Available at: http://publications.ictp.it IC/2010/095

United Nations Educational, Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

SUMMATION OF THE FOURIER TRANSFORM OF MEASURES AND FOUR DENOMINATOR ESTIMATES

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Abstract

In this paper we consider convergence exponent for the Fourier transform of surface-carried measures. We apply the obtained bound for the Fourier transform of measures to so-called four denominator estimate related to the Schrödinger operator on a lattice.

MIRAMARE – TRIESTE
December 2010

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1. Introduction

Oscillatory integrals play an essential role in many branches of mathematics. Especially many problems from mathematical physics, theory of probability and analytic number theory lead to investigate oscillatory integrals connected to Fourier transform of distributions. In particular, distributions associated to surfaces in Euclidean spaces. In this paper we consider some problem connected to summation of oscillatory integrals related to the Fourier transform of surface-carried measures.

Note that behavior of the Fourier transform of measures supported on hyper-surfaces depending on the direction may be quit complicated but, as noted by Duistermaat J.J. [7] and Varchenko A.N. [25] although some average behavior may be much more better. Such results may be helpful in some problems connected to summation of the Fourier transform of the indicator functions. Recently, in the paper [4] the same type of results were obtained for more general class of surfaces requiring only $C^{3/2}$ smoothness condition (see later for the definitions).

By using this results one can obtain summation of the Fourier transform of indicator functions of compact convex domains or the compact domains with $C^{3/2}$ smooth boundaries.

Note that the summation problem for the Fourier transform of surfaces-carried measures is much more subtle problem and in general case it is still one of the open problems of harmonic analysis (see [1] and also [5] and [10]).

In this paper we consider the summation problem for the Fourier transform of surface-carried measures connected to some problem of mathematical physics.

Let $S \subset \mathbb{R}^{n+1}$ be a smooth hyper-surface and $\psi \in C_0^\infty(\mathbb{R}^{n+1})$. Consider the integral

$$\hat{d\mu}(\xi) := \int_S e^{ix\cdot\xi} d\mu(x), \tag{1.1}$$

where $d\mu(x) := \psi(x)dS$ and dS is the induced Lebesgue measure on the surface S and $x \cdot \xi$ is the inner product of the vectors x and ξ. If $S(a) \subset \mathbb{R}^{n+1}$ is a family of smooth hyper-surfaces depending on the parameters $a \in \mathbb{R}^m$ then by the analogy we can define $d\mu_a$ and $\hat{d\mu}_a(\xi)$.

In this paper we consider the problem:

Problem: Find the greatest lower bound p_0 of the set $\{p : \hat{d\mu} \in L^p(\mathbb{R}^n)\}$.

Motivations. This problem has a few motivations. We consider problem from Mathematical physics considered by [8] it is so-called four-denominator estimate related to the summation exponent of the oscillatory integrals.

In the recent paper by Erdös L. and Salmhofer M. [8] is considered the same problem for particular case when the Gaussian curvature of the two-dimensional hyper-surfaces satisfies some non-degeneracy conditions. In this paper we consider analogical problems for wider class of hyper-surfaces in multi-dimensional Euclidean space. Moreover, in the three-dimensional case we obtain much more better results. Then we apply our results for oscillatory integrals to so-called the four-denominator estimate.

Denote by $K_l(x) \subset \mathbb{R}^{n+1}$ the class of smooth hyper-surfaces having at least l non-vanishing principal curvatures at the point x. Further, the relation $S \subset K_l$ means that at every point $x \in S$ we have $S \subset K_l(x)$. Littman W. [17] proved that if a hyper-surface belongs to the class K_l then the integral (1.1) has the following uniform estimate

$$\hat{d\mu}(\xi) = O(|\xi|^{-l/2}) \quad \text{as,} \quad |\xi| \to +\infty.$$

Let's define the Randoll maximal function:

$$M(\omega) := \sup_{r>0} r^{\frac{n}{2}} |\hat{d\mu}(r\omega)|, \tag{1.2}$$

where $\omega \in S^n$ and $\xi = r\omega$. The maximal function corresponding to $\hat{d\mu}_a(\xi)$ is denoted by M_a.

Analogical maximal functions introduced and investigated by B. Randoll [21] for the case of the Fourier transform of the indicator function of the compact convex domains with analytic boundary and by I. Svensson [22] for more general convex domains with finite type smooth boundary.

Suppose $S(a) \subset \mathbb{R}^{n+1}$ (where $a \in U \subset \mathbb{R}^m$ some parameters) be a family of smooth (analytic) hyper-surfaces smoothly (analytically) depending on the parameters a.

Theorem 1.1. *Suppose $S(a)$ is an analytic family of hyper-surfaces satisfying the conditions:*
1) $S(0) \subset \mathbb{R}^{n+1}$ is an analytic hyper-surface and $S(0) \in K_{n-1}$;
2)Denote by $K(x,a)$ the Gaussian curvature at the point $x \in S(a)$. Assume $K(x,0) \not\equiv 0$ then the following statements are hold:
(i) There exist a neighborhood $V \times U \subset \mathbb{R}^{n+1} \times \mathbb{R}^m$ and $p_m > 2$ such that for any $\psi \in C_0^\infty(V)$ the following inclusion $M_a \in L^{p_m}(S^n)$ holds, where M_a is the corresponding to $\hat{d\mu}_a(r\omega)$ maximal function. Moreover, the following integral

$$\int_{S^n} M_a^{p_m} d\omega$$

is uniformly bounded with respect to $a \in U \subset \mathbb{R}^m$.
(ii) If $p_m > 2(n+1)/n$ then $\hat{d\mu}_a \in L^p(\mathbb{R}^{n+1})$ for any $p > 2(n+1)/n$. Moreover, the following integral

$$\int_{\mathbb{R}^{n+1}} |\hat{d\mu}_a(\xi)|^p d\xi$$

is uniformly bounded with respect to $a \in U$ for any fixed $p > 2(n+1)/n$.
(iii) If $2 < p_m \leq 2(n+1)/n$ then $\hat{d\mu}_a \in L^p(\mathbb{R}^{n+1})$ for any $p > (2n+2-p_m)/(n-1)$.

Moreover, the following integral

$$\int_{\mathbb{R}^{n+1}} |\hat{d\mu}_a(\xi)|^p d\xi$$

is uniformly bounded with respect to $a \in U$ for any fixed $p > (2n+2-p_m)/(n-1)$.

Remark 1.2. *If S is the cylinder with the spherical base then we have $S \in K_{n-1}$ and $K \equiv 0$. In this case $M \notin L^2(S^n)$. Moreover, if $n = 2$ then $\hat{d\mu} \notin L^4(\mathbb{R}^3)$ for appropriate amplitude function and $\hat{d\mu} \in L^p(\mathbb{R}^3)$ for any $p > 4$ and $\psi \in C_0^\infty(\mathbb{R}^3)$.*

For hyper-surfaces in $\mathbb{R}^3$ we have more better estimate for any analytic hyper-surfaces. We consider the hyper-surface in a sufficiently small neighborhood of a fixed point say at the origin. Then we can define a hight of the hyper-surface at the origin (see [15] and also [26]).

