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**THE STUDY OF BIREFRINGENT HOMOGENOUS MEDIUM
WITH GEOMETRIC PHASE**

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Abstract

The property of linear and circular birefringence at each point of the optical medium has been evaluated here from differential matrix N using the Jones calculus. This matrix lies on the OAM sphere for $l = 1$ orbital angular momentum. The geometric phase is developed by twisting the medium uniformly about the direction of propagation of the light ray. The circular birefringence of the medium, is visualized through the solid angle and the angular twist per unit thickness of the medium, k , that is equivalent to the topological charge of the optical element.

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The cyclic transformations of polarized light develops both dynamical and geometric phase (GP). The GP has attracted attention in the last 25 years for its robustness, insensitive to vibrations and other mechanical effects. Four manifestations of GP have been reported in optics so far. i) Pancharatnam phase [1] is the first identified GP, $\Omega/2$, originated from the cycle changes of polarization of a plane polarized light with fixed direction over a closed path on the Poincare sphere. Berry found the quantal counterpart [2] of Pancharatnam's phase in case of cyclic adiabatic evolution. He studied the phase two-form (GP) [3] in connection with the dielectric tensor and birefringence of the medium. ii) The second kind of phase was experimentally performed by Chaio and co-workers [4] when the light with fixed polarizations slowly rotate around a closed circuit with the varied directions. The developed GP was the spin-redirection or coiled light phase. iii) The third one was developed by the squeezed state of light through the cyclic changes of Lorentz transformation [5]. iv) The fourth GP was studied by van Enk [6], in case of cyclic change of transverse mode pattern of Gaussian light beam without affecting the direction of propagation or the polarization of light.

Optical elements such as linear and circular polarizer, retarder, rotator, have the representation of (2×2) matrix by Jones [7],[8]. The passage of light through a sequence of optical elements has been found in the Jones calculus through matrix multiplications. Azzam studied [9] the passage of polarized light through anisotropic media with and without depolarization, using both differential 4×4 matrix and the 2×2 calculus of Jones. From the view point of GP, the idea of Jones has been re-established for plane polarized light passing through the polarization matrix M generated from the polarized photon as two-component spinors of spherical harmonics whose plane of polarization is rotated over a closed path by a rotator [10]. The geometric phase has been studied further from the dielectric property through the differential matrix N of Jones in the anisotropic medium [11].

Recently the study with angular momentum of photon plays an interesting role. The first observation of the angular momentum of light was by Beth [12] by making an experiment where a beam of right circularly polarized light was passed through a birefringent medium (quarter-wave plate) and transformed to left circularly polarized light transferring $2\hbar$ of spin angular momentum for each photon to the birefringent plate. When light is circularly polarized each photon carries a spin angular momentum (SAM) given by $\sigma\hbar = \pm 1$ corresponding to left and right-handed circular polarization respectively. These two states are the eigen states of the quantum mechanical spin operator. Apart from the SAM, photons can also carry orbital angular momentum (OAM) arising from the inclination of the phase fronts with respect to beam's propagation axis. An important advancement was to realize the connection between topological charge and the orbital angular momentum of single photon by Allen et.al. [13] for Laguerre-Gaussian (LG) beams with azimuthal phase $\exp(il\phi)$ for OAM l per photon. Galves et.al. gave an experimental measurement of GP [14] in mode space acquired by the optical angular momentum. A detailed description regarding the appearance of OAM and SAM in the field of optics was given by Tiwari [15] along with

their connection with different kinds of GPs. As the exchange of momentum is inevitable by the interaction between radiation and matter, the OAM and SAM of polarized light have acquired growing interest even in Poincare representation [16] in the present days studies. The equivalence between the phase shift introduced by birefringent wave plates and mode converter allows one to treat the mode structure formulation analogous to the Jones matrix. There is an interesting analogy [17] between rotations in one formulation and mode converters in another formulation.

