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**OPTIMAL STRATEGY ANALYSIS OF A COMPETING PORTFOLIO MARKET  
WITH A POLYVARIANT PROFIT FUNCTION**

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## **Abstract**

A competing market model with a polyvariant profit function that assumes “zeitnot” stock behavior of clients is formulated within the banking portfolio medium and then analyzed from the perspective of devising optimal strategies. An associated Markov process method for finding an optimal choice strategy for monovariant and bivariant profit functions is developed. Under certain conditions on the bank “promotional” parameter with respect to the “fee” for a missed share package transaction and at an asymptotically large enough portfolio volume, universal transcendental equations - determining the optimal share package choice among competing strategies with monovariant and bivariant profit functions - are obtained.

# 1 Introduction

Ever since the pioneering work of Markowitz [19], the mathematical analysis of numerous types of portfolio markets - especially from the perspective of formulating optimal choice strategies - has become an increasingly active area of research. Moreover, many of the fruits of this research have been adopted and standardized in a variety of influential financial treatises such as Berezovsky & Gnedin [4] and Brealy & Myers [6]. With the recent successes of financial mathematics and the keen interest (some would say obsession) fueled by the uncertainty and volatility of current economic markets, it is not surprising that there has been something of an explosion in optimal market strategy papers such as Blanchet-Scalliet et al. [5], Bronshtein & Zav'yalova [7], Chan & Yung [8], Davis et al. [9, 10], Gollier [16], Kyshakevych et al. [17, 18], Maslov [20], Okui [24], Reidel [26], Sun et al. [27], and Ye & Peng [28], to name just a few. Here we adapt and extend some of the techniques developed in [17, 18] in order to add another piece to the puzzle of optimal strategy formulation: one that treats "zeitnot" markets with polyvariant profit functions. The zeitnot (not enough time) assumption imposes a strong time-horizon dependence on our model, and establishes a certain commonality with the work in [5], but our work also has some striking differences with this and the other research appearing in the literature.

It is a well known that stock markets within the banking medium have a regulative influence on a country's economic well being. This medium may have within its portfolio large share packages of diverse business-industrial structures, ordered by means of some natural indices of their financial-economic attractiveness or worth to a potential client-buyer. For modern zeitnot stock market processes, both at the fixed time constraint and bounded access to the full resource information about share financial-economic value, an optimal choice strategy [9, 4], identifying the most desirable share package from a particular bank portfolio, assumes a great deal of importance for clients.

The situation becomes much more complicated when many client-buyers are in competition, and then a nontrivial fast choice problem arises subject to the most worth share package within the portfolio. For, as was already mentioned above, the "zeitnot" market character of such share market operations provides a client with only comparative information data about their worth during the choice process. Namely, if a client-buyer chooses some share package from the bank portfolio, he or she can after learning its basic characteristics buy it right away, or return the request back to the portfolio and pass ahead to become familiar with a next share package. If its worth characteristic proves to be equal or lower than those previously considered, the client-buyer will right away pass on to choosing a next share packages until he or she finds a share package with a worth characteristic higher than all those considered previously. In this case the client-buyer should make a decision as to whether this package is potentially the most valuable among all the possible choices and stops the process by purchasing it. If the client-buyer decides not to buy this share package, then he or she should proceed to analyzing the worth characteristics of the next packages, taking into account that the portfolio volume is finite and the market time is fixed.

If there are two or more clients-buyers, a similar choice strategy subject to the most valuable share package is followed, and based on an analysis of the relative characteristics, both decide to buy the

package, then the client-buyer who acts fastest will acquire the package and be most successful. The edge in speed will go to the client-buyer able to evaluate the potential share package in the fewest number of steps. At the same time, the choice process for identifying the most valuable share package is definitely affected by certain additional Financial constraints, which essentially influence the number of steps-requests to the portfolio data base. So, in a "zeitnot" market, a client buyer ought to be charged a progressive amount of money (fee) when using the request procedure subject to the portfolio data base for each share package considered and then returned to the portfolio share package. If, at last, the client-buyer stops at some potentially most valuable share package (from his or her point of view) and buys it, the bank, as a financial promotional-active institution, reimburses some money (gift) for the successful commercial operation, thereby stimulating clients to engage in active cooperation with the bank.

The competing stock market model within a bank portfolio medium under the "zeitnot" market scheme delineated above, which governs the relationships among clients-buyers, represents a fairly typical situation [9, 10, 4] in a modern financial-economical context. As the whole choice process of the most valuable share package tends to be quite casual and unstructured, it is natural to employ stochastic process theory in its description. More specifically, we shall employ certain aspects of minimax optimization strategies and stopping rules associated with stochastic processes. A major component of our analysis will be the construction of a mathematical model capable of accurately reflecting the most important aspects of the stock market processes described above. Once this is obtained as it is in the sequel, we can employ fairly standard mathematical techniques to make predictions of market behavior and formulate optimal strategies.

When analyzing optimal strategies subject to a competing stock market portfolio model within a bank medium, an important problem arises for the "zeitnot" market choice problem for two and more clients-buyers of share packages, parameterized by a certain profit function.

In the first approximation, we assume that customers do not have financial restrictions and have sufficient capital to purchase any stake. Under this condition, when there are several variants of the profit function distributed independently within the portfolio, its analysis is important for modeling the optimal behavior of the clients-buyers, and thus, for ensuring stability of the financial-economic processes.

In the proposed investigation we develop a method of using an associated Markov-processes for the construction of the optimal strategy for the behavior of two clients-buyers from the competing portfolio model described above. The model is assumed to have either a given distributed mono- or bivariant profit function governing [4, 2, 25, 21] the share packages in the bank environment.

## 2 Elements of Optimal Stops Theory

Let  $(\Omega, \mathcal{F}, P)$  be a probability space [13, 23, 1] with probability measure  $P$  on the  $\sigma$ -algebra  $\mathcal{F}$  of subsets of  $\Omega$ , and  $x_t : \Omega \rightarrow \mathcal{H}$  be a Markov process for  $t \in \mathbb{Z}_+$  or  $t \in \mathbb{R}_+$ . For simplicity we shall suppose that  $\mathcal{H}$  is a discrete or finite-dimensional topological space. Let us assume that on the space  $\mathcal{H}$  there are defined two functions:  $f : \mathcal{H} \rightarrow \mathbb{R}$ , which is interpreted as a revenue in the case that the process stops,

and  $c : \mathcal{H} \rightarrow \mathbb{R}$ , which is interpreted as a fee for the next process monitoring. Thus, if an agent monitors the trajectory of the process  $x_t : \Omega \rightarrow \mathcal{H}$  at the moments  $t = 0, \dots, n$ , and when  $\tau = n \in \mathbb{Z}_+$  decides to stop monitoring, then the revenue is the following:

$$f(x_n) - \sum_{j=0}^{n-1} c(x_j). \quad (2.1)$$

Since the value (2.1) is random, we should consider its mathematical expectation and find a time  $\tau := \tau^* \in \mathbb{Z}_+$  when the following equality

$$\tau^* = \arg \sup_{\tau} \left\{ f(x_{\tau}) - \sum_{j=0}^{\tau-1} c(x_j) \right\} \quad (2.2)$$

holds. Let us describe the concept of the *stop moment*  $\tau \in \mathbb{Z}_+$ , which is a *strategy* [4, 23, 21] of the monitoring agent. In order to do this let us define a nondecreasing sequence of the  $\sigma$ -algebras  $\mathcal{F}_n := \sigma\{x_1, x_2, \dots, x_n\}$ ,  $n \in \mathbb{Z}_+$  on the probability space  $(\Omega, \mathcal{F}, P)$ , where  $\mathcal{F}_n$  is a minimal  $\sigma$ -algebra containing all possible sets of the form  $\{\omega : x_i(\omega) \in B, 0 \leq i \leq n\}$ , where  $B \subset \mathcal{H}$  is an arbitrary Borel set.

**Definition 2.1** *The Markov moment is a random quantity  $\tau = \tau(\omega)$  whose value lies in the set  $\mathbb{Z}_+$ , and for any  $n \in \mathbb{Z}_+$*

$$\{\omega : \tau(\omega) = n\} \in \mathcal{F}_n. \quad (2.3)$$

This condition means that decision about the end of monitoring at the moment of time  $n \in \mathbb{Z}_+$  is based only on the results of the monitoring  $\{x_1, x_2, \dots, x_n\}$  of the Markov process up to the moment  $n \in \mathbb{Z}_+$  inclusive.

**Definition 2.2** *The Markov moment  $\tau \in \mathbb{Z}_+$  for which the condition  $P\{\tau < \infty\} = 1$  is satisfied, or when the event  $\{\omega \in \Omega : t < \tau(\omega)\} \subset \mathcal{F}_t$  for all  $t \in \mathbb{Z}_+$ , is referred to as a stop moment.*

**Definition 2.3** *The value*

$$V(x_0) := \sup_{\tau} E_{x_0} \left\{ f(x_0) - \sum_{j=0}^{\tau-1} c(x_j) \right\} \quad (2.4)$$

*is called the price of the optimal stop problem.*

Let us consider the situation when the fee function  $c : \mathcal{H} \rightarrow \mathbb{R}$  of the Markov process monitoring  $x_t : \Omega \rightarrow \mathcal{H}$  with the transformation matrix  $\mathcal{P} := \{p_{ij} : 0 \leq i, j \leq N\}$  is zero, where we took into account that  $\text{card } \mathcal{H} = N + 1$ . For convenience we shall suppose that the function  $f : \mathcal{H} \rightarrow \mathbb{R}$  is non-negative on  $\mathcal{H}$ . In order to simplify the analysis of the value (2.4), let us also introduce the so called *reevaluation coefficient*  $\alpha \in (0, 1]$ , which includes the cost of monitoring changes in time. Then, if the agent-observer uses the Markov moment  $\tau \in \mathbb{Z}_+$  as a strategy, the price of the optimal stop is

$$V(x_0) := \sup_{\tau} E_{x_0} \{ \alpha^{\tau} f(x_{\tau}) \}, \quad (2.5)$$

since the function  $c = 0$ . Now we define the operation

$$(\mathcal{P}f)_i := \sum_{j=0}^N p_{ij}f(j), \quad (2.6)$$

for  $i = 0, \dots, N$ , where by definition  $f(j) := f(x_{n_j} = j)$  for the some  $n_j \in \mathbb{Z}_+, j \in \mathcal{H} \simeq \{0, 1, \dots, N\}$ .

The next definitions are useful for understanding material to be introduced in the sequel:

**Definition 2.4** *The function  $g : \mathcal{H} \rightarrow \mathbb{R}_+$  is called an excessive function, if*

$$g(x) \geq \alpha(\mathcal{P}g)(x) \quad (2.7)$$

for all  $x \in \mathcal{H}$ .

