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ON THE COMPLETE INTEGRABILITY OF NONLINEAR DYNAMICAL SYSTEMS ON DISCRETE MANIFOLDS WITHIN THE GRADIENT-HOLONOMIC APPROACH

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Abstract

A gradient-holonomic approach for the Lax type integrability analysis of differential-discrete dynamical systems is devised. The asymptotical solutions to the related Lax equation are studied and the related gradient identity is stated. The integrability of a discrete nonlinear Schrödinger type dynamical system is treated in detail. The integrability of a generalized Riemann type discrete hydrodynamical system is discussed.

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1. Preliminary notions and definitions

Consider an infinite dimensional discrete-manifold $M \subset l_2(\mathbb{Z};\mathbb{C}^m)$ for some integer $m \in \mathbb{Z}_+$ and a nonlinear dynamical system on it in the form

$$du/dt = K[u], \tag{1.1}$$

where $u \in M$ and $K : M \to T(M)$ is a Frechet smooth nonlinear local functional on M and $t \in \mathbb{R}$ is

the evolution parameter. As an example of dynamical system (1.1) on a discrete manifold $M \subset l_2(\mathbb{Z};\mathbb{C}^2)$ one can consider the following well-known [5, 10] discrete nonlinear Schrödinger equation:

$$\left.\begin{aligned} d\psi_n/dt = i(\psi_{n+1} - 2\psi_n + \psi_{n-1}) - i\psi_n^*\psi_n(\psi_{n+1} + \psi_{n-1}) \\ d\psi_n^*/dt = -i(2\psi_n^* - \psi_{n+1} - \psi_{n-1}) + i\psi_n^*\psi_n(\psi_{n+1}^* + \psi_{n-1}^*) \end{aligned}\right\} := K_n[\psi,\psi^*], \tag{1.2}$$

having interesting applications in diverse physics plasma investigations.

To study the integrability properties of differential-difference dynamical system (1.1) we will develop below a gradient-holonomic scheme previously devised in [6, 14, 12, 7] for nonlinear dynamical systems defined on spatially one-dimensional functional manifolds and extended in [11] for the case of discrete manifolds.

Denote by $(\cdot,\cdot)$ the standard bi-linear form on the space $T^*(M) \times T(M)$ naturally induced by that existing in the Hilbert space $l_2(\mathbb{Z};\mathbb{C}^m)$. Having denoted by $\mathcal{D}(M)$ smooth functionals on M, for any functional $\gamma \in \mathcal{D}(M)$ one can define the gradient $grad\gamma[u] \in T^*(M)$ as follows:

$$grad\gamma[u] := \gamma^{\prime,*}[u] \cdot 1, \tag{1.3}$$

where the dash-sign "$'$" means the corresponding Frechet derivative and the star-sign " $*$ " means the conjugation naturally related with the bracket on $T^*(M) \times T(M)$.

Definition 1.1. A linear smooth operator $\vartheta : T^*(M) \times T(M)$ is called implectic, on the manifold M, if the bi-linear bracket

$$\{\cdot,\cdot\}_\vartheta := (< grad(\cdot), \vartheta\; grad(\cdot) >) \tag{1.4}$$

is Poissonian [1, 2] on the space of functionals $\mathcal{D}(M)$.

This means, in particular, that bracket (1.4) satisfies the standard Jacobi identity on $\mathcal{D}(M)$.

Definition 1.2. A linear smooth operator $\vartheta : T^*(M) \times T(M)$ is called Nötherian subject to the nonlinear dynamical system (1.1), if the following condition

$$L_K\vartheta = \vartheta' K - \vartheta K^{\prime,*} - K'\vartheta = 0 \tag{1.5}$$

holds identically on the manifold M, where we denoted by L_K the corresponding Lie-derivative along the vector field $K : M \to T(M)$.

Assume now that the mapping $\vartheta : T^*(M) \times T(M)$ is invertible, that is there exists the inverse mapping $\vartheta^{-1} := \Omega : T^*(M) \times T(M)$, called symplectic. It then easily follows from (1.5) that the condition

$$L_K\Omega = \Omega' K + \Omega K' + K^{\prime,*}\Omega = 0 \tag{1.6}$$

holds identically on M. Having now assumed that the manifold $M \subset l_2(\mathbb{Z};\mathbb{C}^2)$ is endowed with a smooth implectic structure $\vartheta : T^*(M) \times T(M)$ one can define the Hamiltonian system

$$du/dt := -\vartheta\; grad\; H[u], \tag{1.7}$$

corresponding to a Hamiltonian function $H \in \mathcal{D}(M)$. As a simple corollary of definition (1.7) one obtains that the bdynamical system

$$-\vartheta\; grad\; H[u] := K[u] \tag{1.8}$$

satisfies the Nötherian conditions (1.5). Keeping in mind the study of the integrability problem [2, 4, 12, 8] subject to discrete dynamical system (1.1), we need to construct a priori given set of invariant functions on M called conservation laws commuting to each other with respect to the Poisson bracket (1.3). The following Lax criterion [3, 6, 12] proves to be very useful.

Lemma 1.3. *Any symmetric with respect to the bracket $(\cdot,\cdot)$ smooth solution $\varphi \in T^*(M)$ to the Lax equation*

$$L_K\ \varphi = d\varphi/dt + K^{\prime,*}\varphi = 0,\ \varphi^{'} = \varphi^{'*} \tag{1.9}$$

corresponds to the conservation law

$$\gamma := \int_0^1 d\lambda(\varphi[u\lambda], u). \tag{1.10}$$

The expression above is easily obtained from the well-known Volterra homology expressions:

$$\gamma = \int_0^1 \frac{d\gamma[u\lambda]}{d\lambda} d\lambda = \int_0^1 d\lambda(\gamma^{'}[u\lambda]\cdot u, 1) = \int_0^1 d\lambda(\gamma^{'*}[u\lambda]\cdot 1, u) = \int_0^1 d\lambda(grad\ \gamma[u\lambda], u) \tag{1.11}$$

and

$$(grad\gamma[u])^{'} = (grad\gamma[u])^{'*}, \tag{1.12}$$

holding identically on M, since then one ensures that there exists a function $\gamma \in \mathcal{D}(M)$ such that

$$L_K\gamma = 0 \text{ and } grad\gamma[u] = \varphi[u] \tag{1.13}$$

for any $u \in M$. The Lax lemma naturally arises from the following generalized Nötherian type lemma.

Lemma 1.4. *Let a smooth element $\varphi \in T^*(M)$ satisfy the Nötherian condition*

$$L_K\varphi = d\varphi/dt + K^{\prime,*}\varphi = \nabla\mathcal{L}_\varphi, \tag{1.14}$$

where we denoted $\nabla\mathcal{L}_\varphi := grad\ \mathcal{L}_\varphi$ for some smooth functional $\mathcal{L}_\varphi \in \mathcal{D}(M)$. Then the following Hamiltonian representation

$$K = -\vartheta\ grad\ H_\vartheta, \tag{1.15}$$

where

$$\vartheta := \varphi^{'} - \varphi^{'*},\ H_\vartheta := (\varphi, K) - \mathcal{L}_\varphi, \tag{1.16}$$

and the Hamiltonian function $H_\vartheta \in \mathcal{D}(M)$.

