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EFFECTS OF DEGREE-BIASED TRANSMISSION RATE AND NONLINEAR INFECTIVITY ON RUMOR SPREADING IN COMPLEX SOCIAL NETWORKS

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Abstract

We introduce the generalized rumor spreading model and analytically investigate the epidemic spreading for this model on scale-free networks. To generalize the standard rumor spreading model (rumor model in which each node's infectivity equals its degree and all links have a uniform connectivity strength), we introduce not only the infectivity function to determine the simultaneous contacts that a given node (individual) establishes to its connected neighbors but also the connectivity strength function (CSF) for the direct link between two connected nodes that lead to degree-biased propagation of rumors. In the case of nonlinear functions, the generalization enters the infectivity's exponent α and the CSF's exponent β into the analytical rumor model. We show that one can adjust the exponents α and β to control the epidemic threshold which is absent for the standard rumor spreading model. In addition, we obtain the critical threshold for the generalized model on the finite scale-free network and compare our results with the standard model on the same network. We show that the generalized model has a greater threshold than the standard model.

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1 Introduction

During the last years, network modelling is undoubtedly the favorite tool used by researchers for representing many social, biological and physical systems [1, 2]. In general terms, complex networks are connected graphs with at most a single edge between nodes where nodes stand for individuals ,and an edge for connecting two nodes denotes the interaction between individuals [3, 4]. The connectivity pattern in these networks encodes information about the structure of the system [5]. An important and much studied characteristic of these networks is their degree distribution $p(k)$, defined as the probability that a randomly selected node is connected to k other nodes. It was found that many interesting networks such as technology (the internet [6] and the world wide web [7]), social sciences (sexual contact networks [8], friendship networks [9], scientific collaboration networks [10]) and biology systems (metabolic networks [11], food webs [12]) are very heterogene with scale-free (SF) degree distribution $p(k) \sim k^{-2-\gamma}$ (power-law distribution) where γ is a characteristic degree exponent, usually in the range $0 < \gamma \leq 1$. The study of epidemics in heterogene networks is therefore of practical importance for the control of the spread of viruses, diseases and biological epidemics.

The modelling of infectious diseases is a tool which has been used to study the mechanisms by which diseases spread, to predict the future course of an outbreak and to evaluate strategies to control an epidemic [13]. Among the numerous possible models, the most investigated and classical models are the susceptible-infected-susceptible (SIS) model [14, 15, 16, 17] and the susceptible-infected-removed (SIR) model [18, 19, 20] which can describe the spreading of diseases in social networks, computer virus and trash mail in technological networks. One of the remarkable results for SIS and SIR models in an infinite-size SF network is that the infection will be epidemic regardless of its spreading rate (i.e., the critical threshold of transmission rate is zero).

Rumors have been a basic element of human interaction for as long as people have had questions about their social environment [21]. Rumor spreads rapidly, is difficult to control, is invisible yet nearly impossible to ignore, and can have damaging and perhaps even deadly consequences. We know it is probably bad for us, and we know it can hurt those around us.

Despite its obvious negative connotations, a rumor has the capacity to satisfy certain fundamental personal and social needs, and in this sense can be beneficial to those who participate in its transmission [22, 23, 24]. Rumors help people make sense of what is going on around them. Also rumor-mongering is a way of trying to explain what is happening and why-whether it be a crime in the neighborhood, a political crisis, or a change in a company's management.

Rumor can be interpreted as an *infection of the mind*. The original model of rumor spreading was introduced by Daley and Kendall (DK) [25, 26]. An important variant model of DK is the Maki-Thompson (MK) model [27]. In the past, these models have been used extensively for studying of rumor spreading [28, 29, 30]. In the DK model a closed and homogeneously mixed population is subdivided into three groups: the ignorant, those who have not heard the rumor yet; the spreader, those who have heard the rumor and are willing to transmit it; the stifler, those who heard the rumor but have lost interest in the rumor and do not transmit it. The rumor is propagated through the population by pair-wise contacts between spreaders and others in the society. Any spreader involved in a pair-wise meeting attempts to *infect* the other individual with the rumor. If this other individual is an ignorant, it becomes a spreader; otherwise the spreader meets another spreader or stifler, so they understand that the rumor is known and do not propagate the rumor anymore, therefore turning into stiflers. In the MK model the rumor is spread by directed contacts of the spreaders with others in the population. Furthermore, when a spreader contacts another spreader only the initiating spreader becomes a stifler.