**Definition 2.5** *An excessive function  $g : \mathcal{H} \rightarrow \mathbb{R}_+$  is called an excessive majorant of the function  $f : \mathcal{H} \rightarrow \mathbb{R}_+$ , if*

$$g(x) \geq f(x) \quad (2.8)$$

for all  $x \in \mathcal{H}$ . The following lemma [4, 21] holds.

**Lemma 2.6** *If  $g : \mathcal{H} \rightarrow \mathbb{R}_+$  is excessive function and  $\tau \in \mathbb{Z}_+$  is Markov moment, then for  $\alpha \in (0, 1]$*

$$g(x) \geq \alpha(\mathcal{P}g)(x) \quad (2.9)$$

for all  $x \in \mathcal{H}$ .

**Proof.** Let  $h := g - \alpha\mathcal{P}g$  and  $\alpha \in (0, 1)$ . Then, obviously,  $h(x) \geq 0$  for all  $x \in \mathcal{H}$ , since the conclusion that follows from the condition (2.7) is that

$$g(x) \geq \alpha(\mathcal{P}g)(x). \quad (2.10)$$

Rewriting the expression  $h := g - \alpha\mathcal{P}g$  in the form

$$g := h + \alpha\mathcal{P}g, \quad (2.11)$$

we readily compute that

$$g := h + \alpha\mathcal{P}h + \alpha^2\mathcal{P}^2h + \dots + \alpha^n\mathcal{P}^nh + \dots, \quad (2.12)$$

$n \in \mathbb{Z}_+$ , and expansion (2.12) is convergent under the condition that  $\alpha \in (0, 1)$ . Besides, since the mathematical expectation

$$E_i\{h(x_n)\} = \sum_{j=0}^N \bar{p}_{ij}^{(n)} h(j), \quad (2.13)$$

where  $\mathcal{P}^n := \{p_{ij}^{(n)} : 0 \leq i, j \leq N\}$ , the expression (2.11) can be rewritten as

$$g(x) = E_x \left\{ \sum_{n=0}^{\infty} \alpha^n h(x_n) \right\}. \quad (2.14)$$

Now the mathematical expectation can be calculated as

$$E_x \{ \alpha^\tau g(x_\tau) \} = E_x \{ \alpha^\tau E_{x_\tau} \left\{ \sum_{n=0}^{\infty} \alpha^n h(x_n) \right\} \} = E_x \left\{ \sum_{n=0}^{\infty} \alpha^{\tau+n} h(x_{\tau+n}) \right\}. \quad (2.15)$$

Now comparing (2.15) and (2.14), we conclude that

$$g(x) \geq \alpha(\mathcal{P}g)(x) \quad (2.16)$$

for all  $x \in \mathcal{H}$ ,  $\alpha \in (0, 1)$ . Letting  $\alpha \rightarrow 1$  in (2.16), it is easy to show that inequality (2.9) holds for  $\alpha = 1$ , which completes the proof. ■

The next theorem [21] plays a crucial role in our further analysis.

**Theorem 2.7** *The profit function (2.5) is the smallest excessive majorant of the function  $f : \mathcal{H} \rightarrow \mathbb{R}_+$ .*

**Proof.** Observe that it follows from (2.5) that the price  $V : \mathcal{H} \rightarrow \mathbb{R}_+$  is the excessive majorant of the function  $f : \mathcal{H} \rightarrow \mathbb{R}_+$ . Indeed, since  $V(x) = \sup_{\tau} E_x \{ \alpha^\tau f(x_\tau) \}$ , the Markov moment  $\tau_\epsilon \in \mathbb{Z}_+$  exists for any  $\epsilon > 0$ , and

$$E_x \{ \alpha^{\tau_\epsilon} f(x_{\tau_\epsilon}) \} > V(x) - \epsilon, \quad (2.17)$$

where the value  $x \in \mathcal{H}$  is fixed. Since  $\text{card } \mathcal{H} = N + 1$  is a finite quantity, the inequality (2.17) also holds for all  $x \in \mathcal{H}$ . Let us calculate the mathematical expectation

$$E_x \{ \alpha^{\tau'} f(\tau') \} = \sum_{j=1}^N p_{x,j} E_j \{ \alpha^{1+\tau_\epsilon} f(x_\tau) \} \geq \alpha \sum_{j=1}^N p_{x,j} V(j) - \alpha\epsilon, \quad (2.18)$$

where  $\tau' := 1 + \tau_\epsilon \in \mathbb{Z}_+$ . From the inequality (2.18) we infer that

$$V(x) \geq E_x \{ \alpha^{\tau'} f(\tau') \} \geq \alpha(\mathcal{P}V)(x) - \alpha\epsilon \quad (2.19)$$

for any  $\epsilon > 0$ . Calculating the limit in (2.19) for  $\epsilon \rightarrow 0$ , we find that  $V(x) \geq \alpha(\mathcal{P}V)(x)$  for all  $x \in \mathcal{H}$ , which means that the excess of the profit function is  $V : \mathcal{H} \rightarrow \mathbb{R}_+$ . Now Lemma 4.1 implies that for any Markov moment  $\tau \in \mathbb{Z}_+$  we have the following inequalities:

$$g(x) \geq E_x \{ \alpha^\tau g(x_\tau) \} \geq E_x \{ \alpha^\tau f(x_\tau) \} \quad (2.20)$$

for all  $x \in \mathcal{H}$ . Calculating the supremum of (2.20) for  $\tau \in \mathbb{Z}_+$ , we obtain

$$g(x) \geq \sup_{\tau} E_x \{ \alpha^\tau f(x_\tau) \} = V(x)$$

for all  $x \in \mathcal{H}$ . Thus the proof is complete. ■

To calculate the choice price  $V : \mathcal{H} \rightarrow \mathbb{R}_+$  we use a criterion that can be formulated as the following theorem.

**Theorem 2.8** *The optimal choice price  $V : \mathcal{H} \rightarrow \mathbb{R}_+$  is the least solution of the equation*

$$V(x) = \max\{f(x), \alpha(\mathcal{P}V)(x)\} \quad (2.21)$$

for all  $x \in \mathcal{H}$  and  $\alpha \in (0, 1]$ .

**Proof.** Using the expression (2.21), let us define the operator  $Q_\alpha$  acting on a function  $y : \mathcal{H} \rightarrow \mathbb{R}_+$  for  $x \in \mathcal{H}$ :

$$(Q_\alpha y)(x) := \max\{f(x), \alpha(\mathcal{P}y)(x)\}. \quad (2.22)$$

It is easy to see that the following inequalities

$$y(x) \leq (Q_\alpha y)(x) \leq \dots \leq (Q_\alpha^n y)(x) \quad (2.23)$$

hold for all  $x \in \mathcal{H}$ . We now consider the following expression

$$\tilde{V}(x) := \lim_{n \rightarrow \infty} (Q_\alpha^n f)(x) \quad (2.24)$$

and show that the function (2.24) satisfies the equation (2.21). Indeed, since

$$(Q_\alpha^n f)(x) = \max\{f(x), \alpha(\mathcal{P}Q_\alpha^{n-1} f)(x)\} \quad (2.25)$$

for all  $n \in \mathbb{Z}_+$ , taking the limit as  $n \rightarrow \infty$  yields the result (2.21). As every solution of the equation (2.21) is an excessive majorant, the solution of (2.24) is the same function. It remains only to show that this function is the smallest excessive minorant of the function  $f : \mathcal{H} \rightarrow \mathbb{R}_+$ . Indeed, owing to the inequality (2.21) and the definition of  $Q_\alpha$ -operation, it is easy to show that for all  $x \in \mathcal{H}$ ,

$$\lim_{n \rightarrow \infty} (Q_\alpha^n g)(x) = g(x). \quad (2.26)$$

Since  $g(x) \geq f(x)$  for  $x \in \mathcal{H}$ , it follows that  $(Q_\alpha^n g)(x) \geq (Q_\alpha^n f)(x)$  too for all  $n \in \mathbb{Z}_+$ . Taking the limit of the last inequality as  $n \rightarrow \infty$ , we obtain

$$g(x) \geq \lim_{n \rightarrow \infty} (Q_\alpha^n f)(x) = \tilde{V}(x) \quad (2.27)$$

for all  $x \in \mathcal{H}$ , and this means that the solution (2.24) is the smallest excessive minorant of the function  $f : \mathcal{H} \rightarrow \mathbb{R}_+$ . Thus,  $V(x) = \tilde{V}(x)$ ,  $x \in \mathcal{H}$ , so the proof is complete. ■

In virtue of the properties of the solution (2.24) of the problem (2.21), the validity of the following theorem is demonstrated in [23, 1, 21].

**Theorem 2.9** *The Markov moment  $\tau^* \in \mathcal{H}$ , defined by the condition (2.2) as the moment of the first hit of the process  $x_t : \Omega \rightarrow \mathcal{H}$ ,  $t \in \mathbb{Z}_+$ , into the set  $\Gamma_+ := \{x \in \mathcal{H} : V(x) = f(x)\}$ , is optimal. Besides,  $\lim_{n \rightarrow \infty} P_x\{\tau^* > n\} = 0$ , as well as  $E_x\{\alpha^{\tau_n} V(x_{\tau_n})\} = V(x)$ , where  $\tau_n := \min(\tau^*, n)$ ,  $n \in \mathbb{Z}_+$ , for all  $x \in \mathcal{H}$ .*



Let us consider the case when the fee mapping  $c : \mathcal{H} \rightarrow \mathbb{R}_+$  is nonzero. Then the price function has the following form

$$V_\alpha(x) := \sup_\tau E_x \left\{ \alpha^\tau f(x_\tau) - \sum_{i=0}^{\tau-1} \alpha^i c(x_i) \right\}, \quad (2.28)$$

where  $x \in \mathcal{H}$ ,  $\alpha \in (0, 1]$ . Analogously to Definition 2.4, we can formulate the following [21] definition.

**Definition 2.10** *The function  $g : \mathcal{H} \rightarrow \mathbb{R}_+$  is called an excessive function, if*

$$g(x) \geq \alpha(\mathcal{P}g)(x) - c(x) \quad (2.29)$$

for all  $x \in \mathcal{H}$ .

