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# PARADOXES OF NEUTRINO OSCILLATIONS

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Despite the theory of neutrino oscillations being rather old, some of its basic issues are still being debated in the literature. We discuss a number of such issues, including the relevance of the "same energy" and "same momentum" assumptions, the role of quantum-mechanical uncertainty relations in neutrino oscillations, the dependence of the coherence and localization conditions that ensure the observability of neutrino oscillations on neutrino energy and momentum uncertainties, the question of (in)dependence of the oscillation probabilities on the neutrino production and detection processes, and the applicability limits of the stationary-source approximation. We also develop a novel approach to calculation of the oscillation probability in the wave-packet approach, based on the summation/integration conventions different from the standard one, which allows a new insight into the "same energy" vs. "same momentum" problem. We also discuss a number of apparently paradoxical features of the theory of neutrino oscillations.

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## 1. INTRODUCTION

More than 50 years have already passed since the idea of neutrino oscillations was put forward [1, 2], and over 10 years have passed since the experimental discovery of this phenomenon (see, e.g., [3] for reviews). However, surprisingly enough, a number of basic issues of the neutrino oscillation theory are still being debated. Moreover, some features of the theory appear rather paradoxical. Among the issues that are still under discussion are:

(1) Why do the often used same-energy and same-momentum assumptions for neutrino mass eigenstates composing a given flavor state, which are known to be both wrong, lead to the correct result for the oscillation probability?

(2) What is the role of quantum-mechanical uncertainty relations in neutrino oscillations?

(3) What determines the size of the neutrino wave packets?

(4) How do the coherence and localization conditions that ensure the observability of neutrino oscillations depend on neutrino energy and momentum uncertainties?

(5) Are wave packets actually necessary for a consistent description of neutrino oscillations?

(6) When can the oscillations be described by a universal (i.e., production and detection process independent) probability?

(7) When is the stationary-source approximation valid?

(8) Would recoillessly emitted and absorbed neutrinos (produced and detected in Mössbauer-type experiments) oscillate?

(9) Are oscillations of charged leptons possible?

In the present paper we consider the first seven issues listed above, trying to look at them from different perspectives. We hope that our discussion will help clarify these points and finally put them to rest. For the last two issues, we refer the reader to the recent discussions in [4–6] (for oscillations of Mössbauer neutrinos) and [7] (for oscillations of charged leptons).

## 2. SAME ENERGY OR SAME MOMENTUM?

In most derivations of the so-called standard formula for the probability of neutrino oscillations in vacuum (see Eq. (6) below), usually the assumptions that the neutrino mass eigenstates composing a given flavor eigenstate either have the same momentum [8–11] or the same energy [12–15] are made. The derivation typically proceeds as follows.

First, recall that in the basis in which the mass matrix of charged leptons has been diagonalized the

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fields describing the massive neutrinos $\nu_i$ and flavor-eigenstate neutrinos $\nu_a$ and the corresponding states $|\nu_i\rangle$ and $|\nu_a\rangle$ are related by

$$\nu_a = \sum_i U_{ai}\nu_i, \quad |\nu_a\rangle = \sum_i U^*_{ai}|\nu_i\rangle, \tag{1}$$

where $U$ is the leptonic mixing matrix. If one now assumes that all the mass eigenstates composing the initially produced flavor state $|\nu(0)\rangle = |\nu_a\rangle$ have the same momentum, then, after time $t$ has elapsed, the $i$th mass eigenstate will simply pick up the phase factor $\exp(-iE_i t)$, and the evolved state $|\nu(t)\rangle$ will be given by

$$|\nu(t)\rangle = \sum_i U^*_{ai} e^{-iE_i t}|\nu_i\rangle. \tag{2}$$

Projecting this state onto the flavor state $|\nu_b\rangle$ and taking the squared modulus of the resulting transition amplitude, one gets the probability of the neutrino flavor transition $\nu_a \to \nu_b$ after the time interval $t$:

$$P(\nu_a \to \nu_b; t) = \left|\sum_i U_{bi} e^{-iE_i t} U^*_{ai}\right|^2. \tag{3}$$

Next, taking into account that the energy $E_i$ of a relativistic neutrino of mass $m_i$ and momentum $\mathbf{p}$ is

$$E_i = \sqrt{p^2 + m_i^2} \simeq p + \frac{m_i^2}{2p}, \tag{4}$$

and that for relativistic pointlike particles the distance $L$ they propagate during the time interval $t$ satisfies

$$L \simeq t, \tag{5}$$

one finally finds

$$P(\nu_a \to \nu_b; L) = \left|\sum_i U_{bi} \exp\left(-i\frac{\Delta m_{ij}^2}{2p}L\right) U^*_{ai}\right|^2, \tag{6}$$

where $\Delta m_{ij}^2 = m_i^2 - m_j^2$ and the index $j$ corresponds to any of the mass eigenstates. This is the standard formula describing neutrino oscillations in vacuum. Note that in this approach neutrino states actually evolve only in time (see Eq. (3)); the usual coordinate dependence of the oscillation probability (6)) is only obtained by invoking the additional "time-to-space conversion" assumption (5). Without this conversion, one would come to a paradoxical conclusion that neutrino oscillations could be observed by just putting the neutrino detector immediately next to the source and waiting long enough.

Likewise, one could assume that all the mass-eigenstate neutrinos composing the initially produced flavor state $|\nu_a\rangle$ have the same energy. Using the fact that the spatial propagation of the $i$th mass eigenstate is described by the phase factor $e^{i\mathbf{p}_i\mathbf{x}}$ and that for a relativistic neutrino of mass $m_i$ and energy $E$

$$p_i = \sqrt{E^2 - m_i^2} \simeq E - m_i^2/2E, \tag{7}$$

one again comes to the same standard formula (6) for the oscillation probability. Note that in this case the neutrino flavor evolution occurs in space and it is not necessary to invoke the "time-to-space conversion" relation (5) to obtain the standard oscillation formula.

The above two alternative derivations of the oscillation probability are very simple and transparent, and they allow one to arrive very quickly at the desired result. The trouble with them is that they are both wrong.

In general, there is no reason whatsoever to assume that different mass eigenstates composing a flavor neutrino state emitted or absorbed in a weak-interaction process have either the same energy or the same momentum [5)]. Indeed, the energies and momenta of particles emitted in any process are dictated by the kinematics of the process and by the experimental conditions. Direct analysis of, e.g., 2-body decays with simple kinematics, such as $\pi^\pm \to l^\pm + \nu_l(\bar{\nu}_l)$, where $l = e, \mu$, allows to find the 4-momenta of the emitted particles and shows that neither energies nor momenta of the different neutrino mass eigenstates composing the flavor state $\nu_l$ are the same [16, 17]. One might question this argument on the basis that it relies on the energy-momentum conservation and the assumption that the energies and momenta of the emitted mass eigenstates have well-defined (sharp) values, whereas in reality these quantities have intrinsic quantum-mechanical uncertainties (see the discussion in Section 5.1). However, the inexactness of the neutrino energies and momenta does not invalidate our argument that the "same energy" and "same momentum" assumptions are unjustified, and in fact only strengthens it. It should be also noted that the "same energy" assumption actually contradicts Lorentz invariance: even if it were satisfied in some reference frame (which is possible for two neutrino mass eigenstates, but not in the 3-species case), it would be violated in different Lorentz frames [17, 18]. The same applies to the "same momentum" assumption.

## 3. WHY WRONG ASSUMPTIONS LEAD TO THE CORRECT RESULT?

One may naturally wonder why two completely different and wrong assumptions ("same energy" and

[5)] The only exception we are aware of are neutrinos produced or detected in hypothetical Mössbauer-type experiments, since for them the "same energy" assumption is indeed justified fairly well.

"same momentum") lead to exactly the same and correct result — the standard oscillation formula. To understand that, it is necessary to consider the wave-packet picture of neutrino oscillations.

### *3.1. Wave Packets Approach to Neutrino Oscillations*

In the discussion in the previous section we were actually considering neutrinos as plane waves or stationary states; strictly speaking, this description was inconsistent because such states are in fact non-propagating. Indeed, the probability of finding a particle described by a plane wave does not depend on the coordinate, while for stationary states this probability does not depend on time. In quantum theory propagating particles must be described by moving wave packets (see, e.g., [19]). Let us recall the main features of this approach [20–30].

Let a flavor eigenstate $\nu_a$ be born at time $t = 0$ in a source centered at $\mathbf{x} = 0$. The wave packet describing the evolved neutrino state at a point with the coordinates $(t, \mathbf{x})$ is then

$$|\nu_a(\mathbf{x}, t)\rangle = \sum_i U^*_{ai}\Psi_i(\mathbf{x}, t)|\nu_i\rangle. \quad (8)$$

Here, $\Psi_i(\mathbf{x}, t)$ is wave packet describing a free propagating neutrino of mass $m_i$:

$$\Psi_i(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^{3/2}} f^S_i(\mathbf{p} - \mathbf{p}_i)e^{i\mathbf{p}\mathbf{x}-iE_i(p)t}, \quad (9)$$

where $f^S_i(\mathbf{p} - \mathbf{p}_i)$ is the momentum distribution function with $\mathbf{p}_i$ being the mean momentum, and $E_i(p) = \sqrt{p^2 + m_i^2}$. The superscript "$S$" at $f^S_i(\mathbf{p} - \mathbf{p}_i)$ indicates that the wave packet corresponds to the neutrino produced in the source. We will assume the function $f^S_i(\mathbf{p} - \mathbf{p}_i)$ to be sharply peaked at or very close to zero of its argument ($\mathbf{p} = \mathbf{p}_i$), with the width of the peak $\sigma_p \ll p_i$. [6)] No further properties of $f^S_i(\mathbf{p} - \mathbf{p}_i)$ need to be specified. If $f^S_i(\mathbf{p} - \mathbf{p}_i)$ is normalized according to

$$\int d^3p|f^S_i(\mathbf{p} - \mathbf{p}_i)|^2 = 1, \quad (10)$$

then the wave function $\Psi_i(\mathbf{x}, t)$ has the standard normalization $\int d^3x|\Psi_i(\mathbf{x}, t)|^2 = 1$. We will, however, use a different normalization convention, which yields the correct normalization of the oscillation probability (see Eq. (20) below). Expanding $E_i(p)$ around the mean momentum,

$$E_i(p) = E_i(p_i) + \left.\frac{\partial E_i(p)}{\partial p^j}\right|_{\mathbf{p}_i}(p - p_i)^j + \quad (11)$$
$$+ \frac{1}{2}\left.\frac{\partial^2 E_i(p)}{\partial p^j \partial p^k}\right|_{\mathbf{p}_i}(p - p_i)^j(p - p_i)^k + \ldots,$$

one can rewrite Eq. (9) as

$$\Psi_i(\mathbf{x}, t) \simeq e^{i\mathbf{p}_i\mathbf{x}-iE_i(p_i)t} g^S_i(\mathbf{x} - \mathbf{v}_{gi}t), \quad (12)$$

where

$$g^S_i(\mathbf{x} - \mathbf{v}_{gi}t) = \int \frac{d^3p}{(2\pi)^{3/2}} f^S_i(\mathbf{p})e^{i\mathbf{p}(\mathbf{x}-\mathbf{v}_{gi}t)} \quad (13)$$

is the shape factor and

$$\mathbf{v}_{gi} = \left.\frac{\partial E_i}{\partial \mathbf{p}}\right|_{\mathbf{p}_i} = \left.\frac{\mathbf{p}}{E_i}\right|_{\mathbf{p}_i} \quad (14)$$

is the group velocity of the wave packet. Here we have retained only the first and the second terms in the expansion (11), since the higher order terms are of second and higher order in the small neutrino mass and so can be safely neglected in practically all situations of interest. This approximation actually preserves the shape of the wave packets (and, in particular, neglects their spread). Indeed, the shape factor (13) depends on time and coordinate only through the combination $(\mathbf{x} - \mathbf{v}_{gi}t)$; this means that the wave packet propagates with the velocity $\mathbf{v}_{gi}$ without changing its shape.

