Available at: http://publications.ictp.it IC/2010/075

United Nations Educational, Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

A NOTE ON THE G-CYCLIC OPERATORS OVER A BOUNDED SEMIGROUP

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Abstract

Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space, and $\mathbf{B}(\mathcal{H})$ be the Banach algebra of all linear bounded operators on $\mathcal{H}$. Let $\mathcal{S}$ be a multiplication semigroup of $\mathbb{C}$ with 1, an operator $T \in \mathbf{B}(\mathcal{H})$ is called $\mathcal{G}$-cyclic operator over $\mathcal{S}$ if there is a vector x in $\mathcal{H}$ such that $\{\alpha T^n x | \alpha \in \mathcal{S}, n \geq 0\}$ is dense in $\mathcal{H}$. In this case x is called a $\mathcal{G}$-cyclic vector for T over $\mathcal{S}$. If T is $\mathcal{G}$-cyclic operator and $\mathcal{S} = \{1\}$ then T is a hypercyclic operator.

In this paper, we study the spectral properties of a $\mathcal{G}$-cyclic operators over a bounded $\mathcal{S}$ under the condition that zero is not in the closure of $\mathcal{S}$. We show that the class of all $\mathcal{G}$-cyclic operators is contained in the norm-closure of the class of all hypercyclic operators.

MIRAMARE – TRIESTE
August 2010

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Let $\mathcal{H}$ be an infinite - dimensional separable complex Hilbert space, and $\mathbf{B}(\mathcal{H})$ be the Banach algebra of all linear bounded operator on $\mathcal{H}$. Let $\mathcal{S}$ be a multiplication semigroup of $\mathbb{C}$ with 1, an operator $T \in \mathbf{B}(\mathcal{H})$ is called $\mathcal{G}$-cyclic operator over $\mathcal{S}$ if there is a vector x in $\mathcal{H}$ such that $\{\alpha T^n x | \alpha \in \mathcal{S}, n \geq 0\}$ is dense in $\mathcal{H}$. In this case x is called a $\mathcal{G}$-cyclic vector for T over $\mathcal{S}$ [8]. Recall that a vector x in $\mathcal{H}$ is hypercyclic for T if the set $\{T^n x | n \geq 0\}$ is dense in $\mathcal{H}$, and supercyclic if the set of scalar multiples of $\{T^n x | n \geq 0\}$ is dense in $\mathcal{H}$. A hypercyclic (supercyclic) operator is one that has a hypercyclic (supercyclic) vector. Let T be a $\mathcal{G}$-cyclic operator then clearly one can verify that T is hypercyclic whenever $\mathcal{S} = \{1\}$, and supercyclic if $\mathcal{S} = \mathbb{C}$.

In 1982, C. Kitai showed in her thesis [4], that in order for an operator to be hypercyclic, every component of its spectrum must intersect the unit circle. The question arises whether the converse is true. Salas has constructed in [9] an example of a supercyclic operator which is not hypercyclic but its spectrum is the unit circle.

Naoum and Jamil [8] showed that any component of the spectrum of a $\mathcal{G}$-cyclic operator T over $\mathcal{S} = \{\lambda | \, |\lambda| = 1\}$ intersects the unit circle. Indeed, León-Saavedra and Müller in [5] proved that T is a hypercyclic operator if and only if T is a $\mathcal{G}$-cyclic operator over $\mathcal{S} = \{\lambda | \, |\lambda| = 1\}$. It is natural to ask whether these results extend to $\mathcal{G}$-cyclic operators over bounded multiplication semigroups.

In this paper, we study the spectral properties of a $\mathcal{G}$-cyclic operator over a bounded $\mathcal{S}$ under the condition that zero is not in the closure of $\mathcal{S}$ (i.e. $0 \notin \overline{\mathcal{S}}$). We get results similar to those in hypercyclic operators. This leads us to ask if the class $HC(\mathcal{H})$ of all hypercyclic operators coincides with the class $\mathcal{G}C_{\mathcal{S}}(\mathcal{H})$ of all $\mathcal{G}$-cyclic operators over a bounded $\mathcal{S}$ under the above condition on $\mathcal{S}$. The first inclusion is direct when $\mathcal{S} = \{1\}$, clearly $0 \notin \overline{\mathcal{S}}$ is satisfied. Unfortunately, we can't obtain the second inclusion, but we show that the class $\mathcal{G}C_{\mathcal{S}}(\mathcal{H})$ is included in the norm-closure of the class $HC(\mathcal{H})$.

1. Spectral Properties of $\mathcal{G}$ -Cyclic Operators

In this section we investigate the behavior of the spectrum of any $\mathcal{G}$-cyclic operators over $\mathcal{S}$ when $0 \notin \overline{\mathcal{S}}$. It was shown in [8] that if T is a $\mathcal{G}$-cyclic operator over $\mathcal{S}$, and $T = T_1 \oplus T_2$ then T_1 and T_2 are $\mathcal{G}$-cyclic operators. This fact is an essential tool in this section. In what follows, $\mathbb{B}$ is the open unit disk $\{z \in \mathbb{C} | |z| < 1\}$, $\partial\mathbb{B}$ is the boundary of $\mathbb{B}$ and $\mathcal{S}$ is a bounded multiplication semigroup.