We define

$$f_\alpha(x) := E_x \left\{ \sum_{i=0}^{\infty} \alpha^i c(x_i) \right\}, \quad (2.30)$$

where  $\alpha \in (0, 1)$  and  $x \in \mathcal{H}$ . Whence, it is clear that the choice price (2.28) has the following representation:

$$V_\alpha(x) := \sup_\tau E_x \left\{ \alpha^\tau [f(x_\tau) + f_\alpha(x_\tau)] \right\} - f_\alpha(x) \quad (2.31)$$

for all  $x \in \mathcal{H}$ . This means that the problem of the optimal stop with the nonzero price  $c : \mathcal{H} \rightarrow \mathbb{R}_+$  gives rise to the analogous problem with the zero price for the observation; that is, to the problem

$$\bar{V}_\alpha(x) = \sup_\tau E_x \left\{ \alpha^\tau [f(x) + f_\alpha(x_\tau)] \right\}, \quad (2.32)$$

$x \in \mathcal{H}$ , which solves the equation

$$\bar{V}_\alpha(x) = \max \{ f(x) + f_\alpha(x), \alpha(\mathcal{P}\bar{V}_\alpha)(x) \}. \quad (2.33)$$

Then the corresponding stop Markov moment is that  $\tau^* \in \mathcal{H}_+$  of the first observation in the set  $\Gamma_+ := \{x \in \mathcal{H} : \bar{V}_\alpha(x) = f(x) + f_\alpha(x)\}$ . It is obvious, that  $\Gamma_+ = \{x \in \mathcal{H} : V_\alpha(x) = f(x)\}$ . The result, formulated above, is valid only for the  $\alpha \in (0, 1)$ . In the case when  $\alpha = 1$  the function  $f_\alpha : \mathcal{H} \rightarrow \mathbb{R}_+$  cannot be defined, and we need to find an alternative solution of the problem (2.28).

Let us set  $\alpha = 1$  and consider the powers  $\mathcal{P}^m$  of the transition probabilities matrix as  $m \rightarrow \infty$ . Then, it follows from the ergodic theorem of A.A. Markov [4, 13] that the following asymptotic equality  $\mathcal{P}^m = \mathcal{S} + h^m \mathcal{R}^{(m)}$  holds, where  $|h| < 1$  and for all  $m \in \mathbb{Z}_+$  the quantity  $\sup_{m \in \mathbb{Z}_+} \|\mathcal{R}^{(m)}\| \leq \bar{r} < \infty$ , and the matrix  $\mathcal{S} \in \text{Hom}(\mathbb{R}^{N+1})$  has exactly the same  $(N+1 \in \mathbb{Z}_+)$  positive vector-rows  $q^\top \in \mathbb{R}_+^{N+1}$  of the limit probabilities. Thus, when  $\alpha = 1$  and the choice strategy price  $\tau = n \in \mathbb{Z}_+$ , the choice price (2.28) is

$$\begin{aligned} E_x \left\{ f(x_\tau) - \sum_{i=0}^{\tau-1} c(x_i) \right\} &= f(x_n) - c(x) - \sum_{j=1}^N p_{x,j} c_j - \dots \\ &- \sum_{j=1}^N p_{x,j}^{(n-1)} c_j = f(x_n) - n \langle q, c \rangle - \sum_{j=1}^N r_{x,j} c_j, \end{aligned} \quad (2.34)$$

where  $\langle \cdot, \cdot \rangle$  is the ordinary scalar product in the space  $\mathbb{R}^{N+1}$ . From (2.34) we see that when  $\langle q, c \rangle < 0$ , the choice price can be made arbitrarily large while still stopping the monitoring process. If  $\langle q, c \rangle \geq 0$ , then the situation is opposite to the previous one, and it can be shown [25] that the quantity (2.30) for  $\alpha \in (\alpha_0, 1)$  and some  $\alpha_0 \in (0, 1)$  is limited and positive. Owing to the action of the operator (2.22),  $Q_{\beta_1}(x) \leq Q_{\beta_2}(x)$  for all  $x \in \mathcal{H}$  and  $\beta_1 \leq \beta_2 \in (0, 1)$  is monotonic, there exists a sequence  $\{\alpha_n \in (0, 1) : n \in \mathbb{Z}_+\}$  such that  $\lim_{n \rightarrow \infty} \alpha_n = 1$  and  $\lim_{n \rightarrow \infty} V_{\alpha_n}(x^*) = f(x^*)$  for some state  $x^* \in \mathcal{H}$ . Under the condition  $\text{card } \mathcal{H} = N + 1 < \infty$ , the limit  $\lim_{n \rightarrow \infty} V_{\alpha_n}(x) = f(x)$  exists for every  $x \in \mathcal{H}$ . Thus, the set

$$\Gamma_+ := \{x \in \mathcal{H} : V(x) = f(x)\} \quad (2.35)$$

is not empty when  $\langle q, c \rangle \geq 0$ , which is tantamount to the optimality of the strategy  $\tau^* \in \mathcal{H}$  of the first hitting the observation into the set  $\Gamma_+$ .

To formulate the concluding proposition for the case  $\alpha = 1$ , we partition the phase space  $\mathcal{H}$  of the Markov process states  $x_t : \Omega \rightarrow \mathcal{H}, t \in \mathbb{Z}_+$ , into the subsets of the nonessential states  $\mathcal{H}_0$  and the classes  $\{\mathcal{H}_i : 1 \leq i \leq m_N\}$  of the essential states with nontrivial transition probabilities. Then the every essential class  $\mathcal{H}_i \subset \mathcal{H}, 1 \leq i \leq m_N$ , corresponds to the vector of boundary probabilities  $q_i \in \mathbb{R}^{N+1}$  and vector  $c_i \in \mathbb{R}^{N+1}$ , for which one can verify the following result.

**Proposition 2.11** *If for some  $i \in \{1, \dots, m_N\}$  the quantity  $\langle q_i, c_i \rangle \geq 0$ , then the moment  $\tau^* \in \mathcal{H}$  of the first hit into the set  $\Gamma_+$  in the form (2.35) is an optimal strategy. Nonessential states  $x \in \mathcal{H}_0$ , for which we can find at least one set  $\mathcal{H}_l \subset \mathcal{H}$ , and  $\langle q_l, c_l \rangle < 0$ , belong to the subset  $\mathcal{H}_0 \setminus \Gamma_+$ .*

In the next section we shall consider the problem of the optimal choice of the competing portfolio model of the share market with a mono-variant profit function, where the price function is defined by a constructive method together with the associated Markov process.

### 3 Mathematical Market Model with a Monovariant Profit Function

We begin with a constructive formulation of the model.

#### 3.1 Model description

Let  $(\Omega, \mathcal{F}, P)$  define [11, 4] a probability space, where  $\Omega$  is the set of the elementary events with a selected  $\sigma$ -algebra  $\mathcal{F}$  of its subsets with probability measure  $P$ , defined on the subsets of  $\mathcal{F}$ . Suppose that on the space  $\Omega$  there is a discrete Markov [22, 1] process  $x : \mathbb{Z}_+ \times \Omega \rightarrow \mathcal{H}$  with the values in some topological space  $\mathcal{H}$ . For all  $t \in \mathbb{Z}_+$  the quantity  $x_t(\omega) \in \mathcal{H}$  is random, and the set  $\{x_t(\omega) \in \mathcal{H} : t \in \mathbb{Z}_+\}$  forms the virtual trajectory of the possible states of the process.

We suppose that there exists an increasing family of the  $\sigma$ -algebras  $\{\mathcal{F}_t \subset \mathcal{F} : t \in \mathbb{Z}_+\}$  such that

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F} \quad (3.1)$$

for all  $t > s \in \mathbb{Z}_+$ . Then the process  $x : \mathbb{Z}_+ \times \Omega \rightarrow \mathcal{H}$  is called the *adapted process* to the family  $\{\mathcal{F}_t \subset \mathcal{F} : t \in \mathbb{Z}_+\}$ , if the mapping  $x_t : \Omega \rightarrow \mathcal{H}$  is  $\mathcal{F}_t$ -measurable for every  $t \in \mathbb{Z}_+$ . For the process  $x : \mathbb{Z}_+ \times \Omega \rightarrow \mathcal{H}$  we introduce the important definition of the Markov stop moment [23, 1]. It is a mapping  $\tau : \Omega \rightarrow \mathbb{Z}_+$  such that the event  $\{\omega \in \Omega : t < \tau(\omega)\} \subset \mathcal{F}_t$  for all  $t \in \mathbb{Z}_+$ .

Now consider an arbitrary mapping  $f : \mathcal{H} \rightarrow \mathbb{R}$ , and find the mathematical expectation [11, 1] of the process  $f(x_t) : \Omega \rightarrow \mathbb{R}$  regarding the  $\sigma$ -algebra  $\mathcal{F}_s \subset \mathcal{F}$ , which we denote as  $E_s(f(x_t)) := E\{f(x_t)|\mathcal{F}_s\}, t > s \in \mathbb{Z}_+$ . Then, by definition,

$$\int_{A \in \mathcal{F}_s} E_s(f(x_t)) dP_s := \int_{A \in \mathcal{F}_s \subset \mathcal{F}} f(x_t) dP \quad (3.2)$$

for all subsets  $A \in \mathcal{F}_s$ , where the measure  $dP_s$  on  $\mathcal{F}_s$  is defined as an induced measure  $i_s^* dP$  with respect to the embedding mapping  $i_s : \mathcal{F}_s \rightarrow \mathcal{F}, s \in \mathbb{Z}_+$ . If we define the mathematical expectation  $E_s(x_t)$  of the process  $x_t : \Omega \rightarrow \mathcal{H}$  for  $t > s \in \mathbb{Z}_+$  and find that  $E_s(x_t) = x_s$ , then this process is called [11, 4] a *martingale process*.

Let  $f : \mathcal{H} \rightarrow \mathbb{R}_+$  be a mapping that characterizes the degree of usefulness of the choice of the element  $x \in \mathcal{H}$ , which models the database of the share package of the bank portfolio. Then the function

$$V(a) := \sup_{\tau} E_a(f(x_t)), \quad (3.3)$$

where the supremum is taken over all possible Markov stop moments of the process  $x_t : \Omega \rightarrow \mathcal{H}, t \in \mathbb{Z}_+$ , under the condition that  $x_0 = a \in \mathcal{H}$ , is called the *price of the problem* of the optimal stop of the probability process, and can serve as a client-buyer's choice price of the most valuable share package from the bank portfolio in the "zeitnot" stock market. For the competing model of the stock market in the bank portfolio environment, we need to construct the corresponding price function of the optimal choice [3, 4] of the most wanted share package for every client-buyer, using the "zeitnot" stock conditions of this process.

