{H}^{(p,q)}_{\overline{\partial}}(X,\, E) = \mathrm{H}^{q}_{\overline{\partial}}(X,\, \Omega^p_X(E)) = \frac{\mathrm{Ker}\left(\Omega^{(p,q)}(X,\, E) \xrightarrow{\overline{\partial}_E} \Omega^{(p,q+1)}(X,\, E)\right)}{\mathrm{Im}\left(\Omega^{(p,q-1)}(X,\, E) \xrightarrow{\overline{\partial}_E} \Omega^{(p,q)}_X(X,\, E)\right)}$$

which we take more as a definition.

EXERCISE 8.6.1. Show that

$$\{d,\, d^c\} = 0, \quad (d^c)^2 = 0$$

CHAPTER 9

Kähler Manifolds

Kähler manifolds are a, rather special, class of complex manifolds. These manifolds have compatible symplectic structures (so like a phase space) and complex structures. Needless to say, there many examples of Kähler manifolds. The $\mathbb{CP}^n$ are all Kähler as are analytic submanifolds of $\mathbb{CP}^n$. Manifolds which have $b_2(X) = 0$ are never Kähler neither are those that do not allow for a complex structure nor those for which $b_1(X) = 2m+1$. The Kodaira-Thurston surface S that we introduced before is not Kähler even though $b_1(S) = 3$.

9.1. The Kähler form

Definition 9.1.1. From the Hermitian metric we can define a 2-form

$$\omega = \frac{i}{2}\, g_{I\overline{J}}\, dz^I \wedge d\overline{z}^{\overline{J}}, \tag{9.1.1}$$

and if this 2-form is closed $d\omega = 0$, then it is called the Kähler form, the metric is said to be a Kähler metric and the manifold with such a metric is said to be Kähler.

One sees that this form is real (just complex conjugate the definition, or write it in real coordinates).

Proposition 9.1.1.

There is still one more useful combination of operators to consider namely

Definition 9.1.2. Let X be a complex manifold and let the operator $d^c : \Omega^p(X, \mathbb{C}) \longrightarrow \Omega^{p+1}(X, \mathbb{C})$ be given by

$$d^c = i(\overline{\partial} - \partial)$$

While this definition makes sense on a complex manifold it does not on a real manifold, much in the same way that both ∂ and $\overline{\partial}$ are coordinate patch invariant in the complex case (holomorphic transition functions) but not for the underlying real manifold.

CHAPTER 10

Hyper-Kähler Manifolds

10.1. K3 Compactification

Now I am going to change the name of the game a little. Rather than looking for compactifications of the form $M_4 \times K$ for K a six dimensional compact manifold we will take space time to be six dimensional (it could be $M_4 \times T^2$ for example) and denote it by M_6 and the compact space K is four dimensional. None of the general structure that we have discussed in previous lectures changes at all. However, there is one very important novelty, and that is that the only compact closed four-manifolds admitting a covariant constant spinor (so that they are Ricci flat) are T^4 and K3! Which narrows the field down a bit. In the list of the second lecture, $K3$ is the unique example of a compact four dimensional manifold with holonomy $SU(2)$. Here we will just consider the K3 manifold (complex surface) with Ricci flat metrics. There is a small catch to this, I will not be able to give you the Ricci flat metric on K3 and that is because, at the moment, nobody can write it down explicitly. That such a metric should be there was conjectured, in a more general context, by E. Calabi in 1957, an existence proof was supplied by S-T. Yau somewhat later and to date there is not one example!

In any case as the hard work has been done by the mathematicians we can simply take it for granted that we have a metric g^0_{ij} such that

$$R_{ij}(g^0) = 0, \tag{10.1.1}$$

where $i, j = 1, \ldots, 4$.

10.1.1. Metric Deformations

Finally we come to the one point that we have been consistently postponing, namely how many scalars coming from the reduction of the metric in ten 10 dimensions to 6 dimensions (as now we end up with M_6) are there when one compactifies on K3? Before answering that lets answer the “simpler” question of how many vectors $g_{\mu i}$ we pick up on M_6, the answer is zero. The reason is that $H^1(\mathrm{K3}) = 0$ so that there are no “one-forms” that the “i” component can correspond to on K3[1].

