Available at: http://publications.ictp.it IC/2010/074

United Nations Educational, Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

SOME RESULTS ON THE INTERSECTION GRAPHS OF IDEALS OF RINGS

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Abstract

Let R be a ring with unity and $I(R)^*$ be the set of all non-trivial left ideals of R. The intersection graph of ideals of R, denoted by $G(R)$, is a graph with the vertex set $I(R)^*$ and two distinct vertices I and J are adjacent if and only if $I \cap J \neq 0$. In this paper, we study some connections between the graph-theoretic properties of this graph and some algebraic properties of rings. We characterize all rings whose intersection graphs of ideals are not connected. Also we determine all rings whose clique number of the intersection graphs of ideals are finite. Among other results, it is shown that for every ring, if the clique number of $G(R)$ is finite, then the chromatic number is finite too and if R is a reduced ring both are equal.

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August 2010

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1. Introduction

The interplay between ring-theoretic and graph-theoretic properties was first studied in [8], and this approach has since became increasingly popular. Many researchers have obtained ring-theoretic properties in terms of graph-theoretic properties by suitable defining graph structure on some elements of a ring, for example, the zero-divisor graph and the total graph [1], [2], [3], [4], [6] and [14].

The field of graph theory and ring theory both benefit from the study of algebraic concepts using graph theoretic concepts. For instance, knowledge of algebraic structures of rings can innovate new ideas for studying the graphs. Usually after translating of algebraic properties of rings into graph-theoretic language, some problems in ring theory might be more easily solved. When one assigns a graph to an algebraic structure numerous interesting algebraic problems arise from the translation of some graph-theoretic parameters such as clique number, chromatic number, independence number and so on. The main purpose of this paper is the study of the intersection of ideals in a ring using graph-theoretic concepts.

Throughout this paper all rings have unity. Let R be a ring. By $I(R)$ and $I(R)^*$ we mean the set of all left ideals of R and the set of all non-trivial left ideals of R, respectively. A ring R is said to be *local* if it has a unique maximal left ideal. The ring of $n \times n$ matrices over R is denoted by $M_n(R)$. The ring R is said to be *reduced* if it has no non-zero nilpotent element. The *socle* of ring R, denoted by $soc(R)$, is the sum of all minimal left ideals of R. If there are no minimal ideals, this sum is defined to be zero. A prime ideal $\mathfrak{p}$ is said to be an *associated prime ideal* of a commutative Noetherian ring R, if there exists a non-zero element x in R such that $\mathfrak{p} = Ann(x)$. By $\mathrm{Ass}(R)$ and $\mathrm{Min}(R)$ we denote the set of all associated prime and minimal prime ideals of R, respectively. The set of nilpotent elements of R is denoted by $\mathrm{Nil}(R)$. The intersection of all maximal left ideas of R is called the *Jacobson radical* of R and is denoted by $J(R)$. A ring R is said to be *semisimple*, if $J(R) = 0$. Let M be a left R-module. A chain of left submodules of *length* n is a sequence M_i $(0 \leq i \leq n)$ of left submodules of M such that $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$. A *composition series* of M is a maximal chain, that is one in which no extra left submodules can be inserted. It is known that every pair of composition series for M are equivalent. The length of composition series of M is denoted by $l(M)$. An R-module M is said to be *finite length* if $l(M) < \infty$.

Let G be a graph with the vertex set $V(G)$. The *Complement graph* of G, denoted by $\overline{G}$, is a graph with the same vertices such that two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *degree* of a vertex v in a graph G is the number of edges incident with v. The degree of a vertex v is denoted by $deg(v)$. Let r be a non-negative integer. The graph G is said to be r*-regular*, if the degree of each vertex is r. If u and v are two adjacent vertices of G, then we write u — v. The *complete graph* of order n, denoted by K_n, is a graph with n vertices in which any two distinct vertices are adjacent. A *star graph* is a graph with a vertex adjacent to all other vertices and has no other edges. Recall that a graph G is *connected* if there is a path between every two distinct vertices. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y and if there is no such a path we define $d(x, y) = \infty$. The *diameter* of G, $diam(G)$, is the supremum of the set $\{d(x, y) : x$ and y are distinct vertices of $G\}$. A *clique* of G is a complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. For a graph G, let $\chi(G)$ denote the *chromatic number* of G, i.e., the minimum number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. An *independent set* of G is a subset of the vertices of G such that no two vertices in the subset represent an edge of G. *The independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set. The *intersection graph of ideals* of a ring R, denoted by $G(R)$, is a graph with the vertex set $I(R)^*$ and two distinct vertices I and J are adjacent if and only if $I \cap J \neq 0$. This graph was first defined in [10] and the intersection graph of ideals of $\mathbb{Z}_n$ was studied. They determined the values of n for which the graph of $\mathbb{Z}_n$ is complete, Eulerian or Hamiltonian. Since the most properties of a ring are closely tied to the behavior of its ideals, one may expect that the intersection graph of ideals reflect many properties of a ring.