It is easy to see that Lemma (1.3) follows from Lemma (1.4) if the conditions $\varphi^{'} = \varphi^{'*}$ and $\mathcal{L}_\varphi = 0$ are imposed in (1.16).

Assume now that equation (1.14) allows an additional not symmetric smooth solution $\phi \in T^*(M)$:

$$L_K\phi = d\phi/dt + K^{\prime,*}\phi = grad\ \mathcal{L}_\phi. \tag{1.17}$$

This means that our system (1.1) is bi-Hamiltonian:

$$-\vartheta grad H_\vartheta = K = -\eta\ grad\ H_\eta, \tag{1.18}$$

where, by definition,

$$\eta := \phi^{'} - \phi^{'*},\quad H_\eta = (\phi, K) - \mathcal{L}_\phi. \tag{1.19}$$

Definition 1.5. One says that two implectic structures $\vartheta, \eta : T^*(M) \to T(M)$ on M are compatible [9, 17, 6, 8] , if for any $\lambda, \mu \in \mathbb{R}$ the linear combination $\lambda\vartheta + \mu\eta : T^*(M) \to T(M)$ will be too implectic on M.

It is easy to derive that this condition is satisfied if, for instance, there exists the inverse operator $\vartheta^{-1} : T(M) \to T^*(M)$ and the expression $\eta(\vartheta^{-1}\eta) : T^*(M) \to T(M)$ is also implectic on M.

Concerning the integrability problem posed for the infinite-dimensional dynamical system (1.1) on the discrete manifold M it is, in general, necessary, but not enough [4, 6, 12], to prove the existence of an infinite hierarchy of commuting to each other with respect to the Poissonian structure (1.3) conservation laws.

Since in the case of the Lax type integrability almost of (1.1) there exist compatible implectic structures and related hierarchies of conservation laws, we will further constrain our analysis by devising an integrability algorithm under the a priori assumption thata given nonlinear dynamical system (1.1) on the manifold M is Lax type integrable. This means that it possesses a related Lax type representation in the following, generally written form:

$$\triangle f_n := f_{n+1} = l_n[u;\lambda] f_n, \tag{1.20}$$

where $f := \{f_n \in \mathbb{C}^r : n \in \mathbb{Z}\} \subset l_2(\mathbb{Z};\mathbb{C}^r)$ for some integer $r \in \mathbb{Z}_+$ and $l_n[u;\lambda] \in End\mathbb{C}^r$ in (1.20) is a local matrix-valued functional on M, depending on the "spectral" parameter $\lambda \in \mathbb{C}$, invariant with respect to our dynamical system (1.1).

Taking into account that the Lax representation (1.20) is "local" with respect to the discrete variable $n \in \mathbb{Z}$, we will assume for convenience, that our manifold $M := M_{(N)} \subset l_\infty(\mathbb{Z};\mathbb{C}^m)$ is periodic with respect to the discrete index $n \in \mathbb{Z}$, that is for any $n \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$

$$l_n[u;\lambda] = l_{n+N}[u;\lambda] \tag{1.21}$$

for some integer $N \in \mathbb{Z}_+$. In this case the smooth functionals on $M(N)$ can be represented as

$$\gamma := \sum_{n=0}^{N-1} \gamma_n[u] \tag{1.22}$$

for some local "Frechet smooth densities" $\gamma_n : M_{(N)} \to \mathbb{C}$, $n = \overline{0, N-1}$.

2. The integrability analysis: gradient-holonomic scheme

Consider the representation (1.20) and define its fundamental solution $F_{n,n_0}^{(\lambda)} \in Aut(\mathbb{C}^r)$, $n \in \mathbb{Z}$, satisfying the condition $F_{n,n_0}^{(\lambda)}\Big|_{n=n_0} = \mathbf{1}$ for some arbitrarily fixed index $n_0 \in \mathbb{Z}$ and all $\lambda \in \mathbb{C}$:

$$F_{n+1,n_0}^{(\lambda)} = l_n[u;\lambda] F_{n,n_0}^{(\lambda)}. \tag{2.1}$$

Then the matrix function

$$S_{n_0}(\lambda) := F_{n_0+N,n_0}(\lambda) \in Aut(\mathbb{C}^r) \tag{2.2}$$

is called the monodromy matrix for the linear equation (1.21) and satisfies at $n_0 = n \in \mathbb{Z}$ the following Novikov-Lax type relationship:

$$S_{n+1}(\lambda) l_n = l_n S_n(\lambda). \tag{2.3}$$

It is easy to compute that $S_n(\lambda) := \prod_{k=\overleftarrow{0,N-1}} l_{n+k}[u;\lambda]$ owing to the periodical condition (1.21). Construct the following generating functional:

$$\overline{\gamma}(\lambda) := tr S_n(\lambda) \tag{2.4}$$

and assume that there exists its asymptotical expansion

$$\overline{\gamma}(\lambda) \simeq \sum_{j\in\mathbb{Z}_+} \overline{\gamma}_j \lambda^{-j+j_0} \tag{2.5}$$

as $|\lambda| \to \infty$ for some fixed $j_0 \in \mathbb{Z}_+$. Then, owing to the evident condition

$$\frac{d}{dn}\overline{\gamma}(\lambda) = 0 \tag{2.6}$$

for all $n \in \mathbb{Z}$, where we put, by definition, the "discrete" derivative

$$d/dn := \triangle - 1, \tag{2.7}$$

we obtain that all functionals $\overline{\gamma}_j \in \mathcal{D}(M_{(N)})$, $j \in \mathbb{Z}_+$, are independent of the discrete index $n \in \mathbb{Z}$.

Assume now, additionally, that the following natural condition holds: the gradient vector

$$\varphi(\lambda) := grad\ \ \overline{\gamma}(\lambda) = tr(l^{'*} S_n(\lambda)) \tag{2.8}$$

satisfies, owing to (2.3), for all $\lambda \in \mathbb{C}$ the next "spectral"-gradient relationship:

$$z(\lambda)\vartheta\varphi(\lambda) = \eta\varphi(\lambda), \tag{2.9}$$

where $z : \mathbb{C} \to \mathbb{C}$ is some meromorphic mapping, ϑ and $\eta : T^*(M_{(N)}) \to T(M_{(N)})$ are compatible implectic on the manifold $M_{(N)}$ and Nötherian operators with respect to the dynamical system (1.1).