For the sake of convenience we suppose that there are only two clients-buyers competing with each other at the time when the choice of the most valuable share package from the proposed portfolio with the finite number  $N \in \mathbb{Z}_+$  of the elements is made. All the share packages  $A_i, i = 1, \dots, N$  will be enumerated in such a way that

$$W(A_1) < W(A_2) < \dots < W(A_N), \quad (3.4)$$

where  $\{W(A_i) : 1 \leq i \leq N\}$  are share package values whose specific expression is not important. The probability space  $\Omega$  obviously consists of all possible permutations  $\omega := \{\omega_1, \omega_2, \dots, \omega_N\}$  of the set of numbers  $\{1, 2, \dots, N\}$ , and we assume that all of them have the same probability, since under the "zeitnot" stock market situation conditions preliminary information is not important. Thus, we denote the process of making the choice of the share package  $\omega_n, n = 1, \dots, N$  by the client-buyers in the  $n$ -time round as  $X_n^{(p)}(\omega) = \omega_n, p = 1, 2$ . In addition, we also denote the stop moments of the process, which will result in the largest values of the mathematical expectations of the corresponding price functions of the share packages choice, as  $\tau_p(\omega) \in \mathcal{H} := \{0, 1, 2, \dots, N\}, p = 1, 2$ . The choice

process of the most desirable share package  $A_N$ , which implicitly has the number  $N$ , is complicated by the fact, that after the share packages  $(X_1^{(p)}, X_2^{(p)}, \dots, X_n^{(p)})$ ,  $p = 1, 2$ , are chosen and returned to the portfolio  $n(\in \{1, \dots, N\})$  times, because each client-buyer lacks the information about their true prescribed price values, and can only see their relative placement in the choice process, that is,  $X_i^{(p)} < X_j^{(p)}$ , if  $W(A_i) < W(A_j)$   $i \neq j \leq n$ ,  $p = 1, 2$ . Consequently, it is natural to introduce families of  $\sigma$ -algebras of the events  $\mathcal{F}_n^{(p)}$ ,  $n = 1, \dots, N$ ,  $p = 1, 2$ , induced the events  $(X_i^{(p)} < X_j^{(p)}, i \neq j \leq n) := \mathcal{F}_n^{(p)}$ , where  $\mathcal{F}_1^{(p)} := \{\emptyset, \Omega\}$ ,  $p = 1, 2$ , and to define two sets of new characteristic random quantities, taking into account the above competition process involving the choice of the most valuable share package. Let the mathematical expectations

$$\begin{aligned} V_{\tau_1}^{(1)}(\tau_2) &:= c_\alpha E\{\chi_{\{X_{\tau_1}^{(1)}=N, X_{\tau_2}^{(2)} \neq N\}} + \chi_{\{X_{\tau_1}^{(1)}=N, X_{\tau_2}^{(2)}=N, \tau_1 < \tau_2\}}\} - \\ &- \alpha \sum_{k=1}^{\tau_1-1} (k/N^2) E\{\chi_{\{X_k^{(1)} \neq N, X_{\tau_2}^{(2)} \neq N\}} + \chi_{\{X_k^{(1)} \neq N, X_{\tau_2}^{(2)}=N, k < \tau_2\}}\}, \end{aligned} \quad (3.5)$$

and

$$\begin{aligned} V_{\tau_2}^{(2)}(\tau_1) &:= c_\alpha E\{\chi_{\{X_{\tau_2}^{(2)}=N, X_{\tau_1}^{(1)} \neq N\}} + \chi_{\{X_{\tau_2}^{(2)}=N, X_{\tau_1}^{(1)}=N, \tau_2 < \tau_1\}}\} - \\ &- \alpha \sum_{k=1}^{\tau_2-1} (k/N^2) E\{\chi_{\{X_k^{(2)} \neq N, X_{\tau_1}^{(1)} \neq N\}} + \chi_{\{X_k^{(2)} \neq N, X_{\tau_1}^{(1)}=N, k < \tau_1\}}\} \end{aligned} \quad (3.6)$$

define the corresponding price functions of the choice process of the most desirable share package for both client-buyers, where  $c_\alpha > 0$  is a fixed parameter representing the "promotional" bank encouragement for the client-buyer to purchase the share package from the portfolio,  $\alpha \in (0, 1)$  is a corresponding coefficient of the "fee" for every refusal of purchase of the share packages, and  $\tau_1, \tau_2 \in \mathcal{H}$  are the corresponding Markov stop moments of the processes. Since the choice processes for every client-buyer are analogous, it suffices to consider in detail only the first problem of choosing the most valuable share package from the following two problems:

$$\arg \sup_{\tau_1} V_{\tau_1}^{(1)}(\tau_2) = \tau_1^*, \quad \arg \sup_{\tau_2} V_{\tau_2}^{(2)}(\tau_1) = \tau_2^*. \quad (3.7)$$

In order to the extremum problems (3.7) we shall use the method of the associated Markov processes for the Markov stop moments of the choice process, which we describe next.

### 3.2 Associated Markov process

Let us consider the following sequence of the price function of the choice of the most valuable share package by the first client-buyer:

$$\begin{aligned} V_n^{(1)}(\tau_2) &:= c_\alpha (P\{X_n^{(1)} = N, X_{\tau_2}^{(2)} \neq N\} + P\{X_n^{(1)} = N, X_{\tau_2}^{(2)} = N, n < \tau_2\}) - \\ &- \alpha \sum_{k=1}^{n-1} (k/N^2) (P\{X_k^{(1)} \neq N, X_{\tau_2}^{(2)} \neq N\} + P\{X_k^{(1)} \neq N, X_{\tau_2}^{(2)} = N, k < \tau_2\}), \end{aligned} \quad (3.8)$$

where  $n = 1, \dots, \tau_1, \alpha \in (0, 1), c_\alpha > 0$ , and it is assumed that the second client-buyer follows the optimal (so called "threshold") strategy with the Markov stop moment  $\tau_2(l) > l$  under the condition that Markov stop moment of the choice of the first client-buyer is  $\tau_1(l) = l \in \mathcal{H}$ . To add specificity to the choice strategy of the most valuable shares package by the first client-buyer, let us calculate the corresponding probabilities (3.8) taking into account the family of the associated  $\sigma$ -algebras  $\mathcal{F}_n^{(p)}, n = 1, \dots, \tau_1, p = 1, 2$ :

$$\begin{aligned} V_n^{(1)}(\tau_2) &= c_\alpha P\{X_n^{(1)} = N | \mathcal{F}_n^{(1)}\} [P\{X_{\tau_2}^{(2)} \neq N\} \\ &+ P\{X_{\tau_2}^{(2)} = N, n < \tau_2\}] \\ -\alpha \sum_{k=1}^{n-1} \frac{k}{N^2} P\{X_k^{(1)} \neq N\} &[P\{X_{\tau_2}^{(2)} \neq N\} + P\{X_{\tau_2}^{(2)} = N, k < \tau_2\}] \end{aligned} \quad (3.9)$$

It should be mentioned, that for  $n = 1, \dots, \tau_1$  the conditional probability

$$\begin{aligned} P\{X_n^{(1)} = N | \mathcal{F}_n^{(1)}\} &= P\{X_n^{(1)} = N : X_n^{(1)} > \max(X_1^{(1)}, X_2^{(1)}, \dots, X_{n-1}^{(1)})\} \\ &= P\{X_n^{(1)} = N\} / P\{X_n^{(1)} > \max(X_1^{(1)}, X_2^{(1)}, \dots, X_{n-1}^{(1)})\} \\ &= \frac{1}{N} / \left( \frac{(n-1)!}{n!} \right) 1_{\{X_n^{(1)} > \max(X_1^{(1)}, X_2^{(1)}, \dots, X_{n-1}^{(1)})\}}, \end{aligned} \quad (3.10)$$

and for every  $k = 1, \dots, n$  the conditional probability

$$\begin{aligned} P\{X_{\tau_2}^{(2)} = N, k < \tau_2\} &+ P\{X_{\tau_2}^{(2)} \neq N, n < \tau_2\} \\ &= 1 - P\{X_{\tau_2}^{(2)} = N, \tau_2 \leq k\}. \end{aligned} \quad (3.11)$$

Thus, the price function of the choice (3.9) for the first client-buyer for  $n = 1, \dots, \tau_1$  has the following form:

$$\begin{aligned} V_n^{(1)}(\tau_2) &= \frac{c_\alpha n}{N} (1 - P\{X_{\tau_2}^{(2)} = N, \tau_2 \leq n\}) \\ - \frac{\alpha(N-1)}{N} \sum_{k=1}^{n-1} \frac{k}{N^2} &(1 - P\{X_{\tau_2}^{(2)} = N, \tau_2 \leq k\}). \end{aligned} \quad (3.12)$$

To calculate the probabilities  $P\{X_{\tau_2}^{(2)} = N, \tau_2 \leq k\}, k = 1, \dots, n$ , in the expression (3.12) we need to consider the random sequences of the Markov stop moments associated with the process of choosing the most valuable share package by the client-buyers:

$$x_n^{(p)} := \min\{t > x_{n-1}^{(p)} : X_t > \max(X_{t-1}, \dots, X_1)\}, \quad (3.13)$$

where  $x_n^{(p)} \in \mathcal{H}$  is a moment of choice of the next candidate for the most valuable share package by the corresponding client-buyer. The random sequences (3.13) are figure definitively for the price function (3.12), whose main properties are defined [15] by the following lemma.

**Lemma 3.1** *The sequences  $x_n^{(p)} \in \mathcal{H}, n = 1, \dots, N, p = 1, 2$ , in the form (3.13) are discrete Markov chains on the phase space  $\mathcal{H}$  with the transition probabilities*

$$p_{ij} = \begin{cases} \frac{i}{j(j-1)}, & 0 \leq i < j; & 0, & i \geq j \geq 0, \\ 1, & i = 0, j = 1; & 0, & i = 0, j > 1, \\ \frac{i}{N}, & j = 0; & 0, & i \geq j > 0 \end{cases} \quad (3.14)$$

for all  $0 \leq i, j \leq N$ , where the additional state  $\{0\}$  of the sequences break is added, which the process settles into after receiving the most valuable share package.

Let us denote the optimal stop moments of the consequences (3.13) as  $\hat{\tau}_p \in \mathcal{H}$ ,  $p = 1, 2$ . Then the following relationships

$$\tau_p = x_{\hat{\tau}_p}, \quad (3.15)$$

hold, where  $p = 1, 2$ . Now consider the arbitrary Markov sequence in the form of (3.13) and the following decomposition of the phase space  $\mathcal{H}$  into the direct sum of the subspaces associated with the sequence of price functions (3.12), which is

$$\begin{aligned} \mathcal{H}_+ &:= \{j \in \mathcal{H} : (\mathcal{P}V^{(1)}(\tau_2))_j > V_j^{(1)}(\tau_2)\}, \\ \mathcal{H}_- &:= \{j \in \mathcal{H} : (\mathcal{P}V^{(1)}(\tau_2))_j \leq V_j^{(1)}(\tau_2)\}, \end{aligned} \quad (3.16)$$

where  $\mathcal{P} := \{p_{ij} : 0 \leq i, j \leq N\}$  is a matrix of the transition probabilities (3.14). Then the following theorem [25] obtains.