Theorem 1.3. *Suppose $S \subset \mathbb{R}^3$ is an analytic hyper-surface and $d\mu(x) := \psi(x)dS$. Assume ψ is a smooth function concentrated in a sufficiently small neighborhood of the origin and h is the hight of the hyper-surface at the origin. Then we have the inclusion $\hat{d\mu} \in L^p(\mathbb{R}^3)$, for any $p > 2 + h$. In particular, if $h = 2$ and also $K \not\equiv 0$ then there exists a positive number $\varepsilon > 0$ such that $\hat{d\mu} \in L^{(4-\varepsilon)}(\mathbb{R}^3)$.*

Remark 1.4. *Note that the last conclusion of the Theorem 1.3 does not follow from the three dimensional case of Theorem 1.1. Because, both principal curvatures may vanish at the origin in the case $h = 2$.*

The following results shows sharpness of our results.

Proposition 1.5. *For any positive number ε there exists a hyper-surface $S \in K_1$ in $\mathbb{R}^3$ with $K(x) \not\equiv 0$ such that $\hat{d\mu} \notin L^{(4-\varepsilon)}(\mathbb{R}^3)$.*

2. Summation of the Fourier transform of the indicator functions

Let $D \in \mathbb{R}^{n+1}$ be a compact domain and let $u \in C^\infty(\mathbb{R}^{n+1})$. Consider an oscillatory integral of the following form:

$$\hat{u}_D(r\omega) = \int_D u(x) exp(irx \cdot \omega) dx \quad r \in \mathbb{R}_+, \quad \omega \in S^n,$$

where S^n is the unit sphere in $\mathbb{R}^{n+1}$ centered at the origin and $x \cdot \omega$ is the inner product of the vectors x and ω. For $u(x) \equiv 1$ the function $\hat{u}_D(r\omega)$ coincides with the Fourier transform of the indicator function of the set D.

First, we give some definition introduced in the paper [4].

Definition 2.1. *Let $S \subset \mathbb{R}^{n+1}$ be a C^1- hyper-surface satisfying the condition: for any given pair of points P, Q the following inequality*

$$|(P-Q)\cdot n(Q)| \lesssim |P-Q|^{3/2} \tag{2.1}$$

holds, where $n(Q)$ is the unite normal to the surface at the point Q and the symbol $\lesssim$ means that the constant in the estimate (2.1) does not depend on P, Q so it is uniformly with respect to points of the surface.

Remark 2.2. *Note that if the hyper-surface is C^2 smooth then for any points P, Q we have the estimate*

$$|(P-Q)\cdot n(Q)| \lesssim |P-Q|^2$$

and the constant in the last inequality depends on only on C^2 norm of the function which defines the hyper-surface.

First, we give some easy application of results of the paper [4].

Proposition 2.3. *Let D be either a bounded convex domain with C^1 smooth boundary or with boundary which can be decomposed into finitely many neighborhoods satisfying the condition (2.1). Then for any $p > 2(n+1)/(n+2)$ we have the inclusion $\hat{u}_D \in L^p(\mathbb{R}^{n+1})$.*

Remark 2.4. *Explicit computation of the Fourier transform of the indicator function of the unite ball show that the result of the Proposition 2.3 is sharp.*

Proof. Actually the proof follows from the main results proved in [4]. Indeed, by using integration by parts arguments the Fourier transform $\hat{u}_D(r\omega)$ is reduced to surfaces integral over the boundary. Then we have the estimate (see [4]):

$$\int_{S^n} |\hat{u}_D(r\omega)|^2 d\omega \lesssim r^{-(n+2)}.$$

The constant in the last estimate does not depend on ω. Further, we use the symbol $\lesssim$ for estimates up to constants. Note that if $1 \le p \le 2$ then by using Hölder's inequality we have

$$\int_{S^n} |\hat{u}_D(r\omega)|^p d\omega \lesssim \left(\int_{S^n} |\hat{u}_D(r\omega)|^2 d\omega\right)^{p/2},$$

where in the last inequality constant depends only on n and p. Thus, by using the results of [4] we obtain

$$\int_{\mathbb{R}^{n+1}} |\hat{u}_D(\xi)|^p d\xi = \int_{|\xi|\le 1} + \int_1^\infty dr r^n \int_{S^n} |\hat{u}_D(r\omega)|^p d\omega \lesssim c + \int_1^\infty r^{n-p(n+2)/2} dr.$$

The last integral converges whenever $2 \ge p > 2(n+1)/(n+2)$. Since the Fourier transform of distribution with compact support is a bounded function the Fourier transform $\hat{u}_D(\xi)$ belongs to the $L^p(\mathbb{R}^{n+1})$ whenever $p > 2$. □

Further, we consider the problem on summation of the $\hat{d\mu}(\xi)$. First, we obtain some estimates for one-dimensional oscillatory integrals.

3. On estimates for one dimensional oscillatory integrals

Let $V(\mathbb{R})$ be the space of functions of bounded variation on $\mathbb{R}$. Denote by $\|.\|_V$ the natural norm on the space i.e.

$$\|a\|_V = |a(-\infty)| + V_{\mathbb{R}}[a],$$

where $V_{\mathbb{R}}[a]$ is the total variation of the function a on $\mathbb{R}$. Consider the $\mathbb{R}_+-$versal deformation of the singularities A_n of the form [2]:

$$p(x,s) = x^{n+1} + s_{n-1}x^{n-1} + ... + s_1 x.$$

Denote by $J(t,s)$ the associated oscillatory integral with the phase function $p(x,s)$ and an amplitude function $a(x,s)$:

$$J(t,s) = \int_{\mathbb{R}} a(x,s) exp(itp(x,s))dx. \tag{3.1}$$

Integral (3.1) is conditionally convergent (see [7]). Introduce the maximal function associated to the oscillatory integral (3.1):

$$m(s) = \sup_{t>0} |t|^{1/2} |J(t,s)|.$$

The main result of this section is the following.

Proposition 3.1. *There exists a number $p > 2$ and a function $\psi \in L^p_{loc}(\mathbb{R}^{n-1})$ such that the maximal function $m(s)$ has the estimate*

$$m(s) \le \psi(s)\|a\|_V,$$

where $L^p_{loc}(\mathbb{R}^{n-1})$ is the space of functions whose pth power is locally integrable.