If the momentum dispersion corresponding to the momentum distribution function $f^S_i(\mathbf{p} - \mathbf{p}_0)$ is $\sigma_p$, then, according to the Heisenberg uncertainty relation, the length of the wave packet in the coordinate space $\sigma_x$ satisfies $\sigma_x \gtrsim \sigma_p^{-1}$; the shape-factor function $g^S_i(\mathbf{x} - \mathbf{v}_{gi}t)$ decreases rapidly when $|\mathbf{x} - \mathbf{v}_{gi}t|$ exceeds $\sigma_x$. Equation (12) actually justifies and corrects the plane-wave approach: the wave function of a propagating mass-eigenstate neutrino is described by the plane wave corresponding to the mean momentum $\mathbf{p}_i$, multiplied by the shape function $g^S_i(\mathbf{x} - \mathbf{v}_{gi}t)$ which makes sure that the wave is strongly suppressed outside a finite space–time region of width $\sigma_x$ around the point $\mathbf{x} = \mathbf{v}_{gi}t$.

In the approximation, where the wave packet spread is neglected, the evolved neutrino state is given by Eq. (8) with $\Psi_i(\mathbf{x}, t)$ from Eq. (12). Note that the wave packets corresponding to different neutrino mass eigenstates $\nu_i$ are in general described by different momentum distribution functions $f^S_i(\mathbf{p} - \mathbf{p}_i)$ and therefore by different shape factors $g^S_i(\mathbf{x} - \mathbf{v}_{gi}t)$.

[6)] For symmetric wave packets (i.e., when $f^S_i(\mathbf{p} - \mathbf{p}_i)$ is an even function of its argument), the position of the center of the peak coincides with the mean momentum $\mathbf{p}_i$. For asymmetric wave packets, it may be displaced from $\mathbf{p}_i$.

Next, we define the state of the detected flavor neutrino $\nu_b$ as a wave packet peaked at the coordinate $\mathbf{L}$ of the detector:

$$|\nu_b(\mathbf{x}-\mathbf{L})\rangle = \sum_i U_{bi}^* \Psi_i^D(\mathbf{x}-\mathbf{L})|\nu_i\rangle. \quad (15)$$

It will be convenient for us to rewrite $\Psi_i^D(\mathbf{x}-\mathbf{L})$ pulling out the factor $e^{i\mathbf{p}_i(\mathbf{x}-\mathbf{L})}$:

$$|\nu_b(\mathbf{x}-\mathbf{L})\rangle = \quad (16)$$
$$= \sum_i U_{bi}^*[e^{i\mathbf{p}_i(\mathbf{x}-\mathbf{L})} g_i^D(\mathbf{x}-\mathbf{L})]|\nu_i\rangle,$$

Here, $g_i^D(\mathbf{x}-\mathbf{L})$ is the shape factor of the wave packet corresponding to the detection of the $i$th mass eigenstate. The transition amplitude $\mathcal{A}_{ab}(\mathbf{L},t)$ is obtained by projecting the evolved state (8) onto (15):

$$\mathcal{A}_{ab}(\mathbf{L},t) = \int d^3x \langle \nu_b(\mathbf{x}-\mathbf{L})|\nu_a(\mathbf{x},t)\rangle = \quad (17)$$
$$= \sum_i U_{ai}^* U_{bi} \int d^3x \Psi_i^{D*}(\mathbf{x}-\mathbf{L})\Psi_i(\mathbf{x},t).$$

Substituting here expressions (12) and (16) yields

$$\mathcal{A}_{ab}(\mathbf{L},t) = \quad (18)$$
$$= \sum_i U_{ai}^* U_{bi} G_i(\mathbf{L}-\mathbf{v}_{gi}t) e^{-iE_i(p_i)t + i\mathbf{p}_i\mathbf{L}},$$

where

$$G_i(\mathbf{L}-\mathbf{v}_{gi}t) = \quad (19)$$
$$= \int d^3x g_i^S(\mathbf{x}-\mathbf{v}_{gi}t) g_i^{D*}(\mathbf{x}-\mathbf{L}).$$

That this integral indeed depends on $\mathbf{L}$ and $\mathbf{v}_{gi}t$ only through the combination $(\mathbf{L}-\mathbf{v}_{gi}t)$ can be easily shown by shifting the integration variable in (19). $G_i(\mathbf{L}-\mathbf{v}_{gi}t)$ is an effective shape factor whose width $\sigma_x$ depends on the widths of both the production and detection wave packets $\sigma_{xS}$ and $\sigma_{xD}$. Indeed, since the moduli of the shape-factor functions $g_i^{S,D}$ quickly decrease when the argument exceeds the corresponding wave-packet widths $\sigma_{xS}$ or $\sigma_{xD}$, from Eq. (19) it follows that the function $G_i(\mathbf{L}-\mathbf{v}_{gi}t)$ decreases when $|\mathbf{L}-\mathbf{v}_{gi}t|$ becomes large compared to $\max\{\sigma_{xS}, \sigma_{xD}\}$. Actually, since $\sigma_x$ characterizes the overlap of the wave packets describing the production and detection states, it exceeds both $\sigma_{xS}$ and $\sigma_{xD}$. In particular, for Gaussian and Lorentzian (in the coordinate space) wave packets one has $\sigma_x = \sqrt{\sigma_{xS}^2 + \sigma_{xD}^2}$ and $\sigma_x = \sigma_{xS} + \sigma_{xD}$, respectively. If the production and detection wave packets are symmetric in the coordinate space, i.e., the shape factors $g_i^S(\mathbf{x}-\mathbf{v}_{gi}t)$ and $g_i^D(\mathbf{x}-\mathbf{L})$ are even functions of their arguments, then so is $G_i(\mathbf{L}-\mathbf{v}_{gi}t)$. In that case the modulus of $G_i(\mathbf{L}-\mathbf{v}_{gi}t)$ reaches its maximum at $\mathbf{L} = \mathbf{v}_{gi}t$; otherwise the maximum may be displaced from this point.

The probability $P_{ab}(\mathbf{L},t) \equiv P(\nu_a \to \nu_b; \mathbf{L}, t)$ of finding a flavor eigenstate neutrino $\nu_b$ at the detector site at time $t$ is given by the squared modulus of the amplitude $\mathcal{A}_{ab}(\mathbf{L},t)$ defined in Eq. (17). Since in most experiments the neutrino emission and arrival times are not measured, the standard procedure in the wave packet approach to neutrino oscillations is then to integrate $P_{ab}(\mathbf{L},t)$ over time. In doing so one has to introduce a normalization factor which is usually not calculated[7)], and in fact is determined by imposing "by hand" the requirement that the probabilities $P_{ab}(L)$ satisfy the unitarity condition. This is an *ad hoc* procedure which is not entirely consistent; the proper treatment would require to consider the temporal response function of the detector and would automatically lead to the correct normalization of the oscillation probabilities. Since we are primarily interested here in the oscillation phase which is practically insensitive to the detector response function, we follow the same procedure of integration over time. The proper normalization of the oscillation probability is achieved by imposing the normalization condition

$$\int_{-\infty}^{\infty} dt |G_i(\mathbf{L}-\mathbf{v}_{gi}t)|^2 = 1. \quad (20)$$

For simplicity, from now on we neglect the transverse components of the neutrino momentum, i.e., the components orthogonal to the line connecting the centers of the neutrino source and detector; this is a very good approximation for neutrinos propagating macroscopic distances. The probability of finding a $\nu_b$ at the detector site, provided that a $\nu_a$ was emitted by the source at the distance $L$ from the detector, is then

$$P_{ab}(L) = \int_{-\infty}^{\infty} dt |\mathcal{A}_{ab}(L,t)|^2 = \quad (21)$$
$$= \sum_{i,k} U_{ai}^* U_{bi} U_{ak} U_{bk}^* I_{ik}(L),$$

where

$$I_{ik}(L) \equiv \int_{-\infty}^{\infty} dt G_i(L - v_{gi}t) \times \quad (22)$$
$$\times G_k^*(L - v_{gk}t) e^{-i\Delta\phi_{ik}(L,t)}.$$

Here, $G_i(L - v_{gi}t)$ is the effective shape factor corresponding to the $i$th neutrino mass eigenstate defined

[7)]For an exception, see [26].

in (the 1-dimensional version of) Eq. (19). The quantity $\Delta\phi_{ik}(L,t)$ is the phase differences between the $i$th and $k$th mass eigenstates:

$$\Delta\phi_{ik} = (E_i - E_k)t - (p_i - p_k)L \equiv \equiv \Delta E_{ik}t - \Delta p_{ik}L, \quad (23)$$

where

$$E_i = \sqrt{p_i^2 + m_i^2}. \quad (24)$$

Note that this phase difference is Lorentz invariant.

To calculate the observable quantities — the numbers of the neutrino detection events — one has to integrate the oscillation probability (folded with the source spectrum, detection cross section and detector efficiency and energy resolution functions) over the neutrino spectrum and over the macroscopic sizes of the neutrino source and detector.

### *3.2. Oscillation Phase: the Answer to the Question*

We are now in a position to present our argument. To simplify the notation, we suppress the indices $i$ and $k$, where it cannot cause a confusion, so that $\Delta E \equiv \Delta E_{ik}$, $\Delta m^2 \equiv \Delta m^2_{ik}$, etc. Consider the case $\Delta E \ll E$ which corresponds to relativistic or quasi-degenerate neutrinos. In this case one can expand the difference of the energies of two mass eigenstates in the differences of their momenta and masses. Retaining only the leading terms in this expansion, one gets

$$\Delta E = \frac{\partial E}{\partial p}\Delta p + \frac{\partial E}{\partial m^2}\Delta m^2 = = v_g\Delta p + \frac{1}{2E}\Delta m^2, \quad (25)$$

where $v_g$ is the average group velocity of the two mass eigenstates and $E$ is the average energy. Substituting this into Eq. (23) yields [31]

$$\Delta\phi = \frac{\Delta m^2}{2E}t - (L - v_g t)\Delta p. \quad (26)$$

Note that our use of the mean group velocity and mean energy of the two mass eigenstates in Eq. (26) is fully legitimate. Indeed, going beyond this approximation would mean retaining terms of second and higher order in $\Delta m^2$ in the expression for $\Delta\phi$. These terms are small compared to the leading $\mathcal{O}(\Delta m^2)$ terms; moreover, though their contribution to $\Delta\phi$ can become of order one at extremely long distances, the leading contribution to $\Delta\phi$ is then much greater than one, which means that neutrino oscillations are in the averaging regime and the precise value of the oscillation phase is irrelevant.