In the above mentioned models of rumor spreading, the authors have assumed that the rumors spread across homogene networks (with degree distribution peaked around the average value), on the other hand their calculations were in the limit of highly simplified models of the topology [28, 29]. While in the real world, the topology of such large social networks shows highly complex connectivity patterns in which the degree distribution is skewed and may present heavy-tails, or more generally, large fluctuation around the average value [30, 31]. Recently, in Ref. [33], the authors introduced a new model of rumor spreading on complex networks which, in comparison with previous models, provides a more realistic description of this process. Their model unifies the MK model of rumor spreading with SIR models of epidemics, and has both of these models as its limiting cases. They have used approximate analytical and exact numerical solution of mean-field equation to examine both the steady-state and the time-dependent behavior of the model on several models of social networks (homogeneous networks, Erdős-Rény (ER) random graphs and uncorrelated scale-free networks). They found that their model shows a new critical behavior on networks with bounded degree fluctuations, such as random graphs, and that this behavior is absent in scale-free networks with unbounded fluctuations in node degree distribution.

We know that in the studied rumor spreading models the transmission rate of rumor is a constant, but in the real world it should be different among individuals depending on their intimacy. Thus, in order to make transmission rate accords with realistic cases much more, we introduce a connectivity strength function (CSF) between connected nodes. For epidemic spreading, when two nodes are connected, the larger CSF, the more possible that the two nodes communicate.

On the other hand, in the classical epidemic models, each spreader can establish contacts with all his (her) neighbors within one time step, that is to say, each node's infectivity (a rumor as an infection of the mind) equals its degree. But in the real case, an individual can't make contact with all acquaintances (connected neighbors) at one time step. In the case of SIS and SIR models, recently in Ref. [34] the authors have left this assumption and they assumed that the infectivity is identical (a constant A) for all nodes regardless of their different degrees. Also in [35] the authors have proposed a piecewise linear infectivity, which means: if the degree k of a node is relatively small, its infectivity is αk; if k is big, i.e., surpasses a constant A/α, then its infectivity will be A. In terms of these new assumptions, the heterogeneous infectivity of nodes with different degrees can't consider as adequately as possible in scale-free networks, it means that there may be some nodes with different degrees which have the same infectivity, and there will be a large number of such nodes according to an irrelevantly selected constant A or the size of underlying networks is infinite. So, in order to solve these problems, in Ref. [36] the authors introduced the nonlinear infectivity function that controls the number of contacts that a node generates within one time step. We follow the latter argument and use the nonlinear function for the infectivity of nodes that spread the rumor to their neighbors.

The rest of this paper is organized as follows: In section 2 we introduce the standard model of rumor spreading and shortly review epidemic dynamics of this model. In section 3 we introduce the generalized rumor spreading model and analytically study in detail the dynamics of this model on infinite scale-free networks. In section 4 we compare the epidemic behavior of the standard and generalized model on finite scale-free networks. Finally, our conclusions are presented in last section.

2 Standard rumor spreading model

The rumor model is defined as follows. Each of the individuals (the nodes in the network) can be in three different categories with respect to the rumor. Individuals are in this way classified as I-ignorant (who are ignorant of the rumor), S-spreader (who have heard the rumor and actively spread it), R-stifler (who knew the rumor but have ceased to spread it). According to Maki and Thompson [27], the spreading process evolves by directed contact of the spreaders with others of the population. However, these contacts can only occur along the links of an undirected social interaction network $G = (V, E)$, where V and E denote the nodes and the edges of the network, respectively. The model that we call the standard model has been studied in Ref. [33]. By following [33], the possible events that can occur between the spreaders and the rest of population are

- $SI \longrightarrow SS$ whenever a spreader meets an ignorant, the ignorant becomes a spreader at a rate λ.
- $SS \longrightarrow RS$ when a spreader contacts another spreader or a stifler, the initiating spreader becomes a stifler at a rate σ.
- $S \longrightarrow R$ there is also a rate δ for a spreader to cease spreading a rumor spontaneously (i.e., without any contact).