Let $\{x_1(s), x_2(s), ..., x_n(s)\} \subset \mathbb{C}$ be the set of all solutions to the equation $p'(x,s) = 0$. We consider the following sum

$$\sigma(s) = \sum_{k=1}^{n} |p''(x_k(s))|^{-1/2}.$$

Lemma 3.2. *There exists a number C depending only on n such that the following estimate holds:*

$$m(s) \le C\sigma(s)\|a\|_V. \tag{3.2}$$

The estimate (3.2) is some version of Colin de Verdier estimate (see [6]), and also [10], [12]). It is worth mentioning that if we consider instead of $\sigma(s)$ sum only over the real critical points as stated in [6] then such estimate fails to be true (see [13] and also [20]).

Proof. A proof of the Lemma (3.2) follows from the Phong D.H. and Stein E.M. theorem [18]. ☐

Let $s \in \mathbb{R}^{n-1}$. Consider the function defined by

$$F(x, s) := x^k \psi(x, s_1, ..., s_{n-1}) + s_k x^{k-1} + ... + s_1,$$

where $\psi(x, s_1, ..., s_{k-1})$ is some real analytic function and $\psi(0, 0) \neq 0$. It can be shown in the usual way that for every $s \in \{|s| < \varepsilon\}$ the function $F(x, s)$ has k zeros (by count of multiplicity) $x_1(s), ..., x_k(s) \in U \subset \mathbb{C}$. We consider the sum

$$\Sigma(s) := \sum_{l=1}^{k} |F'(x_l(s), s)|^{-1/2}.$$

Lemma 3.3. *There exists a positive number $\varepsilon > 0$ such that for every fixed number $p < 2k/(k-1)$ the following estimate holds:*

$$\int \Sigma^p(s_1, ..., s_{n-1}) ds_1 < C,$$

where C is some fixed positive number which does not depend on $(s_2, \ldots, s_{n-1})$.

Proof. . The proof is by induction over k. For $k = 1$ there is nothing to prove. Let $k \geq 2$ be some fixed positive integer number and assume that the statement of the Lemma 3.3 holds for every $l < k - 1$. There exist positive numbers δ_1, δ_2 and ε such that for every $|x| < \delta_1$ and $|s| < \varepsilon$ the following estimate holds $|\psi(x, s_1, ..., s_{n-1})| > \delta_2 > 0$ and the equation $F(x, s) = 0$ has k zeros $|x_l(s)| < \delta_1, l = 1, k$. Let $\rho = |s_2|^{(k+1)/2} + ... + |s_k|^{(k+1)/k}$. First, we consider the case $s \in \{s \in \mathbb{R}^{n-1} : |\rho/s_1| < \delta_3\}$, where δ_3 is some fixed small number.

In this case we apply change of variables $x_1 \mapsto |s_1|^{1/k} y$ and have

$$\Phi(y, s) = y^k \psi(|s_1|^{1/k} y, s_2, ..., s_{n-1}) + s_k s_1^{-2/k} y^{k-1} + \cdots + s_2 s_1^{-(k-1)/k} y + sgn(s_1).$$

It is obvious that there exists a positive number M such that for every $|s| < \varepsilon$ and $M \leq |y| \leq \delta_1 |s_1|^{-1/k}$ the equation $\Phi(y, s) = 0$ has no solutions. If $s_2 = ... = s_{n-1} = 0$ then the equation $\Phi(y, s) = 0$ has exactly k roots in the set $\{|y| \leq M\}$. Consequently, there exist positive numbers δ_3 and δ_4 such that for every $s \in \{|s| < \varepsilon, |\rho/s_1| < \delta_3\}$ the equation $\Phi(y, s) = 0$ has k roots $y_l(s)$, $l = \overline{1, k}$. Moreover, the following estimate holds

$$\left| \frac{\partial \Phi(y_l(s), s)}{\partial y} \right| \geq \delta_4 > 0, \quad l = 1, \ldots, k,$$

whenever $s \in \{|s| < \varepsilon, |\rho/s_1| < \delta_3\}$. Thus, the sum $\Sigma(s)$ has the estimate

$$\Sigma(s) \leq \frac{C}{\delta_4 |s_1|^{\frac{k-1}{2k}}}.$$

Let δ_3 be any fixed positive number. We consider the case $s \in \{|s| < \varepsilon, \, |s_1/\rho| < \delta_3^{-1}\}$. In this case we apply the following change of variables $x \mapsto \rho^{1/k}y$ and have

$$F(\rho^{\frac{1}{k}}y, s) = \rho\Phi_1(y, \, \sigma, \, \rho),$$

where

$$\Phi_1(y, \, \sigma, \, \rho) := y^k\psi(\rho^{1/k}y, \sigma_2\rho^{\frac{k-1}{k}}, \ldots, \sigma_k\rho^{\frac{2}{k}}, \, s_{k+1}, \ldots, s_{n-1})+$$

$$\sigma_{k-1}y^{k-2} + \cdots + \sigma_2 y + \sigma_1, \quad \sigma_l = \rho^{\frac{l-k-1}{k}}s_l, \quad l = 1, 2, \ldots, k-1.$$

Let $s_{k+1} = ... = s_{n+1} = 0$, $\sigma = \sigma^0$ be a fixed point on the quasisphere $\{\rho(\sigma_{k-1}, \ldots, \sigma_2) = 1\}$ and $\sigma_1 = \sigma_1^0 \in [-\delta_3^{-1}, \, \delta_3^{-1}]$.

Note that every solution to the equation $\Phi_1(y, \sigma) = 0$ (where $\{\rho(\sigma_{k-1}, \ldots, \sigma_2) = 1\}$ and $\sigma_1 \in [-\delta_3^{-1}, \, \delta_3^{-1}]$ belongs to the compact set $[-c, c]$ and has multiplicity $l < k$. By the induction hypothesis we have

$$\int_{-\delta_3^{-1}}^{\delta_3^{-1}} |\Sigma_1(s_{k+1}, \ldots, s_{n-1}, \sigma_k, \ldots, \sigma_1)|^p d\sigma_1 \leq c,$$

where $p < \frac{2k}{k-1}$ is some fixed positive number. It follows that

$$\int_{|\frac{s_1}{\rho}| < \delta_3^{-1}} \Sigma^p(s) ds_1 = \int_{-\delta_3^{-1}}^{\delta_3^{-1}} \rho^{1-\frac{p(k-1)}{2k}} |\Sigma_1(s_{k+1}, \ldots, s_{n-1}, \sigma_k, \ldots, \sigma_1)|^p d\sigma_1 \leq$$

$$c\int_{-\delta_3^{-1}}^{\delta_3^{-1}} |\Sigma_1(s_{k+1}, \ldots, s_{n-1}, \sigma_k, \ldots, \sigma_1)|^p d\sigma_1 \leq c.$$

We see that the function

$$\Sigma(s) := \frac{c}{\delta_4|s_1|^{\frac{k-1}{2k}}} + \frac{1}{\rho^{\frac{k-1}{2k}}}\sigma_1(\frac{s_1}{\rho}, \ldots, s_k\rho^{\frac{-2}{k}}, s_{k+1}, \ldots, s_{n-1})\chi_{|s_1| < \delta_3\rho}(s),$$

satisfies the conclusion of Lemma 3.3. This completes the proof of Lemma 3.3. □

Proof of Proposition 3.1. Let $J(t, s)$ be the oscillatory integral. By using Lemma 3.2 we have the estimate

$$m(s) \leq c\|a\|_V\psi(s).$$

Then due to the Lemma 3.3 the maximal function satisfies the conclusion of Proposition 3.1.