Our present study will be at first to focus on the appearance of GP by the passage of polarized light through the non-absorbing birefringent anisotropic optical medium. The optical property of the medium represented by differential matrix N , varies along the direction of propagation of the incident photon. This causes an exchange of the optical power between the two states of the light along two directions for which a natural twist is visible. The appearance of GP will be studied in two ways by

i) uniformly twisted crystal ii) arbitrarily twisted crystal Here we would specially like to study the variation of the direction of the polarized light passing through uniformly twisted optical medium. All these recent findings, in relation with angular momentum of photon, indicate that our previous studies [18],[8] and [9] on GP of a polarized photon, was an obvious new representation where the helicity of photon plays the crucial role. In this communication we have in mind to give an angular momentum interpretation of GP appeared in our present work and its comparison with previous findings also.

I. THE JONES MATRIX REPRESENTATION OF BIREFRINGENT MEDIUM

A light beam is said to be polarized whenever it is transmitted through a certain crystalline medium that allows electrical anisotropy [7]. This change of polarization state can be written as

$$\varepsilon = M\varepsilon_o \quad (1)$$

If the polarization of light remains unaltered with the passage through any optical element then the state can be the eigenvector of the optical component and in the language of matrix, Jones had shown the condition

$$M^n \varepsilon_i = d_i \varepsilon_i \quad (2)$$

where d_i is the eigenvalue corresponding to the eigenvectors ε_i of a particular polarization matrix $M = \begin{pmatrix} m_1 & m_4 \\ m_3 & m_2 \end{pmatrix}$.

Whenever the optical properties such as birefringence and dichroism of a homogeneous medium varies with distance, one can study the light vector ε between the planes z and $z+dz$ by the differential matrix N which refers to the optical element for a given infinitesimal path length within the element. For particular wavelength and direction, the evolution of a light vector ε

becomes

$$\frac{d\varepsilon}{dz} = \frac{dM}{dz} \varepsilon_0 = \frac{dM}{dz} M^{-1} \varepsilon = N \varepsilon \quad (3)$$

where it is evident that N is the operator that determines dM/dz from M as follows

$$N = \frac{dM}{dz} M^{-1} = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix} \quad (4)$$

When N is independent of z , then on integration the dependence of polarization matrix M on z is seen from

$$M = M_0 \exp\left(\int N dz\right) \quad (5)$$

Jones had shown further that the eigenvectors of M are equal to that of N when it is independent of z .

Any homogeneous crystal without optical activity could be considered for normal incidence as a laminated crystal. According to the lamellar representation suggested by Jones [7], a thin slab of a given medium is equivalent to a pile of retardation plates and partial polarizers. Eight constants are required to specify the real and imaginary parts of the four matrix elements of 2×2 N matrix, each possessing one and only one of the eight fundamental properties. The eight optical properties are paired [9] and reduce to four.

- i) Isotropic refraction and absorption
- ii) Linear birefringence and linear dichroism along the xy coordinate axis.
- iii) Linear birefringence and linear dichroism along the bisector of xy coordinate axes.
- iv) Circular birefringence and circular dichroism.

The optical medium that has circular birefringence and linear birefringence will be our point of interest, and could have the following matrix form

$$\theta_{cb} = \eta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (6)$$

$$\theta_{lb} = \rho \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (7)$$

In that case the required differential matrix will be

$$N = \theta_{cb} + \theta_{lb} = \begin{pmatrix} 0 & -\eta + i\rho \\ \eta + i\rho & 0 \end{pmatrix} = \begin{pmatrix} 0 & n_2 \\ n_3 & 0 \end{pmatrix} \quad (8)$$

where η is the circular birefringence that measures the rotation of the plane polarized light per unit thickness and ρ is the part of linear birefringence which measures the difference between the two principal constants with the coordinate axes.

