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LIFTING PRIME IDEALS AND KRULL DIMENSION

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Abstract

Let R, T be two commutative rings and $\varphi : R \longrightarrow T$ be a ring homomorphism. In this paper we are concerned with the study of the lifting of chains of prime ideals from R to T . More precisely, we seek a necessary and sufficient condition for a such chain of prime ideals of R to lift to a chain of prime ideals of T .

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INTRODUCTION

All rings considered in this paper are assumed to be commutative with a unity element. If R is a subring of a ring S , we assume that the unity element of S belongs to R , and hence is the unity element of R . We let $\text{Spec}(R)$, $Z(R)$, $\text{Inv}(R)$ and $\text{Ann}_R(I)$, respectively, denote the spectrum of R (the set of prime ideals of R), the set of zero-divisors of R , the set of invertible elements of R , and the annihilator of a subset I of R . We use the term dimension of R , denoted $\dim R$, to refer to the Krull dimension of R . Thus $\dim R$ is the non-negative integer n if there exists a chain $P_0 \subset P_1 \subset \dots \subset P_n$ of proper prime ideals of R , but no longer such chain; if there is no upper bound on the lengths of such chains, we write $\dim R = \infty$.

Several papers in the literature have dealt with lifting chains of prime ideals under ring-homomorphisms, that is, if $\varphi : R \rightarrow T$ is a ring-homomorphism and $P_0 \subset P_1 \subset \dots \subset P_n$ is a chain of prime ideals of R , we see if there exists a chain $Q_0 \subset Q_1 \subset \dots \subset Q_n$ such that $Q_i \cap R = P_i$ for each $i = 0, \dots, n$. We use that the dimension is preserved under integral ring-homomorphisms (cf.[2, 11.8]). In particular, an integral extension ring of a zero-dimensional ring is zero-dimensional.

1. PRELIMINARIES AND GENERAL RESULTS

Let $\varphi : R \rightarrow T$ be a ring-homomorphism and $P_0 \subset P_1 \subset \dots \subset P_n$ be a chain of prime ideals of R . Our purpose in this article is to see a necessary and sufficient condition in order to exist a chain $Q_0 \subset Q_1 \subset \dots \subset Q_n$ such that $Q_i \cap R = P_i$ for each $i = 0, \dots, n$. The following result is a source of motivation for this paper:

Lemma 1.1. [1, Lemma 10.11] *Let R and T be two commutative rings and $\varphi : R \rightarrow T$ be a ring homomorphism. If $\varphi : R \rightarrow T$ is integral, then for each chain $P_0 \subset P_1 \subset \dots \subset P_n$ in R there exists a chain $Q_0 \subset Q_1 \subset \dots \subset Q_n$ in T such that $Q_i \cap R = P_i$ for each $i = 0, \dots, n$.*

This fact leads us to ask:

(Q) Let $\varphi : R \rightarrow T$ be a ring homomorphism, and $P_0 \subset P_1 \subset \dots \subset P_n$ be a chain in R .

Under what conditions does there exist a chain $Q_0 \subset Q_1 \subset \dots \subset Q_n$ in T such that $Q_i \cap R = P_i$ for each $i = 0, \dots, n$?

Lemma 1.2. [3, Theorem 1] *Let I be an ideal of R , and S a submonoid of R (a subset of R , containing and closed under multiplication) disjoint from I . Then there exists a prime ideal P of R containing I and disjoint from S .*

Definition 1.1. If I is an ideal of R and S a multiplicative submonoid of R . We denote by

$$\mathcal{P}(I, S) = \{P \in \text{Spec}(R) : I \subseteq P \text{ and } P \cap S = \emptyset\}$$

(the prime ideals of R containing I and disjoint from S .)

Example 1.3. (1) $\mathcal{P}(0, 1) = \text{Spec}(R)$;

(2) $\mathcal{P}(0, R \setminus P) = \text{Spec}(R_P)$, where $P \in \text{Spec}(R)$;

(3) $\mathcal{P}(P, 1) = \text{Spec}(\frac{R}{P})$, where $P \in \text{Spec}(R)$;

Lemma 1.2 implies that $\mathcal{P}(I, S) \neq \emptyset$ if and only if $I \cap S = \emptyset$.

We define:

$$I + S = \{x + y : x \in I, y \in S\}$$

$$I \div S = \{r \in R : (\exists y \in S) ry \in I\}$$

We can easily show that $I + S$ is a multiplicative submonoid of R and $I \div S$ is an ideal of R .

We note that a prime ideal $P \in \mathcal{P}(I, S)$ must contain $I \div S$ and must be disjoint from $I + S$.