4. Some auxiliary statements

Suppose that $\Phi : (\mathbb{R} \times \mathbb{R}^n, 0) \mapsto (\mathbb{R}, 0)$ is a non-zero real-analytic function. In the case $n = 1$, the assertion of the following Lemma which is proved in the paper [11] is an immediate consequence of the Weierstrass-Malgrange preparation theorem [18].

Lemma 4.1. *There exists a real-analytic manifold Y and a mapping $\pi : Y \mapsto \mathbb{R}^n$ which is the composition of finitely many σ-processes such that, for any point $y_0 \in Y$, there is a chart $(y_1, ..., y_n)$ centered at y^0 for which the following relation holds:*

$$\Phi(x, \pi(y)) = y^{k_1} \dots y_n^{k_n} g(x, y) p(x, y),$$

where $g(x, y)$ is a real-analytic function with $g(0, y^0) \neq 0$, and $p(x, y)$ is a unitary pseudo-polynomial, i.e.,

$$p(x, y) = x^m + d_1(y)x^{m-1} + \dots + d_m(y);$$

here $d_1, \dots, d_m$ are real-analytic functions at the point y^0 and $d_l(y^0) = 0$, $l = 1, \dots, m$.

Suppose the function Φ depends on some additional parameters $\Phi : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m, 0) \mapsto (\mathbb{R}, 0)$.

Consider the phase function given by

$$F(x, \sigma, s_1, a) := \Phi(x, \sigma, a) + s_1 x, \tag{4.1}$$

and the associated oscillatory integral defined by

$$J(\lambda, \sigma, s_1, a) := \int_{\mathbb{R}} e^{\lambda F(x, \sigma, s_1, a)} b(x, \sigma, s_1, a) dx. \tag{4.2}$$

Lemma 4.2. *Assume the function Φ is a real analytic function at the origin and also it satisfies the condition $\Phi(x, \sigma, 0) \not\equiv 0$. Then there exists a neighborhood $W \times V \times U \subset \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^m$ of the origin such that for any amplitude function $b \in C_0^\infty(W \times V \times U)$ for the oscillatory integral (4.2) the following estimate*

$$|J(\lambda, \sigma, s_1, a)| \leq \frac{\psi(\sigma, s_1, a)}{|\lambda|^{1/2}} \tag{4.3}$$

holds, where ψ is a function satisfying the condition: there exists a positive number ε such that for any fixed a the function $\psi(\cdot, a) \in L^{2+\varepsilon}(V)$ and also

$$\|\psi(\cdot, a)\|_{L^{2+\varepsilon}(V)}$$

is uniformly bounded with respect to $a \in U$.

Proof. We follow method of proof of the Lemma 4.1 [10]. Let's write the function Φ in the form:

$$\Phi(x, \sigma, a) = \sum_{k=2}^{\infty} c_k(\sigma, a) x^k,$$

where $\{c_k\}_{k=2}^\infty$ are analytic function in a fixed neighborhood of the origin.

Let's consider the ideal of the algebra of analytic functions at the origin generated by that functions $I :=< \{c_k\}_{k=2}^\infty >$. By our assumption $I \neq 0$. If I coincides with algebra of analytic functions then the function $\Phi(x, 0, 0)$ would be a non-zero analytic function so it is a deformation of A_k type singularity with some finite k see [2]. In this case the proof of Lemma 4.2 follows from Lemma 3.3. Further, we assume that I is a proper ideal. Then by Hilbert's Theorem there exists an N such that the ideal I generates by the elements $\{c_k\}_{k=2}^N$. Also we naturally define an ideal $I_0 :=< \{c_k(.,0)\}_{k=2}^\infty >$. Then obviously it is generated by $\{c_k(.,0)\}_{k=2}^N$. In particular, by our condition we have $c^2(\sigma, 0) \not\equiv 0$ for the analytic function c^2 defined by

$$c^2(\sigma, a) := \sum_{l=2}^{N} c_l^2(\sigma, a).$$

Note that there exists a positive number $\delta > 0$ such that for any $a \in U$ the integral

$$\int_V \frac{d\sigma}{|c(\sigma,\, a)|^\delta} \tag{4.4}$$

is uniformly bounded with respect to a.

We remark that from the convergence of the last integral for $a = 0$ does not follow uniformly boundedness of that integral with respect to a as can be seen from Varchenko A.N. example [26]. But, for sufficiently small δ the last integral converges and uniformly bounded with respect to a. For, example if

$$c^2(\sigma, 0) = \sum_{k=m}^{\infty} p_k(\sigma)$$

is a Taylor development of the function c^2, where p_k are homogeneous polynomial of degree k and also $p_m \not\equiv 0$ then it is easy to show that if $\delta < 2/m$ then the integral (4.4) converges for any sufficiently small a and it is uniformly bounded. For more refined results on the direction see [16] (also for more generalization see [19]).

First, consider the case $c(\sigma, a) < |s_1|$ then the function $\Phi(x,\, \sigma,\, a)/(x^2 c(\sigma, a))$ and its derivatives are bounded. Therefore the function $\Phi(x,\, \sigma,\, a)/|s_1|$ and its derivative are sufficiently small in a neighborhood of the origin. Therefore, we can use integration by parts arguments and obtain:

$$|J(\lambda,\, \sigma,\, a,\, s_1)| \le \frac{C\chi_{c(\sigma,a)<|s_1|}(\sigma,\, a,\, s_1)}{|\lambda s_1|^{1/2}},$$

with some constant C, where $\chi_{c(\sigma,a)<|s_1|}$ is the indicator function of the set $c(\sigma, a) < |s_1|$.

Further, we consider the case $c(\sigma, a) > |s_1|$. In this case we use the Lemma 4.1. Let's define

$$\Phi_1(x,\, \sigma,\, a) := \frac{\Phi(x,\, \sigma,\, a)}{c(\sigma,\, a)} \quad \text{and} \quad \varsigma_1 := \frac{s_1}{c(\sigma,\, a)}.$$

By using the Lemma 4.1 we may assume that the function $\Phi_1(x,\, \sigma,\, a)$ is a smooth perturbation of the singularity A_k (although may be different k depending on the parameters). Now, we fix $\varsigma_1 = \varsigma_1^0 \in [-1, 1]$ and consider the function

$$\Phi^0(x, y, \varsigma_1) := \Phi_1(x, \pi(y)) + \varsigma_1^0 x + (\varsigma_1 - \varsigma_1^0)x,$$

where y are local coordinates on the manifold Y defined from the Lemma 4.1. Hironaka [9] (see also [3]) resolution of singularities arguments show that between the manifold Y and $V \times U \subset \mathbb{R}^n \times \mathbb{R}^m$ there is one-to-one correspondence π outside of an analytic set and π is a proper analytic mapping $\pi : Y \mapsto V \times U$.