[1]In detail: A non-zero $g_{\mu i}$ requires a Killing vector field on K, the equation for which is $\nabla_i V_j + \nabla_j V_i = 0$ which implies $\nabla^i V_i = 0$ or $*d*V = 0$. Acting with ∇^j on the Killing equation gives $0 = \nabla^j\nabla_i V_j + \nabla^j\nabla_j V_i = [\nabla^j, \nabla_i]V_j + \nabla^j\nabla_j V_i = R^j{}_i V_j + \nabla^j\nabla_j V_i$, so we have learnt that on a Ricci flat manifold the Killing vectors satisfy $\nabla^j\nabla_j V_i = 0$. However, $0 = \int_K \sqrt{g} V^i \nabla^j\nabla_j V_i$ is again the integral

The zero modes for the metric on K3 can be determined in a linear approximation. Write a general metric on K3 as $g_{ij} = g^0_{ij} + h_{ij}$ for very small h demanding

$$R_{ij}(g^0 + h) = 0 \tag{10.1.2}$$

leads to the Lichnerowicz Laplacian

$$0 = \nabla^k \nabla_k h_{ij} + R_{ikjl} h^{kl}, \tag{10.1.3}$$

in the gauge, $\nabla^i h_{ij} - \frac{1}{2}\nabla_j h^i{}_i = 0$. The total number of non-zero solutions to this equation is 58. I will try to motivate this presently, though it will require a bit more machinery. For the moment we simply note that, this implies that the g_{ij} count as 58 scalars from the point of view of M_6.

Let us write the field content of the IIA theory of massless fields on M_6

Ten Dimensions	Fields on M_6
ϕ	ϕ
g_{MN}	$g_{\mu\nu}$ + 58 scalars: g_{ij}
B_{MN}	$B_{\mu\nu}$ + 22 scalars: B_{ij}
A_M	A_μ
C_{MNR}	$C_{\mu\nu\rho}$ + 22 vectors: $C_{\mu ij}$

From the perspective of super-gravity[2] the scalars actually fit together neatly to form a coset space. The ones coming from the geometry, that is from the metric in ten dimensions, live on the coset (modulo a discrete group)

$$\frac{SO(19,3)}{SO(19) \times SO(3)} \times \mathbb{R}^+, \tag{10.1.4}$$

where the dilaton corresponds to the factor $\mathbb{R}^+$. However, once one incorporates the scalars coming from the antisymmetric tensor field, the scalars actually span a larger coset (also modulo a discrete group),

$$\frac{SO(20,4)}{SO(20) \times SO(4)} \times \mathbb{R}^+. \tag{10.1.5}$$

10.1.2. Metric Deformation Count

OK its time to "roll up our sleeves" and get down to the real work. Well actually there is not that much to do.... We already have all the pieces at our disposal, its just a question of sticking them together. In what ways can we deform the metric? Remember K3 is a complex and Kähler manifold. We still have a K3 if we change the complex structure (and make sure its Kähler) or if we change the Kähler structure (and keep fixed the complex structure).

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of a square and so we conclude that $\nabla_j V_i = 0$. This last equality implies $dV = 0$, and together with $*d*V = 0$, we have that $V \in H^1(K)$.

[2]From our, linear, perspective it is not possible to see that the scalars live on such complicated manifolds.

Firstly the metric is Kähler so if we can change it by deforming the Kähler form and at the same time keep the complex structure fixed then we have a deformation of the metric. Well a Kähler form is a $(1,1)$-form and lives in $\mathrm{H}^{(1,1)}(K3)$. So we can move the Kähler form around in $\mathrm{H}^{(1,1)}(K3)$. The dimension of the vector space $\mathrm{H}^{(1,1)}(K3)$ is 20. Since the Kähler form is real it can be moved around in a 20-dimensional real vector space (= 20 scalar fields)[3]. Each Kähler form gives us a Kähler metric and so we get a 20 dimensional space of Kähler metrics this way.

What of deformations that come by changing the complex structure? Remember that complex structure deformations are measured by $\mathrm{H}^1(T^{(1,0)}K3)$, which is the same as being given a tensor $\delta J^I_{\bar{J}}$ which is $\overline{\partial}$ closed but not exact. On K3 there exists a very special $(2,0)$-form, ϵ, which is holomorphic and point wise invertible. Since this form is invertible we have a one to one correspondence between $\delta^I_{\bar{J}}$ and

$$K_{I\bar{J}} = \epsilon_{IJ}\, \delta J^J_{\bar{J}}. \tag{10.1.6}$$

Notice that K is a $(1,1)$-form and that it is holomorphic (since both ϵ and δJ are), so that in fact it lives in $\mathrm{H}^{(1,1)}(K3)$. We have shown that

$$\mathrm{H}^1(T^{(1,0)}K3) \equiv \mathrm{H}^{(1,1)}(K3) \tag{10.1.7}$$

and so the number of complex deformations is equal to the dimension of $\mathrm{H}^{(1,1)}(K3)$, which is again 20. This time since the δJ can be complex we get 20 complex directions or 40 real.

All in all we seem to have 60 real possible deformations (and so 60 scalars). But this is not quite right. We have over counted because of the following fact. For a given Ricci-flat metric on K3 there is a spheres worth of complex structures consistent with it. That's 2 dimensions too many, so the count is 60 - 2 = 58.

[3] At least infinitesimally.

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