Proposition 1.4. *Let I be an ideal of R and S a multiplicatively closed subset of R . Then*

(i) $\mathcal{P}(I, S) = \mathcal{P}(I, S + I)$;

(ii) $\mathcal{P}(I, S) = \mathcal{P}(I \div S, S)$;

(iii) $\mathcal{P}(I, S) = \mathcal{P}(I \div S, S + I)$;

Proof. (i) Let $P \in \mathcal{P}(I, S)$, by definition we have $I \subseteq P$ and $P \cap S = \emptyset$. Let $x \in P \cap (S + I)$ then $x = p = s + i$, where $p \in P$, $s \in S$ and $i \in I$. In other words, $p + i = s$ a contradiction with $P \cap S = \emptyset$. As $P \cap (S + I) = \emptyset$, we may state that $P \in \mathcal{P}(I, I + S)$. Thus $\mathcal{P}(I, S) \subseteq \mathcal{P}(I, I + S)$. Now let $P \in \mathcal{P}(I, I + S)$, then $P \cap (I + S) = \emptyset$ and hence $P \cap S = \emptyset$, otherwise, there exist $p \in P$ and $s \in S$ such that $p = s$. Since $0 \in I$, we have $p = 0 + s \in I + S$ and we obtain a contradiction with $P \cap (S + I) = \emptyset$. It follows that $P \in \mathcal{P}(I, S)$.

(ii) Let $P \in \mathcal{P}(I \div S, S)$, then $I \div S \subseteq P$ and $P \cap S = \emptyset$. Since $I \subseteq I \div S$, we have $I \subseteq P$ and $P \cap S = \emptyset$ and hence $P \in \mathcal{P}(I, S)$. It follows that $\mathcal{P}(I \div S, S) \subseteq \mathcal{P}(I, S)$. Conversely, let $P \in \mathcal{P}(I, S)$, then $P \cap S = \emptyset$ and $I \subseteq P$. Let $r \in I \div S$, then there exists $s \in S$ such that $sr \in I \subseteq P$. That means that $r \in P$, since $s \in S \setminus P$. This shows that $I \div S \subseteq P$. Thus, $P \in \mathcal{P}(I \div S, S)$.

(iii) Let $P \in \mathcal{P}(I, S)$, then $I \subseteq P$ and $P \cap S = \emptyset$. We have already shown that $P \cap (I + S) = \emptyset$ and $I \div S \subseteq P$. This means that $P \in \mathcal{P}(I \div S, I + S)$. Let $P \in \mathcal{P}(I \div S, I + S)$. It is easy to show that $P \in \mathcal{P}(I, S)$. Hence $\mathcal{P}(I \div S, I + S) = \mathcal{P}(I, S)$.

□

Corollary 1.5. *Let I be an ideal of R and S a multiplicatively closed subset of R . Let $P \in \text{Spec}(R)$ then*

- (i) *If $I \subseteq P$, then $P \cap S = \emptyset$ if and only if $P \cap (S + I) = \emptyset$.*
- (ii) *If $P \cap S = \emptyset$, then $I \subseteq P$ if and only if $I \div S \subseteq P$.*

Proof. The proof of this corollary is a consequence of Proposition 1.4. □

Let $Q \in \mathcal{P}(I, S)$, then $I \cap R \setminus Q = \emptyset$ and hence $I \cap (R \setminus Q)S = \emptyset$. By Lemma 1.2, we can find a prime ideal P of R such that $I \subseteq P$ and $P \cap (R \setminus Q)S = \emptyset$. In other words, $P \cap S = \emptyset$ and $P \cap R \setminus Q = \emptyset$. Hence $P \in \mathcal{P}(I, S)$ and $P \subseteq Q$.

Lemma 1.6. *Let I be an ideal of R and S a multiplicatively closed subset of R . Then*

- (i) *A prime ideal Q of R contains the ideal $I \div S$ if and only if it contains a prime $P \in \mathcal{P}(I, S)$.*
- (ii) *A prime ideal P of R is disjoint from the multiplicatively closed subset $S + I$ if and only if it is contained in a prime ideal $Q \in \mathcal{P}(I, S)$.*

Proof. (i) Let $Q \in \text{Spec}(R)$ such that $I \div S \subseteq Q$. Then $(I \div S) \cap (R \setminus Q) = \emptyset$. Therefore $I \cap S(R \setminus Q) = \emptyset$. By Lemma 1.2, there exists a prime ideal $P \in \text{Spec}(R)$ such that $I \subset P$ and $P \cap S(R \setminus Q) = \emptyset$, i.e., $P \cap S = \emptyset$ and $P \cap R \setminus Q = \emptyset$. It follows that $I \subset P$, $P \cap S = \emptyset$ and $P \subseteq Q$. Thus $P \in \mathcal{P}(I, S)$ and $P \subseteq Q$. Conversely let $P \in \mathcal{P}(I, S)$. If $P \subseteq Q$, where Q is a prime ideal of R . As $I \div S \subseteq P$, we have $I \div S \subseteq Q$.