Now, due to the Lemma 3.3 there exists a function $\Psi(y, \varsigma_1)$ such that for the oscillatory integral we have the estimate:

$$|J(\lambda, \sigma, a, s_1)| \leq \frac{C\chi_{c(\sigma,a)\geq|s_1|}(\sigma, a, s_1)\Psi(\sigma, \frac{s_1}{a})}{|c(\sigma, a)\lambda|^{1/2}},$$

with the following property:

$$\int_{-1}^{1} (\Psi(y, \varsigma_1)^p d\varsigma_1$$

is uniformly bounded on the manifold Y, due to compactness arguments. We set

$$\psi(\sigma, a, s_1) := \frac{C\chi_{c(\sigma,a)<|s_1|}(\sigma, a, s_1)}{|s_1|^{1/2}} + \frac{C\chi_{c(\sigma,a)\geq|s_1|(\sigma, a, s_1)\Psi(\sigma, \frac{s_1}{a})}}{|c(\sigma, a)|^{1/2}}.$$

It is easy to see that the last function satisfies condition of the Lemma 4.3. □

5. Proof of the main theorem

Proof. We first, prove the Theorem 1.1. **Proof of (i)**. Suppose $S(a)$ is a family of real-analytic hyper-surfaces. By using stationary phase arguments we reduce our integral to one dimensional integral satisfying the conditions of the Lemma 4.3 [2]. For the one dimensional integral we use estimate of the Lemma 4.3. Then we obtain

$$|\hat{\mu}_a(\xi)| \leq M(\omega)|\xi|^{-n/2},$$

where $M \in L^{p_m}(S^n)$ for some number $p_m > 2$.

Moreover, since $S \in K_{n-1}$ due to Littman W. [17] Theorem for the oscillatory integral the following uniform estimate

$$|\hat{\mu}_a(\xi)| \leq c|\xi|^{-(n-1)/2}$$

holds.

Proof of (ii). If $p_m > 2(n+1)/n$ then by using spherical coordinates we obtain

$$\int_{\mathbb{R}^{n+1}} |\hat{\mu}_a(\xi)|^p d\xi = c + \int_1^\infty r^n dr \int_{S^n} |\hat{\mu}_a(r\omega)|^p d\omega \leq c+$$
$$\int_1^\infty r^{n-pn/2} dr \int_{S^n} |M_a(\omega)|^p d\omega < \infty.$$

The last integral converges provided $p_m \geq p > 2(n+1)/n$.

Proof of (iii). Now, we consider the case $2 < p_m \leq 2(n+1)/n$. In this case by using classical uniform estimate we have

$$\int_{\mathbb{R}^{n+1}} |\hat{\mu}_a(\xi)|^p d\xi \lesssim c + \int_1^\infty r^{n-p_m n/2-(p-p_m)(n-1)/2} dr \int_{S^n} |M_a(\omega)|^{p_m} d\omega.$$

The last integral converges provided $p > (2n + 2 - p_m)/(n-1)$.

$\square$

Proof. Now, we prove the Theorem 1.3. If $S \subset \mathbb{R}^3$ is a real analytic hyper-surfaces then due to the results [12] we have the relation $M \in L^{2-0}(S^2)$ for the maximal function. Take $\varepsilon > 0$ sufficiently small, then $M \in L^{2-\varepsilon}(S^2)$. Due to Karpushkin V.N [16] results for the Fourier transform we have the estimate

$$|\hat{\mu}(\xi)| \lesssim \frac{\log(2+|\xi|)^m}{(1+|\xi|)^{1/h}},$$

where m is co-called Varchenko exponent $m = 0, 1$. An analog of the uniform estimates for the smooth case is proved in [14]. In particular, we may write the estimate

$$|\hat{\mu}(\xi)| \lesssim \frac{1}{(1+|\xi|)^{1/h-\varepsilon}},$$

for any fixed positive real number $\varepsilon > 0$.

Thus, we have

$$\int_{\mathbb{R}^3} |\hat{\mu}(\xi)|^p d\xi \le c + \int_1^\infty r^2 dr \int_{S^2} |\hat{\mu}(r\omega)|^p d\omega \le$$
$$c + c_1 \int_1^\infty r^{2-(2-\varepsilon)-(p-(2-\varepsilon))(1/h-\varepsilon)} dr \int_{S^2} |M(\omega)|^{2-\varepsilon} d\omega.$$

The last integral converges provided

$$p > 2 + \frac{h(1+\varepsilon+\varepsilon^2)-\varepsilon}{1-h\varepsilon}.$$

Since ε is any positive real number we are done. In particular, if $h < 2$ then there exists $\varepsilon > 0$ such that $\hat{d\mu} \in L^{4-\varepsilon}(\mathbb{R}^3)$.

Finally, we consider the case $h = 2$.

After possible linear change of variables and by using partition of unity arguments we may suppose that the surface is given by

$$S := \{x \in \mathbb{R}^3 : x_3 = \Phi(x_1, x_2)\},$$

where Φ is a real analytic function at the origin and $\Phi(0,0) = 0$, $\nabla\Phi(0,0) = 0$.

Proposition 5.1. *If $h(\Phi) = 2$ and $K(0,0) \not\equiv 0$ then there exists a positive number ε such that $M \in L^{2+\varepsilon}(S^2)$.*

Proof. We use methods of the paper [14]. Due to results of the paper [14] after possible linear change of variables the function Φ can be written as

$$\Phi(x_1, x_2) = \Phi_\kappa(x_1, x_2) + \Phi_{r\kappa}(x_1, x_2), \tag{5.1}$$

where Φ_κ is a weighted homogeneous polynomial with weights (κ_1, κ_2)(with $\kappa_2 \ge \kappa_1$) satisfying the condition $h(\Phi_\kappa) = 2$ and $\Phi_{r\kappa}$ is a remainder term. We suppose that $|\xi_3| \ge |\xi_1| + |\xi_2$, otherwise we can use integration by parts arguments provided the amplitude function ψ is concentrated in a sufficiently small neighborhood of the origin. Following the paper [14] we consider dyadic decomposition of the Fourier transform of measure

$$\hat{d\mu}(\xi) = \sum_{k=k_0}^{\infty} \hat{d\mu}_k(\xi),$$

where after scaling the measure $\hat{d\mu}_k(\xi)$ has the form:

$$\hat{d\mu}_k(\xi) := 2^{-|\kappa|k} \int e^{i2^{-k}\lambda(\Phi_\kappa(x_1,x_2)+\tilde{\Phi}_{r\kappa}(x_1,x_2,2^{-k})+\sigma_1 x_1+\sigma_2 x_2)}\chi(x)\psi(\delta_{2^{-k}}(x))dx,$$

here $\delta_{2^{-k}}$ is scaling corresponding to the weights κ and $\sigma_j := 2^{(1-\kappa_j)k}s_j (j = 1, 2)$, $\lambda := \xi_3$, $s_j := \xi_j/\lambda$ and $\tilde{\Phi}_{r\kappa}(x_1, x_2, 2^{-k})$ is the remainder term after scaling. It can be considered as a small perturbation provided k_0 is sufficiently big. If the amplitude function is concentrated in a sufficiently small neighborhood of the origin then we may assume that k_0 is sufficiently big.