Let us now consider expression (26) for the phase difference $\Delta\phi$. If one adopts the same-momentum assumption for the mean momenta of the wave packets, $\Delta p = 0$, the second term on the right-hand side disappears, which leads to the standard oscillation phase in the "evolution in time" picture. If, in addition, one assumes the "time-to-space conversion" relation (5), the standard formula for neutrino oscillations is obtained.

Alternatively, instead of expanding the energy difference of two mass eigenstates in the differences of their momenta and masses, one can expand the momentum difference of these states in the differences of their energies and masses:

$$\Delta p = \frac{\partial p}{\partial E}\Delta E + \frac{\partial p}{\partial m^2}\Delta m^2 = = \frac{1}{v_g}\Delta E - \frac{1}{2p}\Delta m^2, \quad (27)$$

where $p$ is the average momentum. Substituting this into Eq. (23) yields [32]

$$\Delta\phi = \frac{\Delta m^2}{2p}L - \frac{1}{v_g}(L - v_g t)\Delta E. \quad (28)$$

Note that this relation could also be obtained directly from Eq. (26) by making use of Eq. (25). If one now adopts the same energy assumption for the mean energies of the wave packets, $\Delta E = 0$, the second term on the right-hand side vanishes, and one arrives at the standard oscillation formula.

However, as we shall show now, Eqs. (26) and (28) actually lead to the standard oscillation phase even without the same-energy or same-momentum assumptions. For this purpose, let us generically write Eqs. (26) and (28) in the form

$$\Delta\phi = \Delta\phi_{\rm st} + \Delta\phi', \quad (29)$$

where $\Delta\phi_{\rm st}$ is the standard oscillation phase either in "evolution in time" or in "evolution in space" approach, and $\Delta\phi'$ is the additional term (the second term in Eq. (26) or (28)). The first thing to notice is that $\Delta\phi'$ vanishes not only when $\Delta p = 0$ (in Eq. (26)) or $\Delta E = 0$ (in Eq. (28)), but also at the center of the wave packet, where $L = v_g t$. Away from the center, the quantity $L - v_g t$ does not vanish, but it never exceeds substantially the length of the wave packet $\sigma_x$, since otherwise the shape factors would strongly suppress the neutrino wave function; thus, $|L - - v_g t| \lesssim \sigma_x$. The physical meaning of the two terms in Eq. (29) is then clear: $\Delta\phi_{\rm st}$ is the oscillation phase acquired over the distance $L$ between the centers of the neutrino emitter and absorber wave functions, whereas $\Delta\phi'$ is the additional phase variation along of the wave packet. Note that in the wave-packet approach the effective spatial length of the neutrino wave packet is determined by the sizes of the neutrino production and detection regions[8], so that the phase $\Delta\phi'$ is related to the condition of localization of the neutrino emitter and absorber. In what follows we shall show that explicitly.

Since $\Delta E$ and $\Delta p$ are the differences of, respectively, the mean energies and mean momenta of different neutrino mass eigenstates, for relativistic neutrinos they are both of the order of $\Delta m^2/2E$ or smaller; recalling that the neutrino oscillation length is $l_{\rm osc} = 4\pi E/\Delta m^2$, we conclude that the additional phase $\Delta\phi'$ can be safely neglected (and the oscillation phase takes its standard value) when the effective length of the wave packets is small compared to the neutrino oscillation length, i.e., $\sigma_x \ll l_{\rm osc}$.

We shall show now that under very general assumptions the oscillation phase takes the standard value. For this we need the wave packet treatment of neutrino oscillations. We start with the expression for $\Delta\phi$ in Eq. (28). Substituting it into (22), we find

$$I_{ik}(L) = \exp\left(-i\frac{\Delta m^2_{ik}}{2p}L\right) \times \tag{30}$$
$$\times \int\limits_{-\infty}^{\infty} dt G_i(L - v_{gi}t)G_k^*(L - v_{gk}t) \times$$
$$\times \exp\left(i\frac{1}{v_g}\Delta E_{ik}(L - v_g t)\right).$$

Let us first neglect the difference between the group velocities of the wave packets describing different mass eigenstates, i.e., take $v_{gi} = v_{gk} = v_g$ and $G_k = G_i$. In this approximation (which neglects the decoherence effects due to the wave-packet separation, see Appendix), the integral in (30) does not depend on $L$; this can be readily shown by changing the integration variable according to $t \to (L - v_g t)$. Thus, in the case when the difference of the group velocities of different mass eigenstates is negligible, we arrive at the standard oscillation phase. The integral in Eq. (30) is then just the Fourier transform of the squared modulus of the shape factor:

$$\frac{1}{v_g}\int\limits_{-\infty}^{\infty} dx'|G_i(x')|^2 \exp\left(i\frac{1}{v_g}\Delta E_{ik}x'\right). \tag{31}$$

It is essentially a localization factor, which takes into account effects of suppression of oscillations in the case when the localization regions of the neutrino emitter and/or absorber are not small compared to the neutrino oscillation length (see Section 5.2 and Appendix for a more detailed discussion). If localization condition (A.8) is fulfilled, one can replace the oscillating phase factor in the integrand by unity; the resulting integral is then simply the normalization factor, which has to be set to 1 in order for the oscillation probability to be properly normalized (see Eq. (20)). If, on the contrary, the condition opposite to the localization condition (A.8) is satisfied, the integral (31) is strongly suppressed due to the fast oscillations of the factor $\exp(i\Delta E_{ik}x'/v_g)$ in the integrand, leading to the suppression of the oscillations. In the borderline case $\Delta E_{ik}\sigma_X/v_g \sim 1$, a partial decoherence due to the lack of localization occurs.

If one now allows for $v_{gi} \neq v_{gk}$, then, as shown in Appendix, the dependence of the integral in Eq. (30) on $L$ is still negligible provided that $L\Delta v_g/v_g \ll \sigma_x$, or

$$L \ll l_{\rm coh} = \sigma_x \frac{v_g}{\Delta v_g}, \tag{32}$$

where $\Delta v_g = |v_{gi} - v_{gk}|$. This is nothing but the condition of the absence of decoherence due to the wave packet separation: the distance traveled by neutrinos should be smaller than the distance over which the wave packets corresponding to different mass eigenstates separate, due to the difference of their group velocities, to such an extent that they can no longer interfere in the detector[9]. If the condition opposite to that in Eq. (32) is satisfied, the integral in Eq. (30) is strongly suppressed because of the lack of overlap between the factors $G_i(L - v_{gi}t)$ and $G_k^*(L - v_{gk}t)$ in the integrand (if $I_{ik}$ is written as a momentum-space integral, the suppression is due to the fast oscillations of the integrand, see Eq. (A.7) of Appendix). The $L$ independence of the integral in Eq. (30) in the absence of decoherence due to the wave packet separation means that the neutrino oscillation phase takes its standard value in that case.

Thus we conclude that *the standard oscillation phase is obtained if neutrinos are relativistic or quasi-degenerate in mass and the decoherence effects due to the wave-packet separation or lack of the emitter or absorber localization are negligible. No unjustified "same energy" or "same momentum" assumptions are necessary to arrive at this result.*

One may wonder why these assumptions are actually so popular in the literature and even made their way to some textbooks, though there is no good reason to believe that the different mass eigenstates have

[8] In the more general quantum field theoretic framework the length of the neutrino wave packet is related to the time scales of the neutrino emission and detection processes rather than to the localization properties of the neutrino emitter and absorber, see Section 5.4.

[9] We reiterate that $\sigma_x$ is an *effective* spatial width of the wave packet, which depends on the widths of both the production and detection wave packets $\sigma_{xS}$ and $\sigma_{xD}$. Condition (32) therefore automatically takes into account possible restoration of coherence at detection, as discussed in Section 5.3.

either same momentum or same energy. One possible reason could be the simplicity of the derivation of the formula for the oscillation probability under these assumptions. However, we believe that simplicity is no justification for using a wrong argument to arrive at the correct result.

## 4. ANOTHER STANDPOINT

In this section we outline a more general approach to the calculation of the oscillation probability, which gives an additional insight into the issues discussed in the present paper. Here we just illustrate some points relevant to our discussion rather than presenting a complete formalism, which is beyond the scope of our paper.

The flavor transition amplitude is given by (the 1-dimensional version of) Eq. (17). The function $\Psi_i^S(x,t)$ can be represented as the integral over momenta of Eq. (9). Inserting this expression and a similar formula for $\Psi_i^D(x-L)$ into the expression for the amplitude and performing the integral over the coordinate, we obtain

$$\mathcal{A}_{ab}(L,t) = \sum_i U_{ai}^* U_{bi} \int dq_i h_i(q_i) e^{iq_iL - iE_i(q_i)t}, \tag{33}$$

where

$$h_i(q) \equiv f_i^S(q-p_i) f_i^{D*}(q-p_i). \tag{34}$$

Here, the momenta $q_i$ are the integration variables, whereas $p_i$ are, as before, the mean momenta of the corresponding wave packets.

The amplitude (33) is the sum of plane waves with different momenta and different masses. The integration over momenta can be formally substituted by a summation to make this point clearer. The oscillation probability is then

$$P_{ab}(L,t) \equiv |\mathcal{A}_{ab}(L,t)|^2 = \left| \sum_i \sum_{q_i} U_{ai}^* U_{bi} h_i(q_i) e^{iq_iL - iE_i(q_i)t} \right|^2. \tag{35}$$

The waves with all momenta and masses should be summed up; the resulting expression for the oscillation probability includes the interference of these waves.

The standard approach to the calculation of the oscillation amplitude (33) (or probability (35)) is to sum up first the waves with different momenta but the same mass, and then sum over the mass eigenstates. In this way first the wave packets corresponding to different mass eigenstates are formed, and then the interference of these wave packets is considered. Since

$$\int dq_i h_i(q_i) e^{iq_iL - iE_i(q_i)t} = G_i(L - v_{gi}t) e^{ip_iL - iE_i(p_i)t}, \tag{36}$$

Eq. (33) then directly reproduces the results of Section 3.

Another possibility is to sum up first the plane waves with different masses but equal (or related) momenta and then perform the integration over the momenta. In particular, one can select the waves with equal energies. Clearly, the final result should not depend on the order of summation if no approximations are made, and should be almost independent of this order if the approximations are well justified. However, different summation conventions allow different physical interpretations of the result. In what follows we will perform computations using the "equal momenta" and "equal energy" summation rules and identify the conditions under which they lead to the standard result for the oscillation probability.