As a corollary of the above condition one easily follows that the generating functional $\overline{\gamma} \in \mathcal{D}(M_{(N)})$ is a conservation law to (1.1), there by giving rise, owing to (2.5), to the existence of the infinite hierarchy of conservation laws $\overline{\gamma}_j \in \mathcal{D}(M_{(N)})$, $j \in \mathbb{Z}_+$, on the periodic manifold $M_{(N)}$.

The latter property, being characteristic concerning the Lax type integrability of the nonlinear dynamical system (1.1), can be effectively implemented into the direct scheme of our analysis if we take into account the defining relationships (1.20) and (2.2). Namely, the following theorem holds.

Theorem 2.1. . *The Lax equation (1.9) allows the following asymptotical as* $|\lambda| \to \infty$ *periodical solution* $\varphi(\lambda) \in T^*(M_{(N)})$*, where for all* $n \in \mathbb{Z}$

$$\varphi_n(\lambda) \simeq a_n(\lambda) \exp[\omega(t;\lambda)] \prod_{j=0}^{n} \sigma_j(\lambda), \tag{2.10}$$

$$\begin{aligned} a_n(\lambda) \quad &: \quad = (1, a_{(1),n}[u;\lambda], a_{(2),n}[u;\lambda], ..., a_{(m-1),n}[u;\lambda])^\tau, \\ a_{(k),n}(\lambda) \quad &\simeq \quad \sum_{s\in\mathbb{Z}_+} a^{(s)}_{(k),n}[u]\lambda^{-s+\tilde{a}}, \sigma_j(\lambda) \simeq \sum_{s\in\mathbb{Z}_+} a^{(s)}_j[u]\lambda^{-s+\tilde{\sigma}}, \end{aligned} \tag{2.11}$$

$k = \overline{1, m-1}$ *and* $\omega(t;\cdot) : \mathbb{C} \to \mathbb{C}$, $t \in \mathbb{R}$, *is some dispersion function. Moreover, the functional* $\gamma(\lambda) := \sum_{n=\overline{0,N}} \ln \sigma_n[u;\lambda] \in \mathcal{D}(M_{(N)})$ *is a generating function of conservation laws to the dynamical system (1.1).*

Proof. The condition $L_K\vartheta = 0 = L_K\eta$ gives rise to the equality $L_K\varphi = 0$, coinciding with (1.9). The latter, owing to Lemma (1.3) and (2.8) means that the functional (2.4) is a conservation law to our dynamical system (1.1). Based now on the expression (2.2) and equation (1.20)we arrive at the solution representation (2.10) to the Lax equation (1.9). Making use of the periodicity of the solution (2.10) we obtain that the functional

$$\gamma(\lambda) := \sum_{n=\overline{0,N}} \ln \sigma_j[u;\lambda] \in \mathcal{D}(M_{(N)}) \tag{2.12}$$

is a conservation law to (1.1), finishing the proof. □

Thus, if we begin the Lax type integrability analysis of a priori given nonlinear dynamical system (1.1), it is necessary, as the first step, to study the asymptotical solutions (2.10) to the corresponding Lax equation (1.9) and construct a related hierarchy of conservation laws in the form

$$\gamma_j := \sum_{j\in\mathbb{Z}_+} \gamma_j[u]\lambda^{-j}, \tag{2.13}$$

taking into account the expansions of (2.11) and definition (2.12).

Remark 2.2. It is easy to observe that owing to the arbitrariness of the period $N \in \mathbb{Z}_+$ of the manifold $M_{(N)}$, all of the finite-sum expressions obtained above are generalized to the corresponding infinite dimensional manifold $M \subset l_2(\mathbb{Z};\mathbb{C}^m)$, if the corresponding infinite series persist to be convergent.

Since our dynamical system (1.1) under the above conditions is a bi-Hamiltonian flow on the manifold $M_{(N)}$, as the next step we need to find the related compatible implectic (or symplectic) structures, satisfying, respectively, the equality (1.5) or (1.6). Before doing this, we need to formulate the following lemma.

Lemma 2.3. *All functionals (2.13) are commuting to each other with respect to both implectic structures* $\vartheta, \eta : T^*(M_{(N)}) \to T(M_{(N)})$*, satisfying the "spectral"-gradient relationship (2.14).*

Proof. Based on the representations (2.10) and (2.8) one obtains that there holds the asymptotical as $|\lambda| \to \infty$ relationship

$$\ln \overline{\gamma}(\lambda) \simeq \gamma(\lambda). \tag{2.14}$$

Since the generative function of conservation laws $\overline{\gamma}(\lambda) \in \mathcal{D}(M_{(N)})$ is, owing to the "spectral"-gradient relationship (2.9) and the Poisson bracket definition (1.3), commuting to each other'for any parameter $\lambda \in \mathbb{C}$, the same, evidently, holds, owing to the relationship (2.14), for the generalizing function $\gamma(\lambda) \in \mathcal{D}(M_{(N)})$, finishing the proof. □

Proceed now to constructing the related to dynamical system (1.1) implectic structures $\vartheta, \eta : T^*(M_{(N)}) \to T(M_{(N)})$.Note here, that these implectic structures are Nötherian also for the whole hierarchy of dynamical systems

$$du/dt_j := -\vartheta \; grad \; \gamma_j := K^{(j)}[u], \tag{2.15}$$

where $t_j \in \mathbb{R}, j \in \mathbb{Z}_+$,are the corresponding evolution parameters, and which commute to each other on the manifold $M_{(N)}$. The latter makes it possible to apply Lemma (1.4) to any of the dynamical systems (2.15), if the related vector fields commuting with (1.1) are supposed to be found before.

To solve analytically equation (1.14) subject to an element $\varphi \in T^*(M_{(N)})$, we can, in the case of a polynomial dynamical system (1.1), make use of the well known asymptotical small parameter method [14]. Applying this approach it is necessary to take into account the following expansions at zero - element $\overline{u} = 0 \in M_{(N)}$ with respect to the small parameter $\mathbb{C} \ni \mu \to 0$:

$$
\begin{aligned}
u &: = \mu u^{(1)},\ \varphi[u] = \varphi^{(0)} + \mu\varphi^{(1)}[u] + \mu^2\varphi^{(2)}[u] + ...,\\
d/dt &= d/dt_0 + \mu d/dt_1 + \mu^2 d/dt_2 + ...,\\
K[u] &= \mu K^{(1)}[u] + \mu^{(2)}K^{(2)}[u] + ...,\\
K'[u] &= K'_0 + \mu K'_1[u] + \mu^2 K'_2[u] + ...,\\
\nabla L[u] &= \nabla L^{(0)} + \mu\nabla L^{(1)}[u] + \mu^2\nabla L^{(2)}[u] + ...
\end{aligned}
\tag{2.16}
$$

having solved the corresponding linear nonuniform functional equations

$$
\begin{aligned}
d\varphi^{(0)}/dt_0 + K_0'^{*}\varphi^{(0)} &= \nabla L^{(0)},\\
d\varphi^{(1)}/dt_0 + K_0'^{*}\varphi^{(1)} &= \nabla L^{(1)} - K_0'^{*}\varphi^{(0)},\\
d\varphi^{(2)}/dt_0 + K_0'^{*}\varphi^{(2)} &= \nabla L^{(2)} - K_1'^{*}\varphi^{(1)} - K_2'^{*}\varphi^{(0)}
\end{aligned}
\tag{2.17}
$$

by means the standard Fourier transform applied to the suitable N-periodical functions, one can obtain the related implectic structure as follows

$$
\vartheta^{-1} = \varphi^{(0),\prime} - \varphi^{(0),\prime *} + \mu(\varphi^{(1),\prime} - \varphi^{(1),\prime *}) + ... \tag{2.18}
$$

where it is necessary put in (2.18) $\mu = 1$.