(ii) Let $P \in \text{Spec}(R)$ such that $P \cap S + I = \emptyset$ which means that $P + I \cap S = \emptyset$. Then by Lemma 1.2, there exists a prime ideal $Q \in \text{Spec}(R)$ which satisfy $P + I \subset Q$ and $Q \cap S = \emptyset$, i.e., $Q \in \mathcal{P}(I, S)$ and $P \subseteq Q$. Conversely, if a prime ideal $Q \in \text{Spec}(R)$ verifies $I \subset Q$ and $S \cap Q = \emptyset$ then $Q \cap S + I = \emptyset$. Therefore, for any prime ideal $P \subseteq Q$, we have $P \cap S + I = \emptyset$. □

Remark 1.7. (a) If $P \in \mathcal{P}(I, S)$, then $I \div S \subseteq P$.

- (b) If $Q \cap S = \emptyset$, then $Q \cap (S + I) = \emptyset$.
- (c) $\mathcal{P}(I, S) \cap \mathcal{P}(I', S') = \mathcal{P}(I + I', SS')$

Lemma 1.8. *The following conditions are equivalent:*

- (i) *There exist prime ideals $P \in \mathcal{P}(I, S)$ and $Q \in \mathcal{P}(I', S')$ such that $P \subseteq Q$;*
- (ii) *$I \cap S(S' + I') = \emptyset$;*
- (iii) *$(I + I') \div S \cap S' = \emptyset$.*

Proof. (i) \Rightarrow (ii). Assume that there exist $P \in \mathcal{P}(I, S)$ and $Q \in \mathcal{P}(I', S')$ such that $P \subseteq Q$. We have $I \subset P \subseteq Q$ and $Q \cap (S' + I') = \emptyset$, i.e., $I \cap (S' + I') = \emptyset$. Since $I \cap S = \emptyset$, we have $I \cap S(S' + I') = \emptyset$.

(ii) \Rightarrow (iii). Suppose that $(I + I') \div S \cap S' \neq \emptyset$. Then there exist $i \in I$, $s \in S$, $s' \in S'$ and $x \in R$ such that $i' + s' + x = 0$ and $sx = i$. Therefore, $si' + ss' = -sx = -i$ that means that $I \cap S(S' + I') \neq \emptyset$, a contradiction with our hypothesis.

(iii) \Rightarrow (i). By Lemma 1.2, there exists a prime ideal $Q \in \text{Spec}(R)$ such that $(I + I') \div S \subseteq Q$ and $Q \cap S' = \emptyset$, i.e., $Q \in \mathcal{P}(I', S')$. Since $(I + I') \div S \subseteq Q$, by Lemma 1.2, there exists a prime ideal $P \in \mathcal{P}(I + I', S)$ with $P \subseteq Q$. Then $P \in \mathcal{P}(I, S)$ and $P \subseteq Q$.

□

2. LIFTING CHAINS OF PRIME IDEALS

The aim of this section is to give an answer to our previous question (Q). We denote by $\text{Int}(R, T)$ the set of all intermediate rings between R and T .

Theorem 2.1. *Let $\{(I_j, S_j)\}_{j \in J}$ be a directed family of pairs indexed by a directed set (I, \leq) , where I_j is an ideal of R and S_j is a multiplicatively closed subsets in R such that $I_j \cap S_j = \emptyset$. Then*

- (1) *There exist prime ideals $P_i \subseteq P_j$ in R such that $P_k \in \mathcal{P}(I_k, S_k)$ for $k = i, j$ if and only if $(I_i \div S_i) \cap (I_j + S_j) = \emptyset$.*
- (2) *There exists a family of prime ideals $\{P_j \in \mathcal{P}(I_j, S_j)\}_{j=1}^s$ such that $P_1 \subseteq \dots \subseteq P_s$ in R if and if only if $(I_j \div S_j) \cap (I_{j+1} + S_{j+1}) = \emptyset$ for each $j = 1, \dots, s$.*

Proof. The proof of this result follows from Lemma 1.8. □

Remark 2.2. We can replace the condition (2) of the Theorem with

- (2') *There exists a family of prime ideals $\{P_j \in \mathcal{P}(I_j, S_j)\}_{j=1}^\infty$ such that $P_1 \subseteq \dots \subseteq P_i \subseteq \dots$ in R if and if only if $(I_j \div S_j) \cap (I_{j+1} + S_{j+1}) = \emptyset$ for each $j \in \mathbb{N}^*$.*

Theorem 2.3. *Let R and T be commutative rings and let $\varphi : R \hookrightarrow T$ be an injective ring homomorphism such that $\varphi(\mathfrak{m})T \neq T$ holds for each $\mathfrak{m} \in \text{Max}(R)$. Then*

- (i) *For every chain $P \subset Q$ in R there exists a chain of primes $P_1 \subset Q_1$ in $\text{Spec}(W)$ such that $P_1 \cap R = P$ and $Q_1 \cap R = Q$, where $W \in \text{Int}(R, T)$.*
- (ii) *In particular, if $P_0 \subset \dots \subset P_r$ is a chain of $\text{Spec}(R)$ then for any intermediate ring $W \in \text{Int}(R, T)$ there exists a chain of $\text{Spec}(W)$ $Q_0 \subset \dots \subset Q_r$ such that $Q_i \cap R = P_i$ for each $i = 0, \dots, r$.*

The proof of this theorem requires the following preparatory lemmas.