Let us first consider the "equal momenta" summation. Setting $q_i = p$ for all $i$, one can write the amplitude (33) as

$$\mathcal{A}_{ab}(L,t) = \int dp \sum_i U_{ai}^* U_{bi} h_i(p) e^{ipL - iE_i(p)t}. \tag{37}$$

Next, we note that, for ultrarelativistic neutrinos, to an extremely good approximation

$$h_i(p) = h(p, E_i(p)), \tag{38}$$

i.e., the functions $h_i(p)$ depend on the index $i$ only through their dependence on the neutrino energy $E_i(p) = \sqrt{p^2 + m_i^2}$. Indeed, $h_i(p)$ depend on $i$ through the neutrino mass dependence of the phase-space volumes and amplitudes of the neutrino emission and detection processes. The phase-space volumes depend on $m_i$ only through $E_i(p)$; the production and detection amplitudes can in principle depend directly on $m_i$ due to the chiral suppression, as, e.g., in the case of $\pi^\pm$ or $K^\pm$ decays. However, the corresponding contributions to the total amplitudes are completely negligible compared to the main contributions, which in this case are proportional to the masses of charged leptons.

Let us expand $h_i$ in power series in $\Delta m_{i3}^2$:

$$h_i(p) = h(p, E_i) = h(p, E_3) + E_i \frac{\partial h_i}{\partial E_i}\bigg|_{m_i = m_3} \frac{\Delta m_{i3}^2}{2E_3^2} + \ldots \tag{39}$$

Inserting this expression into (37), we obtain

$$\mathcal{A}_{ab}(L,t) = \int dp h(p, E_3(p)) e^{ipL - iE_3(p)t} \times \tag{40}$$

$$\times \sum_i U_{ai}^* U_{bi} e^{-i[E_i(p)-E_3(p)]t} + \mathcal{A}_{ab}^{\Delta}(L,t),$$

where

$$\mathcal{A}_{ab}^{\Delta}(L,t) \equiv \int dp e^{ipL-iE_3(p)t} \times \tag{41}$$
$$\times \sum_i U_{ai}^* U_{bi} e^{-i[E_i(p)-E_3(p)]t} \left.\frac{E_i \partial h_i}{\partial E_i}\right|_{m_i=m_3} \frac{\Delta m_{i3}^2}{2E_3^2(p)}.$$

It is easy to see that the term $\mathcal{A}_{ab}^{\Delta}(L,t)$ is typically very small:

$$\mathcal{A}_{ab}^{\Delta}(L,t) \sim \frac{\Delta m_{i3}^2}{E\sigma_E}\mathcal{A}_1, \tag{42}$$

where $\mathcal{A}_1$ is the first term on the right-hand side of Eq. (40). Indeed, if the width of the effective momentum distribution function $h(p)$ is $\sigma_p$, one has $\partial h/\partial p \sim h/\sigma_p$, so that

$$\frac{E\partial h}{\partial E} = \frac{E}{v_g}\frac{\partial h}{\partial p} \sim \frac{Eh}{v_g\sigma_p} = \frac{Eh}{\sigma_E}, \tag{43}$$

which immediately leads to (42). Thus, if

$$\sigma_E \gg \frac{\Delta m^2}{2E}, \tag{44}$$

$\mathcal{A}_{ab}^{\Delta}(L,t)$ can safely be neglected.

Consider now the first term in Eq. (40). For a fixed momentum the phase difference is $\Delta\phi_{i3} = (E_i - E_3)t \approx \Delta m_{i3}^2 t/2E_3$[10]. If the width of the effective momentum distribution function $h(p,E_3)$ is small enough, so that the change of the phase within the wave packet is small, we can pull the oscillatory factor out of the integral at some effective momentum (corresponding to an energy $E$):

$$\mathcal{A}_{ab}(L,t) \simeq \left[\sum_i U_{ai}^* U_{bi} \exp\left(-i\frac{\Delta m_{i3}^2}{2E}t\right)\right] \times \tag{45}$$
$$\times \int dp h(p, E_3(p)) e^{ipL-iE_3(p)t} =$$
$$= \left[\sum_i U_{ai}^* U_{bi} \exp\left(-i\frac{\Delta m_{i3}^2}{2E}t\right)\right] \times$$
$$\times G_3(L - v_{g3}t) e^{ip_3L-iE_3(p_3)t}.$$

Here, the factor in the square brackets gives the standard oscillation amplitude in the "evolution in time" approach. Integrating the squared modulus of amplitude (45) over time and using once again Eq. (44), one arrives at the standard expression for the oscillation probability.

Thus, we obtain the standard oscillation formula by first summing up the waves with equal momenta and different masses and then integrating over the momenta provided that the following two conditions are satisfied:

(i) The variation of the oscillation phase within the wave packet due to the energy spread is small: $\sigma_E \ll (2\pi E^2/\Delta m_{ik}^2)L^{-1}$; this condition allows one to pull the oscillatory factor out of the integral over the momenta, as discussed above. Note that it is actually equivalent to the condition of no wave packet separation, Eq. (32) (recall that $\Delta v_g \simeq \Delta m^2/2E^2$ and $\sigma_x \simeq v_g/\sigma_E$).

(ii) The momentum distribution functions $h_i(p)$ are not too narrow: $\sigma_E \gg \Delta m^2/2E$. This condition, in particular, allows one to neglect the contribution $\mathcal{A}_{ab}^{\Delta}(L,t)$ in Eq. (40). It is essentially the localization condition in the wave-packet picture, as it actually ensures that the neutrino wave packet length $\sigma_x \simeq v_g/\sigma_E$ is small compared to the neutrino oscillation length.[11]

These conditions for obtaining the standard oscillation formula coincide with the conditions found in a different framework in Section 3.2.

To describe the possible decoherence effects due to the separation of the wave packets and reproduce the localization factor in the oscillation probability explicitly, one should lift conditions (i) and (ii) and consider the corresponding corrections to the oscillation amplitude. This is discussed in detail in Section 5.

Similarly, we can consider summation of waves with equal energies and different masses, with the subsequent integration over energies (or momenta). Requiring $E_i(q_i) = E_3(p)$ yields $q_i = \pm\sqrt{p^2 + \Delta m_{3i}^2}$, $q_3 = p$. Taking into account that $dq_i = \pm dp/\sqrt{1 + \Delta m_{3i}^2/p^2}$, we obtain

$$\mathcal{A}_{ab}(L,t) = \tag{46}$$
$$= \int dp \sum_i U_{ai}^* U_{bi} \frac{h_i(q_i(p), E_3(p))}{\sqrt{1 + \Delta m_{3i}^2/p^2}} e^{iq_i(p)L - iE_3(p)t}.$$

Here, it is assumed that $\nu_3$ is the heaviest mass eigenstate, so that $\Delta m_{3i}^2 \geq 0$ and no singularities appear in the integrand. Note that our change of

[10] This approximation breaks down at very small momenta. Note, however, that the small-$p$ contribution to the integral in (40) is strongly suppressed because of the effective momentum distribution function $h(p,E_3)$, which is strongly peaked at a relativistic momentum $p = p_3$. This justifies using the approximation for $\Delta\phi_{i3}$ in (40).

[11] Note that the oscillation probability may take the standard form even if this condition is violated, as it is the case, e.g., for Mössbauer neutrinos [4, 6].

the integration variables $q_i \to p$ excludes, for $i = 1, 2$, the small regions of momenta $q_i$ around zero; this, however, introduces only a tiny error, because the main contributions to the integral come from the regions around the points $q_i = p_i$, where the functions $h_i(q_i)$ are strongly peaked.

Expanding $h_i(q_i(p), E_3(p))$ as

$$\frac{h_i(q_i, E_3(p))}{\sqrt{1 + \Delta m_{3i}^2/p^2}} \simeq h(p, E_3(p)) + \qquad (47)$$
$$+ \left(h(p, E_3(p)) - p\frac{\partial h(p, E_3(p))}{\partial p}\right) \frac{\Delta m_{3i}^2}{2p^2}$$

and inserting this expression into (46), we obtain

$$\mathcal{A}_{ab}(L, t) = \int dp h(p, E_3(p)) e^{ipL - iE_3(p)t} \times \qquad (48)$$
$$\times \sum_i U_{ai}^* U_{bi} e^{i[q_i(p) - p]L} + \bar{\mathcal{A}}_{ab}^{\Delta},$$

where

$$\bar{\mathcal{A}}_{ab}^{\Delta}(L, t) \equiv \int dp e^{ipL - iE_3(p)t} \times \qquad (49)$$
$$\times \sum_i U_{ai}^* U_{bi} e^{i[q_i(p) - p]L} \left(h - \frac{p\partial h}{\partial p}\right) \frac{\Delta m_{3i}^2}{2p^2}.$$

Just as in the previous case, one can show that the amplitude $\bar{\mathcal{A}}_{ab}^{\Delta}(L, t)$ can be neglected if condition (44) is satisfied. Assuming this to be the case and that the variation of the oscillation phase within the wave packet due to the momentum spread is small, and taking into account that $(q_i - p)L = (q_i - q_3)L \simeq \simeq \Delta m_{3i}^2 L/2p$, one finds

$$\mathcal{A}_{ab}(L, t) \simeq \qquad (50)$$
$$\simeq \left[\sum_i U_{ai}^* U_{bi} \exp\left(i\frac{\Delta m_{3i}^2}{2p} L\right)\right] \times$$
$$\times G_3(L - v_{g3}t) e^{ip_3 L - iE_3(p_3)t},$$

where $p$ is the average neutrino momentum. This is the standard oscillation amplitude multiplied by the effective shape factor of the wave packet of $\nu_3$ (note that in our current approximation we actually neglect the difference between the wave packets of different mass eigenstates). Integrating the squared modulus of the amplitude in Eq. (50) over time and using normalization condition (20), we again arrive at the standard expression for the oscillation probability, just as in the previous case when we first summed the terms with equal momenta and different masses and then integrated over the momenta. In deriving this result we once again used conditions (i) and (ii) discussed above.

Our discussion of the new summation rules for calculating the oscillation probability presented here leads to an alternative explanation of why the "same energy" and "same momentum" assumptions eventually lead to the correct physical observables, as discussed in Section 6.

## 5. QUANTUM-MECHANICAL UNCERTAINTY RELATIONS AND NEUTRINO OSCILLATIONS

Neutrino oscillations, being a quantum-mechanical interference phenomenon, owe their very existence to quantum-mechanical uncertainty relations. The coordinate–momentum and time–energy uncertainty relations are implicated in the oscillations phenomenon in a number of ways. First, it is the energy and momentum uncertainties of the emitted neutrino state that allow it to be a coherent superposition of the states of well-defined and different mass. The same applies to the detection process — for neutrino detection to be coherent, the energy and momentum uncertainties inherent in the detection process should be large enough to prevent a determination of the absorbed neutrino's mass in this process. The uncertainty relations also determine the size of the neutrino wave packets and therefore are crucial to the issue of the loss of coherence due to the wave-packet separation. In addition, these relations are important for understanding how the produced and detected neutrino states are disentangled from the accompanying particles. Let us now discuss these issues in more detail.

### *5.1. Uncertainty Relations and Disentanglement of Neutrino States*

In the majority of analyses of elementary-particle processes it is assumed that the energies and momenta of all the involved particles have well-defined (sharp) values and obey the exact conservation laws. However, for this description to be exact, the considered processes (and the particles involved) should be completely delocalized in space and in time, whereas in reality these processes occur in finite and relatively small spatial volumes and during finite time intervals. For this reason, the energy and momenta of all the participating particles have intrinsic quantum mechanical uncertainties, and the particles should be described by wave packets rather than states of definite momentum — plane waves (see, e.g., [19]). The conservation of energy and momentum for these particles is also fulfilled up to these small uncertainties.