A crystal will be considered homogeneous if any one of the optical property will be visible instead of eight (as if all the eight lamina are sandwiched). The evolution of the ray vector $\varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$ when passes through such medium N , as in eq.(3) could be re-written into the components as

$$\frac{d\varepsilon_1}{dz} = n_1\varepsilon_1 + n_2\varepsilon_2 \quad (9)$$

$$\frac{d\varepsilon_2}{dz} = n_3\varepsilon_1 + n_4\varepsilon_2 \quad (10)$$

For pure birefringent medium represented by eq.(8), one has to use for the evolution of ray vector

$$\frac{d\varepsilon_1}{dz} = n_2\varepsilon_2, \quad \frac{d\varepsilon_2}{dz} = n_3\varepsilon_1 \quad (11)$$

This shows that as the light enters into the birefringent plate, the spatial variation of component of electric vector in one direction gives the effect in the other perpendicular direction. It means that there is an exchange of optical power between the two component states of the polarized light indicating the rotation of the ray vector after entering the medium.

Geometrically this state ε is a point P on the surface of the Poincare sphere that defines a position vector \vec{p} in three dimensional space. The evolution of the vector \vec{p} is equivalent to the cyclic change of the state vector during the passage of infinitesimal distance dz of the optical medium. Huard pointed out in his book [19] that the spatial change of vector as it passes through the crystal becomes

$$\frac{d\vec{p}}{dz} = \vec{\Omega} \times \vec{p} \quad (12)$$

This shows a natural twist by the instantaneous rotation vector Ω along the axis oz about which the \vec{P} makes an elementary angle $d\alpha = \Omega dz$, for thickness dz . The magnitude and direction of the rotation vector depends on the inherent property of the medium, in other words on the element of the N matrix.

(i) **The uniformly twisted crystal**

It had been pointed out [10] that the N -matrices have special dependence upon z and are transformed upon rotation when an originally homogeneous crystal twisted uniformly about an axis parallel to the direction of transmission where k is the angular twist per unit thickness. It could be noted here that this k has similar definition of Ω in eq.(12) but the basic difference lies in their appearance. The former appear in the external space where as the later is in connection with the internal/natural rotation of ray in the medium. The external twist of N_0 results twisted matrix N by

$$N = S(kz)N_0S(-kz) \quad (13)$$

where S is the rotation matrix having developed after uniform rotation about an angle $\theta = kz$. Jones realized the simultaneous rotation of the twisted state ε' in the opposite direction of N

matrix,

$$\varepsilon' = S(-kz)\varepsilon \quad (14)$$

so that it satisfies the following equation with the twisted matrix N'

$$\frac{d\varepsilon'}{dz} = N'\varepsilon' \quad (15)$$

It has been shown [7] after few steps that this twisted matrix N' can be expressed

$$N' = N_0 - kS(\pi/2) \quad (16)$$

in terms of $N_0 =$ the matrix for untwisted crystal and $S(\pi/2)$ denotes the rotation matrix for normal incidence of light. The solution of the above eq. may be written $\varepsilon' = \exp(N'z)\varepsilon'_0$ where ε'_0 is the value of the vector ε' at $z = 0$.

(ii) **The arbitrarily twisted crystal**

The transformation illustrated above may also be applied with an arbitrarily twisted crystal. Jones showed [7] that the N matrix of the twisted crystal is

$$N = S(\omega(z))N_0S(\omega(z)) \quad (17)$$

where $\omega(z)$ specifies the angle of twist, that is the arbitrary function of z . From the similar transformation of the uniformly twisted crystal, one finds

$$N' = N_0 - \left(\frac{d\omega(z)}{dz}\right)S(\pi/2) \quad (18)$$

It has been pointed out by Jones that the derivative $\frac{d\omega(z)}{dz}$ is a constant for a uniformly twisted crystal.

The polarization of the transmitted light will not change if the medium is twisted. Due to the inherent property of the optical medium (N) a natural twist is realized by the light. Further, external twist of the medium might develop an additional phase in fixed OAM sphere. We realize this phase in the next section as OAM holonomy which will visualize the circular dichroism of the medium.