2.1 Dynamics of standard model

Let $I_k(t)$, $S_k(t)$ and $R_k(t)$ denote the densities of ignorant, spreader and stifler nodes (individuals) with connectivity (degree) k at time t, respectively. These quantities satisfy the normalization condition $I_k(t) + S_k(t) + R_k(t) = 1$, for all k-classes. We shortly review some classical results from [33], where Nekovee et al. described a formulation of this model on networks in terms of interacting Markov chains, and used this framework to derive, from first-principles, mean-field equations for the dynamics of rumor spreading on complex networks with arbitrary degree correlations as follows:

$$\frac{dI_k(t)}{dt} = -k\lambda I_k(t) \sum_l S_l(t) P(l|k) \tag{1}$$

$$\frac{dS_k(t)}{dt} = k\lambda I_k(t) \sum_l S_l(t) P(l|k) - k\sigma S_k(t) \sum_l (S_l(t) + R_l(t)) P(l|k) - \delta S_k(t) \tag{2}$$

$$\frac{dR_k(t)}{dt} = k\sigma S_k(t) \sum_l (S_l(t) + R_l(t)) P(l|k) + \delta S_k(t) \tag{3}$$

where the conditional probability $P(l|k)$ means that a randomly chosen link emanating from a node of degree k leads to a node of degree l. Moreover, we suppose that the degrees of nodes in the whole network are uncorrelated, i.e., $P(l|k) = lp(l)/\langle k \rangle$ where $p(k)$ is the degree distribution and $\langle k \rangle$ is the average degree. In this case, Nekovee et al. showed that to leading order in σ, the critical threshold is independent of the stifling mechanism, i.e., $\frac{\lambda}{\delta} \geq \frac{\langle k \rangle}{\langle k^2 \rangle}$, so in particular, for $\delta = 1$ the critical threshold is given by $\lambda_c = \langle k \rangle / \langle k^2 \rangle$ and it is the same as for the SIR model [18, 20]. This result implies the absence of the epidemic threshold in a wide range of scale-free networks ($\langle k^2 \rangle \to \infty$, $\lambda_c \to 0$). This is a bad message for epidemic controlling, since the epidemic will prevail in many real networks with any nonzero value of transmission rate λ.

3 Generalized rumor spreading model

In order to make the transmission rate fit with realistic cases much more, we take into account the effect of strength of connectivity between individuals. For example, in social networks it can present the intimacy, confidence, kinship etc. between individuals. So different from the previous studies (that each individual can spread the rumor with constant transmission rate λ), in this paper, we mainly focus on the rumor spreading model in which the transmission rate between two connected nodes is a function of their degrees. Based on this assumption, we define $g(k, l)$ as a connectivity strength function (CSF) for the link (k, l). Let G_k denote the total strength of connectivity for a node with degree k, which can be obtained by summing the CSFs of the links that are connected to it, i.e., $G_k = k \sum_l P(l|k) g(k, l)$.

Here, for each node with degree k we keep constant a total rumor transmission rate and a total stifling process rate, which are given by λk and σk, respectively. The rumor transmission rate from k-degree node to l-degree node, will be determined by the proportion of the $g(k,l)$ to G_k, therefore the λ_{kl} can be written as follows:

$$\lambda_{kl} = \lambda k \frac{g(k,l)}{G_k}, \tag{4}$$

from which we know that the more proportion of $g(k,l)/G_k$, the more possible that a rumor transmits through the edge. For the same reason, the stifling process rate is $\sigma_{kl} = \sigma k \frac{g(k,l)}{G_k}$. In this paper, for simplicity, we focus on uncorrelated networks, where the conditional probability satisfies $P(l|k) = lp(l)/\langle k\rangle$, and we assume the $g(k,l)$ is a symmetric multiplicative function of the degrees at the edge's endpoints, namely $g(k,l) = \eta(k)\eta(l)$. Later we will show that this assumption leads to introducing the biased spreading of the rumors. So one can obtain $G_k = \frac{k\eta(k)\langle k\eta(k)\rangle}{\langle k\rangle}$ and λ_{kl} and σ_{kl} are reduced to

$$\lambda_{kl} = \lambda \frac{\langle k\rangle \eta(l)}{\langle k\eta(k)\rangle}, \qquad \sigma_{kl} = \sigma \frac{\langle k\rangle \eta(l)}{\langle k\eta(k)\rangle}. \tag{5}$$

On the other hand, one of the inappropriate assumptions in the details of the standard rumor spreading model, is that each spreader can establish contacts with all his (or her) neighbors within one time step, that is to say, each node's infectivity equals its degree. But in the real case, an individual can't contact all his (or her) friends simultaneously. So considering epidemic in real cases, we leave this assumption and we introduce the infectivity function $\varphi(k)$ to take control of the number of contacts that a spreading node generates within one time step. To rewrite Eq.(1-3) for the generalized model, we should replace the λ, σ and $P(l|k)$ by λ_{kl}, σ_{kl} and $\frac{\varphi(l)P(l|k)}{l}$, respectively. So we have

$$\frac{dI_k(t)}{dt} = -\frac{\lambda k\eta(k)}{\langle k\eta(k)\rangle} I_k(t) \sum_l S_l(t) p(l)\varphi(l), \tag{6}$$

$$\begin{aligned} \frac{dS_k(t)}{dt} &= \frac{\lambda k\eta(k)}{\langle k\eta(k)\rangle} I_k(t) \sum_l S_l(t) p(l)\varphi(l) \\ &- \frac{\sigma k\eta(k)}{\langle k\eta(k)\rangle} S_k(t) \sum_l (S_l(t) + R_l(t)) p(l)\varphi(l) - \delta S_k(t), \end{aligned} \tag{7}$$

$$\frac{dR_k(t)}{dt} = \frac{\sigma k\eta(k)}{\langle k\eta(k)\rangle} S_k(t) \sum_l (S_l(t) + R_l(t)) p(l)\varphi(l) + \delta S_k(t). \tag{8}$$

Eq. (6) can be integrated exactly to yield:

$$I_k(t) = I_k(0) e^{-\frac{\lambda k\eta(k)}{\langle k\eta(k)\rangle}\phi(t)}, \tag{9}$$

where $I_k(0)$ is the initial density of ignorant nodes with connectivity k, and we have used the auxiliary function

$$\phi(t) = \sum_k p(k)\varphi(k) \int_0^t S_k(t')dt' \equiv \int_0^t \langle \varphi(k) S_k(t')\rangle dt'. \tag{10}$$

In order to get a closed relation for the final size of the rumor, R, it is more useful to focus on the time evolution of $\phi(t)$. Assuming a homogeneous initial distribution of ignorant, i.e., $I_k(0) = I_0$ (without loss of generality, we can put $I_0 \approx 1$), we can obtain a differential expression for $\phi(t)$ by multiplying (7) with $p(k)\varphi(k)$ and summing over all k's. After some elementary manipulations, one finds

$$\begin{aligned} \frac{d\phi}{dt} &= \langle\varphi(k)\rangle - \langle\varphi(k)e^{-\frac{\lambda k\eta(k)}{\langle k\eta(k)\rangle}\phi(t)}\rangle - \delta\phi \\ &- \frac{\sigma}{\langle k\eta(k)\rangle}\int_0^\infty [\langle\varphi(k)\rangle - \langle\varphi(k)e^{-\frac{\lambda k\eta(k)}{\langle k\eta(k)\rangle}\phi(t')}\rangle]\langle k\eta(k)\varphi(k)S_k(t')\rangle dt', \end{aligned} \quad (11)$$