This does not, of course, mean that the energy–momentum conservation, which is a fundamental law of nature, is violated: it is satisfied exactly when one applies it to all particles in the system, including those whose interactions with the particles directly involved in the process localize the latter in a given space–time region. Schematically speaking, if we consider the process as occuring in a box, the interactions with the walls of the box and the contributions of these walls to the energy–momentum balance have to be taken into account. In reality, this is never done; however, the resulting inaccuracy of the energy and momentum conservation as well as the intrinsic quantum-mechanical uncertainties of the energies and momenta of the involved particles are usually completely negligible compared to their energies and momenta themselves, and therefore can be safely ignored in most processes.

This is, however, not justified when neutrino oscillations are considered, since the neutrino energy and momentum uncertainties, as tiny as they are, are crucially important for the oscillation phenomenon. In this respect, we believe that the attempts to use the exact energy–momentum conservation in the analyses of neutrino oscillations are inconsistent. In some analyses the exact energy–momentum conservation is assumed for the neutrino production and detection processes in order to describe neutrinos as being entangled with accompanying particles. The subsequent disentanglement, which is necessary for neutrino oscillations to occur, is assumed to be due to the interaction of these accompanying particles (such as, e.g., electrons or muons produced in decays of charged pions) with medium. This localizes those particles and creates the necessary energy and momentum uncertainties for the neutrino state. The described approach misses the fact that the parent particles are already localized in the neutrino production and detection processes, and so no additional disentanglement through the interaction of the accompanying particles with medium is necessary. Indeed, it is clear that neutrinos produced, for example, in $\pi^\pm$ decays oscillate even if the accompanying charged leptons do not interact with medium, i.e., are not "measured". The measurement of the flavor of these charged leptons that discriminates between $e^\pm$ and $\mu^\pm$ and makes neutrino oscillations possible is actually provided by the decoherence of the charged leptons due to their very large mass difference [7].

### *5.2. Coherence of the Produced and Detected Neutrino States*

In order for a neutrino state produced in a charged-current weak interaction process to be a coherent superposition of different neutrino mass eigenstates, it should be in principle impossible to determine which mass eigenstate has been emitted. This means that the intrinsic quantum-mechanical uncertainty of the squared mass of the emitted neutrino state $\sigma_{m^2}$ must be larger than the difference $\Delta m^2$ of the squared masses of different neutrino mass eigenstates [21, 23]: $\sigma_{m^2} \gtrsim \Delta m^2$. Conversely, if $\sigma_{m^2} \ll \Delta m^2$, one can determine which mass eigenstate has been emitted, i.e., the coherence of different mass eigenstates is destroyed. This situation is quite similar to that with the electron interference in double slit experiments: If there is no way to find out which slit the detected electron has passed through, the detection probability will exhibit an interference pattern, but if such a determination is possible, the interference pattern will be washed out.

Assume that by measuring energies and momenta of the other particles involved in the production process we can determine the energy $E$ and momentum $p$ of the emitted neutrino state, and that the intrinsic quantum-mechanical uncertainties of these quantities are $\sigma_E$ and $\sigma_p$. From the energy–momentum relation $E^2 = p^2 + m^2$ we can then infer the squared mass of the neutrino state with the uncertainty $\sigma_{m^2} = [(2E\sigma_E)^2 + (2p\sigma_p)^2]^{1/2}$, where it is assumed that $\sigma_E$ and $\sigma_p$ are uncorrelated. Therefore the condition that the neutrino state be emitted as a coherent superposition of different mass eigenstates is [21, 23]

$$\sigma_{m^2} \equiv \left[(2E\sigma_E)^2 + (2p\sigma_p)^2\right]^{1/2} \gg \Delta m^2. \quad (51)$$

This condition has a simple physical meaning. Note first that in many cases the two terms in the square brackets in Eq. (51) are of the same order of magnitude: $E\sigma_E \sim p\sigma_p$ (more generally, $E\sigma_E \le p\sigma_p$, see Section 5.4). Therefore the condition in Eq. (51) essentially reduces to $2p\sigma_p \gg \Delta m^2$, or $\sigma_p \gg \Delta m^2/2p = 2\pi/l_{\rm osc}$. Since the momentum uncertainty $\sigma_p$ is related to the coordinate uncertainty of the neutrino source $\sigma_{xS}$ by $\sigma_p \sim \sigma_{xS}^{-1}$, we finally get

$$\sigma_{xS} \ll l_{\rm osc}. \quad (52)$$

This is nothing but the obvious requirement that the neutrino production be localized in a spatial region that is small compared to the neutrino oscillation length; if it is not satisfied, then neutrino oscillations will be averaged out upon the integration over the neutrino emission coordinate in the source, which is equivalent to decoherence. For this reason the condition in Eq. (51) is often called the localization condition. Similar argument applies to the detection process, i.e., the spatial size of the neutrino absorber $\sigma_{xD}$ should satisfy

$$\sigma_{xD} \ll l_{\rm osc}. \quad (53)$$

Since both (52) and (53) have to be fulfilled for neutrino oscillations to be observable, one can reformulate the localization condition as the requirement that the oscillation length $l_{\rm osc}$ be large compared to

$\sigma_X \equiv \max\{\sigma_{xS}, \sigma_{xD}\}$, i.e., essentially compared to the effective size of the wave packet $\sigma_x$ which is of order of $\sigma_X$:

$$\sigma_x \ll l_{\rm osc}. \qquad (54)$$

That this directly follows from the wave packet formalism is shown in Appendix.

In real situations one always deals with large ensembles of neutrino emitters, and the detectors also consist of a large number of particles. Therefore, in calculating the observable quantities — the numbers of the neutrino detection events — one always has to integrate over the macroscopic volumes of the neutrino source and detector. In some situations (e.g., for solar or reactor neutrinos) the source and detector are much larger than the localization domains of the wave functions of individual neutrino emitters and absorbers. In these cases the integration over the source and detector volumes modifies the localization conditions: instead of depending on the spatial sizes of the individual neutrino emitter and absorber, $\sigma_{xS}$ and $\sigma_{xD}$, they contain the macroscopic lengths of the source and detector in the direction of the neutrino beam, $L_S$, and $L_D$. In other words, a necessary condition for the observability of neutrino oscillations is

$$L_{S,D} \ll l_{\rm osc}, \qquad (55)$$

which is much more restrictive than the conditions in Eqs. (52) and (53). If this condition is violated, neutrino oscillations are averaged out.

### *5.3. Wave Packet Separation and Restoration of Coherence at Detection*

Let us now assume that a neutrino flavor state was produced coherently in a weak interaction process and consider its propagation. The wave packet describing a flavor state is a superposition of the wave packets corresponding to different mass eigenstates. Since the latter propagate with different group velocities, after some time $t_{\rm coh}$ they will separate in space and will no longer overlap. If the spatial width of the wave packet is $\sigma_x$, this time is $t_{\rm coh} \simeq \sigma_x/\Delta v_g$. The distance $l_{\rm coh}$ that the neutrino state travels during this time is $l_{\rm coh} \simeq v_g(\sigma_x/\Delta v_g)$. If the distance $L$ between the neutrino emission and detection points is small compared to the coherence length, i.e., if condition (32) is satisfied, then the coherence of the neutrino state is preserved, and neutrino oscillations can be observed. If, on the contrary, $L \sim l_{\rm coh}$ or $L \gg l_{\rm coh}$, partial or full decoherence should take place.

If the wave-packet width $\sigma_x$ in the above discussion is assumed to be fully determined by the emission process, then this is not the full story yet: even if the coherence is lost on the way from the source to the detector, it still may be restored in the detector if the detection process is sufficiently coherent, i.e., is characterized by high enough energy resolution [33]. According to the quantum-mechanical time—energy uncertainty relation, high energy resolution requires the detection process to last sufficiently long; in that case, the different wave packets may arrive during the detection time interval and interfere in the detector even if they have spatially separated on their way from the source to the detector. One can take into account this possible restoration of coherence in the detector by considering $\sigma_x$ to be an effective width of the wave packet, which exceeds the width of the emitted wave packet and takes the coherence of the detection process into account. This effective wave-packet width is actually the width that characterizes the shape factors $G_i(L - v_{gi}t)$, as discussed in Section 3.1. With $\sigma_x$ being the effective wave-packet width, coherence condition (32) includes the effects of possible coherence restoration at detection.

It is well known that coherence plays a crucial role in observability of neutrino oscillations. It is interesting to note, however, that even non-observation of neutrino oscillations at baselines that are much shorter than the oscillation length is a consequence of and a firm evidence for coherence of the neutrino emission and detection processes: if it were broken (i.e., if the different neutrino mass eigenstates were emitted and absorbed incoherently), the survival probability of neutrinos of a given flavor, instead of being practically equal to 1, would correspond to averaged neutrino oscillations.

### *5.4. What Determines the Size of the Wave Packet?*

According to the quantum-mechanical uncertainty relations, the energy and momentum uncertainties of a neutrino produced in some process are determined by, correspondingly, the time scale of the process and spatial localization of the emitter. These two quantities are in general independent; on the other hand, for a free on-shell particle of definite mass the dispersion relation $E^2 = p^2 + m^2$ immediately leads to

$$E\sigma_E = p\sigma_p. \qquad (56)$$

Since this relation is satisfied for each mass eigenstate component of the emitted flavor state, it must also be satisfied for the state as a whole (provided that the energies and momenta of different components as well as their uncertainties are nearly the same, which is the case for relativistic or quasi-degenerate neutrinos). Thus, we have an apparently paradoxical situation: on the one hand, $\sigma_E$ and $\sigma_p$ should be essentially independent, while on the other hand they must satisfy Eq. (56).

The resolution of this paradox comes from the observation that at the time of their production neutrinos are actually not on the mass shell and therefore do not satisfy the standard dispersion relation. Therefore their energy and momentum uncertainties need not satisfy (56). However, as soon as neutrinos move away from their production point and propagate distances $x$ such that $px \gg 1$, they actually go on the mass shell, and their energy and momentum uncertainties start obeying Eq. (56). This happens because the bigger of the two uncertainties shrinks towards the smaller one, so that Eq. (56) gets fulfilled. Indeed, when the neutrinos go on the mass shell, the standard relativistic dispersion relation connecting the energies and momenta of their mass eigenstate components allows to determine the less certain of these two quantities through the more certain one, thus reducing the uncertainty of the former. As a result, the two uncertainties get related by Eq. (56), with the one that was smaller at production retaining its value also on the mass shell. Note that for neutrino energies in the MeV range neutrinos go on the mass shell as soon as they propagate distances $x \gtrsim 10^{-10}$ cm from their birthplace.

From the above discussion it follows that the spatial width of the neutrino wave packet, which determines the neutrino coherence length $l_{\rm coh}$ through Eq. (32), is determined by the smaller of the two uncertainties at production, $\sigma_E$ and $\sigma_p$. This comes about because $l_{\rm coh}$ is a characteristic of neutrinos that propagate macroscopic distances and therefore are on the mass shell. At the same time, localization condition (54) always depends on the spatial localization properties of the neutrino emitter and detector, which are related to the corresponding momentum uncertainties at production and detection.