Lemma 2.4. *Let R and T be commutative rings and let $\varphi : R \hookrightarrow T$ be a ring homomorphism such that for every maximal ideal $\mathfrak{m} \in \text{Max}(R)$, we have $\varphi(\mathfrak{m})T \neq T$. Then for every chain of prime ideals $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ in R , there exists a chain of prime ideals of T , $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ such that $\mathfrak{q}_j \cap R = \mathfrak{p}_j$ for each $j = 1, \dots, n$.*

Proof. Let $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ be a chain of R and suppose that \mathfrak{p}_n is a maximal ideal of R . We denote by $\mathfrak{a}_i = \mathfrak{p}_i T = \mathfrak{p}_i^e$ is the extended ideal of \mathfrak{p}_i in T for each $i = 1, \dots, n$. By our assumption on φ , we have $\mathfrak{a}_n \neq T$. Let $S_i = \varphi(R \setminus \mathfrak{p}_i)$ be a multiplicatively closed subset of T for each $i = 1, \dots, n$. We state that $\mathfrak{a}_i \cap S_i = \emptyset$ for each $i = 1, \dots, n$. We have constructed a finite family $\{\mathfrak{a}_i, S_i\}_{i=1}^n$ of pairs of ideals \mathfrak{a}_i of T and multiplicatively closed subset S_i of T . We claim that $(\mathfrak{a}_i \div S_i) \cap (\mathfrak{a}_{i+1} + S_{i+1}) = \emptyset$, for each $i = 1, \dots, n-1$. Suppose that there exist elements $a \in \mathfrak{a}_i$, $s \in S_i$, $b \in \mathfrak{a}_{i+1}$, $t \in S_{i+1}$ and $x \in R$ satisfying $t + b + x = 0$ and $sx = a$, i.e., $x = -t - b$ and $sx = a$. Then $sx = s(-t - b) = a \in \mathfrak{a}_i$ and hence $st + sb = -a$, i.e., $st = -a - sb$. As $S_i = \varphi(R \setminus \mathfrak{p}_i)$ and $S_{i+1} = \varphi(R \setminus \mathfrak{p}_{i+1})$, we have to note that $S_{i+1} \subset S_i$ for each $i = 1, \dots, n$ and also $\mathfrak{a}_i \subset \mathfrak{a}_{i+1}$ for each i . Since $st \in S_{i+1}$ and $-a - sb \in \mathfrak{a}_{i+1}$, we have a contradiction with $S_{i+1} \cap \mathfrak{a}_{i+1} = \emptyset$ for each $i = 0 \dots, n-1$. By Theorem 2.1, there exists a chain in T of prime ideals, that means, $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for each $i = 1, \dots, n$. \square

Lemma 2.5. *Let $\varphi : R \hookrightarrow T$ be an extension ring such that $\varphi(\mathfrak{m})T \neq T$ for each maximal ideal $\mathfrak{m} \in \text{Max}(R)$. Then $\varphi(\mathfrak{m})W \neq W$ for each $W \in \text{Int}(R, T)$.*

Proof. Suppose that there exists a ring $W_1 \in \text{Int}(R, T)$ such that $\mathfrak{m}W_1 = W_1$, which means that $1 = m_1w_1 + \cdots + m_sw_s$ for some m_1, \dots, m_s elements of \mathfrak{m} and w_1, \dots, w_s elements of W_1 . Since $w_1, \dots, w_s \in W_1 \subset T$, we may state that $T \subseteq \mathfrak{m}T$, which is a contradiction with $\varphi(\mathfrak{m})T \neq T$. \square

Proof of Theorem 2.3. Since (i) is a particular case of (ii), we just give the proof of (ii).

(ii) Let $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_r$ be a chain of prime ideals of R . Since $\varphi(\mathfrak{m})T \neq T$ for each maximal ideal $\mathfrak{m} \in \text{Max}(R)$, by Lemma 2.5, $\varphi(\mathfrak{m})W \neq W$ for each maximal ideal $\mathfrak{m} \in \text{Max}(R)$. Now, let $\varphi_W : R \hookrightarrow W$ be an extension ring and we note that $\varphi|_W = \varphi_W$ is the reduction of φ over W . According to Lemma 2.4, there exists a chain $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_r$ of prime ideals of W such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for each $i = 1, \dots, r$, as required. \square

Corollary 2.6. *Let R and T be commutative rings and let $\varphi : R \rightarrow T$ be a ring homomorphism such that $\varphi(\mathfrak{m})T \neq T$ holds for each $\mathfrak{m} \in \text{Max}(R)$. If $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_r$ is a chain of $\text{Spec}(R)$ then*

for any intermediate ring $W \in \text{Int}(\varphi(R), T)$ there exists a chain of $\text{Spec}(W)$ $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_r$ such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for each $i = 0, \dots, r$.