On the infinite time limit, i.e., at the end of the epidemics, we have that $S_k(\infty) = 0$ and consequently $lim_{t\to\infty} d\phi(t)/dt = 0$, so Eq. (11) becomes:

$$\begin{aligned} 0 &= \langle\varphi(k)\rangle - \langle\varphi(k)e^{-\frac{\lambda k\eta(k)}{\langle k\eta(k)\rangle}\phi_\infty}\rangle - \delta\phi_\infty \\ &- \frac{\sigma}{\langle k\eta(k)\rangle}\int_0^\infty [\langle\varphi(k)\rangle - \langle\varphi(k)e^{-\frac{\lambda k\eta(k)}{\langle k\eta(k)\rangle}\phi(t')}\rangle]\langle k\eta(k)\varphi(k)S_k(t')\rangle dt', \end{aligned} \quad (12)$$

For $\sigma = 0$, one can solve explicitly Eq. (12) to find a closed relation for ϕ_∞. For $\sigma \neq 0$ we solve Eq. (12) to leading order in σ. For this purpose, it is sufficient to obtain $S_k(t)$ to zero order in σ. $S_k(t)$ or Eq. (7), to zero order in σ, is a first order linear differential equation that has the form $\frac{dy}{dt} + p(t)y = q(t)$ and it can be easily solved to obtain

$$S_k(t) = 1 - e^{-\frac{\lambda k\eta(k)}{\langle k\eta(k)\rangle}\phi(t)} - \delta\int_0^\infty e^{\delta(t-t')}[1 - e^{-\frac{\lambda k\eta(k)}{\langle k\eta(k)\rangle}\phi(t')}]dt' + O(\sigma). \quad (13)$$

Close to the critical threshold both $\phi(t)$ and ϕ_∞ are small. Writing $\phi(t) = \phi_\infty f(t)$, where $f(t)$ is a finite function. Thus if we only hold the leading order of ϕ_∞, we obtain

$$S_k(t) \simeq -\delta\frac{\lambda k\eta(k)}{\langle k\eta(k)\rangle}\phi_\infty I + O(\phi_\infty^2) + O(\sigma), \quad (14)$$

where I is a finite and positive integral that has the form $I = \int_0^t e^{\delta(t-t')}f(t')dt'$. Putting this in Eq. (12) and expanding the exponential to the relevant order in ϕ_∞ we obtain

$$\begin{aligned} 0 &= \left[\lambda\frac{\langle k\eta(k)\varphi(k)\rangle}{\langle k\eta(k)\rangle} - \delta\right]\phi_\infty \\ &- \frac{\lambda^2\langle k^2\eta^2(k)\varphi(k)\rangle}{\langle k\eta(k)\rangle^2}\left[\frac{1}{2} + \sigma\delta\frac{\langle k\eta(k)\varphi(k)\rangle}{\langle k\eta(k)\rangle}\right]\phi_\infty^2 + O(\sigma^2) + O(\phi_\infty^3), \end{aligned} \quad (15)$$

The non-zero solution of above equation is given by:

$$\phi_\infty = \frac{\lambda\frac{\langle k\eta(k)\varphi(k)\rangle}{\langle k\eta(k)\rangle} - \delta}{\lambda^2\frac{\langle k^2\eta^2(k)\varphi(k)\rangle}{\langle k\eta(k)\rangle^2}\left(1/2 + \sigma\delta\frac{\langle k\eta(k)\varphi(k)\rangle}{\langle k\eta(k)\rangle}I\right)}. \quad (16)$$

In order to have a positive value for ϕ_∞ the condition

$$\frac{\lambda}{\delta} \geq \frac{\langle k\eta(k)\rangle}{\langle k\eta(k)\varphi(k)\rangle}, \quad (17)$$

must be fulfilled. Thus, to leading order in σ the critical threshold is independent of the stifling mechanism and for $\delta = 1$ the critical threshold is given by

$$\lambda_c = \frac{\langle k\eta(k)\rangle}{\langle k\eta(k)\varphi(k)\rangle}. \tag{18}$$

if λ is below λ_c, the rumor will die out, while if λ is above λ_c, the rumor will spread on the network. For different rumor spreading models in the real world, different $\varphi(k)$ and $\eta(k)$ should be adopted. But in this work we only take the nonlinear (power-law) function form into account in the next section.