Which of the two uncertainties, $\sigma_p$ or $\sigma_E$, is actually the smaller one at production and so determines the spatial width of the neutrino wave packet? Quite generally, this happens to be the energy uncertainty $\sigma_E$. Indeed, consider, e.g., an unstable particle, the decay of which produces a neutrino. In reality, such particles are always localized in space, so one can consider them to be confined in a box of a linear size $L_S$. The localizing "box" is actually created by the interactions of the particle in question with some other particles. Assume first that the time interval $T_S$ between two subsequent collisions of the decaying particle with the walls of the box (more precisely, the interval between its collisions with the surrounding particles) is shorter than its lifetime $\tau = \Gamma^{-1}$. Then the energy width of the state produced in the decay is given by the so-called collisional broadening and is actually $\simeq T_S^{-1}$. This width directly gives the neutrino energy uncertainty, i.e., $\sigma_E \simeq T_S^{-1}$. On the other hand, the neutrino momentum uncertainty is $\sigma_p \simeq \simeq L_S^{-1}$. Since $T_S$ is related to $L_S$ through the velocity of the parent particle $v$ as $T_S \simeq L_S/v$, one finds

$$\sigma_E < \sigma_p, \qquad (57)$$

which is actually a consequence of $v < 1$.

Consider now the situation when the lifetime of the parent particle is shorter than the interval between two nearest collisions with the walls of the box. In this case the decaying particle can be considered as quasifree, and the energy uncertainty of the produced neutrino is given by the decay width of the parent particle: $\sigma_E \simeq \Gamma$[12]. The momentum uncertainty of neutrino is then the reciprocal of its coordinate uncertainty $\sigma_x$, which in turn is just the distance traveled by neutrino during the decay process: $\sigma_x \simeq (p/E)\tau = = (p/E)\Gamma^{-1} \simeq (p/E)\sigma_E^{-1}$. Thus we find $p\sigma_p \simeq E\sigma_E$, i.e., Eq. (56) is approximately satisfied in this case. Once again condition (57) is fulfilled. It can be shown that this inequality is also satisfied when neutrinos are produced in collisions rather than in decays of unstable paricles [34].

Thus, we conclude that the spatial width $\sigma_x$ of the wave packets describing the flavor neutrino states is always determined by the energy uncertainty of the state as the smaller one between $\sigma_p$ and $\sigma_E$, i.e., $\sigma_x \sim v_g/\sigma_E$. This is in accord with the known fact that for stationary neutrino sources (for which $\sigma_E = 0$) the neutrino coherence length is infinite [33, 35]. On the other hand, the localization conditions for neutrino production and detection are determined by the corresponding momentum uncertainties.

It then follows that the parameters $\sigma_x$ that enter into the coherence condition and into the localization condition, discussed in Sections 3.2 and 5.2 and in Appendix (see Eqs. (32) and (54)), are in general different; they only coincide (or nearly coincide) when at production $\sigma_p \sim \sigma_E$, so that Eq. (56) is approximately satisfied from the very beginning. As we discussed above, such a situation is not uncommon, but it is not the most general one.

This observation underlines a shortcoming of the simple wave-packet approach to neutrino oscillations: it does not take into account the neutrino production and detection processes, except by assigning to the neutrino state a momentum uncertainty, which is supposed to be determined by these processes. In particular, the neutrino wave-packet picture assumes the mass eigenstate components of the flavor neutrino

[12] This only holds for slow parent particles. In the case of relativistic decaying particles, $\sigma_E$ depends on the angle between the momenta of the parent particle and of neutrino [28]. Still, condition (57) holds in that case as well.

states to be always on the mass shell, so that their energy and momentum uncertainties are always related by Eq. (56). Obviously, this approach is adequate in the cases when at production and detection $\sigma_p \simeq \sigma_E$, but is not satisfactory when this condition is strongly violated. In the latter cases one has to resort to the more consistent quantum field theoretical (QFT) treatment of neutrino oscillations, in which neutrino production, propagation and detection are considered as one single process. For QFT approach to neutrino oscillations see, e.g., [4, 23, 26, 34–38].

An interesting case in which the condition $\sigma_p \simeq \simeq \sigma_E$ is strongly violated both at production and detection is the proposed Mössbauer neutrino experiment, for which $\sigma_E \sim 10^{-11}$ eV and $\sigma_p \sim 10$ keV are expected [4, 6], so that $\sigma_E \sim 10^{-15}\sigma_p$. This case also gives a very interesting example of coherence restoration at detection, which resolves another paradox of neutrino oscillations. In a Mössbauer neutrino experiment neutrinos are produced coherently due to their large momentum uncertainty [4]. However, as soon as the emitted neutrino goes on the mass shell, its momentum uncertainty shrinks to satisfy Eq. (56), i.e., essentially becomes equal to the tiny energy uncertainty. Therefore, for on-shell Mössbauer neutrinos $\sigma_p \sim 10^{-11}$ eV $\ll \Delta m^2/2E \simeq 10^{-7}$ eV, i.e., the momentum uncertainty is much smaller than the difference of the momenta of different mass eigenstates. This means that coherence of different mass eigenstates in momentum space is lost. However, the fact that in this case both the energy and momentum uncertainties of the propagating neutrino state are much smaller than $\Delta m^2/2E$ does not mean that oscillations cannot be observed. In fact, it has been shown in [4] that the Mössbauer neutrinos should exhibit the usual oscillations. The resolution of the paradox lies in the detection process: the large momentum uncertainty at detection, $\sigma_{pD} \sim 10$ keV $\gg \Delta m^2/2E$, restores the coherence by allowing the different mass eigenstates composing the flavor neutrino state to be absorbed coherently.

Our final comment in this section is on the case when $\sigma_p \sim \sigma_E$ at production, in which the coherence and localization conditions depend on the same parameter $\sigma_x \sim \sigma_p^{-1}$. While the localization condition requires relatively large $\sigma_p$ ($\sigma_p \gg \Delta m^2/2p$) for the emitted and detected neutrino states to be coherent superpositions of mass eigenstates, the condition of no decoherence due to the wave-packet separation, on the contrary, requires long wave packets, i.e., relatively small $\sigma_p$. Is there any clash between these two requirements? By combining the two conditions we find $\Delta m^2/2p \ll \sigma_p \ll (v_g/\Delta v_g)L^{-1}$, which can only be satisfied if

$$\Delta m^2/2p \ll (v_g/\Delta v_g)L^{-1}. \tag{58}$$

This can be rewritten as the following condition on the baseline $L$:

$$2\pi \frac{L}{l_{\mathrm{osc}}} \ll \frac{v_g}{\Delta v_g}. \tag{59}$$

Since $v_g/\Delta v_g \gg 1$, this condition is expected to be satisfied with a large margin in any experiment which intends to detect neutrino oscillations: if it were violated, neutrino oscillations would have been averaged out because of the very large oscillation phase (except for unrealistically good experimental energy resolution $\delta E/E < \Delta v_g/v_g \sim \Delta m^2/2E^2$).

## 6. WHEN IS THE STATIONARY-SOURCE APPROXIMATION JUSTIFIED?

It has been pointed out in [23, 33] and greatly elaborated and exploited in [14] that for stationary neutrino sources the following two situations are physically indistinguishable:

(a) a beam of plane-wave neutrinos, each with a definite energy $E$ and with an overall energy spectrum $\Phi(E)$;

(b) a beam of neutrinos represented by wave packets, each of them having the energy shape factor $h(E)$ such that $|h(E)|^2 = \Phi(E)$.

As was stressed in [14], this actually follows from the fact that in stationary situations the spectrum $\Phi(E)$ fully determines the neutrino density matrix and therefore contains the complete information on the neutrino system.

This, in fact, gives an alternative explanation of why the "same energy" approach, though based on an incorrect assumption, leads to the correct result. It has been shown in Section 4 that, within the proper wave-packet formalism, one can choose to sum up first the states of different mass but the same energy, and then integrate over the energy (or momentum) distributions described by the effective energy or momentum shape factors of the wave packets, $h(E)$ or $h(p)$. In the light of the physical equivalence of situations ($a$) and ($b$), it is obvious that the integration over the spectrum of neutrinos, which is inherent in any calculation of the event numbers, leads to the same result as the integration over the energy spread within the wave packets (provided that the corresponding energy distributions coincide). Thus, the "same energy" assumption, though by itself incorrect, leads to the correct number of events upon the integration over the neutrino energy spectrum. The same is true for the "same momentum" assumption. This actually means that the wave-packet description becomes unnecessary in stationary situations, when the temporal structure of the neutrino emission and detection processes is irrelevant and the complete information on neutrinos is contained in their spectrum $\Phi(E)$, as was first pointed out in [14].

Let us derive the results of stationary source approximation in terms of the wave-packet picture described in this paper. We start with Eq. (17) for the oscillation amplitude and Eq. (9) for the evolved wave function $\Psi_i^S(x,t)$. Notice that the shape factor $f_i^S(p-p_i)$ in (9) does not depend on time, and furthermore this expression is determined for all moments $t$ from $-\infty$ to $+\infty$. The only time dependence in (9) is in the form of the plane waves in the integrand. This is precisely what corresponds to the stationarity condition: the source has no special time feature, and there is no tagging of neutrino emission and detection times.

Substituting Eqs. (9) and (15) into Eq. (17) we obtain, upon neglecting the transverse components of the neutrino momenta,

$$\mathcal{A}_{ab}(L,t) = \tag{60}$$
$$= \sum_i U_{ai}^* U_{bi} \int dp f_i^S(p-p_i) \tilde{f}_i^D(p) e^{ipL - iE_i(p)t},$$

where

$$\tilde{f}_i^D(p) \equiv \int dx \Psi_i^{D*}(x-L) e^{ip(x-L)} \tag{61}$$

is the Fourier transform of the detection state. Note that $R_i \equiv |\tilde{f}_i^D(p)|^2$ characterizes the momentum (energy) resolution of the detector.