Proof. The proof of this corollary is immediate, by using Lemma 2.4 and Lemma 2.5. \square

Corollary 2.7. Let $\varphi : R \rightarrow T$ be a ring homomorphism such that $\varphi(\mathfrak{m})T \neq T$ for each maximal ideal $\mathfrak{m} \in \text{Max}(R)$. Then $\dim(R) \leq \dim(W)$, where $W \in \text{Int}(\varphi(R), T)$. In particular, $\dim(R) \leq \dim(T)$.

Proof. Assume that $\dim(R) = s$, by definition of Krull dimension there exists a chain of prime ideals of R , i.e., $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_s$ satisfying the dimension of R . By Corollary 2.6, there exists a chain $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_s$ of $\text{Spec}(W)$ such that $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ for each $i = 0, \dots, s$. It follows that $s \leq \dim(R)$ and hence $\dim(R) \leq \dim(W)$. \square

Remark 2.8. Let $\varphi : R \hookrightarrow T$ be an extension ring such that $\varphi(\mathfrak{m})T \neq T$ for each maximal ideal $\mathfrak{m} \in \text{Max}(R)$. Then $\dim(R) \leq \dim(W)$ for every intermediate ring $W \in \text{Int}(\varphi(R), T)$. In particular, $\dim(R) \leq \dim(T)$. Indeed, the proof of this remark is straightforward, by applying Corollary 2.7.

In the next example, we give a ring-homomorphism which illustrates Corollary 2.6.

Example 2.9. Let (R, \mathfrak{m}) be a local domain of dimension 1. We have as a chain $(0) \subset \mathfrak{m}$. Let $B = S^{-1}R[X]$, where $S = R[X] \setminus (\mathfrak{m}[X] \cup XR[X])$, we note that $\dim(B) = 1$, since $\text{Spec}(B) = \{(0), S^{-1}(XR[X]), S^{-1}(\mathfrak{m}R[X])\}$. Let T be the quotient ring of B by the product ideal $X\mathfrak{m}[X]$ in B . Let $\varphi : R \rightarrow R[X] \rightarrow T$ the natural homomorphism. First, we note that $\dim(T) = 0$ since $\text{Spec}(T) = \{\overline{S^{-1}(XR[X])}, \overline{S^{-1}(\mathfrak{m}R[X])}\}$. Therefore, the ring T has exactly two maximal ideals $\overline{S^{-1}(XR[X])}$ and $\overline{S^{-1}(\mathfrak{m}R[X])}$. In other words, $\mathfrak{m} = \varphi^{-1}(\overline{S^{-1}(\mathfrak{m}R[X])})$ and $(0) = \varphi^{-1}(\overline{S^{-1}(XR[X])})$. Thus the chain of prime ideals $(0) \subset \mathfrak{m}$ in R cannot be lifted to a chain of prime ideals in T .

3. RELATIONSHIP BETWEEN RING-HOMOMORPHISMS AND MAXIMAL IDEALS

Proposition 3.1. Let $\varphi : R \rightarrow T$ be a ring-homomorphism. Then $\varphi^* : \text{Spec}(T) \rightarrow \text{Spec}(R)$ is surjective if and only if for every maximal ideal \mathfrak{m} of R we have $\varphi(\mathfrak{m})T \neq T$, where φ^* is the mapping associated with φ from $\text{Spec}(T)$ to $\text{Spec}(R)$.

Proof. Suppose that $\varphi^* : \text{Spec}(T) \rightarrow \text{Spec}(R)$ is surjective and let \mathfrak{m} be a maximal ideal of R , by our assumption there exists a prime ideal \mathfrak{q} of T such that $\varphi^*(\mathfrak{q}) = \mathfrak{m}$ then $\varphi(\mathfrak{m})T \subset \mathfrak{q}$

and hence $\varphi(\mathfrak{m})T \neq T$. Conversely, suppose that $\varphi(\mathfrak{m})T \neq T$ for each maximal ideal \mathfrak{m} of R . Let $\mathfrak{p} \in \text{Spec}(R)$, there exists a maximal ideal of R such that $\mathfrak{p} \subset \mathfrak{m}$. Since $\varphi(\mathfrak{m})T \neq T$, we may obtain $\varphi(\mathfrak{p})T \neq T$. Let \mathfrak{q} be a minimal prime ideal of T over $\varphi(\mathfrak{p})T$. We denote by \mathfrak{q} the pre-image of \mathfrak{p} . In other words, $\varphi^*(\mathfrak{q}) = \mathfrak{p}$. It follows that $\varphi^* : \text{Spec}(T) \rightarrow \text{Spec}(R)$ is surjective. \square

Proposition 3.2. *Let $\varphi : R \rightarrow T$ be a ring-homomorphism. Then $\varphi^* : \text{Spec}(T) \rightarrow \text{Spec}(R)$ is surjective if and only if $\varphi_X^* : \text{Spec}(T[X]) \rightarrow \text{Spec}(R[X])$, where φ_X is the homomorphism defined from $R[X]$ to $T[X]$.*