II. THE GEOMETRIC PHASE BY TWISTING HOMOGENEOUS MEDIUM

The polarization state of light traveling in the z direction in the xy plane can be written by a two component spinor [2]

$$|\psi\rangle = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

where $\psi_{\pm} = (d_x \pm id_y)$ the intensity is $I = |d_x|^2 + |d_y|^2$ while the complex ratio d_x/d_y defines its polarization state. The passage of light through an optical element such as birefringent, absorbing

or dichroic plate would be to change both d_x and d_y so that the effect may be represented by $|\psi_f\rangle = M|\psi_i\rangle$. For a non-absorbing plate, there is no change in the intensity and the matrix M is therefore unitary $\det M = 1$ which makes $|\psi_f| = |\psi_i|$.

Every polarization matrix M can be constructed [3] from the eigenvector $|\psi\rangle$ with eigenvalue $+1/2$ through the relation $(|\psi\rangle\langle\psi| - 1/2)$. If we consider the incident polarized eigenvector by [10]

$$|\psi\rangle = \begin{pmatrix} \cos\theta/2e^{i\phi} \\ \sin\theta/2 \end{pmatrix} \quad (19)$$

we can calculate the polarization matrix

$$M = 1/2 \begin{pmatrix} \cos\theta & \sin\theta e^{i\phi} \\ \sin\theta e^{-i\phi} & -\cos\theta \end{pmatrix} \quad (20)$$

having eigenvalues $\pm 1/2$.

To find the N matrix as in the eq.(4), of optical medium along the direction of propagation of the incident light at a particular position z , we have to use M matrix of eq.(20). Since the eigenvalues of helicities for polarized photons is ± 1 , we omitted the factor $1/2$ from the polarization matrix M for following calculations.

$$N = \left(\frac{dM}{d\theta}\right)\left(\frac{d\theta}{dz}\right)M^{-1} \quad (21)$$

If θ is directly related to the thickness z of the optical medium by $\theta = kz$, where k is the angular twist per unit thickness we find the differential matrix N as follows

$$N = k \begin{pmatrix} 0 & -e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix} \quad (22)$$

having complex eigenvalues $\pm ik$ with eigenvectors

$$\begin{pmatrix} \pm ie^{i\phi} \\ 1 \end{pmatrix} \quad (23)$$

which is different from that of M as N has dependence on the thickness z . The nature of the optical medium could be identified comparing the above N matrix in eq.(22) with eq.(8). It is seen that our N matrix is homogenous and has circular birefringence $k \cos\phi$ and linear birefringence $(-k \sin\phi)$. It may be noted that in case one observes the eigenvalues of N are opposite and imaginary, the optical medium possesses the property of purely circular birefringence.

We are now interested initially to rotate the light ray considering the angular twist per unit thickness zero, $k = 0$ (for $\theta = 0$), then for a uniformly twisted optical medium, using eq.(16), the twisted matrix becomes

$$N' = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \quad (24)$$

for the untwisted matrix $N_0 = 0$ and the rotation matrix $S(\pi/2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. As the optical medium is twisted, the twisted ray will be obtained from eq.(14)

$$\varepsilon' = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} ie^{i\phi} \\ 1 \end{pmatrix} \quad (25)$$

in other words

$$\varepsilon' = \begin{pmatrix} ie^{i\phi} \cos \theta + \sin \theta \\ -ie^{i\phi} \sin \theta + \cos \theta \end{pmatrix} \quad (26)$$

If an another opposite rotation is given to ε' , the initial eigenvector ε appears again. Now the optical medium could be changed by changing the initial value of N in eq.(22), by using $k = 1$ and as a result the twisted matrix N' becomes

$$N' = \begin{pmatrix} 0 & -e^{i\phi} + k \\ e^{-i\phi} - k & 0 \end{pmatrix} \quad (27)$$

It can be realized looking at eqs.(22),(24) and (27) that k , the angular twist per unit thickness of the optical medium plays the important role in the birefringent medium as similar as Ω for twisting ray vector.