**Theorem 3.2** *Let the matrix  $\mathcal{P}$  of the transition probabilities (3.14) be such that  $p_{ij} = 0$  for all  $i \in \mathcal{H}_+$  and  $j \in \mathcal{H}_-$ . Then the moment  $\hat{\tau}_1 \in \mathcal{H}$  of the first entrance of the random sequence  $\{x_n^{(1)} : n = 0, \dots, N\}$  into the set  $\mathcal{H}_-$  is optimal for the sequence of price function  $\{V_n^{(1)}(\tau_2) : n = 0, \dots, N\}$ .*

In order to apply theorem 3.2, we calculate the probabilities  $P\{X_{\tau_2}^{(2)} = N, \tau_2 \leq k\}$  in the (3.12) for all  $0 \leq k \leq N$  under the condition that  $\tau_1(l) = x_{\hat{\tau}_1}^{(1)}(l) := l \in \mathcal{H}$ . Then, if  $k = 1, \dots, l-1$ , the probability

$$P\{X_{\tau_2}^{(2)} = N, \tau_2 \leq k\} = P\{X_{\tau_2(l)}^{(2)} = N, \tau_2(l) \leq k\} = 0, \quad (3.17)$$

since  $\tau_2(l) \geq l$ , and if  $k = l, \dots, N$ ,

$$\begin{aligned} P\{X_{\tau_2(l)}^{(2)} = N, \tau_2(l) \leq k\} &= \sum_{j=l}^k P\{X_{\tau_2(l)}^{(2)} = N, \tau_2(l) \leq j\} \\ &= \sum_{j=l}^k P\{X_{\tau_2(l)}^{(2)} = N | \tau_2(l) = j\} P\{\tau_2(l) = j\} \\ &= \sum_{j=l}^k P\{X_j^{(2)} = N : X_j^{(1)} > \max(X_1^{(2)}, X_2^{(2)}, \dots, X_{j-1}^{(2)})\} P\{\tau_2(l) = j\} \\ &= \sum_{j=l}^k P\{\tau_2(l) = j\} \frac{j}{N}. \end{aligned} \quad (3.18)$$

In order to calculate the probability  $P\{\tau_2(l) = j : j \in \mathcal{H}\}$ , we note that it follows the direct Kolmogorov equation

$$P\{\tau_2(l) = j\} = \begin{cases} 1, & j = 1, \\ \sum_{i=1}^{j-1} P\{x_{\hat{\tau}_2}^{(2)} = i\} p_{ij}, & j = \overline{2, l-1}, \\ \sum_{i=1}^{l-1} P\{x_{\hat{\tau}_2}^{(2)} = i\} p_{ji}, & j = \overline{l, N}, \end{cases} \quad (3.19)$$

[9, 21] and (3.19) that

$$P\{\tau_2(l) = j\} = \begin{cases} \frac{1}{j}, & j = \overline{1, l-1}, \\ \frac{l-1}{j(j-1)}, & j = \overline{l, N}, \end{cases} \quad (3.20)$$

From (3.20) and (3.18) we can find for  $k = \overline{l, N}$ , that

$$\{X_{\tau_2(l)}^{(2)} = N, \tau_2(l) \leq k\} = \sum_{j=l}^k \frac{l-1}{N(j-1)}. \quad (3.21)$$

Thus, substituting the result of (3.2) into (3.22), we can get the final expression for the price function for the first client-buyer:

$$\begin{aligned} V_n^{(1)}(\tau_2) &= c_\alpha n \left(1 - \frac{l-1}{N} \sum_{j=l}^n \frac{1}{j-1}\right) - \frac{\alpha(N-1)}{N} \sum_{k=1}^{n-1} \frac{k}{N^2} \\ &\quad - \frac{\alpha(N-1)}{N} \sum_{k=l}^{l-1} \frac{k}{N^2} - \frac{\alpha(N-1)}{N} \sum_{k=l}^n \frac{k}{N^2} \left(1 - \frac{l-1}{N} \sum_{j=l}^k \frac{1}{j-1}\right) \\ &= c_\alpha n \left(1 - \frac{l-1}{N} \sum_{j=l}^n \frac{1}{j-1}\right) - \frac{\alpha(N-1)n(n+1)}{2N^3} + \frac{\alpha(N-1)(l-1)}{N^2} \sum_{k=l}^n \frac{k}{N^2} \sum_{j=l}^k \frac{1}{j-1} \end{aligned} \quad (3.22)$$

for all  $n = 1, \dots, N$ . Now in order to solve the first equation in (3.7) it is easy to calculate  $\tau_1^* = \arg V_{\tau_1}^{(1)}(\tau_2) \in \mathcal{H}$  using Theorem 3.2. Thus, the obtained sequence (3.22) of the optimal choice of the most valuable shares package by the first client must be stopped at the moment  $\tau_1(l) = l = x_{\tau_1(l)}^{(1)} \in \mathcal{H}$ , which we can find solving the inequalities

$$\begin{aligned} (\mathcal{P}V^{(1)}(\tau_2))_{l-1} &> V_{l-1}(\tau_2), \\ (\mathcal{P}V^{(1)}(\tau_2))_l &\leq V_l^{(1)}(\tau_2). \end{aligned} \quad (3.23)$$

Let  $l \in \mathcal{H}$  satisfy the inequalities (3.23). The the following lemma is readily verified.

**Lemma 3.3** *Under the condition that promotional coefficient  $c_\alpha \geq \alpha/2 > 0$ , the sequence (3.22) induces the decomposition of the phase space  $\mathcal{H}$  with*

$$\mathcal{H}_+ = \{1, \dots, l-1\}, \quad \mathcal{H}_- = \{l, \dots, N\}. \quad (3.24)$$

It follows from Lemma 3.3 that  $\tau_1(l) = l \in \mathcal{H}$ , which satisfies the inequalities (3.23), and yields the optimal choice strategy of the most valuable share package by the first client-buyer. It is obvious from symmetry considerations that the competing choice problem involving the behavior strategy of the second client-buyer must be the same.

### 3.3 Asymptotic analysis

The main equation of the choice process of the most valuable share package for the optimal strategy (3.23) has the form:

$$\begin{aligned}
c_\alpha - \frac{\alpha(N-1)(l+1)}{2N^3} + \frac{\alpha(N-1)}{N^2} &= \\
= c_\alpha \left( \sum_{j=l-1}^{N-1} \frac{i}{j} - \frac{(l-1)}{N} \sum_{j=l}^{N-1} \frac{1}{j} \sum_{k=l-1}^j \frac{1}{k} \right) - \frac{\alpha l(N-1)}{2N^3} \sum_{j=l+1}^N \frac{j+1}{j-1} + \\
+ \frac{\alpha(N-1)l(l-1)}{N^2} \sum_{j=l+1}^N \frac{1}{j(j-1)} \sum_{k=l}^j \frac{k}{N^2} \sum_{j=l}^k \frac{1}{j-1}.
\end{aligned} \tag{3.25}$$

In order to simplify the analysis of the equation (3.25), we suppose that the bank portfolio contains a large number  $N \in \mathbb{Z}_+$  of share packages. Thus, for the optimal choice strategy of the first client-buyer the stop moment  $\tau_1(l) := l(N) \in \mathcal{H}$  satisfies asymptotic condition  $\lim_{N \rightarrow \infty} l(N)/N := z \in (0, 1)$ . Taking this into account, using asymptotic analysis [14, 12], we find that the relation (3.25) at  $N \rightarrow \infty$  turns into the following transcendental equation for finding the stop parameter  $z^* \in (0, 1)$ :

$$c_\alpha(1 + \ln z + \frac{z}{2} \ln^2 z) + \frac{\alpha}{2} z(1-z) = \frac{\alpha}{2} z^2 [\ln z \frac{1}{2} (1-z)(3-z)]. \tag{3.26}$$

The solution  $z^* \in (0, 1)$  depends heavily on the choice of a bank “gift”-parameter  $c_\alpha \in \mathbb{R}_+$ , which is naturally limited by the positiveness of the price function (3.22). Namely, it is easy to see that

$$c_\alpha - \alpha/2 \geq 0 \tag{3.27}$$

must hold for every  $\alpha \in (0, 1)$ . If we assume the lowest risk condition of losses of the bank shares seller, then the optimal choice is  $c_\alpha = \alpha/2$ . In this case equation (3.26) takes an invariant form with respect to the interest rate of the “fee“  $\alpha \in (0, 1)$  for purchases the potential desired share package that have yet to be made by the client-buyer:

$$1 + \ln z + \frac{z}{2} \ln^2 z + z(1-z) = z^2 [\ln z + \frac{1}{2} (1-z)(3-z)]. \tag{3.28}$$

This transcendental equation (3.28) has the only one real solution  $z^* \simeq 0,21 \in (0, 1)$ . Accordingly we can now formulate the next behavior strategy as follows: When the number  $N \in \mathbb{Z}_+$  of the share packages in the bank portfolio is large enough, the optimal strategy of the choice of the most valuable share package by the first client-buyer is to compare the relative value of the first  $l = z^* N \in \mathbb{Z}_+$  shares, and then to choose the first shares package whose value is greater then all of those previously compared.

### 3.4 Some conclusions

Our portfolio competing share market model under the condition of “zeitnot” stock choice of potentially the most valuable share package by client-buyers appears to be a well known discrete Markov process on the phase space  $\mathcal{H} = \{0, 1, \dots, N\}$ . As it has been shown, when the bank chooses the most useful “promotional“ parameter  $c_\alpha = \alpha/2 \in (0, 1)$ , the client-buyer’s optimal strategy choice of the most



valuable share package is defined by the universal transcendental equation (3.27) independent the "fee"-parameter  $\alpha \in (0, 1)$  and under the condition that the values of the number of packages within the portfolio are large.

It should be noticed that our model is a somewhat simplified version of the "zeitnot" stock behavior of clients/share buyers when they do not dispose of a priori information about the qualitative characteristics of the portfolio. Moreover, we assumed that every client-buyer possesses sufficient financial capital for the purchase of any share package of the bank portfolio.

In the case if there exist either some financial constraints on clients funds subject to portfolio share packages prices prescribed by a bank or several quality parameters, the corresponding clients optimal behavior strategies are essentially more complicated, and is a subject of analysis in the next section.

## 4 Mathematical Model of the Market with a Bivariant Profit Function

Our construction of the model with a bivariant profit function has some similarities with the monovariant case, but there are some striking differences as well.