Finally, R is given by

$$R = \sum_k p(k)(1 - e^{-\frac{\lambda k\eta(k)}{\langle k\eta(k)\rangle}\phi_\infty}). \tag{19}$$

The solution to the above equation depends on the form of $p(k)$.

3.1 The epidemic threshold for the generalized model of rumor with nonlinear infectivity and nonlinear CSF on scale-free networks

In this model, we assume that $\varphi(k) = k^\alpha$ where $0 < \alpha \leq 1$, it means that each spreader can establish contacts with k^α neighbors within one time step. The exponent α will control the infectivity among nodes with different degrees. Since $0 < \alpha \leq 1$, it can be balanced to make the contacts fall on a more realistic range. Also the node infectivity will grow nonlinearly by increasing degree k.

Furthermore, we suppose that $\eta(k) = ak^\beta$, where a is a positive quantity and β is a real exponent. So, according to Eq. (5) the spreading rate is $\lambda_{kl} = \lambda\frac{\langle k\rangle l^\beta}{\langle k^{1+\beta}\rangle}$. The exponent β allows to tune the dependence of the transmission process on the node's degree. When $\beta \neq 0$ we are introducing in the random transmission of the rumor a bias towards high-degree ($\beta > 0$) or low-degree (when $\beta < 0$) neighbors. Also, when $\beta = 0$ the standard (unbiased) spreading process is recovered. By putting the above mentioned $\varphi(k)$ and $\eta(k)$ in Eq. (18) we get the λ_c, the epidemic threshold, of degree-biased transmission of the rumor on the network:

$$\lambda_c = \frac{\langle k^{\beta+1}\rangle}{\langle k^{\alpha+\beta+1}\rangle}, \tag{20}$$

Now, we consider the epidemic threshold in the case of general scale-free networks in which the degree distribution is $p(k) = ck^{-2-\gamma}$, $0 < \gamma \leq 1$, where c is the normalization constant. For this purpose, we obtain $\langle k^{\beta+1}\rangle = c(k_c^{\beta-\gamma} - m^{\beta-\gamma})/(\beta-\gamma)$ and $\langle k^{\alpha+\beta+1}\rangle = c(k_c^{\alpha+\beta-\gamma} - m^{\alpha+\beta-\gamma})/(\alpha+\beta-\gamma)$, where $k_c(m)$ denotes the largest (smallest) degree in the underlying network. By substituting these into Eq. (20), one can rewrite the epidemic threshold as follows:

$$\lambda_c = \frac{\alpha+\beta-\gamma}{\beta-\gamma} \times \frac{k_c^{\beta-\gamma} - m^{\beta-\gamma}}{k_c^{\alpha+\beta-\gamma} - m^{\alpha+\beta-\gamma}}, \tag{21}$$

From (21), one can see that the infinite of the largest degree ($k_c \to \infty$ or equally $N \to \infty$, since $k_c \propto N^{1/(\gamma+1)}$[37]) will make the epidemic threshold λ_c tends toward zero if $\gamma < \alpha + \beta$; on the other hand, if $\gamma > \alpha + \beta$, the epidemic threshold λ_c is approximated to be a finite value, given by

$$\lambda_c = m^{(-\alpha)} \frac{\alpha + \beta - \gamma}{\beta - \gamma}. \tag{22}$$

Thus, the critical border is $\gamma = \alpha + \beta$. One can adjust the infectivity's exponent α and the CSF's exponent β to obtain nonzero threshold for the given networks (a fixed value of γ).