The following theorem says that any component of $\sigma(T)$, the spectrum of a $\mathcal{G}$-cyclic operator T over a bounded $\mathcal{S}$, such that $0 \notin \overline{\mathcal{S}}$ meets the unit circle.

Theorem 1.1. *If $T \in \mathcal{G}C_{\mathcal{S}}(\mathcal{H})$ and $0 \notin \overline{\mathcal{S}}$, then each component of $\sigma(T)$ intersects the unit circle.*

Proof. Let σ be a component of $\sigma(T)$ such that $\sigma \cap \partial\mathbb{B} = \emptyset$. Then either $\sigma \subset \mathbb{B}$ or $\sigma \subset \mathbb{C} - \overline{\mathbb{B}}$. One can find a clopen set $\sigma_1 \subset \sigma(T)$ such that $\sigma \subset \sigma_1 \subset \mathbb{B}$ or $\sigma \subset \sigma_1 \subset \mathbb{C} - \overline{\mathbb{B}}$. Suppose that $\sigma \subset \sigma_1 \subset \mathbb{B}$, by Riesz decomposition theorem applied for σ_1 and $\sigma_2 := \sigma(T) - \sigma_1$, one can write $T = T_1 \oplus T_2$ where $\sigma(T_1) = \sigma_1$. Remember that T_1 and T_2 are $\mathcal{G}$-cyclic operators. Now, for all $x \in H$, $\|T_1^n x\| \to 0$ as $n \to \infty$. But $\mathcal{S}$ is bounded, hence $|\lambda_n| \, \|T_1^n x\| \to 0$ as $n \to \infty$ for each sequence λ_n in $\mathcal{S}$ which is a contradiction.

On the other hand, if $\sigma \subset \mathbb{C} - \overline{\mathbb{B}}$, then by the same argument one can prove that T is not a $\mathcal{G}$-cyclic operator over $\mathcal{S}$, otherwise, T_2 is an invertible $\mathcal{G}$-cyclic operator over $\mathcal{S}$. But $0 \notin \overline{\mathcal{S}}$, thus T_2^{-1} is $\mathcal{G}$-cyclic operator over $\mathcal{S}^{-1}$. Clearly $\mathcal{S}^{-1}$ is also bounded. But $\sigma\left(T_2^{-1}\right) \subset \mathbb{B}$, a contradiction. $\square$

In fact, the union of the unit circle and the spectrum of any $\mathcal{G}$-cyclic operator T over $\mathcal{S}$ where $0 \notin \overline{\mathcal{S}}$ is a connected set. Recall that if X and Y are connected topological spaces such that $X \cap Y \neq \emptyset$, then $X \cup Y$ is connected.

Proposition 1.2. *Let $T \in \mathcal{G}C_{\mathcal{S}}(\mathcal{H})$ and $0 \notin \overline{\mathcal{S}}$, then $\sigma(T) \cup \partial\mathbb{B}$ is connected.*

Proof. It is enough to prove that $\sigma(T) \cap \overline{\mathbb{B}}$ and $\sigma(T) \cap (\mathbb{C} - \mathbb{B})$ are connected.

Assume $\sigma(T) \cap \mathbb{D}$ is not connected. Then there is a clopen subset σ of $\sigma(T) \cap \mathbb{D}$. Thus $\sigma(T) = \sigma \cup (\sigma(T) - \sigma)$. By the Riesz decomposition theorem $T = T_1 \oplus T_2$ where $\sigma(T_1) = \sigma$. But $\sigma \cap \partial\mathbb{B} = \emptyset$, otherwise $\partial\sigma \neq \emptyset$, a contradiction with theorem 1.1.

By the same arguments we get $\sigma(T) \cap (\mathbb{C} - \mathbb{B})$ is connected. $\square$

It is natural to ask whether the unit circle contains the spectrum of $\mathcal{G}$-cyclic operator T over $\mathcal{S}$ such that $0 \notin \overline{\mathcal{S}}$?