2. Diameter and Some Finiteness Conditions

In this section, all rings whose the intersection graphs are not connected will be characterized. We prove that if $G(R)$ is a connected graph, then its diameter is at most 2. Next, some conditions under which the intersection graph of ideals of a ring is finite are given. Furthermore, the regularity of the intersection graph of ideals of a ring is studied. First we need the following results.

Theorem 1. [10, Corollary 2.5] *For any graph $G = G(R)$ of a ring R, whenever G is not connected, it is a null graph (i.e., it has no edge).*

Lemma 2. [12, p.232] *Let D be a division ring and n be a natural number. If H_r, $0 \leq r \leq n$, denotes the left ideal of $M_n(D)$ containing all matrices whose $d_{ij} = 0$, for every $r < j \leq n$, then every left ideal of $M_n(D)$ is similar to H_r, for some r, $0 \leq r \leq n$.*

Theorem 3. *Let R be a ring. Then $G(R)$ is not connected if and only if either $R \cong D_1 \times D_2$, where D_1 and D_2 are two division rings or $R \cong M_2(D)$, where D is a division ring.*

Proof. First suppose that $G(R)$ is not connected. By [10, Theorem 2.4], R has at least two maximal left ideals and every left ideal is a minimal left ideal. So the intersection of every two maximal left ideals is zero and hence $J(R)$=0. Also, it is clear that R is a left Artinian ring. Now, by Wedderburn-Artin Theorem (see [13, 3.5]), $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$, where D_i is a division ring for every i, $1 \leq i \leq k$. If $k \geq 3$, then R has a maximal left ideal which is not minimal and this contradicts [10, Theorem 2.4]. Therefore, $k \leq 2$. If $R \cong M_{n_1}(D_1) \times M_{n_2}(D_2)$, we show that both of $n_1 = n_2 = 1$. If $n_1 > 1$, then $M_{n_1}(D_1)$ has at least a non-zero left ideal, say I. Then $(0, M_{n_2}(D_2))$ and $(I, M_{n_2}(D_2))$ are adjacent. Now, Theorem 1, implies that $G(R)$ is a connected graph, a contradiction. Thus $n_1 = 1$. Similarly, $n_2 = 1$. Now, assume that $R \cong M_n(D)$, where D is a division ring. By Lemma 2, the dimension of every maximal left ideal of $M_n(D)$ over D is $n^2 - n$. So if $n > 2$, then every two maximal left ideals of $M_n(D)$ has non-zero intersection, a contradiction. Thus $R \cong M_2(D)$, where D is a division ring. Note that if $R \cong D_1 \times D_2$, where D_1 and D_2 are two division rings, then it is clear that $G(R)$ is not connected. Finally, assume that $R \cong M_2(D)$. By Lemma 2, the dimension of every non-trivial left ideal of $M_2(D)$ over D is 2. So the intersection of every two distinct non-trivial left ideals of $M_2(D)$ is zero. Thus $G(R)$ is not a connected graph. □

Theorem 4. *Let R be a ring and $G(R)$ be a connected graph. Then diam$(G(R)) \leq 2$.*

Proof. Let I and J be two left ideals of R. If $I \cap J$ is non-zero, then I and J are adjacent. Thus assume that $I \cap J = 0$. If $I + J \neq R$, then consider the path $I — (I + J) — J$. Thus suppose that $I + J = R$. If there exists a left ideal $0 \neq L \subset I$, and $L + J \neq R$, then consider the path $I — (L + J) — J$. Thus assume that $L + J = R$. Let $x \in I$. Thus there exists $a \in L$ and $b \in J$ such that $x = a + b$. We have $x - a = b \in I \cap J$. Thus $x = a \in L$. This implies that $L = I$. Thus I is a minimal left ideal. Now, since $G(R)$ is connected, there exists a non-trivial left ideal I_1 such that $I_1 \neq I$ and $I_1 \cap I \neq 0$. If $I_1 \cap J$ is non-zero, then consider the path $I — I_1 — J$. Hence assume that $I_1 \cap J = 0$. By a similar argument one can see that $I_1 + J = R$ and I_1 is a minimal left ideal of R. This implies that $I \cap I_1 = 0$, a contradiction. The proof is complete. □

Lemma 5. *Let R be a ring and I be a left ideal of R. If deg(I) is finite, then R is a left Artinian ring.*