We show that there exists a function $\varphi \in L^{2+\varepsilon}(V)$ such that

$$|\hat{d\mu}(\lambda, s)| \le \frac{\varphi(s_1, s_2)}{|\lambda|},$$

where V is sufficiently small neighborhood of the origin.

If $|\sigma| \gtrsim 1$ then integration by parts arguments show that

$$|\hat{d\mu}_k(\xi)| \lesssim \frac{2^{-|\kappa|k}}{1 + |2^{-k}\lambda\sigma|}.$$

It is easy to see that if ε is a sufficiently small positive number then there exists a positive number δ such that

$$2^{(1-|\kappa|)k} \left(\int_{|\sigma|>1} |\sigma|^{(\varepsilon+2)} ds_1 ds_2 \right)^{1/(2+\varepsilon)} \lesssim 2^{-\delta k}.$$

Now, we suppose that $|\sigma| \lesssim 1$. In this case note that $x \in support(\chi) \subset \{1/2 < |x| < 2\}$. Then arguments of the paper [15] show that on that set the surface defined by

$$\{x \in \mathbb{R}^3 : \, x_3 = \Phi_\kappa(x_1, \, x_2) + \tilde{\Phi}_{r\kappa}(x_1, x_2, 2^{-k})\}$$

can be considered as family of analytic hyper-surfaces with one non-vanishing principal curvature and it satisfies the condition of the first part of Theorem 1.1. Therefore, there exists a function $\varphi_k(\sigma)$ such that $\varphi_k \in L^{2+\varepsilon}(V)$. Note that

$$\int_V \varphi_k^p(\sigma)ds = 2^{(|\kappa|-2)k} \int_V \varphi_k^p(\sigma)d\sigma.$$

Combining the obtained estimates we have

$$|\hat{d\mu}_k(\xi)| \le \frac{\tilde{\varphi}_k(s)}{\lambda}.$$

Moreover, the series

$$\varphi(s) := \sum_{k=k_0}^{\infty} \tilde{\varphi}_k(s)$$

converges in $L^{2+\varepsilon}$ space, because

$$\sum_{k=k_0}^{\infty} \|\tilde{\varphi}_k\|_{2+\varepsilon} \lesssim \sum_{k=k_0}^{\infty} 2^{-\delta k} < \infty,$$

where

$$\|\tilde{\varphi}\|_p := \left(\int_V |\tilde{\varphi}|^p ds \right)^{1/p}.$$

Thus, we obtain a proof of Proposition 5.1.

□

The Proposition 5.1 finishes the last part of Theorem 1.3. Indeed, we use arguments of the proof of Theorem 1.1 for $n = 2$ and obtain a required estimate.

□

6. Sharpness of the estimates.

We prove the following Lemma which shows that our bounds close to the sharp estimates.

Lemma 6.1. *Consider the hyper-surface $S \subset \mathbb{R}^3$ given by the relation*

$$S := \{x \in \mathbb{R}^3 : \quad x_3 = x_2^2 + x_1^n\}, \tag{6.1}$$

where $n \geq 3$. If μ is the associated measure on the surface (6.1) then $\hat{d\mu} \in L^p(\mathbb{R}^3)$ for any $p > p_n := 2(2n+1)/(n+2)$ and if $\psi(0) > 0$ then $\hat{d\mu} \notin L^p(\mathbb{R}^3)$ for any $p \leq p_n$.

Remark 6.2. *The Lemma 6.1 proves the Proposition 1.5.*

Proof. Suppose $d\mu := \psi(x)dS(x)$ and the amplitude function ψ is concentrated in a sufficiently small neighborhood of the origin and without loss of generality we assume that $\psi(0,0) = 1$. Let's assume $|\xi_2| + |\xi_1| < |\xi_3|$ otherwise integration by parts arguments show that the oscillatory integral decays faster. We can use stationary phase method in x_2 variables and obtain

$$\hat{d\mu}(\xi) = c\frac{e^{-i\frac{\xi_2}{4\xi_3^2}}}{|\xi_3|^{1/2}} \int e^{i(\xi_3 x_1^n + \xi_1 x_1)} a\left(x_1, -\frac{\xi_2}{2\xi_3}\right) dx_1 + R(\xi),$$

where $R(\xi)$ is a remainder term satisfying the condition $R(\xi) = O(|\xi|^{-3/2})$(as $|\xi| \to +\infty$). Therefore we have $R \in L^{2+\varepsilon}(\mathbb{R}^3)$ for any positive number ε.

Thus, we consider the principal part given by the oscillatory integral

$$J(\xi) := \frac{e^{-i\frac{\xi_2}{4\xi_3^2}}}{|\xi_3|^{1/2}} \int e^{i(\xi_3 x_1^n + \xi_1 x_1)} a\left(x_1, -\frac{\xi_2}{2\xi_3}\right) dx_1.$$

First, we claim that if $p \leq p_n$ then $J \notin L^p(\mathbb{R}^3)$. We use method used by [1]. Indeed, if $|\xi_1| << |\xi_3|^{1/n}$ then we have the following lower estimate

$$|J(\xi)| \geq \frac{c}{|\xi_3|^{(n+2)/(2n)}}.$$

Therefore, it is easy to see that integral of the function $J(\xi)|^p$ over the set

$$\Omega := \{(\xi \in \mathbb{R}^3 : |\xi_2| \leq |\xi_3|, |\xi_1| << |\xi_3|^{1/n}\}$$

diverges provided $p \leq p_n$.

Now, we show that if $p > p_n$ then the integral of that function over $\mathbb{R}^3$ converges.