Proposition 1.3. *Let $T \in \mathcal{G}C_{\mathcal{S}}(\mathcal{H})$ such that $0 \notin \overline{\mathcal{S}}$. If $|\sigma(T)|$ is countable, then $\sigma(T) \subset \partial\mathbb{B}$.*

Proof. Since $T \in \mathcal{G}C_{\mathcal{S}}(\mathcal{H})$, then by theorem(1.1) every component σ intersects the unit circle $\partial\mathbb{B}$. But $|\sigma(T)|$ is countable and compact, hence $|\sigma(T)|$ has an isolated point say λ. Let (K_1, K_2) be the Riesz decomposition of $\sigma(T)$ such that $K_1 = \sigma(T) \cap L$, where L is the circle of centre 0 and radius λ, and $K_2 = \sigma(T) - K_1$. Since K_1 is component of $\sigma(T)$, therefore by theorem(1.1) $K_1 \cap \partial\mathbb{B} \neq \emptyset$, hence $\lambda = 1$. Now K_2 is also component of $\sigma(T)$ which does not intersect the unit circle, thus K_2 must be empty. Consequently, $\sigma(T) \subset \partial\mathbb{B}$. $\square$

Note that if $\sigma(T)$ is countable, then $|\sigma(T)|$ is so, hence we get the following corollary

Corollary 1.4. *Let $T \in \mathcal{G}C_{\mathcal{S}}(\mathcal{H})$ such that $0 \notin \overline{\mathcal{S}}$. If $\sigma(T)$ is countable, then T is invertible.*

2. The relation between $\mathcal{G}C_{\mathcal{S}}(\mathcal{H})$ and $HC(\mathcal{H})$

In this section we discuss the relation between the class $\mathcal{G}C_{\mathcal{S}}(\mathcal{H})$ of all $\mathcal{G}$-cyclic operators under a given condition on $\mathcal{S}$ and the class $HC(\mathcal{H})$ of all hypercyclic operators.

Recall that the complex number λ is a normal eigenvalue for $T \in \mathbf{B}(\mathcal{H})$ if $\{x|Tx = \lambda x\}$ coincides with $\left\{x|T^*x = \overline{\lambda}\right\}$ [3, p. 294], the set of all normal eigenvalues for T is defined by $\sigma_0(T)$. Herrero in [2] characterized the norm-closure of the class $HC(\mathcal{H})$ by the following theorem.

Theorem 2.1 (Herrero, 1991)**.** *$\overline{HC(\mathcal{H})}$ is the class of all those operators T in $\mathbf{B}(\mathcal{H})$ satisfying the conditions*

(1) *$\sigma_w(T) \cup \partial\mathbb{B}$ is connected (σ_w is the Weyl spectrum of T);*
(2) *$\sigma_0(T) = \phi$; and*
(3) *$ind(\lambda - T) \geq 0$ for all λ in the semi-Fredholm domain of T.*

It is easy to check that if each component of an operator T meets the unit circle and T^* has no eigenvalue then T satisfies the above three spectral. Hence $T \in \overline{HC(\mathcal{H})}$. To give the main result in this section we first need to prove the following proposition

Proposition 2.2. *Let $T \in \mathcal{G}C_{\mathcal{S}}(\mathcal{H})$, and $0 \notin \overline{\mathcal{S}}$. Then $\sigma_p(T^*) = \phi$.*

Proof. Suppose that $T^*y = \mu y$ for some $\mu \in \mathbb{C}$ and some $y \in \mathcal{H}, y \neq 0$. If x is a $\mathcal{G}$-cyclic vector. Then $\{\lambda T^n x|\lambda \in \mathcal{S}$ and $n \in \mathbb{N}\}$ is dense in $\mathcal{H}$. Hence $< y, \lambda T^n x >$ is dense in $\mathbb{C}$. Note that $< y, \lambda T^n x >= \mu^n < y, \lambda x >$ for all $n \in \mathbb{N}, \lambda \in \mathcal{S}$. Since $\mathcal{S}$ is bounded and $0 \notin \overline{\mathcal{S}}$, then there exists positive numbers m and M such that $m \leq |\lambda| \leq M$. Now, if $\mu \leq 1$ then $|\mu^n < y, \lambda x > | \leq |M| < y, x >$, a contradiction. If $\mu > 1$ then $|\mu^n < y, \lambda x > | > m| < y, x > |$ which is a contradiction. □

We are now in the position to give the main result. The proof follows from Theorem (1.1) and Proposition (2.2).

Theorem 2.3. *Let $T \in \mathcal{G}C_s(\mathcal{H})$ such that $0 \notin \overline{\mathcal{S}}$, then $T \in \overline{HC(\mathcal{H})}$.*

Salas has constructed in [9] an example of supercyclic operator T which is not hypercyclic such that the spectrum of T, $\sigma(T)$ is the unit circle. Naoum and Jamil [6] showed that any component of the spectrum of a $\mathcal{G}$-cyclic operator T over $\mathcal{S} = \{\lambda|\,|\lambda| = 1\}$ intersects the unit circle. Indeed, León-Saavedra and Müller in [5] proved that T is hypercyclic operator if and only if T is $\mathcal{G}$- cyclic operator over $\mathcal{S} = \{\lambda|\,|\lambda| = 1\}$. This observation suggests the following problem.

Problem 2.4. If $T \in \mathcal{G}C_s(\mathcal{H})$ such that $0 \notin \overline{\mathcal{S}}$, is T a hypercyclic operator?

Acknowledgments

The authors would like to thank the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, for hospitality and support.

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