Proof. Assume that $\varphi^* : \text{Spec}(T) \rightarrow \text{Spec}(R)$ is surjective, let $\mathcal{P} \in \text{Spec}(R[X])$, it is well known that \mathcal{P} is either an extension of a prime ideal p of R , that means, $\mathcal{P} = p[X]$ or \mathcal{P} is an upper to p . First, suppose that $\mathcal{P} = p[X]$, where $p = \mathcal{P} \cap R$. Since φ^* is surjective, there exists a prime ideal \mathfrak{q} of T such that $\varphi^*(\mathfrak{q}) = p$. Now, suppose that \mathcal{P} is an upper to p and let $\mathfrak{q} \in \text{Spec}(T)$ such that $\varphi^*(\mathfrak{q}) = p$ as φ^* is surjective. We consider \mathcal{Q} as an upper to \mathfrak{q} and hence we can associate to \mathcal{Q} an upper to p says \mathcal{P} . Thus $\varphi_X^*(\mathcal{Q}) = \mathcal{P}$. It follows that φ_X^* is also surjective. Conversely, suppose that φ_X^* is surjective and let p be a prime ideal of R . We consider $p^* = p[X]$ as the extension ideal of p in $R[X]$ and since φ_X^* is surjective, there is a prime ideal \mathcal{Q} of $T[X]$ such that $\varphi_X^*(\mathcal{Q}) = p^*$. Let $q = \mathcal{Q} \cap T$ be a prime ideal of T . Then $\varphi^*(q) = p$ and hence φ^* is surjective. \square

Corollary 3.3. *As notation below, then $\varphi^* : \text{Spec}(T) \rightarrow \text{Spec}(R)$ is surjective if and only if $\varphi_n^* : \text{Spec}(T[n]) \rightarrow \text{Spec}(R[n])$, where φ_n is the homomorphism defined from $R[n]$ to $T[n]$, φ_n^* is the mapping associated with φ_n and $R[n]$ denotes the polynomial ring $R[X_1, \dots, X_n]$.*

Proof. The proof of this result is made by induction on n from Proposition 1.4 \square

In the next example, we have to check a pair of rings which verifies:

- (1) $\varphi : R \rightarrow T$ is a ring-homomorphism and $\dim(R) \leq \dim(T)$;
- (2) for each maximal ideal \mathfrak{m} of R , we have $\mathfrak{m}^e \neq T$.

Example 3.4. Let R be a commutative ring and X be an indeterminate over R . It is well known that $\dim(R[X]) \geq \dim(R) + 1$, that means that, $\dim(R) < \dim(R[X])$ and for each maximal ideal \mathfrak{m} of R , $\mathfrak{m}^e = \mathfrak{m}R[X] \neq R[X]$.

Corollary 3.5. *If $\varphi^* : \text{Spec}(T) \rightarrow \text{Spec}(R)$ is surjective, then*

- (1) For every $\mathfrak{p} \in \text{Spec}(R)$, we have $\mathfrak{p}^{ec} = \varphi^{-1}(\mathfrak{p}T) = \mathfrak{p}$ and hence $\dim(R) \leq \dim(T)$.

(2) $\varphi_n^* : \text{Spec}(T[n]) \rightarrow \text{Spec}(R[n])$. The mapping associated with φ_n is surjective and $\dim(R[X]) \leq \dim(T[X])$.

Proof. (1) Suppose that $\varphi^* : \text{Spec}(T) \rightarrow \text{Spec}(R)$ is surjective. Let $\mathfrak{p} \in \text{Spec}(R)$ then there exists $\mathcal{Q} \in \text{Spec}(T)$ such that $\varphi(\mathfrak{p}) = \mathcal{Q}$. Let S be the image of $R \setminus \mathfrak{p}$ in T . Then S is a multiplicatively closed subset of T and hence \mathcal{Q} does not meet S , i.e., $\mathcal{Q} \cap S = \emptyset$ and so $S^{-1}\mathcal{Q}$ is a proper ideal of $S^{-1}T$. Therefore, $S^{-1}\mathcal{Q}$ is contained in a maximal ideal \mathfrak{m} of $S^{-1}T$. Then $\mathfrak{m}^c \in \text{Spec}(R)$, $\mathfrak{m}^c \supseteq \mathcal{Q}^c$ and $\mathfrak{m}^c \cap S = \emptyset$. Hence $\mathfrak{p} = \mathfrak{m}^c = \mathcal{Q}^c$. Thus $\mathfrak{p} = \varphi^{-1}(\varphi(\mathfrak{p})) = \mathcal{Q}^c = \mathfrak{p}^{ec}$.

(2) We state this statement by Corollary 3.3. \square

Proposition 3.6. Let $\varphi : R \rightarrow T$ be a ring homomorphism such that $\varphi(\mathfrak{q}^c) = \mathfrak{q}$ for each $\mathfrak{q} \in \text{Spec}(T)$. Let $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ be a strict chain of prime ideals in T . Then $\mathfrak{q}_1^c \subset \cdots \subset \mathfrak{q}_n^c$ is a strict chain of prime ideals in R .