The oscillation probability is obtained by integrating the squared modulus of the amplitude over time. We have

$$|\mathcal{A}_{ab}(L,t)|^2 = \sum_{i,k} U_{ai}^* U_{bi} U_{bk}^* U_{ak} \times \tag{62}$$
$$\times \int dp \int dp' f_i^S(p-p_i) f_k^{S*}(p'-p_k) \tilde{f}_i^D(p) \times$$
$$\times \tilde{f}_k^{D*}(p') e^{i(p-p')L} e^{-i[E_i(p)-E_k(p')]t}.$$

The integration over time is trivial:

$$\int_{-\infty}^{+\infty} dt\, e^{-i[E_i(p)-E_j(p')]t} = \tag{63}$$
$$= 2\pi\delta[E_i(p) - E_k(p')],$$

which means that only the waves with equal energies interfere. We stress once again that this is a consequence of the fact that no time structure appears in the detection and production processes, which is reflected in the time independence of the momentum distribution functions $f_i^S(p-p_i)$ and $\tilde{f}_i^D(p)$, and in the integration over the infinite interval of time. The equality $E_i(p) = E_k(p')$ leads to

$$p - p' = \Delta m_{ki}^2/2p \tag{64}$$

to the leading order in the momentum difference. The $\delta$ function (63) can be used to remove one of the momentum integrations in (62), so that we finally obtain

$$P_{ab}(L) = \int_{-\infty}^{+\infty} dt |\mathcal{A}_{ab}(L,t)|^2 = \tag{65}$$
$$= 2\pi \int dp |f^S(p-\bar{p})|^2 |\tilde{f}^D(p)|^2 \times$$
$$\times \sum_{i,k} U_{ai}^* U_{bi} U_{bk}^* U_{ak} \exp\left(-i\frac{\Delta m_{ik}^2}{2p} L\right).$$

Here we have neglected the dependence of the shape factors on the neutrino mass. Replacing the integration over momenta by the integration over energies, we can rewrite the oscillation probability as

$$P_{ab}(L) = \int_{-\infty}^{+\infty} dt |\mathcal{A}_{ab}(L,t)|^2 = \tag{66}$$
$$= \frac{2\pi}{v_g} \int dE \Phi(E) R(E) P_{ab}(E,L),$$

where

$$P_{ab}(E,L) = \tag{67}$$
$$= \sum_{i,k} U_{ai}^* U_{bi} U_{bk}^* U_{ak} \exp\left(-i\frac{\Delta m_{ik}^2}{2E} L\right)$$

is the standard expression for the oscillation probability, $\Phi(E) \equiv |f^S(E-\bar{E})|^2$ is the energy spectrum of the source, and $R(E) \equiv |\tilde{f}^D(E)|^2$ is the resolution function of the detector (note that we have substituted the momentum dependence of these quantities by the energy dependence using the standard on-shell dispersion relation).

The expression in Eq. (66) corresponds to the stationary-source approximation: the oscillation probability is calculated as an incoherent sum of the oscillation probabilities, computed for the same-energy plane waves, over all energies.

Notice that we have performed integration over the spatial coordinate at the level of the amplitude and over time at the probability level. Apparently, such an asymmetry of space and time integrations is not justified from the QFT point of view. In QFT computations the integration over time is performed in the amplitude, and this leads (in the standard setup) to the $\delta$ function which expresses the conservation of energy in the interaction process. To match our picture with that of QFT we need to consider the detection process and take into account the energies of all the particles that participate in the process. Suppose that the algebraic sum of the energies of all the accompanying particles (taken with the "$-$" sign for all incoming paricles and the "$+$" sign for the outgoing ones) is $E_D$. Then instead of (63) we will have in the probability

$$\int_{-\infty}^{+\infty} dt e^{-i[E_i(p)-E_D]t} \int_{-\infty}^{+\infty} dt' e^{i[E_k(p')-E_D]t'} = \tag{68}$$
$$= 4\pi^2 \delta[E_i(p) - E_k(p')]\delta[E_i(p) - E_D],$$

where the second $\delta$ function on the right-hand side reflects the energy conservation in the detection process. Using (68) we again obtain the "same energy" interference, as before.

In the above calculation we have not introduced any time structure at detection and performed the integration over $t$ from $-\infty$ to $+\infty$. In reality, certain time scales are always involved in the detection processes (even if we do not perform any time tagging in the emission process). For example, we can measure with some accuracy the appearance time of a charged lepton produced by the neutrino capture in the detection process. In this case, the neutrino state of detection will have a time dependence:

$$\Psi_i^D = \Psi_i^D(x - L, t - t_0), \tag{69}$$

where $\Psi_i^D$ has a peak at $t_0$ with a width $\sigma_t$ that is determined by the accuracy of the measurement of the time of neutrino detection. Since in practice the spatial characteristics of neutrino detection do not change with time, the dependences of $\Psi_i^D$ on $x$ and $t$ factorize: $\Psi_i^D = \Psi_{xi}^D(x - L)\Psi_{ti}^D(t - t_0)$. Integrating over time in the amplitude, we will have

$$\int dx \int dt \Psi_i^{D*}(x - L, t - t_0) e^{ipx - iE_i t} = \tag{70}$$
$$= e^{ipL - iE_i t_0} \tilde{f}_i^D(p) f_{ti}^D(E),$$

where $\tilde{f}_i^D(p)$ was defined in (61) and

$$f_{ti}^D(E) \equiv \int dt \Psi_{ti}^{D*}(t - t_0) e^{-iE_i t} \tag{71}$$

is the Fourier transform of $\Psi_{ti}^{D*}(t - t_0)$. As we have mentioned, $\Psi_{ti}^D(t - t_0)$ has a peak of width $\sigma_t$ at $t = t_0$. Taking for $\sigma_t$ the value $\sigma_t \sim 10^{-9}$ s (which is probably the best currently achievable time resolution), we obtain $\delta E \sim \sigma_t^{-1} \sim 10^{-6}$ eV. This is many orders of magnitude smaller than the typical energy resolution in the oscillation experiments. Therefore one can substitute $f_{ti}^D(E) \to \delta(E - E_D)$, which brings us back to our previous consideration.

## 7. WHEN CAN NEUTRINO OSCILLATIONS BE DESCRIBED BY PRODUCTION AND DETECTION INDEPENDENT PROBABILITIES?

In most analyses of neutrino oscillations it is assumed that the oscillations can be described by universal, i.e., production and detection process independent probabilities. In other words, it is assumed that by specifying the flavor of the initially produced neutrino state, its energy and the distance between the neutrino source and detector, one fully determines the probability of finding neutrinos of all flavor at the detector site (for known neutrino mass squared differences and leptonic mixing matrix). The standard formula for neutrino oscillations in vacuum, Eq. (6), is actually based on this assumption. Such an approach is very often well justified, but certainly not in all cases. It is, therefore, interesting to study the applicability limits and the accuracy of this approximation.

A natural framework for this is that of QFT, which provides the most consistent approach to neutrino oscillations. In this approach the neutrino production, propagation, and detection are considered as a single process with neutrinos in the intermediate state. This allows one to avoid any discussion of the properties of the neutrino wave packets since neutrinos are actually described by propagators rather than by wave functions. The properties of neutrinos in the intermediate state are fully determined by those of the "external" particles, i.e., of all the other particles that are involved in the neutrino production and detection processes. The wave functions of these external particles have to be specified. Usually, these particles are assumed to be described by wave packets; for this reason the QFT-based treatment is often called the "external wave packets" approach [30], as opposed to the usual, or "internal wave packets" one, which was discussed in Sections 3, 4, 6 and Appendix and which does not include neutrino production and detection processes. The results of the QFT-based approach turn out to be similar, but not identical, to those of a simple wave-packet one; in particular, possible violations of on-shell relation (56) between the neutrino energy and momentum uncertainties is now automatically taken into account. Moreover, the values of these uncertainties, which specify the properties of the neutrino wave packet in the "internal wave packets" approach and which have to be estimated in that approach, are now directly derived from the properties of the external particles.

The results of the QFT approach can be summarized as follows. For neutrinos propagating macroscopic distances the overall probability of the production — propagation — detection process for relativistic or quasi-degenerate neutrinos can to a very good accuracy be represented as a product of the individual probabilities of neutrino production, propagation (including oscillations), and detection[13]. The oscillation probability, however, is not in general independent of production and detection processes, which means that the factorizability of the probability of the entire production—propagation—detection process and the universality of the oscillation probability (or lack thereof) are in general independent issues. The oscillation probability can be generically represented as

$$P(\nu_a \to \nu_b; L) = \sum_{i,k} U_{ai}^* U_{bi} U_{ak} U_{bk}^* \times \qquad (72)$$

$$\times \exp\left(-i\frac{\Delta m_{ik}^2}{2p}L\right) S_{\rm coh}(L/l_{ik}^{\rm coh}) S_{\rm loc}(\sigma_x^{\rm loc}/l_{ik}^{\rm osc}).$$

Here, $S_{\rm coh}(L/l_{ik}^{\rm coh})$ and $S_{\rm loc}(\sigma_x^{\rm loc}/l_{ik}^{\rm osc})$ are, respectively, the coherence and localization factors, which account for possible suppression of the oscillations due to wave-packet separation and violation of the localization conditions. They both are equal to unity at zero argument and quickly decrease (typically exponentially) when their arguments become large. Note that these factors depend on the indices $i$ and $k$ because so do the partial oscillation and coherence lengths $l_{ik}^{\rm osc}$ and $l_{ik}^{\rm coh}$. The simple "internal wave packets" approach leads to an expression for the oscillation probability that is similar in form to that in Eq. (72); however, unlike in that simple approach, in the QFT-based framework the coherence and localization lengths entering into Eq. (72) depend in general on different length parameters, $\sigma_x^{\rm coh}$ and $\sigma_x^{\rm loc}$. This is related to the fact that the energy and momentum uncertainties at production and detection need not satisfy (56), as discussed in Section 5.4. In addition, the form of the coherence and localization factors $S_{\rm coh}(L/l_{ik}^{\rm coh})$ and $S_{\rm loc}(\sigma_x^{\rm loc}/l_{ik}^{\rm osc})$ in Eq. (72), rather than being postulated, is derived from the properties of the external particles and of the detection and production processes.

For the oscillation probability to be independent of the production and detection processes, the following conditions have to be fulfilled:

(i) decoherence effects due to wave-packet separation and due to violation of the localization conditions should be negligible;

(ii) the energy release in the production and detection reactions should be large compared to the neutrino mass (or compared to mass differences).

[13] A notable exception from this rule is the case of the Mössbauer effect with neutrinos, in which the probabilities of neutrino production and detection do not factorize but are instead entangled with each other [4]. Still, even in this case, the oscillation (actually, $\bar\nu_e$ survival) probability can to a very good accuracy be factored out of the expression for the overall probability of the process.

The necessity of (i) is clear from the discussion above: if this condition is fulfilled, the coherence and localization factors in Eq. (72) are both equal to unity, and the standard neutrino-oscillation formula is recovered. If, on the contrary, (i) is violated, the oscillations will suffer from the production- and detection-dependent decoherence effects. As to the condition (ii), it ensures that the production- and detection-probabilities are essentially the same for all mass-eigenstate components of the emitted or detected flavor neutrino states (modulo the different values of $|U_{ai}|^2$); if this condition is violated, the phase space available in the production or detection process will depend on the mass of the participating neutrino mass eigenstate, and the mass eigenstate composition of the flavor eigenstates will no longer be given by simple formula (1).

## 8. DISCUSSION AND SUMMARY

In the present paper we discussed a number of subtle issues of the theory of neutrino oscillations which are still currently under debate or have not been sufficiently studied yet. For each problem we discussed, we were trying to present our analysis from different perspectives and obtained consistent results. We have also developed a new approach to calculating the oscillation probability in the wave-packet picture, in which we changed the usual order of integration over the momenta (or energies) and summation over the mass eigenstate components of the wave packets representing the flavor neutrino states. This allowed a new insight into the question why the unjustified "same energy" and "same momentum" assumptions lead to the correct result for the oscillation probability.

We have also presented an alternative derivation of the equivalence between the results of the sharp-energy plane-wave formalism and of the wave-packet approach in the case of stationary neutrino sources, as well as discussed the applicability conditions for the stationary-source approximation.

Below we give a short summary of our answers to the first seven questions listed in Introduction.