Another direct way of obtaining an implectic operator $\vartheta : T^*(M_{(N)}) \to T(M_{(N)})$ for (1.1) is to solve by means of the same asymptotical small parameter approach the Nötherian equation (1.5), having reduced it to the following set of linear nonuniform equations:

$$
\begin{aligned}
\frac{d}{dt_0}(\vartheta_0\varphi^{(0)}) &= K'_0(\vartheta_0\varphi^{(0)}),\\
\frac{d}{dt_0}(\vartheta_1\varphi^{(0)}) &= K'_0(\vartheta_1\varphi^{(0)}) + \vartheta_0 K_1'^{,*}\varphi^{(0)} + K'_1\vartheta_0\varphi^{(0)},\\
\frac{d}{dt_0}(\vartheta_2\varphi^{(0)}) &= K'_0(\vartheta_2\varphi^{(0)}) - \varphi^{(0)\prime}K^1 + \vartheta_0 K_2'^{,*}\varphi^{(0)} +\\
&\quad +\vartheta_1 K_1'^{,*}\varphi^{(0)} + \vartheta_2 K_0'^{,*}\varphi^{(0)} + K'_1\vartheta_1\varphi^{(0)} + K'_2\vartheta_0\varphi^{(0)}.
\end{aligned}
\tag{2.19}
$$

Based now on the analytical expressions for actions $\vartheta_j : \varphi^{(0)} \to \vartheta_j\varphi^{(0)}$, $j \in \mathbb{Z}_+$, one can easily retrieve them in operator form, giving rise to the implectic operator

$$
\vartheta = \vartheta_0 + \mu\vartheta_1 + \mu^2\vartheta_2 + ... \tag{2.20}
$$

if to put in (**??**) $\mu = 1$. Similarly one can construct the second implectic operator $\eta : T^*(M_{(N)}) \to T(M_{(N)})$ for the nonlinear dynamical system (1.1).

Resuming up all this analysis described above, we can formulate the following proposition.

Proposition 2.4. *Let a nonlinear dynamical system (1.1) on a discrete manifold M allow a nontrivial asymptotical as $|\lambda| \to \infty$ symmetric solution $\varphi \in T^*(M)$ to the Lax equation (1.9) in the form (2.10), generating an infinite hierarchy of nontrivial functionally independent conservation laws (2.12). Then this dynamical system is a Lax type integrable bi-Hamiltonian flow on M with respect to two compatible implectic structures $\vartheta, \eta : T^*(M) \to T(M)$, whose adjoint Lax type representation*

$$
d\Lambda/dt = [K'^{,*}, \Lambda], \tag{2.21}
$$

where $\Lambda := \vartheta^{-1}\eta$ is the so-called recursion operator, can be transformed to the standard discrete Lax type form as

$$
\frac{dl_n}{dt} = [p_n(l), l_n] + \frac{d}{dn}p_n(l) \tag{2.22}
$$

for all $n \in \mathbb{Z}$, where $p_n(l) \in End\mathbb{C}^r$ gives rise to the associated with the discrete linear Lax type problem (1.20) temporal evolution equation

$$
df_n/dt = p_n(l)f_n \tag{2.23}
$$

for $f_n \in C^\infty(\mathbb{R};\mathbb{C}^r)$, $n \in \mathbb{Z}$.

Remark 2.5. Based on the property that all Hamiltonian flows (2.15) commute to each other and with dynamical system (1.1) and using the fact that they possess the same implectic and compatible (ϑ,η)-pair, the analytical algorithm described above can be successfully applied to any other flow, commuting with (1.1).

Concerning the discrete linear Lax type problem (1.20), it can be constructed by means of the gradient-holonomic algorithm, first devised in [6, 7, 12] for studying the integrability nonlinear dynamical sytems on functional manifolds. Namely, making use of the preliminary found analytical expressions for the related compatible implectic structures $\vartheta,\eta : T^*(M) \to T(M)$ on the manifold M and using the fact that the recursion operator $\Lambda := \vartheta^{-1}\eta : T^*(M) \to T^*(M)$ satisfies the dual Lax type commutator equality

$$d\Lambda/dt = [\Lambda, K^{\prime,*}], \tag{2.24}$$

one can retrieve the standard "spectral" Lax representation for (2.24) by means of sutably derived algebraic relationships. As a corollary of Proposition (2.4), formulated above, one can claim that the existence of a nontrivial asymptotic as $|\lambda| \to \infty$ solution to the Lax equation (1.9) can serve as an effective Lax type integrability criterion for a given nonlinear dynamical system (1.1) on the manifold M.

3. The Bogoyavlensky-Novikov finite-dimensional reduction

Assume that our dynamical system (1.1) on the periodic manifold $M_{(N)}$ is Lax type integrable and possesses two compatible implectic structures $\vartheta,\eta : T^*(M_{(N)}) \to T(M_{(N)})$. Thus, we have the nonlinear finite-dimensional dynamical system

$$du_n/dt := K_n[u] = -\vartheta\ grad\ H_n[u] \tag{3.1}$$

for indexes $n = \overline{0, N-1}$ owing to its $N-$periodicity. The finite dimensional dynamical system (3.1) can be equivalently considered as that on the finite-dimensional space $M_{(N)} \simeq (\mathbb{C}^m)^N$ parametrized by integer index $n \in \mathbb{Z}$, and whose Liouville integrability analysis is our next task. To proceed with the study of the flow (3.1) on the manifold $M_{(N)}$, we will make use of the Bogoyavlensky-Novikov [15, 4] reduction scheme, first adapted for discrete dynamical systems in [11, 6].