### 4.1 Model description

We take as a base the mathematical model of the bank share portfolio and the process of client-buyer's choice of the share package described above and developed in [17]. Let us suppose that there are two competing client-buyers in the process of choosing the most valuable share package with a finite number  $N \in \mathbb{Z}_+$  of elements. All share packages  $A_i, i = 1, \dots, N$ , are a priori numbered in such a way that

$$W_1(A_i) < W_1(A_2) < \dots < W_1(A_N), \quad W_2(A_{\sigma(1)}) < W_2(A_{\sigma(2)}) < \dots < W_2(A_{\sigma(N)}), \quad (4.1)$$

where  $\{W_i(A_j) : 1 \leq j \leq N\}, i = 1, 2$ , are rankings of usefulness characteristics of share packages, which are distributed independently within a given portfolio; that is, the permutation  $\sigma \in S_N$  of the ordered set of numbers  $\{1, 2, \dots, N\}$  is random. The probability space  $\Omega$  consists of all possible pairs of permutations  $\{\omega_1, \dots, \omega_N\} \times \{\sigma(\omega_1), \dots, \sigma(\omega_N)\}$  of the set of numbers  $\{1, 2, \dots, N\}$ , naturally assumed to have equal probability. Thus, we denote the result of a client-buyer's choice of the share packages  $A_n, n = 1, \dots, N$ , preceded by an  $n$ -time examination as a  $\Omega_n^{(s)} := (X_n^{(s)}(\omega), (Y_n^{(s)}(\omega))) \in \{1, 2, \dots, N^{(x)} := N\} \times \{\sigma(1), \sigma(2), \dots, \sigma(N^{(y)} := N)\}, s = 1, 2$ , and the Markov stop moments of the process of choice of the most desired share package by client-buyers, under the conditions that the values of mathematical expectations of the respective choice price functions will be the largest, as  $\tau_s(\omega) \in \mathcal{H} := \{0, 1, 2, \dots, N\}, s = 1, 2$ . We choose the price function for the first client-buyer in the following form:

$$\begin{aligned} V_{\tau_1}^{(1)}(\tau_2) = & c_\alpha [E\{\chi_{\{\Omega_{\tau_1}^{(1)}=(N^{(x)}, N^{(y)}), \Omega_{\tau_2}^{(2)} \neq (N^{(x)}, N^{(y)}) \vee \Omega_{\tau_1}^{(1)}=(\sigma^{-1}(N^{(x)}), N^{(y)}), \Omega_{\tau_2}^{(2)} \neq (\sigma^{-1}(N^{(x)}), N^{(y)})\}}\} + \\ & + E\{\chi_{\{\Omega_{\tau_1}^{(1)}=(N^{(x)}, N^{(y)}), \Omega_{\tau_2}^{(2)}=(N^{(x)}, N^{(y)}), \tau_1 < \tau_2 \vee \Omega_{\tau_1}^{(1)}=(\sigma^{-1}(N^{(x)}), N^{(y)}), \Omega_{\tau_2}^{(2)}=(\sigma^{-1}(N^{(x)}), N^{(y)}), \tau_1 < \tau_2\}}\} - \\ & - \alpha \sum_{k=1}^{\tau_1-1} \frac{k}{N^2} [E\{\chi_{\{\Omega_{\tau_1}^{(1)} \neq (N^{(x)}, N^{(y)}), \Omega_{\tau_2}^{(2)} \neq (N^{(x)}, N^{(y)}) \vee \Omega_k^{(1)} \neq (\sigma^{-1}(N^{(x)}), N^{(y)}), \Omega_{\tau_2}^{(2)} \neq (\sigma^{-1}(N^{(x)}), N^{(y)})\}}\} + \\ & + E\{\chi_{\{\Omega_k^{(1)} \neq (N^{(x)}, N^{(y)}), \Omega_{\tau_2}^{(2)} \neq (N^{(x)}, N^{(y)}), k < \tau_2 \vee \Omega_k^{(1)} \neq (\sigma^{-1}(N^{(x)}), N^{(y)}), \Omega_{\tau_2}^{(2)}=(\sigma^{-1}(N^{(x)}), N^{(y)}), k < \tau_2\}}\} ], \end{aligned} \quad (4.2)$$

where  $c_\alpha > 0$  is a corresponding bank gift-coefficient, and  $\alpha > 0$  is a "fee" -coefficient for the unmade transaction of purchase-sale of the shares package. The choice price function for the second client is obtained in the same way. In order to calculate, for example, the quantity

$$\tau_1^* := \arg \sup_{\tau_1 \in \mathcal{H}} V_{\tau_1}^{(1)}(\tau_2), \quad (4.3)$$

which characterize the most optimal share package choice strategy of the first client-buyer, we need to construct [4, 17, 21] the basic associated Markov sequences

$$x_{n+1}^{(s)} := \min\{t > x_n^{(s)} : X_t^{(s)} > \max(X_{t-1}^{(s)}, \dots, X_1^{(s)}) \vee (Y_t^{(s)} > \max(Y_{t-1}^{(s)}, \dots, Y_1^{(s)})\}, \quad (4.4)$$

where the quantities  $x_n^{(s)} \in \mathcal{H}, n = 1, \dots, N, s = 1, 2$ , are the moments of the most valuable shares package for the corresponding clients-buyers. The Markov sequences (4.4) are characterized [4, 25, 21] by the following lemma.

**Lemma 4.1** *The integer sequences (4.4) are the discrete Markov chains on the phase space  $\mathcal{H}$  with the transition probabilities*

$$p_{ij}^{(s)} = \begin{cases} \frac{[2j(j-1)-i]i^2}{j^2(j-1)^2(2i-1)}, & 1 \leq i < j; & 0, & i \geq j \geq 0; \\ 1, & i = 0, j = 1; & 0, & i = 0, j > 1, & 0; \\ 1 - \sum_{k=i+1}^N \frac{[2k(k-1)-i]i^2}{k^2(k-1)^2(2i-1)}, & j = 0, \end{cases} \quad (4.5)$$

for  $s = 1, 2$  and  $i, j \in \mathcal{H}$ .

Thus, we have constructed two Markov sequences (4.4) associated with the most valuable share package choice process by means of which we can calculate the quantity (4.3), using the following result [25] as the criterion.

**Theorem 4.2** *Let the matrix  $\mathcal{P} := \{p_{ij}^{(1)} : i, j \in \mathcal{H}\}$  of the transition probabilities be such that  $p_{ij}^{(1)} = 0$  for all  $i \in \mathcal{H}_+, j \in \mathcal{H}_-$ , where*

$$\begin{aligned} \mathcal{H}_+ &:= \{j \in \mathcal{H} : (\mathcal{P}V^{(1)}(\tau_2))_j > V_j^{(1)}(\tau_2)\}, \\ \mathcal{H}_- &:= \{j \in \mathcal{H} : (\mathcal{P}V^{(1)}(\tau_2))_j \leq V_j^{(1)}(\tau_2)\}. \end{aligned} \quad (4.6)$$

*Then the Markov sequence (4.4) for optimal choice of the most valuable share package by the first client-buyer can be broken at the moment  $\tau_1(l) = l = x_{\tau_1(l)}^{(1)} \in \mathcal{H}$ , which can be found solving the inequalities (4.6).*

The corresponding choice price function of the share package in (4.6) is given as

$$\begin{aligned}
V_n^{(1)}(\tau_2) &= c_\alpha [P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} \neq N^{(x)} \vee Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} \neq N^{(y)}\} + \\
&+ P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} = N^{(x)}, n < \tau_2 \vee Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} \neq N^{(y)}\} + \\
&+ P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} \neq N^{(x)} \vee Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} = N^{(y)}, n < \tau_2\} + \\
&+ P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} = N^{(x)}, n < \tau_2 \vee Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} = N^{(y)}, n < \tau_2\} - \\
&- \alpha \sum_{k=1}^{n-1} \frac{k}{N^2} [P\{X_k^{(1)} \neq N^{(x)}, X_{\tau_2}^{(2)} \neq N^{(x)} \wedge Y_k^{(1)} \neq N^{(y)}, Y_{\tau_2}^{(2)} \neq N^{(y)}, n < \tau_2\} + \\
&+ P\{X_k^{(1)} \neq N^{(x)}, X_{\tau_2}^{(2)} \neq N^{(x)} \wedge Y_k^{(1)} \neq N^{(y)}, Y_{\tau_2}^{(2)} = N^{(y)}, k < \tau_2\} + \\
&+ P\{X_k^{(1)} \neq N^{(x)}, X_{\tau_2}^{(2)} = N^{(x)}, k < \tau_2 \wedge Y_k^{(1)} \neq N^{(y)}, Y_{\tau_2}^{(2)} \neq N^{(y)}\} + \\
&+ P\{X_k^{(1)} \neq N^{(x)}, X_{\tau_2}^{(2)} = N^{(x)}, k < \tau_2 \wedge Y_k^{(1)} \neq N^{(y)}, Y_{\tau_2}^{(2)} = N^{(y)}, k < \tau_2\}],
\end{aligned} \tag{4.7}$$

where the bank gift-parameter  $c_\alpha > 0$  is chosen from the condition  $V_n^{(1)}(\tau_2) > 0$  for all  $n = 1, \dots, N$ . Thus, after calculating the value of the function of price of choice (4.7) of the most valuable share package by the first client-buyer by means of Theorem 4.2, the structure of the sets  $\mathcal{H}_+$  and  $\mathcal{H}_-$  on the transition probabilities (4.5) needs to be analyzed, as we do in the next subsection.

## 4.2 Associated Markov process and structural analysis of the model

Taking into account the structure of the independent family of associated  $\sigma$ -algebras  $\{\mathcal{F}_n^{(s)}, 1 \leq n \leq \tau_1\}$ ,  $s = 1, 2$ , let us rewrite expression (4.7) in the following form:

$$\begin{aligned}
V_n^{(1)}(\tau_2) &= c_\alpha [P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} \neq N^{(x)}\} + P\{Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} \neq N^{(y)}\} - \\
&- P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} \neq N^{(x)}\} P\{Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} \neq N^{(y)}\} + \\
&+ P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} = N^{(x)}, n < \tau_2\} + P\{Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} \neq N^{(y)}\} - \\
&- P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} = N^{(x)}, n < \tau_2\} P\{Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} \neq N^{(y)}\} + \\
&+ P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} \neq N^{(x)}\} + P\{Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} = N^{(y)}, n < \tau_2\} - \\
&- P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} \neq N^{(x)}\} P\{Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} = N^{(y)}, n < \tau_2\} + \\
&+ P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} = N^{(x)}, n < \tau_2\} + P\{Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} = N^{(y)}, n < \tau_2\} - \\
&- P\{X_n^{(1)} = N^{(x)}, X_{\tau_2}^{(2)} = N^{(x)}, n < \tau_2\} P\{Y_n^{(1)} = N^{(y)}, Y_{\tau_2}^{(2)} = N^{(y)}, n < \tau_2\} - \\
&- \alpha \sum_{k=1}^{n-1} \frac{k}{N^2} [P\{X_k^{(1)} \neq N^{(x)}, X_{\tau_2}^{(2)} \neq N^{(x)}\} P\{Y_k^{(1)} \neq N^{(y)}, Y_{\tau_2}^{(2)} \neq N^{(y)}\} + \\
&+ P\{X_k^{(1)} \neq N^{(x)}, X_{\tau_2}^{(2)} \neq N^{(x)}\} P\{Y_k^{(1)} \neq N^{(y)}, Y_{\tau_2}^{(2)} = N^{(y)}, k < \tau_2\} + \\
&+ P\{X_k^{(1)} \neq N^{(x)}, X_{\tau_2}^{(2)} = N^{(x)}, k < \tau_2\} P\{Y_k^{(1)} \neq N^{(y)}, Y_{\tau_2}^{(2)} \neq N^{(y)}\} + \\
&+ P\{X_k^{(1)} \neq N^{(x)}, X_{\tau_2}^{(2)} = N^{(x)}, k < \tau_2\} P\{Y_k^{(1)} \neq N^{(y)}, Y_{\tau_2}^{(2)} = N^{(y)}, k < \tau_2\}],
\end{aligned} \tag{4.8}$$