Note that if

$$\xi \in \Omega_M := \{(\xi \in \mathbb{R}^3 : |\xi_2| \leq |\xi_3|, |\xi_1| \leq M|\xi_3|^{1/n}\},$$

where M is a fixed positive real number. Then we can use Van der Corpute type estimate and have

$$|J(\xi)| \leq \frac{C}{|\xi_3|^{(n+2)/(2n)}}.$$

Therefore, if $p > p_n$ then integral of the function $|J(\xi)|^p$ over the set Ω_M converges. Further, we consider integral over the set $\mathbb{R}^3 \setminus \Omega_M$. Take a smooth function χ with properties:

$$\chi(x) = \{ \begin{array}{ccc} 1 & \text{if} & |x| \leq 1 \\ 0 & \text{if} & |x| \geq 2. \end{array}$$

Then define the new function $\beta(x) := \chi(x) - \chi(2x)$ (see [23]). By using the function we obtain the following decomposition:

$$J(\xi) = \sum_{k=k_0}^{\infty} \frac{e^{-i\frac{\xi_2}{4\xi_3^2}}}{|\xi_3|^{1/2}} \int e^{i(\xi_3 x_1^n + \xi_1 x_1)} a\left(x_1, -\frac{\xi_2}{2\xi_3}\right) \beta(2^k x_1) dx_1,$$

where we can take k_0 as large as we wish depending on the support of the amplitude function a (see [14]).

Further, we consider estimate for the oscillatory integral J_k defined by:

$$J_k(\xi) := \frac{e^{-i\frac{\xi_2}{4\xi_3^2}}}{|\xi_3|^{1/2}} \int e^{i(\xi_3 x_1^n + \xi_1 x_1)} a\left(x_1, -\frac{\xi_2}{2\xi_3}\right) \beta(2^k x_1) dx_1.$$

After scaling we obtain:

$$J_k(\xi) := 2^{-k} \frac{e^{-i\frac{\xi_2}{4\xi_3^2}}}{|\xi_3|^{1/2}} \int e^{i(2^{-kn}\xi_3 x_1^n + 2^{-k}\xi_1 x_1)} a\left(2^{-k}x_1, -\frac{\xi_2}{2\xi_3}\right) \beta(x_1) dx_1.$$

First, suppose $|2^{-kn}\xi_3| \lesssim 1$ in this case we use the estimate

$$O\left(\frac{1}{1 + |\xi_1 2^{-k}|}\right)$$

for the integral in x_1. Indeed, if $|2^{-k}\xi_1| \lesssim 1$ then we use a trivial estimate for the integral and otherwise we use integration by parts in x_1. Then we get

$$\int_{0\leq \max\{|\xi_1|,\,|\xi_2|\}\leq|\xi_3|\leq 2^{kn}} |J_k(\xi)|^p d\xi \lesssim 2^{-k(p-1)} \int_1^{2^{kn}} |\xi_3|^{1-p/2} d\xi_3 \lesssim 2^{-k(p-1)+(2-p/2)kn} \lesssim 2^{-\delta k},$$

where $\delta > 0$ is a positive number provided $p > p_n$.

Further, we consider the case $|\xi_3| \gtrsim 2^{kn}$. Let's use notation

$$\sigma_1 := 2^{-k(n-1)} \frac{\xi_1}{\xi_3}.$$

If $|\sigma_1| \gtrsim 1$ then the phase function of the oscillatory integral has no critical points. Therefore, we can use integration by parts in x_1 and obtain

$$|J_k(\xi)| \lesssim \frac{2^{k(n-1)}}{|\xi_3|^{3/2}|\sigma_1|}.$$

Then integration in ξ yields

$$\int_{\{0\leq \max\{|\xi_1|,\,|\xi_2|\}\leq|\xi_3|\}\cap\{|\xi_3|\gtrsim 2^{kn}\}\cap\{|\sigma_1|\gtrsim 1\}} |J_k(\xi)|^p d\xi \lesssim$$

$$2^{-k(p-1)} \int_{2^{kn}}^{\infty} |\xi_3|^{2-3p/2} d\xi_3 \lesssim 2^{-k((2n+1)-p(n+2)/2)} \lesssim 2^{-\delta k},$$

where again $\delta > 0$ provided $p > p_n$.

Finally, we consider the case $|\sigma_1| \lesssim 1$. Note that if $|\sigma_1| << 1$ then the phase function of the oscillatory integral has no critical points. Therefore, by using integration by parts in x_1, we have again the estimate

$$|J_k(\xi)| \lesssim \frac{2^{k(n-1)}}{|\xi_3|^{3/2}}.$$

Then by using integration in ξ over the set

$$\{0 \leq \max\{|\xi_1|,\, |\xi_2|\} \leq |\xi_3|\} \cap \{|\xi_3| \gtrsim 2^{kn}\} \cap \{|\sigma_1| < 1\}$$

we obtain an estimate $O(2^{-\delta k})$.

Thus, we have to consider the case $|\sigma_1| \backsim 1$. In this case the phase function has at most two non-degenerate critical points and we use classical van der Corpute type estimate and obtain:

$$|J_k(\xi)| \lesssim \frac{2^{k(n/2-1)}}{|\xi_3|}.$$

So, we get

$$\int_{\{0\leq \max\{|\xi_1|,\,|\xi_2|\}\leq|\xi_3|\}\cap\{|\xi_3|\gtrsim 2^{kn}\}\cap\{|\sigma_1|\backsim 1\}} |J_k(\xi)|^p d\xi \lesssim 2^{k(p(n/2-1)-(n-1)+(3-p)n)} \lesssim 2^{-\delta k},$$

where again $\delta > 0$ provided $p > p_n$. Thus we can sum up the obtained estimates. This finishes a proof of Lemma 6.1.

□

7. The Four Denominator Estimate

Now, we apply the obtained results to so-called four denominator estimate (see [8]).

Let's consider the function

$$e(p) = \sum_{j=1}^{3} [1 - \cos p_j]. \tag{7.1}$$

It is connected to Schrödinger operator on a lattice. The function $e(p)$ is called to be a dispersion relation of the Laplace operator on a lattice. It can be considered as a smooth (analytic) function on $\mathbb{R}^3$ or T^3.

For any $\alpha \in \mathbb{R}$, $u \in \mathrm{T}^3$ and $\eta > 0$ we define

$$I_{\alpha,\eta}(u) := \int_{(\mathrm{T}^3)^3} \frac{dp^1 dp^2 dp^3}{|\alpha - e(p^1 + p^2 + p^3 - u) + i\eta| \prod_{j=1}^{3} |\alpha - e(p^j) + i\eta|}. \tag{7.2}$$

Our goal in this section is to estimate the integral (7.2) for small η uniformly in u Following [8] we define:

$$|||\alpha||| := \min\{|\alpha|,\, |\alpha - 2|,\, |\alpha - 3|,\, |\alpha - 4|,\, |\alpha - 6|\} \tag{7.3}$$

for any real number α.

The following result improves Theorem 2.4 of the paper [8].

Theorem 7.1. *Let $0 < \eta \le \frac{1}{2}$. For any $\Lambda > \eta$ there exists a positive constant C_Λ such that for any α with $|||\alpha||| \ge \Lambda$ the following estimate*

$$\sup_{u \in \mathrm{T}^3} I_{\alpha,\eta}(u) \le C_\Lambda |\log \eta|^4 \tag{7.4}$$

holds.