Let $\Lambda(M_{(N)}) := \underset{j\in\mathbb{Z}_+}{\oplus} \Lambda^j(M_{(N)})$ be the standard Grassmann algebra [2, 6, 12] of differential forms on the manifold $\bar{M}_{(N)}$. Then the following differential complex

$$\Lambda^0(M_{(N)}) \stackrel{d}{\to} \Lambda^1(M_{(N)}) \stackrel{d}{\to} ... \stackrel{d}{\to} \Lambda^j(M_{(N)}) \stackrel{d}{\to} \Lambda^{j+1}(M_{(N)}) \stackrel{d}{\to} ...., \tag{3.2}$$

where $d : \Lambda(M_{(N)}) \to \Lambda(M_{(N)})$ is the external differentiation, is finite and exact. Since the discrete "derivative" $D_n := \Delta - 1$ commutes with the differentiation $d : \Lambda(M_{(N)}) \to \Lambda(M_{(N)})$, $[D_n, d] = 0$ for all $n = \overline{0, N-1}$, and for any element $a \in \Lambda^0(M_{(N)})$

$$grad(\sum_{n=0}^{N-1} \frac{d}{dn} a_n[u]) = 0 \tag{3.3}$$

one can formulate the following Gelfand-Dikiy type [16] lemma.

Lemma 3.1. *4.1 Let $\mathcal{L}[u] \in \Lambda^0(M_{(N)})$ be a Frechet smooth local Lagrangian functional on the manifold $M_{(N)}$. Then there exists a 1-form $\alpha^{(1)} \in \Lambda^1(M_{(N)})$ such that the equality*

$$d\mathcal{L}_n[u] =< grad\ \mathcal{L}_n[u], du_n > + D_n\alpha^{(1)}[u] \tag{3.4}$$

is satisfied for all $n \in \mathbb{Z}$.

Proof. We can easily observe that

$$\begin{aligned} d\mathcal{L}_n[u] &= \sum_{n=0}^{N-1} < \frac{\partial \mathcal{L}_n[u]}{\partial u_{n+j}}, du_{n+j} >= \sum_{n=0}^{N-1} < \frac{\partial \mathcal{L}_n[u]}{\partial u_{n+j}}, \Delta^j du_n >= \\ &= < \sum_{n=0}^{N-1} \Delta^{-j} \frac{\partial \mathcal{L}_n[u]}{\partial u_{n+j}}, du_n > + D_n \left(\sum_{n=0}^{N-1} < p_j, du_{n+j} > \right), \end{aligned} \tag{3.5}$$

where, by definition, for $j=\overline{0,N-1}$

$$p_j := \sum_{k=0}^{N-1} \Delta^{-k} \frac{\partial \mathcal{L}_n[u]}{\partial u_{n+j+k+1}}. \tag{3.6}$$

Having denoted the expression

$$grad \mathcal{L}_n[u] := \sum_{j=0}^{N-1} \Delta^{-j} \frac{\partial \mathcal{L}_n[u]}{\partial u_{n+j}}, \tag{3.7}$$

one obtains the result (3.4), where

$$\alpha^{(1)}[u] := \sum_{j=0}^{N-1} < p_j, du_{n+j} > \tag{3.8}$$

is the corresponding 1-form on the manifold $M_{(N)}$. □

Applying now the $d-$differentiation to expression (3.4) we obtain that

$$-D_n \omega^{(2)}[u] =< d\ grad\ \mathcal{L}_n[u], \wedge du_n > \tag{3.9}$$

for any $n \in \mathbb{Z}$, where the 2-form

$$\omega^{(2)}[u] := d\alpha^{(1)}[u] \tag{3.10}$$

is nondegenerate on $M_{(N)}$, if the Hessian $\partial_n^2 \mathcal{L}[u]/\partial^2 u_n$ is nondegenerate as well.

Assume now that the submanifold

$$\bar{M}_{(N)} := \{grad\ \mathcal{L}_n[u] = 0;\ u \in M\}, \tag{3.11}$$

where, by definition, the Lagrangian functional

$$\mathcal{L} := -\gamma_{N+1} + \sum_{k=0}^{N} c_j, \gamma_j \tag{3.12}$$

and $\gamma_k \in \mathcal{D}(M)$, $k=\overline{0,N+1}$, are the suitable nontrivial conservation laws for the dynamical system (1.1), constructed before. As a result of (3.11) and (3.9) we obtain that the closed 2-form $\omega_n^{(2)} \in \Lambda^2(\bar{M}_{(N)})$ is invariant with respect to the index $n \in \mathbb{Z}$ on the manifold $\bar{M}_{(N)}$. Moreover, the submanifold (3.11) is also invariant with respect to both the index $n \in \mathbb{Z}$ and the evolution parameter $t \in \mathbb{R}$. Really, for any $n \in \mathbb{Z}$ the Lie derivative

$$L_K grad \mathcal{L}_n = (grad\ \mathcal{L}_n)' K + K'^{,*}(grad\ \mathcal{L}_n) = 0, \tag{3.13}$$

since owing to the functional $\mathcal{L} \in \mathcal{D}(M_{(N)})$ is a conservation law for the dynamical system (1.1), whose gradient satisfies the Lax condition (1.9). Applying now the Lie derivative L_K to the expression $D_n\ grad\ \mathcal{L}_n[u], n \in \mathbb{Z}$, we easily obtain, owing to (3.13), that if $D_n\ grad\ \mathcal{L}_n[u] = 0, n \in \mathbb{Z}$, at $t=0$, then

$$D_n\ grad\ \mathcal{L}_n[u] = 0 \tag{3.14}$$

for all $t \in \mathbb{R}$ and $n \in \mathbb{Z}$. Thus, we have reduced via Bogoyavlensky-Novikov the initially Lax type given integrable dynamical system (1.1) upon the invariant submanifold $\bar{M}_{(N)}$.

A question arises: how are the dynamical system (1.1), reduced upon the submanifold $M_{(N)}$,and the dynamical system (1.1), reduced upon the finite dimensional submanifold $\bar{M}_{(N)} \subset M$, related. To analyze these reductions we will consider the following equality:

$$< grad\ \mathcal{L}_n[u], K_n[u] >= -D_n h^{(t)}[u], \tag{3.15}$$

for some local functional $h^{(t)}[u] \in \Lambda^0(M)$, following from conditions (3.3) and (1.9):

$$\begin{aligned} grad \quad < \quad & grad\ \mathcal{L}_n[u], K_n[u] >= (grad\ \mathcal{L}_n[u])'^{,*} K_n[u] + K_n'^{,*}[u] grad\ \mathcal{L}_n[u] = \\ = \quad & (grad\ \mathcal{L}_n[u])' K_n[u] + K_n'^{,*}[u] grad\ \mathcal{L}_n[u] = L_K\ grad\ \mathcal{L}_n[u] = 0, \end{aligned} \tag{3.16}$$

giving rise to (3.15). Since on the submanifold $\bar{M}_{(N)}$ the gradient $grad\ \mathcal{L}_n[u] = 0$ for all $n \in \mathbb{Z}$, we obtain from (3.15) that the local functional $h^{(t)}[u] \in \Lambda^0(\bar{M}_{(N)})$ does not depends on the index $n \in \mathbb{Z}$.