where we use the fact that the respective traces of both observations of the values of usefulness are distributed independently. It follows from the results of [17] that (4.8) can be rewritten as

$$\begin{aligned}
V_n^{(1)}(\tau_2) &= c_\alpha \{2P\{X_n^{(1)} = N^{(x)} | \mathcal{F}_n^{(1)} \vee \mathcal{F}_n^{(2)}\} [P\{X_{\tau_2}^{(2)} \neq N^{(x)}\} + \\
&+ P\{X_{\tau_2}^{(2)} = N^{(x)}, n < \tau_2\}] + 2P\{Y_n^{(1)} = N^{(y)} | \mathcal{F}_n^{(1)} \vee \mathcal{F}_n^{(2)}\} \times \\
&\times [P\{Y_{\tau_2}^{(2)} \neq N^{(y)}\} + P\{Y_{\tau_2}^{(2)} = N^{(y)}, n < \tau_2\}] - \\
&- P\{X_n^{(1)} = N^{(x)} | \mathcal{F}_n^{(1)} \vee \mathcal{F}_n^{(2)}\} P\{Y_n^{(1)} = N^{(y)} | \mathcal{F}_n^{(1)} \vee \mathcal{F}_n^{(2)}\} \times \\
&\times [P\{X_{\tau_2}^{(2)} \neq N^{(x)}\} + P\{X_{\tau_2}^{(2)} = N^{(x)}, n < \tau_2\}] \times \\
&\times [P\{Y_{\tau_2}^{(2)} \neq N^{(y)}\} + P\{Y_{\tau_2}^{(2)} = N^{(y)}, n < \tau_2\}] - \\
&- \alpha \sum_{k=1}^{n-1} \frac{k}{N^2} P\{X_k^{(1)} \neq N^{(x)}\} P\{Y_k^{(1)} \neq N^{(y)}\} [P\{X_{\tau_2}^{(2)} \neq N^{(x)}\} + \\
&+ P\{X_{\tau_2}^{(2)} = N^{(x)}, k < \tau_2\}] \times \\
&\times [P\{Y_{\tau_2}^{(2)} \neq N^{(y)}\} + P\{Y_{\tau_2}^{(2)} = N^{(y)}, k < \tau_2\}],
\end{aligned} \tag{4.9}$$

or the equivalent form

$$\begin{aligned}
V_n^{(1)}(\tau_2) &= c_\alpha \{2P\{X_n^{(1)} = N^{(x)} | \mathcal{F}_n^{(1)} \vee \mathcal{F}_n^{(2)}\} [P\{X_{\tau_2}^{(2)} \neq N^{(x)}\} + \\
&+ P\{X_{\tau_2}^{(2)} = N^{(x)}, n < \tau_2\}] + 2P\{Y_n^{(1)} = N^{(y)} | \mathcal{F}_n^{(1)} \vee \mathcal{F}_n^{(2)}\} \times \\
&\times [P\{Y_{\tau_2}^{(2)} \neq N^{(y)}\} + P\{Y_{\tau_2}^{(2)} = N^{(y)}, n < \tau_2\}] - \\
&- P\{X_n^{(1)} = N^{(x)} | \mathcal{F}_n^{(1)} \vee \mathcal{F}_n^{(2)}\} P\{Y_n^{(1)} = N^{(y)} | \mathcal{F}_n^{(1)} \vee \mathcal{F}_n^{(2)}\} \times \\
&\times [P\{X_{\tau_2}^{(2)} \neq N^{(x)}\} + P\{X_{\tau_2}^{(2)} = N^{(x)}, n < \tau_2\}] \times \\
&\times [P\{Y_{\tau_2}^{(2)} \neq N^{(y)}\} + P\{Y_{\tau_2}^{(2)} = N^{(y)}, n < \tau_2\}] - \\
&- \alpha \sum_{k=1}^{n-1} \frac{k}{N^2} P\{X_k^{(1)} \neq N^{(x)}\} P\{Y_k^{(1)} \neq N^{(y)}\} [P\{X_{\tau_2}^{(2)} \neq N^{(x)}\} + \\
&+ P\{X_{\tau_2}^{(2)} = N^{(x)}, k < \tau_2\}] \times \\
&\times [P\{Y_{\tau_2}^{(2)} \neq N^{(y)}\} + P\{Y_{\tau_2}^{(2)} = N^{(y)}, k < \tau_2\}].
\end{aligned} \tag{4.10}$$

It should be remarked, that for  $n = 1, \dots, \tau_1$  the conditional probabilities

$$\begin{aligned}
P\{X_n^{(1)} = N^{(x)} | \mathcal{F}_n^{(1)} \vee \mathcal{F}_n^{(2)}\} &= P\{X_n^{(1)} : X_n^{(1)} > \max(X_{n-1}^{(1)}, \dots, X_1^{(1)})\} = \\
&= P\{X_n^{(1)} = N^{(x)}\} / P\{X_n^{(1)} > \max(X_{n-1}^{(1)}, \dots, X_1^{(1)})\} = \\
&= \frac{1}{N} / \left( \frac{(n-1)!}{n!} \right) = \frac{n}{N} \mathbf{1}_{\{X_n^{(1)} > \max(X_{n-1}^{(1)}, \dots, X_1^{(1)})\}},
\end{aligned} \tag{4.11}$$

and analogously,

$$P\{Y_n^{(1)} = N^{(y)} | \mathcal{F}_n^{(1)} \vee \mathcal{F}_n^{(2)}\} = \frac{n}{N} \mathbf{1}_{\{X_n^{(1)} > \max(X_{n-1}^{(1)}, \dots, X_1^{(1)})\}}. \tag{4.12}$$

It also is easy to compute that for every  $k = 1, \dots, n$  the conditional probabilities

$$\begin{aligned}
P\{X_{\tau_2}^{(2)} \neq N^{(x)}\} + P\{X_{\tau_2}^{(2)} = N^{(x)}, k < \tau_2\} &= 1 - P\{X_{\tau_2}^{(2)} = N^{(x)}, \tau_2 \leq k\}, \\
P\{Y_{\tau_2}^{(2)} \neq N^{(y)}\} + P\{Y_{\tau_2}^{(2)} = N^{(y)}, k < \tau_2\} &= 1 - P\{X_{\tau_2}^{(2)} = N^{(x)}, \tau_2 \leq k\}.
\end{aligned} \tag{4.13}$$

Thus, substituting the expressions (4.11)-(4.13) in (4.10) for all  $n = 1, \dots, \tau_1$ , we find that

$$\begin{aligned}
V_n^{(1)}(\tau_2) &= c_\alpha \left[ \frac{2n}{N} (1 - P\{X_{\tau_2}^{(2)} = N^{(x)}, \tau_2 \leq n\}) \right. \\
&+ \frac{2n}{N} (1 - P\{Y_{\tau_2}^{(2)} = N^{(y)}, \tau_2 \leq n\}) - \\
&- \frac{n^2}{N^2} (1 - P\{X_{\tau_2}^{(2)} = N^{(x)}, \tau_2 \leq n\}) (1 - P\{Y_{\tau_2}^{(2)} = N^{(y)}, \tau_2 \leq n\}) \left. \right] - \\
&- \frac{\alpha(N-1)^2}{N^2} \sum_{k=1}^{n-1} \frac{k}{N^2} (1 - P\{X_{\tau_2}^{(2)} = N^{(x)}, \tau_2 \leq k\}) (1 - P\{Y_{\tau_2}^{(2)} = N^{(y)}, \tau_2 \leq k\}).
\end{aligned} \tag{4.14}$$

In order to calculate the expression (4.14) first we need to find the conditional probabilities  $P\{X_{\tau_2}^{(2)} = N^{(x)}, \tau_2 \leq k\}$  and  $P\{Y_{\tau_2}^{(2)} = N^{(y)}, \tau_2 \leq k\}$  for every  $k = 1, \dots, n$  from the Markov sequence (4.4) with the transitional probabilities (4.5). Employing the methods described in [17, 18], it is easy to show that probabilities with the threshold strategy  $\tau_2(l) \in \hat{\mathcal{H}}$  are

$$P\{X_{\tau_2}^{(2)} = N^{(x)}, \tau_2 \leq k\} := P\{X_{\tau_2(l)}^{(2)} = N^{(x)}, \tau_2(l) \leq k\} = 0 \tag{4.15}$$

for  $k = 1, \dots, l-1$  under the condition of optimal choice, when  $\tau_1(l) = x_{\hat{\tau}_1(l)}^{(1)} = l$ , and  $\tau_2 := \tau_2(l) > l \in \mathcal{H}$ . If  $1 \leq k \leq N$ , then

$$\begin{aligned}
P\{X_{\tau_2(l)}^{(2)} = N^{(x)}, \tau_2(l) \leq k\} &= \sum_{j=l}^k P\{X_{\tau_2(l)}^{(2)} = N^{(x)}, \tau_2(l) \leq j\} = \\
&= \sum_{j=l}^k P\{X_{\tau_2(l)}^{(2)} = N^{(x)} | \tau_2(l) = j\} P\{\tau_2(l) = j\} = \\
&= \sum_{j=l}^k P\{X_{\tau_2(l)}^{(2)} = N^{(x)} | \mathcal{F}_j^{(1)} \vee \mathcal{F}_j^{(2)}\} P\{\tau_2(l) = j\} = \\
&= \sum_{j=l}^k P\{X_j^{(2)} = N^{(x)} | \mathcal{F}_j^{(1)} \vee \mathcal{F}_j^{(2)}\} P\{\tau_2(l) = j\} = \\
&= \sum_{j=l}^k \frac{j}{N} P\{\tau_2(l) = j\},
\end{aligned} \tag{4.16}$$

and similarly

$$P\{Y_{\tau_2(l)}^{(2)} = N^{(y)}, \tau_2(l) \leq k\} = \sum_{j=l}^k \frac{j}{N} P\{\tau_2(l) = j\} \tag{4.17}$$

for  $k = l, \dots, N$ . In order to calculate the probabilities  $P\{\tau_2(l) = j\}, j \in \mathcal{H}$ , we remark that by Theorem 4.2 in accordance with the threshold strategy  $\hat{\tau}_2(l) \in \mathcal{H}$  on the basis of the Kolmogorov equation from the relationships