Proof. Let $x \in \mathfrak{q}_{i+1} \setminus \mathfrak{q}_i$ for some $i = 1, \dots, n$. Then there exists $y \in \mathfrak{q}_{i+1}^c$ such that $\varphi(y) = x$ since $\varphi(\mathfrak{q}_{i+1}^c) = \mathfrak{q}_{i+1}$. We must have $y \notin \mathfrak{q}_i^c$, otherwise $x \in \mathfrak{q}_i$. Then $y \in \mathfrak{q}_{i+1}^c \setminus \mathfrak{q}_i^c$ and hence $\mathfrak{q}_i^c \subset \mathfrak{q}_{i+1}^c$. By induction, it follows that $\mathfrak{q}_1^c \subset \cdots \subset \mathfrak{q}_n^c$. \square

Corollary 3.7. With notation as before. If $\mathfrak{q}^{ce} = \mathfrak{q}$ for each $\mathfrak{q} \in \text{Spec}(T)$, then $\dim(R) \geq \dim(T)$.

Proof. Suppose that $\dim(T) = s$, then there exists a chain of T satisfying the dimension of T , i.e., $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_s$. By Proposition 3.6, there exists a strict chain of R , $\mathfrak{q}_0 \subset \cdots \subset \mathfrak{q}_s$ which means that $\dim(R) \geq s$ and hence $\dim(R) \geq \dim(T)$. \square

Corollary 3.8. (1) Let $\varphi : R \rightarrow T$ be a ring-homomorphism such that $\mathfrak{p}^{ec} = \mathfrak{p}$ for each $\mathfrak{p} \in \text{Spec}(R)$ then $\dim(R) \leq \dim(T)$.

(2) If $\mathfrak{p}^{ec} = \mathfrak{p}$ for each $\mathfrak{p} \in \text{Spec}(R)$ and $\mathfrak{q}^{ce} = \mathfrak{q}$ for each $\mathfrak{q} \in \text{Spec}(T)$, then $\dim(R) = \dim(T)$.

Proof. (1) Assume that $\mathfrak{p}^{ec} = \mathfrak{p}$ for each $\mathfrak{p} \in \text{Spec}(R)$. By Corollary 3.5, φ^* the mapping associated with φ is surjective and hence $\dim(R) \leq \dim(T)$.

(2) By (1) we have $\dim(R) \leq \dim(T)$ and by Corollary 3.7, $\dim(R) \geq \dim(T)$. It follows that $\dim(R) = \dim(T)$. \square

This corollary led us to ask the following question:

(Q) It is well known that if $\varphi : R \rightarrow T$ is an integral ring-homomorphism, then $\dim(R) = \dim(T)$. Now, we assume that $\mathfrak{p}^{ec} = \mathfrak{p}$ for each $\mathfrak{p} \in \text{Spec}(R)$ and $\mathfrak{q}^{ce} = \mathfrak{q}$ for each $\mathfrak{q} \in \text{Spec}(T)$. Is it true that the ring-homomorphism $\varphi : R \rightarrow T$ is integral ?

Remark 3.9. If every prime ideal of T is an extended ideal of R , then $\dim(R) \geq \dim(T)$, however, we cannot compare the cardinality of $\text{Spec}(R)$ and $\text{Spec}(T)$, i.e., we can not say that $|\text{Spec}(R)| < |\text{Spec}(T)|$ or $|\text{Spec}(T)| < |\text{Spec}(R)|$.

4. LYING OVER, GOING-UP AND GOING-DOWN

Proposition 4.1. *Let $\varphi : R \rightarrow T$ be a ring-homomorphism such that $\mathfrak{p}^{ec} = \mathfrak{p}$ for each $\mathfrak{p} \in \text{Spec}(R)$. Then*

- (i) φ satisfies the lying over.
- (ii) φ satisfies the going-up.
- (iii) φ satisfies the going-down.

Proof. (i) Since $\mathfrak{p}^{ec} = \mathfrak{p}$ for each $\mathfrak{p} \in \text{Spec}(R)$, the associated mapping with φ , i.e., $\varphi^* : \text{Spec}(T) \rightarrow \text{Spec}(R)$ is surjective. Therefore, for each $\mathfrak{p} \in \text{Spec}(R)$, there exists a prime ideal \mathfrak{q} of T such that $\varphi^*(\mathfrak{q}) = \varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. Thus φ satisfies the lying-over.