Light having fixed polarization and helicity if suffers the slow variation of path in real space that can be mapped on to the surface of unit sphere in the wave vector space. The geometric phase is found to appear as the initial state $|A\rangle$ unite with final $|A'\rangle$.

$$\langle A|A'\rangle = \pm \exp(i\Omega(C)/2) \quad (28)$$

where Ω is the solid angle swept out by e_k on its unit sphere.

Our present work is based on the consideration of polarized light passing normally through a medium N having linear and circular birefringence. Due to the inherent property of the medium, the incident polarized light suffers a natural twist about the axis parallel to the direction of its propagation. We assume Υ and Υ' are the respective phases developed due to the natural and forced rotation of the optical birefringent medium N . These phases can be written considering the initial state $|A\rangle = \varepsilon$, that after passing through the differential matrix N becomes the final state $\frac{d\varepsilon}{dz} = |A'\rangle$ as follows.

$$\Upsilon = \varepsilon^* \frac{d\varepsilon}{d\theta} \frac{d\theta}{dz} = \varepsilon^* N \varepsilon \quad (29)$$

$$\Upsilon' = \varepsilon'^* \frac{d\varepsilon'}{d\theta} \frac{d\theta}{dz} = \varepsilon'^* N' \varepsilon' \quad (30)$$

where we consider $d\theta = d\theta'$. The internal dynamics of the birefringent medium is the very source of developing Υ . The polarized light passing through the twisted medium will acquire Υ' , that

will contain the phase both due to internal and external rotation of the medium. The latter comes due to rotation of plane of polarization by the external twist. We realize that for a particular light and optical element the difference ($\Upsilon' - \Upsilon$), will eliminate the phase due to natural twist, resulting the outcome of geometric phase for normal incidence of the polarized light on the medium. To find Υ , we use eq. (22) and (23) and (11) in (29) in the following equation

$$\begin{aligned}\Upsilon &= \varepsilon^* N \varepsilon = \varepsilon_1^* n_2 \varepsilon_2 + \varepsilon_2^* n_3 \varepsilon_1 \\ &= (-ie^{-i\phi})(-ke^{i\phi}) + (ke^{-i\phi})(ie^{i\phi}) \\ &= 2ik\end{aligned}\tag{31}$$

that will yield the phase of untwisted medium by $2ik$. By varying the value of $k = 0, 1, 2, \dots$, the value of the phase Υ changes. Now to proceed in a similar way for twisted stack, we use the twisted matrix N' eq.(24) for $k = 0$ for the corresponding twisted state by eq.(26) that will satisfy eq.(15) resulting to find Υ'

$$\begin{aligned}\Upsilon' &= \varepsilon'^* N' \varepsilon' = \varepsilon'_1{}^* n'_2 \varepsilon'_2 + \varepsilon'_2{}^* n'_3 \varepsilon'_1 \\ &= (-ie^{-i\phi} \cos \theta + \sin \theta)k(-ie^{i\phi} \sin \theta + \cos \theta) \\ &\quad + (ie^{-i\phi} \sin \theta + \cos \theta)(-k)(ie^{i\phi} \cos \theta + \sin \theta) \\ &= -2ik \cos \phi\end{aligned}\tag{32}$$

Any ray passing through N or N' will suffer a twist. The ray passing through N' has two parts in the phase acquired from the dynamics and geometrical change of the medium. Thus the phase Υ' will express both the dynamical and geometric phase of a uniformly twisted birefringent medium. To grasp the geometric phase of the incident polarized state due to external twist of a birefringent medium in an initial adjustment $k = 0$, we take the difference between the two phases $\Upsilon' - \Upsilon$ that will eliminate the dynamical phase. As a result we find that

$$\Upsilon' - \Upsilon = -2ik \cos \phi\tag{33}$$

as it is seen $\Upsilon = 0$ for $k = 0$. Similarly using the above process the required geometric phase of the other conjugate eigenvector could be evaluated as

$$\Upsilon' - \Upsilon = 2ik \cos \phi\tag{34}$$

Thus it is seen the geometric phase visualize the circular birefringence of the medium by $k \cos \phi$.