The properties of the manifold $\bar{M}_{(N)}$, described above, make it possible to consider it as a symplectic manifold endowed with the symplectic structure $\omega^{(2)} \in \Lambda^2(\bar{M}_{(N)})$, given by expressions (3.8) and (3.10).

From this point of view we can proceed to study the integrability properties of the dynamical system (1.1) reduced upon the invariant finite-dimensional manifold $\bar{M}_{(N)} \subset M$.

First, we observe that the vector field d/dt on $\bar{M}_{(N)}$ is canonically Hamiltonian with respect to the symplectic structure $\omega^{(2)} \in \Lambda^2(\bar{M}_{(N)})$:

$$-i_{\frac{d}{dt}}\omega^{(2)}(u,p) = dh^{(t)}(u,p), \tag{3.17}$$

where $h^{(t)}(u,p) := h^{(t)}(u)$ and $(u,p)^\tau \in \bar{M}_{(N)}$ are canonical variables induced on the manifold $\bar{M}_{(N)}$ by the Liouville 1-form (3.8). Really, from expression (3.15) one obtains that

$$di_{\frac{d}{dt}} < grad\ \mathcal{L}_n[u], du_n >= -D_n dh^{(t)}[u],$$

which, being supplemented with the identity (3.9) in the form

$$i_{\frac{d}{dt}} d < grad\ \mathcal{L}_n[u], du_n >= -D_n i_{\frac{d}{dt}}\omega^{(2)}[u],$$

gives rise to the following:

$$\frac{d}{dt} < grad\ \mathcal{L}_n[u], du_n >= -D_n(dh^{(t)}[u] + i_{\frac{d}{dt}}\omega^{(2)}[u]), \tag{3.18}$$

Since $grad\ \mathcal{L}_n[u] = 0$ on $\bar{M}_{(N)}$, from (3.18) one follows the result (3.17).

The same statement one can claim subject to any of Hamiltonian systems (2.15), commuting with (1.1) on the manifold M. Moreover, owing to the functional independence of invariants $\gamma_j \in \mathcal{D}(M_{(N)})$, $j = \overline{0, N+1}$, entering into the Lagrangian functional (3.12), we can construct the set of functionally independent functions $h_j \in \mathcal{D}(\bar{M}_{(N)})$, $j = \overline{0, N}$, as follows:

$$< grad\ \mathcal{L}_n[u], \vartheta\ grad \gamma_j[u] >= \frac{dh_j[u]}{dn}, \tag{3.19}$$

It is easy to check that these functions $h_j \in \mathcal{D}(\bar{M}_{(N)}), j = \overline{0, N}$, are commuting to each other and to the Hamiltonian function $h^{(t)} \in \mathcal{D}(\bar{M}_{(N)})$ with respect to the symplectic styructure $\omega^{(2)} \in \Lambda^2(\bar{M}_{(N)})$. Thus, if the dimension $\dim \bar{M}_{(N)} = 2(N+1)$, the discrete dynamical system (1.1), when reduced on the finite-dimemsional sub-manifold $\bar{M}_{(N)} \subset M$, will be Liouville integrable.

4. Example: the differential-difference nonlinear Schrödinger dynamical system and its integrability

The differential-difference nonlinear Schrödinger dynamical system given by (1.2) is defined on the discrete infinite-dimensional manifold $M \subset l_2(\mathbb{Z}; \mathbb{C}^2) \subset l_\infty(\mathbb{Z}; \mathbb{C}^2)$. Its Lax type integrability was stated in [5, 10, 13] making use of the simplest discretization of the standard Zakharov-Shabat spectral problem for the well-known nonlinear Schrödinger equation.

In this Section we will try to apply the gradient-holonomic integrability analysis devised above to the system (1.2). From the very beginning we will show the existence of an infinite hierarchy of functionally independent conservation laws, having solved the determining Lax equation (1.9) in the asymptotical form (2.10). The following lemma holds.

Lemma 4.1. *The expression*

$$\varphi_n := \begin{pmatrix} 1 \\ a_n(\lambda) \end{pmatrix} \prod_{j=0}^{n} \sigma_j(\lambda) \exp[it(2-\lambda-\lambda^{-1})], \tag{4.1}$$

where, by definition,

$$\begin{aligned} \sigma_j(\lambda) &\simeq \frac{\lambda}{h_j[\psi^*,\psi]}(1 - \sum_{s\in\mathbb{Z}_+} \sigma_j^{(s)}[\psi^*,\psi]\lambda^{-s-1}), \\ a_n(\lambda) &\simeq \sum_{s\in\mathbb{Z}_+} a_n^{(s)}[\psi^*,\psi]\lambda^{-s}, \end{aligned} \tag{4.2}$$

is an asymptotical solution as $|\lambda\} \to \infty$ to the Lax determining equation (1.9)

$$d\varphi_n/dt + K^{\prime,*}\varphi_n = 0 \tag{4.3}$$

for all $n \in \mathbb{Z}$ with the operator $K'^{,*} : T^*(M) \to T^*(M)$ of the form:

$$K'^{,*} = \begin{pmatrix} i\Delta^{-1}D_n^2 - i\psi_n^*(\psi_{n+1} + \psi_{n-1}) - \\ -i(\Delta + \Delta^{-1}) \cdot \psi_n^*\psi_n & i\psi_n^*(\psi_{n+1}^* + \psi_{n-1}^*) \\ -i\psi_n(\psi_{n+1} + \psi_{n-1}) & -i\Delta^{-1}D_n^2 + i\psi_n(\psi_{n+1}^* + \psi_{n-1}^*) + \\ & +i(\Delta + \Delta^{-1}) \cdot \psi_n^*\psi_n \end{pmatrix}. \tag{4.4}$$

Proof. To prove Lemma 4.1 it is enough to find the corresponding coefficients of the asymptotical expansions (4.2). To do this we will consider the following equations easily obtained from (1.9):

$$\begin{gathered} D_n^{-1}[-\ln h_n + \ln(1 -_{s\in\mathbb{Z}_+} \sigma_n^{(s)}\lambda^{-s-1})]_t + \\ +i\lambda[h_{n+1}^{-1}(1 - \psi_n^*\psi_n)(1 -_{s\in\mathbb{Z}_+} \sigma_n^{(s)}\lambda^{-s-1}) - 1] + \\ +\frac{i}{\lambda}\left[\frac{(1-\psi_{n-1}^*\psi_{n-1})h_n}{(1-_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})} - 1\right] - i\psi_n^*(\psi_{n+1} + \psi_{n-1}) + \\ +i\psi_n^*(\psi_{n+1}^* + \psi_{n-1}^*)_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s}; \end{gathered} \tag{4.5}$$