$$P\{x_{\hat{\tau}_2}^{(2)} = j\} = \begin{cases} 1, & j = 1, \\ \sum_{i=1}^{j-1} P\{x_{\hat{\tau}_2}^{(2)} = i\} p_{ij}^{(1)}; & 2 \leq j \leq l-1, \\ \sum_{i=1}^{l-1} P\{x_{\hat{\tau}_2}^{(2)} = i\} p_{ij}^{(1)}; & 1 \leq j \leq N, \end{cases} \tag{4.18}$$

one obtains (4.18)

$$P\{x_{\hat{\tau}_2}^{(2)} = j\} = \begin{cases} \sum_{k=1}^{j-1} (\mathcal{P}^k)_{1,j}; & 2 \leq j \leq l-1, \\ \sum_{s=1}^l \sum_{k=1}^{s-1} (\mathcal{P}^k)_{1,sp_{s,j}^{(2)}}; & 1 \leq j \leq N, \end{cases} \quad (4.19)$$

where  $\mathcal{P} := \{p_{ij}^{(2)} : i, j = \overline{0, N}\}$  is the matrix of transitional probabilities (4.5) of the associated Markov process for the process of choice of the most desirable share package by the first client-buyer. For convenience, we denote the quantity  $P\{\tau_2(l) = j\} := h_j, j = 0, \dots, N$ , which is defined by means of (4.19). Since  $P\{\tau_2(l) = j\} = 0$  for all  $1 \leq j \leq l-1$ , for the price function of the optimal choice (4.14), one obtains

$$V_n^{(1)} = c_\alpha \left[ \frac{4n}{N} \left( 1 - \sum_{j=l}^n \frac{j}{N} h_j \right) - \frac{n^2}{N^2} \left( 1 - \sum_{j=l}^n \frac{j}{N} h_j \right)^2 \right] - \frac{\alpha(N-1)^2}{N^2} \sum_{k=1}^{n-1} \frac{k}{N^2} \left( 1 - \sum_{j=l}^n \frac{j}{N} h_j \right)^2 \quad (4.20)$$

for all  $n = 1, \dots, \tau_1$ . Using Theorem 4.2, let us find the quantity  $l \in \mathcal{H}$  satisfying the inequalities (4.6), which we can rewrite in the more convenient form:

$$V_{l-1}^{(1)} < (\mathcal{P}V^{(1)})_{l-1}, (\mathcal{P}V^{(1)})_l \leq V_l^{(1)}. \quad (4.21)$$

The determining respective inequalities are given by the following analytical expressions:

$$\begin{aligned} & c_\alpha \sum_{k=l}^N \frac{(l-1)^2 [2k(k-1)-1]}{(2l-3)k^2(k-1)^2} \left[ \frac{4k}{N} \left( 1 - \sum_{j=l}^k \frac{j}{N} h_j \right) - \frac{k^2}{N^2} \left( 1 - \sum_{j=l}^k \frac{j}{N} h_j \right)^2 \right] - \\ & - \frac{\alpha(N-1)^2}{N^2} \sum_{k=l}^N \frac{(l-1)^2 [2k(k-1)-1]}{(2l-3)k^2(k-1)^2} \sum_{s=1}^{k-1} \frac{s}{N^2} \left( 1 - \sum_{j=l}^s \frac{j}{N} h_j \right)^2 > \\ & > c_\alpha \left( \frac{4(l-1)}{N} - \frac{(l-1)^2}{N^2} \right) - \frac{\alpha(N-1)^2}{N^2} \sum_{k=1}^{l-1} \frac{k}{N^2} \left( 1 - \sum_{j=l}^k \frac{j}{N} h_j \right)^2 \end{aligned} \quad (4.22)$$

and

$$\begin{aligned} & c_\alpha \sum_{k=l}^N \frac{l^2 [2k(k-1)-1]}{(2l-1)k^2(k-1)^2} \left[ \frac{4k}{N} \left( 1 - \sum_{j=l}^k \frac{j}{N} h_j \right) - \frac{k^2}{N^2} \left( 1 - \sum_{j=l}^k \frac{j}{N} h_j \right)^2 \right] - \\ & - \frac{\alpha(N-1)^2}{N^2} \sum_{k=l}^N \frac{l^2 [2k(k-1)-1]}{(2l-1)k^2(k-1)^2} \sum_{s=1}^{k-1} \frac{s}{N^2} \left( 1 - \sum_{j=l}^s \frac{j}{N} h_j \right)^2 \leq \\ & \leq c_\alpha \left( \frac{4l}{N} - \frac{l^2}{N^2} \right) - \frac{\alpha(N-1)^2}{N^2} \sum_{k=1}^{l-1} \frac{k}{N^2} \left( 1 - \sum_{j=l}^k \frac{j}{N} h_j \right)^2. \end{aligned} \quad (4.23)$$

Let  $l \in \mathcal{H}$  satisfy the inequalities (4.22) and (4.23). As a result, we obtain the following algebraic equation:

$$\begin{aligned} & \frac{2c_\alpha l}{2l-1} \left[ 4 \frac{l}{N} \ln \frac{N}{l} - \frac{2l^2}{N^2} \ln^2 \frac{N}{l} - \frac{(N-l)l}{N^2} + \frac{2l^3}{N^3} \left( \frac{N}{l} \ln \frac{N}{l} - \frac{N}{l} + 1 \right) - \right. \\ & \quad \left. - \frac{l^4}{N^4} \left( \frac{N}{l} \ln^2 \frac{N}{l} - \frac{2N}{l} \ln \frac{N}{l} + \frac{N}{l} - 1 \right) \right] - \\ & - \alpha \left[ \frac{l^2(N-l)}{2N^3} + \frac{l(N-l)^2}{2N^3} - \frac{l^2}{N^2} \ln \frac{N}{l} + \frac{l^2(N-l)}{N^3} + \frac{l^2(N-l)^2}{2N^4} + \right. \\ & \quad \left. + \frac{l^4}{2N^4} \left( \frac{N}{l} \ln^2 \frac{N}{l} - \frac{3N}{l} \ln \frac{N}{l} + \frac{7N}{2l} - 4 + \frac{l}{2N} \right) \right] = \\ & = c_\alpha \left( \frac{4l}{N} - \frac{l^2}{N^2} \right) - \frac{\alpha(N-1)^2 (l-1)l}{2N^4}, \end{aligned} \quad (4.24)$$

where we have taken into account that in accordance with (4.20),  $h_j = (j - 1)p_{1j}^{(2)}$ ,  $j = l, \dots, N$ . Then it is straightforward to verify the following result.

**Theorem 4.3** *When the procedure of imposing a fine on a buyer is progressive linear and agrees with the portfolio volume, the Markov sequences (4.4) allow the division of the phase space  $\mathcal{H}$  into the direct sum of subspaces  $\mathcal{H}_+ = \{0, \dots, l - 1\}$  and  $\mathcal{H}_- = \{l, \dots, N\}$  under the condition that the promotional parameter  $c_\alpha \geq \alpha/8 > 0$ .*

Since the expression (4.24) is quite complicated when the quantity  $N \in \mathbb{Z}_+$  is finite, we shall next carry out an asymptotic analysis under the condition that  $\lim_{N \rightarrow \infty} l(N)/N := z \in (0, 1)$ , exists, where  $l(N) \in \mathcal{H}$  is a corresponding solution of the given equation.

### 4.3 Asymptotic analysis of the optimal share package choice strategy

Under the condition that  $\lim_{N \rightarrow \infty} l(N)/N = z \in (0, 1)$  we obtain [14, 12] the following transcendental equation from the algebraic expression (4.24):

$$\begin{aligned} & \beta(4z \ln z + 2z^2 \ln^2 z + 2z^2 \ln z + z^3 \ln^2 z + 2z^3 \ln z + 5z - z^3 - z^4) + \\ & + (2z + 2z^2 \ln z + 2z^2(1 - z)^2 + 2z^3 \ln^2 z + 3z^3 \ln z + 10z^3 - z^4 + z^5) = 0, \end{aligned} \quad (4.25)$$

where we denote  $4c_\alpha/\alpha := \beta \geq 1/2$ . The transcendental equation (4.25) has only one solution on the interval  $(0, 1)$ , which can be found by means of numerical methods.

The approximate solutions of the equation (4.25) on the interval  $(0, 1)$  for some values of  $\beta \in [0.5, 1.5]$  are shown in the Table 1.

Table 1. Real solutions of the equation (4.25) for different values of the coefficient  $\beta$ .

$\beta$	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
$z^*$	0.155	.171	.186	.199	.210	.220	.228	.236	.243	.249	.254

As a result, we can formulate the following optimal strategy of the client-buyer behavior in the stock market in terms of bivariate usefulness: At a large enough bulk  $N \in \mathbb{Z}_+$  of share packages within a bank portfolio, the first client's optimal behavior strategy for choosing the best share package is the relative quality value monitoring of  $l = z^*N \in \mathbb{Z}_+$  packages, followed by the choice of the first share package with bivariate quality surpassing all of the preceding.

## 5 Concluding Remarks

In contrast to our monovariant profit model, we have used a rather special discrete Markov process on the phase space  $\mathcal{H} = \{0, 1, \dots, N\}$  to develop a fairly realistic simulation in which to formulate an optimal strategy for choosing the most desirable share package in a zeitnot market with and multiple client-buyers. We showed that when the number of share packages in the bank portfolio is sufficiently large, the buyer's optimal strategy of choice of the most valuable share package is defined by the universal

transcendental equation (4.25) that depends on the parameter  $\beta := 4c_\alpha/\alpha \geq 1/2$ , which characterizes the bank parameter of encouragement and fine (or incentive or disincentive). The loss risk on the part of the bank, the share-seller, is the lowest when  $\beta = 1/2$ , which leads to the invariant form of the equation (4.25) with respect to this parameter. In this case, the buyer can skim only  $\simeq 15.54\%$  of the share package portfolio to optimally choose the most valuable share package in the ordered (by desirability) list of packages following those that are skimmed.

It should be also emphasized that when there is no a priori information about the qualitative characteristics of the portfolios, our model of zeitnot stock behavior of the client-buyers in the bivariant profit function case is a rather simplified version of the real situation. Moreover, it should be noted that we have consciously assumed that every client possesses sufficient financial capital to purchase any bank portfolio share package. If the client-buyers do not have sufficient resources to buy some of the share packages for the price offered by the bank, our model and subsequent analysis would have to be modified. Additional alterations and further development of our approach would also be required if there is a large number of competing client-buyers or when there are more profit related parameters affecting the choice of share packages. But all of these extra degrees of variability are actually quite typical in large scale banking portfolio markets, so we plan to investigate these more complex cases in our future research.

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