If the orientation of the matrix N is changed considering $k = 1$, then using the initial matrix $N_0 = \begin{pmatrix} 0 & -e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}$ one can find the phase $\Upsilon = \pm 2i$ for the respective eigenvectors in eq.(23). To calculate the twisted phase Υ' , the twisted matrix N' as in eq.(27) could be used that intuitively will act on the twisted light ray ε' , in opposite direction for making $k = 1$. As result the original

eigenvector $\varepsilon = \varepsilon'' = S(\theta)\varepsilon'$ is visible. In the language of mathematics

$$\begin{aligned}\Upsilon' &= \varepsilon'_1{}^* n'_2 \varepsilon'_2 + \varepsilon_2{}^* n'_3 \varepsilon'_1 \\ &= (-ie^{-i\phi}, 1) \begin{pmatrix} 0 & -e^{i\phi} + k \\ e^{-i\phi} - k & 0 \end{pmatrix}\end{aligned}\quad (35)$$

$$\begin{pmatrix} ie^{i\phi} \\ 1 \end{pmatrix} = 2i[1 - k \cos \phi]\quad (36)$$

This helps us to recover again the previous form of the geometric phase

$$\Upsilon' - \Upsilon = 2i[1 - k \cos \phi] - 2i = -2i(k \cos \phi)\quad (37)$$

Hence again the GP in terms $k \cos \phi$ is visible where the k is non-zero being associated with the twisted crystal. We choose the light incident at a particular angle θ on the optical medium. Now for the first choice $k = 0$, the twisted matrix N' gives the geometric phase which is identical with the findings for the second choice $k = 1$. The geometric phase visualizes in both cases the circular birefringence of the medium.

Instead if without considering the double rotation of light vector from $\varepsilon \longrightarrow \varepsilon' \longrightarrow \varepsilon''$, we consider only the interaction of light ray ε' in eq.(26) with the optical medium in eq.(27) the outcome of the calculation is

$$\begin{aligned}&\varepsilon'_1{}^* n'_2 \varepsilon'_2 + \varepsilon_2{}^* n'_3 \varepsilon'_1 \\ &= (-ie^{-i\phi} \cos \theta + \sin \theta)(k - e^{i\phi})(-ie^{i\phi} \sin \theta + \cos \theta) \\ &+ (ie^{-i\phi} \sin \theta + \cos \theta)(e^{-i\phi} - k)(ie^{i\phi} \cos \theta + \sin \theta) \\ &= i[2 - 2ik \cos \phi - (1 - \cos 2\theta)(1 - \cos 2\phi)]\end{aligned}\quad (38)$$

As a result one can finally find the phase Υ' for the initial adjustment of the optical medium at $k = 1$

$$\Upsilon' = -i[2 - 2ik \cos \phi - (1 - \cos 2\theta)(1 - \cos 2\phi)]\quad (39)$$

This helps us to identify here for $k = 1$ the geometric phase by twisting the medium

$$\Upsilon' - \Upsilon = 2i(k \cos \phi) + (1 - \cos 2\theta)(1 - \cos 2\phi)\quad (40)$$

Hence it is seen that the GP appeared here is different from eq (37) though the twisted optical medium is same. The very cause of this difference is to consider the proper twist of the light ray passing through the medium. In all cases, the external twist of the system visualizes the circular birefringence of the medium in terms of $k \cos \phi$.

It is seen from the above that the two geometric phases are in the form of the usual solid angle visualizing the circular birefringence of the medium. If the optical medium is twisted arbitrarily, $k = d\theta/dz \neq \text{constant}$, we realize that the nature of the phase will depend not only on the above solid angle but also on the nature of k . If $k = \text{periodic}$, a precessional type motion may be realized that could be studied further extensively.