$$\begin{gathered} (_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s})D_n^{-1}[-\ln h_n + \ln(1 -_{s\in\mathbb{Z}_+} \sigma_n^{(s)}\lambda^{-s-1})]_t + +4i(_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s}) + \\ + \left[i\lambda h_{n+1}(\psi_{n+1}^*\psi_{n+1} - 1)(_{s\in\mathbb{Z}_+} a_{n+1}^{(s)}\lambda^{-s})(_{s\in\mathbb{Z}_+} a_{n+1}^{(s)}\lambda^{-s}) -_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s}\right] + \\ +\frac{d}{dt}{}_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s} + +\frac{i}{\lambda}\left[\frac{(\psi_{n-}^*\psi_{n-1} - 1)(_{s\in\mathbb{Z}_+} a_{n+1}^{(s)}\lambda^{-s})h_n}{(1-_{s\in\mathbb{Z}_+}\sigma_n^{(s)}\lambda^{-s-1})} -_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s}\right] + \\ -i\psi_n(\psi_{n+1} + \psi_{n-1}) + i\psi_n(\psi_{n+1}^* + \psi_{n-1}^*)_{s\in\mathbb{Z}_+} a_n^{(s)}\lambda^{-s}. \end{gathered}$$

Having equated the coefficients of (4.5) at the same degrees of the parameter $\lambda \in \mathbb{C}$, we obtain step by step the functional expression expression for $h_n, \sigma_n^{(s)}$ and $a_n^{(s)}, n \in \mathbb{Z}, s \in \mathbb{Z}_+$:

$$\begin{aligned} h_n &= (1 - \psi_n^*\psi_n), a_n^{(0)} = 0, a_n^{(1)} = \beta, \\ \sigma_n^{(0)} &= \psi_{n-1}^*(\psi_n + \psi_{n-2}) - i\Delta^{-1}D_n^2(\ln h_n)_t, \\ \sigma_n^{(1)} &= i\frac{d}{dt}\sigma_{n-1}^{(0)} + (h_{n-1}h_{n-2} - 1) + a_{n-1}^{(1)}\psi_{n-1}^*(\psi_n + \psi_{n-2}), \\ a_n^{(2)} &= -3a_{n-1}^{(1)} + i\frac{d}{dt}\sigma_{n-1}^{(1)} - ia_{n-1}^{(1)}D_n^{-1}(\ln h_{n-1})_t + \\ &\quad +a_n^{(1)}\sigma_n^{(0)} - \psi_{n-1}(\psi_n^* + \psi_{n-2}^*)a_{n-1}^{(1)}, \\ \frac{d}{dt}h_n &= iD_n(\psi_{n-1}^*\psi_n - \psi_n^*\psi_{n-1}), ..., \end{aligned} \tag{4.6}$$

whence

$$\begin{aligned} \sigma_n^{(0)} &= -(\psi_n^*\psi_{n-1} + \psi_{n-1}^*\psi_{n-2}), \\ \sigma_n^{(1)} &= i\frac{d}{dt}\sigma_{n-1}^{(0)} + (1 - \psi_{n-1}^*\psi_{n-1})(1 - \psi_{n-2}^*\psi_{n-2}) + \beta\psi_{n-1}^*(\psi_n + \psi_{n-2}), ..., \end{aligned} \tag{4.7}$$

and so on. Thus, having stated the corresponding iterative equations solvable for all $s \in \mathbb{Z}_+$, we can claim that expression (4.1) is a true asymptotical solution to the Lax equation (4.3). □

Recalling now that the expression

$$\gamma(\lambda) := -\sum_{n=0}^{N} \ln h_n + \sum_{n=0}^{N} \ln(1 -_{s\in\mathbb{Z}_+} \sigma_n^{(s)}\lambda^{-s-1}) \tag{4.8}$$

as $|\lambda| \to \infty$ is a generating function of conservation laws for the dynamical system (1.2), one finds that functionals

$$\begin{aligned} \bar{\gamma}_0 &= {}^{N}_{n=0}(1 - \psi_n^*\psi_n), \gamma_0 = -\sum_{n=0}^{N} \sigma_n^{(0)}, \\ \gamma_1 &= -\sum_{n=0}^{N}(\sigma_n^{(1)} + \frac{1}{2}\sigma_n^{(0)}\sigma_n^{(0)}), \\ \gamma_2 &= -\sum_{n=0}^{N}(\sigma_n^{(2)} + \frac{1}{3}\sigma_n^{(0)}\sigma_n^{(0)}\sigma_n^{(0)} + \sigma_n^{(0)}\sigma_n^{(1)}), ..., \end{aligned} \tag{4.9}$$

and so on, make up an infinite hierarchy of exact conservative quantities for the discrete nonlinear Schrödinger dynamical system (1.2).

We make here some remarks concerning the complete integrability of the discrete nonlinear Schrödinger dynamical system (1.2). First we can easily enough state, making use of the standard asymptotical small parameter approach [6, 12, 7], that the Nöther equation (1.5) possesses [11, 10] the following exact implectic operator solution:

$$\vartheta = \begin{pmatrix} 0 & ih_n \\ -ih_n & 0 \end{pmatrix}, \tag{4.10}$$

$n \in \mathbb{Z}$, subject to which the dynamical system (1.2) is Hamiltonian:

$$\frac{d}{dt}\begin{pmatrix} \psi_n \\ \psi_n^* \end{pmatrix} = -\vartheta \ grad\ H_\vartheta[\psi^*, \psi]_n \tag{4.11}$$

on the periodic manifold $M_{(N)}$, where the Hamiltonian function

$$H_\vartheta := \sum_{n=0}^{N} \ln h_n^2 - \sum_{n=0}^{N} (\psi_n^* \psi_{n+1} - \psi_n^* \psi_{n+1}) = 2 \ln \bar{\gamma}_0 - \frac{1}{2}(\gamma_0 + \gamma_0^*). \tag{4.12}$$

By means of similar, but more cumbersome calculations, one can find the second implectic operator solution to the Nöther equation (1.5) in the following matrix form:

$$\begin{aligned} \eta \;=\; & \begin{pmatrix} (h_n - \psi_n D_n^{-1} \psi_n)\Delta & (\psi_n^2 + \psi_n D_n^{-1}\psi_n)\Delta^{-1} \\ \psi_n^* D_n^{-1} \psi_n^* \Delta & -(1 + \psi_n^* D_n^{-1} \psi_n)\Delta^{-1} \end{pmatrix} \times \\ & \times \begin{pmatrix} \psi_n D_n^{-1} \psi_n & (h_n - \psi_n D_n^{-1} \psi_n^* \\ 1 + \psi_n^* D_n^{-1} \psi_n & -(\psi_n^* + \psi_n^* D_n^{-1} \psi_n^*) \end{pmatrix}, \end{aligned} \tag{4.13}$$

where the operation $D_n^{-1}(...) := \frac{1}{2}[{}_{k=0}^{n-1}(...)_k - {}_{k=n}^{N}(...)_k]$ is quasi-skew-symmetric with respect to the usual bi-linear form on $T^*(M_{(N)}) \times T(M_{(N)})$, satisfying the operator identity $(D_n^{-1})^* = -\Delta^{-1} D_n^{-1} \Delta$, $n \in \mathbb{Z}$.