Proof. Let's first, remind auxiliary Lemma proved in the paper [8]

Lemma 7.2. *There exists a constant C such that for any $0 < \eta \le \frac{1}{2}$*

$$\sup_{\alpha \in \mathbb{R}} \int_{\mathrm{T}^3} \frac{dp}{|\alpha - e(p) + i\eta|} \le C|\log \eta|. \tag{7.5}$$

Following [8] we fix α, η, Λ and u through the proof. C_Λ and c_Λ will denote large and small universal constants depending only on Λ. We will mostly omit the α and $u-$dependence in the notation, all estimates are uniform for $u \in \mathbb{T}^3$ and $\alpha \in \mathbb{R}$ with $|||\alpha||| \ge \Lambda$.

We know that the range of the function $e(p)$ defined by (7.1) is $[0, 6]$. Let $0 \le \chi(t) \le 1$ be a smooth cutoff function on $[0, 6]$ such that $\chi(t) \equiv 1$ if $|||t||| \ge 2\Lambda/3$ and $\chi(t) \equiv 0$ if $|||t||| \le \Lambda/3$, $\|\partial^\nu \chi\|_{C[0,6]} \le C_\nu \Lambda^{-\nu}$, where $\|.\|_{C[0,6]}$ is a natural norm of the space of continuous functions and $|||.|||$ is defined by (7.3).

We insert $1 \equiv \chi(e(p)) + [1 - \chi(e(p))]$ in the integral (7.2). It is easy to see that [8] on the set where $1 - \chi(e(p)) \neq 0$ we can estimate

$$\frac{1}{|\alpha - e(p) + i\eta|} \leq C\Lambda^{-1},$$

and once one of the denominators is eliminated, the rest can be integrated out at the expense of $C|\log \eta|^3$, see (7.5).

So, we can focus on the term with $\chi(e(p))$. Similarly we can insert $\chi(e(p^1 + p^2 + p^3 - u) \prod_{j=1}^3 \chi(e(p^j))$ as well and we define

$$I = \int \frac{\chi(e(p^1 + p^2 + p^3 - u) \prod_{j=1}^3 \chi(e(p^j)) dp^1 dp^2 dp^3}{|\alpha - e(p^1 + p^2 + p^3 - u) + i\eta| \prod_{j=1}^3 |\alpha - e(p^j) + i\eta|}.$$

Then we have

$$I_{\alpha,\eta}(u) \leq C_\Lambda |\log \eta|^3 + I. \tag{7.6}$$

We set

$$I(\xi) = \int_{\mathrm{T}^3} \frac{e^{ip\xi} \chi(e(p)) dp}{|\alpha - e(p) + i\eta|} = \int_{\mathrm{R}^3} \frac{e^{ip\xi} \chi(e(p)) dp}{|\alpha - e(p) + i\eta|} \tag{7.7}$$

for $\xi \in \mathbb{R}^3$, then clearly $I(\xi) = I(-\xi)$ and it is real, we naturally continue the function $h(p) := \chi(e(p))|\alpha - e(p) + i\eta|^{-1}$ from $\mathrm{T}^3 =: [-\pi, \pi]^3$ to $\mathbb{R}^3$ as zero. Moreover,

$$I = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (I(\xi))^4 e^{-iu\xi} d\xi \leq \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} (I(\xi))^4 d\xi.$$

Actually, the LUB with respect to u of the last function defined by integral coincides with the last integral.

The function $h(p)$ in the oscillatory integral (7.7) is regular on scale η,

$$|\partial^\beta h(p)| \leq \frac{C_{\Lambda,\beta} \eta^{-|\beta|}}{|\alpha - e(p) + i\eta|}$$

for any multi-index β. Thus, by a standard stationary phase estimate and (7.2), we easily see that

$$|I(\xi)| \leq \frac{C_\Lambda |\log \eta|}{\eta |\xi|},$$

therefore

$$I \leq C_\Lambda |\log \eta|^4 + \int_{|\xi| \leq \eta^{-4}} |I(\xi)|^4 d\xi. \tag{7.8}$$

By the co-area formula

$$I(\xi) = \int_0^6 \frac{\chi(a) \hat{\mu}_a(\xi) da}{|\alpha - a + i\eta|}$$

with

$$\hat{\mu}_a(\xi) = \int_{\Sigma_a} \frac{e^{ip\xi}}{|\nabla e(p)|} dm_a(p),$$

where we recall that $dm_a(p)$ is the uniform surface measure on the set

$$\Sigma_a = \{p : e(p) = a\} \subset \mathrm{T}^3.$$

Clearly $\hat{\mu}_a(\xi)$ is an integral of the form (1.1) with $\psi(x) = |\nabla e(p)|^{-1}$. Note that

$$|||a|||^{1/2} \leq C \left(\sum_{j=1}^{3} \sin^2 p_j \right) = C|\nabla e(p)|.$$

Thus for $|||a||| \geq \Lambda/3$, the function $|\nabla e(p)|$ on the set Σ_a is separated away from zero and is smooth with derivatives bounded uniformly in a (depending only on Λ), so $|\nabla e(p)|^{-1}$ is smooth.

Proposition 7.3. *Let* $0 < \Lambda < 1/2$. *For any* a *with* $|||a||| \geq \Lambda$, *we have*

$$\int_{|\xi| \leq \eta^{-4}} |\hat{\mu}_a(\xi)|^4 d\xi \leq C_\Lambda$$

Proof. The proof of Proposition (7.3) follows from observations of the paper [8] and from Theorem (1.1). Actually, it follows that if at some point the Gaussian curvature of the surface $\{p : e(p) = a\}$ (with $|||a||| \neq 0$) vanish then its first differential is nonzero. Then it is easy to see that at least one of the principal curvatures is non-zero and surely Gaussian curvature can not be identically zero. □

Now, from Proposition 7.3 by using Hölder's inequality we get

$$\int_{|\xi| \leq \eta^{-4}} |I(\xi)|^4 d\xi = \int_{|\xi| \leq \eta^{-4}} \left| \int_0^6 \frac{\chi(a)da}{|\alpha - a + i\eta|} \hat{\mu}_a(\xi) \right|^4 d\xi \leq$$

$$\left(\int_0^6 \frac{\chi(a)da}{|\alpha - a + i\eta|} \right)^3 \int_{|\xi| \leq \eta^{-4}} \int_0^6 \frac{\chi(a)da}{|\alpha - a + i\eta|} |\hat{\mu}_a(\xi)|^4 d\xi \leq C_\Lambda |\log \eta|^4.$$

□

Acknowledgments. This research was carried out while the author was visiting ICTP (Trieste, Italy) under the Associateship Scheme. The author would like to thank the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, for kind hospitality and support.

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