(ii) Let $\mathfrak{p}_1 \subset \mathfrak{p}_2$ be a chain of $\text{Spec}(R)$. We denote by $S_2 = \varphi(R \setminus \mathfrak{p}_2)$ a multiplicatively closed subset of T which verifies $S_2 \cap \mathfrak{p}_2^e = \emptyset$. According to lying over, there exists a prime ideal \mathfrak{q}_1 of T such that $\mathfrak{q}_1 \cap R = \mathfrak{p}_1$. Suppose that $\mathfrak{q}_1 \cap S_2 \neq \emptyset$, then there exists $a \in \mathfrak{q}_1 \cap S_2$, i.e., $a = \varphi(b)$ and $a = \varphi(c)$ with $b \in R \setminus \mathfrak{p}_2$ and $c \in \mathfrak{p}_1$, a contradiction with $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$. Thus $\mathfrak{q}_1 \cap S_2 = \emptyset$. It is well known that \mathfrak{p}^e is not necessarily a prime ideal, then we consider \mathfrak{q}' a minimal prime ideal of T such that $\mathfrak{p}^e \subseteq \mathfrak{q}'$. Now, let $S_1 = T \setminus \mathfrak{q}'$ be a multiplicatively closed subset of T and $S = S_1 S_2$. Then S is a multiplicatively closed subset of T . Let \mathfrak{q}_2 be a prime ideal such that $\mathfrak{q}_2 \cap S = \emptyset$. In other words, $\mathfrak{q}_2 \cap S_1 = \emptyset$ and $\mathfrak{q}_2 \cap S_2 = \emptyset$. The statement $\mathfrak{q}_2 \cap S_1 = \emptyset$ implies that $\mathfrak{p}_2^{ec} = \mathfrak{p}_2 \subseteq \mathfrak{q}_2 \cap R$ and the statement $\mathfrak{q}_2 \cap S_2 = \emptyset$ means that $\mathfrak{q}_1 \subset \mathfrak{q}_2$ and $\mathfrak{q}_2 \cap R \subseteq \mathfrak{p}_2^{ec} = \mathfrak{p}_2$. Hence $\mathfrak{q}_2 \cap R = \mathfrak{p}_2$ and $\mathfrak{q}_1 \subset \mathfrak{q}_2$. It follows that φ satisfies the going-up property.

(iii) Let $\mathfrak{p}_1 \subset \mathfrak{p}_2$ be a chain of prime ideals of R and \mathfrak{q}_2 be a prime ideal of T such that $\mathfrak{q}_2 \cap R = \mathfrak{p}_2$. We have $\mathfrak{p}_1^e \subset \mathfrak{p}_2^e$, we know that \mathfrak{p}_1^e is not necessarily a prime ideal then there exists a prime ideal \mathfrak{a} of T which is minimal over \mathfrak{p}_1^e and $\mathfrak{p}_1^e \subseteq \mathfrak{a}$. Assume that $\mathfrak{p}^e \neq \mathfrak{a}$ otherwise we have what it is required. Now, let $I_1 = \mathfrak{p}_1^e$, $S_1 = \varphi(R \setminus \mathfrak{p}_1)$, $I_2 = \mathfrak{p}_2^e$, and $S_2 = \varphi(R \setminus \mathfrak{p}_2)$. We note that I_1 and I_2 are ideals of T , S_1 and S_2 are multiplicatively closed subsets of T which verify, $I_1 \cap S_1 = \emptyset$ and $I_2 \cap S_2 = \emptyset$ and hence $\mathcal{P}(I_1, S_1) \neq \emptyset$ also $\mathcal{P}(I_2, S_2) \neq \emptyset$. We observe that $I_1 \cap S_1(S_2 + I_2) = \emptyset$. By Lemma 1.2, there exists a prime ideal \mathfrak{q}' of T such that $\mathfrak{q}' \subset \mathfrak{q}_2$ since $\mathfrak{q}_2 \in \mathcal{P}(I_2, S_2)$ and $\mathfrak{p}_1^e \subset \mathfrak{q}'$, i.e., $\mathfrak{p}_1 = \mathfrak{p}_1^{ec} \subseteq \mathfrak{q}' \cap R \subseteq \mathfrak{p}_1$ since $\mathfrak{q}' \cap S_1 = \emptyset$. Then $\mathfrak{q}' \cap R = \mathfrak{p}_1$ and hence we can choose for \mathfrak{q}_2 the prime ideal \mathfrak{q}' , i.e., $\mathfrak{q}_2 = \mathfrak{q}'$. Thus, φ satisfies the going-down property. \square

These results lead up to raise the following question

(q') Let $\varphi : R \rightarrow T$ be a ring-homomorphism such that $\mathfrak{p}^{ec} = \mathfrak{p}$ for each $\mathfrak{p} \in \text{Spec}(R)$. Is φ an integral homomorphism?

Counterexample 4.2. Let n be a nonzero integer number. We consider $R = \frac{\mathbb{Z}}{n\mathbb{Z}}$, then R is a zero-dimensional ring. Let $T = R[X]$ be the polynomial ring of R . We denote by $\varphi : R \rightarrow T = R[X]$ the natural homomorphism, and let $\mathfrak{p} \in \text{Spec}(R)$, we have $\mathfrak{p}^e = \mathfrak{p}[X]$ and hence $\mathfrak{p}^{ec} = \mathfrak{p}$. It follows that $\mathfrak{p}^{ec} = \mathfrak{p}$ for each $\mathfrak{p} \in \text{Spec}(R)$. However, φ is not an integral extension since $\dim(R) = 0$ and $\dim(T) = 1$.

REFERENCES

- [1] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Springer - Verlag, 1995.
- [2] R. Gilmer, "*Multiplicative Ideal Theory*," Dekker, New York, 1972.
- [3] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.