The implectic operators (4.10) and (4.13) are compatible, that makes it possible to construct by means of the algebraic gradient-holonomic algorithm the related Lax type representation for the dynamical system (1.2). The corresponding result is as follows: the discrete linear spectral problem

$$\Delta f_n = l_n[\psi^*, \psi; \lambda] f_n, \tag{4.14}$$

where $f \in l_\infty(\mathbb{Z}; \mathbb{C}^2)$ and for $n \in \mathbb{Z}$

$$l_n[\psi^*, \psi; \lambda] = \begin{pmatrix} \lambda & \psi_n \\ \psi_n^* & \lambda^{-1} \end{pmatrix}, \tag{4.15}$$

allows the linear Lax type isospectral evolution

$$df_n/dt = p_n(l) f_n \tag{4.16}$$

for some matrix $p_n(l) \in End\mathbb{C}^2, n \in \mathbb{Z}$, equivalent to the Hamiltonian flow

$$df_n/dt = \{H_\vartheta, f_n\}_\vartheta, \tag{4.17}$$

where $\{.,.\}_\vartheta$ is the Poissonian structure on the manifold $M_{(N)}$, corresponding to (4.10). The equivalence of (4.10) and (4.17) can be easily enough demonstrated, making use of the corresponding to (4.14) monodromy matrix $S_n(\lambda), n \in \mathbb{Z}$, for all $\lambda \in \mathbb{C}$ and to calculate the Hamiltonian evolution

$$\frac{d}{dt} S_n(\lambda) = \{H_\vartheta, S_n(\lambda)\}_\vartheta = [p_n(l), S_n(\lambda)], \tag{4.18}$$

giving rise to the same matrix $p_n(l) \in End\mathbb{C}^2, n \in \mathbb{Z}$, as that entering equation (4.16).

Thus, we have shown that the nonlinear discrete Schrödinger dynamical system (1.2) is Lax type integrable bi-Hamiltonian flow on the manifold $M_{(N)}$. Since the solution $\varphi(\lambda) \in T^*(M_{(N)})$, constructed above, satisfies the gradient-like relationship

$$\lambda \vartheta \varphi(\lambda) = \eta \varphi(\lambda) \tag{4.19}$$

for all $\lambda \in \mathbb{C}$, we derive that the above found conservation laws are commuting to each other with respect to both Poisson brackets $\{.,.\}_\vartheta$ and $\{.,.\}_\vartheta$. The latter gives rise to the classical Liouville intergrability [2, 14] of the discrete nonlinear Schrödinger dynamical system (1.2) on the periodic manifold $M_{(N)}$. Detailed analysis of the integrability procedure, via the previously mentioned Bogoyavlensky-Novikov reduction [15, 4] and the explicit construction of solutions to the dynamical system (1.2), are planned to be presented in a separate work.

5. Conclusion

The gradient-holonomic scheme of direct studying Lax type integrability of differential-difference nonlinear dynamical systems devised in this work appears to be effective enough for applications in the one-dimensional case, similar to the case [6, 14, 12, 7] of nonlinear dynamical systems defined on spatially one-dimensional functional manifolds. The algorithm, developing that previously suggested in [11, 10], makes it possible to construct simply enough an infinite hierarchy of conservation laws as well as to calculate their compatible co-symplectic structures. As it was also shown, via the Bogoyavlensky-Novikov approach the reduced integrable Hamiltonian dynamical systems on the corresponding invariant periodic submanifolds generate finite dimensional Liouville integrable Hamiltonian systems with respect to the canonical Gelfand-Dikiy type symplectic structures. As an example almost complete integrability analysis of the nonlinear discrete Schrödinger dynamical system was presented in detail.

Subject to different not-direct approaches to study the integrability of differential-difference dynamical systems on discrete manifolds it is worth to mention the works [18, 19, 20, 21, 22, 27] based on the inverse spectral transform and related Lie-algebraic methods, where *a priori* Lax type integrable Hamiltonian flows, possessing infinite hierarchies of conservation laws, were constructed. Concerning these approaches, many of their important analytical properties were constructively absorbed by the gradient-holonomic scheme of this work and realized directly as an algorithm.

We would also like to mention here the interesting differential-algebraic approaches [23, 28, 29] proposed for direct algorithmical studying the integrability both of differential and differential-difference dynamical systems. So, in [23, 25, 24] by means of simple enough differential-algebraic tools a generalized (owing to D. Holm and M. Pavlov) Riemann type hydrodynamical hierarchy of dynamical systems

$$D_t^N u = 0, \quad D_t := \partial/\partial t + uD_x, D_x := \partial/\partial x, \tag{5.1}$$

was analyzed on a smooth functional manifold $M \subset C^\infty(\mathbb{R};\mathbb{R})$ for any integer $N \in \mathbb{Z}_+$ and proved both their bi-Hamiltonian structure and Lax type integrability. Having replaced in equations (5.1) the spatial differentiation $D_x, x \in \mathbb{R}$, by its discrete analogue $D_n := \Delta - 1, n \in \mathbb{Z}$, one can similarly construct a generalized Rie

$$\mathfrak{D}_t^N u_n = 0, \quad \mathfrak{D}_t := \partial/\partial t + u_n(D_n + D_{n-1})/2, \; n \in \mathbb{Z}, \tag{5.2}$$

for any fixed integer $N \in \mathbb{Z}_+$ on a suitable discrete manifiold $M \subset l_2(\mathbb{Z};\mathbb{R})$, and whose integrability properties are important from a practical point of view. In particular, it would be interesting both to apply the devised in this work the direct gradient-holonomic integrability approach to hierarchy (5.2) and to find its correct differential-difference analogue using the already known [23, 24, 26] corresponding Lax type representations. As one can easily check, one of the discrete analogues of the corresponding linear Lax type "spectral" problem for (5.1) at $N = 2$ has the form

$$\Delta f_n = l_n[u, z; \lambda] f_n, \quad l_n[u, z; \lambda] = (\mathbf{1} + \mathcal{A}_{n+1} - \mathcal{A}_n), \tag{5.3}$$

where, by definition,

$$\mathcal{A}_n := \begin{pmatrix} -\lambda u_n & -z_n \\ 2\lambda^2 n & \lambda u_n \end{pmatrix} \tag{5.4}$$

and $z_n := \mathfrak{D}_n u_n$ for any $n \in \mathbb{Z}$. Remark here that the linear "spectral" problem (5.3), in spite of the form of matrix (5.4), persists to be independent on the variable index $n \in \mathbb{Z}$.

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