Let us concentrate on quantity β. For $\beta > 0$, if one chooses the exponents α and β provided $0 < \alpha \leq 1$, $\beta > 0$ and $\gamma > \alpha + \beta$, one obtains the model of rumor spreading in which λ_c is a finite value that above it rumor will spread mentioned as before , when $\beta > 0$ we have the model that exhibits the biased spreading of the rumor from low-degree nodes to high-degree nodes. We call this model as a *down-up epidemic model*. On the other side, for $\beta < 0$ together with an allowable α that satisfies the constraint $\gamma > \alpha + \beta$ (for $\beta \leq -\alpha$ the constraint $\gamma > \alpha + \beta$ is always satisfied since $0 < \gamma \leq 1$), the model has a finite critical threshold for the rumor spreading rate that above it a rumor can propagate in the system. In this case, we have the model in which exists the biased spreading of rumor towards low-degree nodes. We call this model as an *up-down epidemic model*. We believe that these models are more remarkable than previous studied models [29, 30, 33].

4 The epidemic threshold for the generalized rumor spreading model on finite scale-free networks

In the real world, an epidemic always occurs on a finite network, though the size of the network may be very large. In paper [38], the authors studied the epidemic threshold $\lambda_c(k_c)$ for the SIS model on bounded scale-free networks with the soft and hard cut-off k_c when $\varphi(k) = k$ and $\eta(k) = 1$. In terms of hard cut-off, the network doesn't possess any nodes with connectivity k larger than k_c and the maximum connectivity k_c of any node is related to network age, measured as the number of nodes N

$$k_c = mN^{1/(1+\gamma)}, \tag{23}$$

where m is the minimum connectivity of the network. In this case the normalized connectivity distribution has the form [38]

$$p(k) = \frac{(1+\gamma)m^{1+\gamma}}{1 - (k_c/m)^{-1-\gamma}} k^{-2-\gamma} \theta(k_c - k). \tag{24}$$

where $\theta(x)$ is the Heaviside step function. Now we apply the same argument for the epidemic threshold of the standard and generalized model of rumor spreading on a finite network. First, the standard model: for this model we have $\varphi(k) = k$ ($\alpha = 1$) and $\eta(k) = 1$ ($a = 1$, $\beta = 0$), so the epidemic threshold $\lambda_c'(k_c)$ is given by

$$\lambda_c'(k_c) = \frac{\langle k \rangle}{\langle k^2 \rangle} = \frac{\int_m^{k_c} k^{-1-\gamma} dk}{\int_m^{k_c} k^{-\gamma} dk} \simeq \frac{1-\gamma}{\gamma m} (k_c/m)^{(\gamma-1)}, \tag{25}$$

By (23) and (25), we obtain

$$\lambda_c'(N) \simeq \frac{1-\gamma}{\gamma m}(N)^{(\gamma-1)/(\gamma+1)}. \tag{26}$$

If $\gamma = 1$, we find

$$\lambda_c'(N) \simeq 2[mln(N)]^{-1}. \tag{27}$$

(26) and (27), we know that the effective epidemic threshold is approaching zero by increasing network size.

Second, the generalized model: we consider the epidemic threshold $\lambda_c^*(k_c)$ for $\varphi(k) = k^\alpha$, $0 < \alpha < 1$, and $\eta(k) = ak^\beta$ where $a > 0$ and β is a real number. After similar calculation one can obtain

$$\lambda_c^*(k_c) = \frac{\int_m^{k_c} k^{\beta-1-\gamma}dk}{\int_m^{k_c} k^{\alpha+\beta-1-\gamma}} = m^{(-\alpha)}\frac{\alpha+\beta-\gamma}{\beta-\gamma}\frac{((k_c/m)^{\beta-\gamma}-1)}{((k_c/m)^{\alpha+\beta-\gamma}-1)}, \tag{28}$$