(1) The standard formula for the probability of neutrino oscillations is obtained if the decoherence effects due to the wave-packet separation are negligible and the neutrino emitter and absorber are sufficiently well localized. Under these conditions the additional oscillation phase $\Delta\phi'$ which is acquired in the neutrino production and detection regions is negligible. The "same energy" and "same momentum" assumptions, which allow one to nullify this additional phase, are then unnecessary. They still lead to the correct result because their main effect is essentially just to remove this extra phase. An alternative explanation of the fact that the "same energy" assumption gives the

correct result comes from the observation that in going from the oscillation probability to the observables such as event numbers, one has to integrate over the neutrino spectra. As discussed in [14, 33] and in Section 6, for stationary neutrino sources this is equivalent to integration over the energy distribution within wave packets (provided that this energy distribution coincides with the spectrum of plane-wave neutrinos). In Section 4 we have shown that the integration over the spectrum of plane-wave neutrinos is just a calculational convention in the wave-packet approach, which does not involve any additional approximations.

(2) Quantum-mechanical uncertainty relations are at the heart of the phenomenon of neutrino oscillations. For neutrino production and detection to be coherent, its energy and momentum uncertainties must be large enough to prevent a determination of the neutrino's mass in these processes. These uncertainties are governed by the quantum-mechanical uncertainty relations, which also determine the size of the neutrino wave packets and therefore are pivotal for the issue of the coherence loss due to the wave-packet separation.

(3) The spatial size of the neutrino wave packets is always determined by their energy uncertainty $\sigma_E$.

(4) The coherence condition ensures that the wave packets corresponding to the different mass eigenstates do not separate to such an extent that they can no longer interfere in the detector. This condition is therefore related to the spatial size of the neutrino wave packets, which is determined by the neutrino energy uncertainty $\sigma_E$. Note that this is an effective uncertainty which depends on the energy uncertainties both at neutrino production and detection. At the same time, the localization conditions are determined by the effective momentum uncertainty $\sigma_p$, which depends on the momentum uncertainties at production and detection. In the simple wave-packet approach, the neutrino energy and momentum uncertainties are related to each other due to the on-shellness of the propagating neutrino, whereas in a more general quantum field theoretic framework they are in general unrelated.

(5) Wave-packet approach (or a superior QFT one) is necessary for a consistent derivation of the expression for the oscillation probability. Once this has been done, wave packets can be forgotten in all situations except when the decoherence effects due to the wave-packet separation or due to the lack of localization of the neutrino source or detector become important. Even in those cases, though, the decoherence effects can in most situations be reliably estimated basing on the standard oscillation formula and simple physical considerations. In addition, the wave packets are unnecessary in the case of stationary problems [14]. Thus, the wave-packet approach is mainly of pedagogical value. It is also useful for analysing certain subtle issues of the neutrino-oscillation theory.

(6) The oscillation probability is independent of the production and detection processes provided the following conditions are satisfied: (i) decoherence effects due to wave-packet separation and due to violation of the localization conditions are negligible, and (ii) the energy release in the production and detection reactions is large compared to the neutrino mass (or compared to mass differences). Note that if the condition opposite to (i) is realized, the probabilities of flavor transitions also take a universal form, as in that case they simply corresponds to averaged oscillations.

(7) The stationary-source approximation is valid when the time-dependent features of the neutrino emission and absorption processes are either absent or irrelevant, so that one essentially deals with steady neutrino fluxes. Integration over the neutrino detection time then results in the equivalence of the oscillation picture to that in the "same energy" approximation.

*Appendix*

## INTEGRAL $I_{ik}$ AND ITS PROPERTIES

Let us consider the properties of the integral $I_{ik}(L)$ defined in Eq. (22). Expressing the shape factors $g_i^{S,D}(x)$ of the wave packets through the corresponding momentum distribution functions according to (the 1-dimensional version of) Eq. (13) and substituting the result into (19), we find the following representation for $G_i(L-v_{gi}t)$:

$$G_i(L-v_{gi}t) = \int_{-\infty}^{\infty} dp f_i^S(p) f_i^{D*}(p) e^{ip(L-v_{gi}t)}. \quad (A.1)$$

Consider now the integral

$$\tilde{I}_{ik}(L) \equiv \int_{-\infty}^{\infty} dt G_i(L-v_{gi}t) G_k^*(L-v_{gk}t) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dp_1 \int_{-\infty}^{\infty} dp_2 f_i^S(p_1) f_k^{D*}(p_1) f_i^{S*}(p_2) \times f_k^D(p_2) e^{ip_1(L-v_{gi}t)-ip_2(L-v_{gk}t)}. \quad (A.2)$$

Performing first the integration over time and making use of the standard integral representation of Dirac's $\delta$ function, we obtain

$$\tilde{I}_{ik}(L) = \frac{2\pi}{v_{gk}} \int_{-\infty}^{\infty} dp f_i^S(p) f_i^{D*}(p) f_k^{S*}(rp) \times \times f_k^D(rp)) e^{ip(1-r)L}, \quad (A.3)$$

where

$$r \equiv \frac{v_{gi}}{v_{gk}} \simeq 1. \quad (A.4)$$

In the limit when the group velocities of the wave packets corresponding to different mass eigenstates are exactly equal to each other, $r = 1$, $\tilde{I}_{ik}$ does not depend on the distance $L$. Since the dominant contribution to the integral in (A.3) comes from the region $|p| \lesssim \sigma_P \equiv \min\{\sigma_{pS}, \sigma_{pD}\}$ of the integration interval, $\tilde{I}_{ik}$ is practically independent of $L$, provided that $|1 - r| L \sigma_P \ll 1$, or

$$L \ll l_{\rm coh} = \sigma_X \frac{v_g}{\Delta v_g}, \quad (A.5)$$

where $\Delta v_g = |v_{gi} - v_{gk}|$ and $\sigma_X = 1/\sigma_p$. This is merely the condition of the absence of the wave-packet separation: the distance traveled by neutrinos should be smaller than the distance over which the wave packets corresponding to different mass eigenstates separate due to the difference of their group velocities and cease to overlap. If the condition opposite to that in Eq. (A.5) is satisfied, the integral $\tilde{I}_{ik}$ is strongly suppressed because of the fast oscillations of the integrand. $\tilde{I}_{ik}$ is actually the overlap integral that indicates how well the wave packets corresponding to the $i$th and $k$th neutrino mass eigenstates overlap with each other upon propagation the distance $L$ from the source.

Consider now the integral $I_{ik}$ that enters into expression (21) for the oscillation probability $P(\nu_a \to \nu_b; L)$ and is given by Eq. (30). The calculation similar to that of the integral $\tilde{I}_{ik}$ in Eq. (A.2) yields

$$I_{ik}(L) = \frac{2\pi}{v_{gk}} e^{-i\Delta E_{ik}[(v_{gi}-v_{gk})/2v_g v_{gk}]L} \times \times \exp\left(-i\frac{\Delta m_{ik}^2}{2p}L\right) \int_{-\infty}^{\infty} dp f_i^S(p) f_i^{D*}(p) f_k^{S*} \times \times (rp + \Delta E_{ik}/v_{gk}) f_k^D(rp + \Delta E_{ik}/v_{gk}) e^{ip(1-r)L}. \quad (A.6)$$

The first exponential factor here must be replaced by unity because the exponent contains the product of the factors $\Delta E_{ik}$ and $(1/v_{gk} - 1/v_g) \simeq [(v_{gi} - v_{gk})/2v_g v_{gk}]$, both of which are $\propto \Delta m_{ik}^2$, and hence is of fourth order in neutrino mass. Thus, we finally get

$$I_{ik}(L) = \exp\left(-i\frac{\Delta m_{ik}^2}{2p}L\right) \frac{2\pi}{v_{gk}} \times \times \int_{-\infty}^{\infty} dp f_i^S(p) f_i^{D*}(p) f_k^{S*}(rp + \Delta E_{ik}/v_{gk}) \times \times f_k^D(rp + \Delta E_{ik}/v_{gk}) e^{ip(1-r)L}. \quad (A.7)$$

Just as in the case of $\tilde{I}_{ik}$, the integral on the right hand side of Eq. (A.7) is practically independent of $L$ when the condition (A.5) is satisfied and is strongly suppressed in the opposite case. In addition, it is quenched if the split of the arguments of $f_i^{S,D}$ and $f_k^{S,D}$ in the integrand exceeds $\sigma_P$; thus, a necessary condition for unsuppressed $I_{ik}$ is

$$\Delta E_{ik} \sigma_X / v_g \ll 1. \quad (A.8)$$

This is often called the localization condition for the following reason. Since $\Delta E_{ik}/v_g \sim \Delta m_{ik}^2/2p \sim \sim l_{\rm osc}^{-1}$ and $\sigma_X$, being the inverse of $\min\{\sigma_{pS}, \sigma_{pD}\}$, is the largest of the sizes of the two spatial localization regions, of the emitter and detector, the condition (A.8) is actually equivalent to the obvious requirement that the neutrino source and detector be localized within spatial regions that are small compared to the oscillation length $l_{\rm osc}$. If this condition is violated, integration over the coordinates of the neutrino emission and detection points within the source and detector results in neutrino oscillations being averaged out.

From the above consideration it follows that the factor $I_{ik}$ in the expression for the oscillation probability yields the standard oscillation phase factor $\exp\left(-i\frac{\Delta m_{ik}^2}{2p}L\right)$ multiplied by the integral which accounts for possible suppression of the oscillating terms due to decoherence caused by wave-packet separation and/or lack of localization of the neutrino source and detector. Note that both decoherence mechanisms lead to exponential suppression of the interference terms in the oscillation probabilities since they come from the infinite-limits integrals of fast oscillating functions. The exact form of these suppression factors depends on the shape of the wave packets, i.e., is model dependent; in particular, for Gaussian and Lorentzian wave packets, these factors are $\sim \exp[-(L/l_{\rm coh})^2] \exp[-(\sigma_X/l_{\rm osc})^2]$ and $\exp(-L/l_{\rm coh}) \exp(-\sigma_X/l_{\rm osc})$, respectively.

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# ПАРАДОКСЫ НЕЙТРИННЫХ ОСЦИЛЛЯЦИЙ

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Несмотря на то что теории нейтринных осцилляций уже немало лет, некоторые основополагающие вопросы этой теории все еще дискутируются в литературе. Мы обсуждаем ряд таких вопросов, включая уместность приближений "одинаковой энергии" и "одинакового импульса", роль квантовомеханических соотношений неопределенности в нейтринных осцилляциях, зависимость условий когерентности и локализации, определяющих возможность наблюдения нейтринных осцилляций, от неопределенностей энергии и импульса нейтрино, условия независимости (или зависимости) вероятности нейтринных осцилляций от процессов рождения и детектирования нейтрино и пределы применимости приближения стационарного источника. Мы также развиваем новый подход к расчету вероятности нейтринных осцилляций исходя из описания с помощью волновых пакетов, основанного на соглашениях о порядке суммирования и интегрирования в амплитуде, отличающихся от стандартного правила. Это позволяет взглянуть с новой точки зрения на проблему "одинаковой энергии" и "одинакового импульса". Мы также обсуждаем ряд парадоксальных особенностей теории нейтринных осцилляций.