III. ANGULAR MOMENTUM INTERPRETATION OF POLARIZATION MATRIX AND GEOMETRIC PHASE

The locus of the electric vector is the identification of polarized light. When the electric vector traces a circle by its rotation then it is the circularly polarized light whose two senses of rotation identifies two spin angular momentum (SAM) of photon by ± 1 that can be visualized by the two opposite directions of helicities. With the change of polarization of the light by some optical element, the parameter in connection with helicity changes. Apart from the SAM, photons carry orbital angular momentum (OAM) [13] that can be identified by the inclination of phase fronts with the propagation axis. As l can take any integer value, there shall be an infinite number of eigenstates of l . The conventional Poincare sphere that define the three kinds of polarized states, is a SAM space. Orthogonal state of two opposite spins are defined in the two hemisphere. Recently the orbital Poincare sphere has been sketched by Galves et.al. [14]. It helps to realize one that there will be two distinct provinces of OAM and SAM sphere in the Poincare representation.

The above discussions help us to specify three variables θ, ϕ and χ to specify a polarized photon in spherical geometry where ϕ and χ identify the operators OAM and SAM by $i\hbar\frac{\partial}{\partial\phi}$ and $i\hbar\frac{\partial}{\partial\chi}$ respectively [10],[18]. The quantities m and μ just represent the eigenvalues of the OAM and SAM operators. We would like to mention further that comprising these three parameters an extended Poincare sphere representation could be given considering the fixed spin vector attached at each point.

For every value of l , there will be values of m from $-l$ to $+l$. We identify the space parameterized by θ and ϕ by the OAM sphere and the SAM sphere by θ and χ . The elements of the 2×2 polarization matrix could be written in terms of the product harmonics Y_1^1, Y_1^{-1} and Y_1^0 for $l = 1$ orbital angular momentum. Then the respective polarization matrix will be equivalent to

$$M \simeq \begin{pmatrix} Y_1^0 & Y_1^1 \\ Y_1^{-1} & Y_1^0 \end{pmatrix}$$

represented by each point on the Poincare sphere parameterized by the angles θ and ϕ . This indicates that for higher OAM states $l = 2, 3..$ the respective product harmonics Y_l^m could be used. On the other hand, if we define our polarization state and the M, N matrix, parameterized by θ and χ , the SAM sphere [10],[11] is needed for their representation.

This shows that we have defined our differential matrix $N = k \begin{pmatrix} 0 & -e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}$ in the OAM sphere for $l = 1$. Both circular and linear birefringence, in terms of angle ϕ , measure the angular momentum of polarized photon. The inherent property of the medium gives a natural twist to the light whose value has been used for $k = 0, 1$, here. The uniform external twist (small) per unit thickness, appeared by k in the twisted matrix of birefringent optical medium. Intuitively k behaves as m because it is the eigenvalues of N . Hence we may consider it as the topological charge of the birefringent medium.

Now the type of geometric phase acquired by the polarized photon depends only on the sphere/space where the path is traced. The place for GP of Pancharatnam (PP) [1] will be in the SAM sphere where polarization changes from point to point. In the OAM sphere the GP is arising for mode transformations [6]. In our representation we have studied PP in the sphere of SAM with the introduction of rotator [10],[11]. Here we have studied the GP in the OAM sphere with use of birefringent medium. Santamato [20] pointed out that a birefringent plates affects the SAM and as well as OAM through some topological charge. He mentioned that a QP is an ordinary birefringent plate rotated at an angle α about the beam z-axis, with α given by $\alpha = \alpha(x, y) = \arctan(y/x) = \phi$ where ϕ is the azimuthal angle in the x, y-plane. If the topological charge of the birefringent plate is q, [12] the OAM of a light beam passing through such a "q-plate" (QP), changes by $\pm 2q$ per photon. Looking at this we may comment here that the matrix N we found may be identified as k -plate having topological charge k that is visualized through the GP of the polarized light passing normally through uniformly twisted birefringent medium. We find that an identical value of GP depending on the final topological charge k of the twisted medium is obtained though the initial medium is chosen different. The novelty of this work also lies in the appearance of the circular birefringence through the geometric phase of the polarized light.

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