Note that by increasing network size, the ratio (k_c/m) is sufficiently large (see Eq. (23)), so when $\alpha+\beta < \gamma$ we have $(k_c/m)^{\beta-\gamma} = (k_c/m)^{\alpha+\beta-\gamma} = 0$ and λ_c^* has a positive value, otherwise λ_c^* is approaching zero. Let us focus on the up-down epidemic model ($\beta < 0$). Thus the above equality can be rewritten as follows

$$\lambda_c^*(k_c) = \begin{cases} m^{(-\alpha)}\frac{\alpha+\beta-\gamma}{\gamma-\beta}(k_c/m)^{\gamma-\alpha-\beta} & \alpha+\beta > \gamma \\ m^{(-\alpha)}\frac{\gamma-\alpha-\beta}{\gamma-\beta} & \alpha+\beta < \gamma \\ m^{(-\alpha)}\frac{1}{\alpha ln(k_c/m)} & \alpha+\beta = \gamma \end{cases} \tag{29}$$

Combining Eq. (23) and Eq. (29), we have

$$\lambda_c^*(N) = \begin{cases} m^{(-\alpha)}\frac{\alpha+\beta-\gamma}{\gamma-\beta}(N)^{(\gamma-\alpha-\beta)/(\gamma+1)} & \alpha+\beta > \gamma \\ m^{(-\alpha)}\frac{\gamma-\alpha-\beta}{\gamma-\beta} & \alpha+\beta < \gamma \\ m^{(-\alpha)}\frac{\gamma+1}{\alpha ln(N)} & \alpha+\beta = \gamma \end{cases} \tag{30}$$

It is obvious from the above equation that the positivity of the critical value $\lambda_c^*(N)$ is unrelated to the size N of the network when $\alpha+\beta < \gamma$ (second term in Eq. (30)), and it is the same as the critical threshold λ_c for infinite scale-free network (see Eq. (22)).

To compare models under condition $\alpha+\beta > \gamma$, we take the ratio of Eq. (26) and the first term in Eq. (30),

$$\frac{\lambda_c'(N)}{\lambda_c^*(N)} = \frac{(1-\gamma)(\gamma-\beta)}{\gamma m^{(1-\alpha)}(\alpha+\beta-\gamma)N^{\frac{1-\alpha-\beta}{\gamma+1}}}. \tag{31}$$

It is straightforward that $\frac{\lambda_c'(N)}{\lambda_c^*(N)} < 1$ when the size of the network $N > N_0$, where N_0 is a positive integer, it means that the epidemic threshold $\lambda_c^*(N)$ is greater than $\lambda_c'(N)$ on finite scale-free networks with the same size $N > N_0$, so an epidemic rumor has more difficulty in propagation for the case $\{\varphi(k) = k^\alpha,\ \eta(k) = ak^\beta,\ \beta < 0,\ \alpha+\beta > \gamma\ \}$, than for the case $\{\varphi(k) = 1,\ \eta(k) = 1\ \}$ on finite scale-free networks with the same size.

Our method to study the generalized model of rumor spreading can be used for studying the SIR [18, 19], SIRS [39] and the suchlike epidemic processes.

5 Conclusion

In this paper, we have studied the dynamical behavior of the generalized model of rumor spreading with degree-biased transmission rate (the result of nonlinear CSF) and nonlinear infectivity. We have shown that one can adjust the infectivity's exponent α and CSF's exponent β to control the epidemic threshold which is absent for the standard rumor spreading model in scale-free networks. In the case of general infinite scale-free networks, we analytically showed that $\beta < 0$ and $\beta > 0$ lead to two new models i.e., the up-down epidemic model (the biased spreading of the rumor towards low-degree neighbors) and the down-up epidemic model (the biased spreading of the rumor towards high-degree neighbors), respectively, in which critical threshold λ_c takes a positive value. Also, in the case of finite scale-free networks, we obtained the epidemic threshold λ'_c of the standard model and the epidemic threshold λ^*_c of the generalized model, we concluded that λ^*_c is a positive value when $\alpha+\beta < \gamma$ and is unrelated to the size N of the network. Finally, we showed that λ^*_c is greater than λ'_c (when $\beta < 0$ and $\alpha + \beta > \gamma$) on a finite scale-free network with the same size.

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