Proof. Since deg(I) is finite, so I and R/I are left Artinian R-modules. Thus by [11, Proposition 4.5], R is a left Artinian R-module and the proof is complete. □

Theorem 6. *Let R be a commutative ring and $\mathfrak{m}$ be a maximal ideal of R. If deg$(\mathfrak{m})$ is finite, then $G(R)$ is finite.*

Proof. By Lemma 5, R is an Artinian ring. So by [7, Theorem 8.7], $R \cong R_1 \times \cdots \times R_n$, where $(R_i, \mathfrak{m}_i)$ is a local Artinian ring, for $i = 1, \ldots, n$. Since $\mathfrak{m}$ is a maximal ideal of R, there exists j, $1 \leq j \leq n$, such that $\mathfrak{m} = R_1 \times \cdots \times R_{j-1} \times \mathfrak{m}_j \times R_{j+1} \times \cdots \times R_n$. Now, if one of the R_i has an infinite number of ideals, then deg$(\mathfrak{m})$ is infinite, a contradiction. Therefore every R_i, $1 \leq i \leq n$, has finitely many ideals and the proof is complete. □

Now, we wish to investigate the properties of a ring (not necessarily commutative) with at least a maximal left ideal of finite degree. Before stating our results we need the following lemma.

Lemma 7. *Let D be an infinite division ring and $n \geq 2$. Then $M_n(D)$ has an infinite number of maximal left ideals. Moreover, if $n \geq 3$, then every two distinct maximal left ideals are adjacent in $G(M_n(D))$ and $\omega(G(M_n(D))$ is infinite.*

Proof. We show that $M_n(D)$ has infinitely many maximal left ideals. For every $x \in D$, let $A_x = I_n + xE_{1n}$, where E_{1n} is an n by n matrix whose $(1, n)$-th entry is 1 and other entries are zero. Clearly, $A_x H_{n-1}(A_x)^{-1}$ is a maximal left ideal of $M_n(D)$ (see Lemma 2). By an easy calculation one can see that if $a_{21} \neq 0$, and

$$A = \begin{bmatrix} a_{12} & \dots & a_{1,n-1} & 0 \\ a_{21} & \dots & a_{2,n-1} & 0 \\ \vdots & \vdots & \dots & \vdots \\ a_{n1} & \dots & a_{n,n-1} & 0 \end{bmatrix},$$

then the inverse of $(2, n)$th entry times $(2, 1)$th of the matrix $A_x A(A_x)^{-1}$ is $-x$. So by Lemma 2, $M_n(D)$ has an infinite number of maximal left ideals. Now, since every two maximal left ideals have a non-zero intersection, we conclude that $\omega(G(M_n(D))$ is infinite. □

Theorem 8. *Let R be a ring and $\mathfrak{m}$ be a maximal left ideal of R of finite degree. If $G(R)$ is infinite and it is not null, then the following hold:*
(i) *The number of maximal left ideals of R is finite.*
(ii) $\chi(G(R)) < \infty$.
(iii) *There exists a two sided ideal of infinite degree.*

Proof. (i) Clearly, R is a left Artinian ring. Toward a contradiction, let $\mathfrak{m}, \mathfrak{m}_1, \mathfrak{m}_2, \dots$ be an infinite number of maximal left ideals of R. Since $deg(\mathfrak{m}) < \infty$, there exists some j such that $\mathfrak{m} \cap \mathfrak{m}_j = 0$. This implies that $J(R) = 0$ and so R is a semisimple left Artinian ring. Thus by Wedderburn-Artin Theorem (see [13, 3.5]), $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$, where D_i is a division ring for every i, $1 \leq i \leq k$. Since $G(R)$ is infinite, at least one of the D_i, say D_1 is infinite and $n_1 \geq 2$. Hence by Lemma 7, $M_{n_1}(D_1)$ has infinitely many left ideals. If $k \geq 2$, then it is not hard to see that the degree of every maximal left ideal of R is infinite, a contradiction. So $R \cong M_{n_1}(D_1)$. If $n_1 = 2$, then $G(R)$ is a null graph, a contradiction. If $n_1 > 2$, then by Lemma 7, the degree of every maximal left ideal of R is infinite, a contradiction. Thus the number of maximal left ideals of R is finite.
(ii) and (iii) Since $deg(\mathfrak{m})$ is finite and $G(R)$ is infinite, there are non-trivial left ideals $\{I_i\}_{i=1}^{\infty}$ such that for every i, $i \geq 1$, $I_i \cap \mathfrak{m} = 0$. Thus $I_i + \mathfrak{m} = R$. We prove that each I_i is a minimal left ideal. Let $I'_i \subset I_i$ be a non-zero left ideal. Clearly, $I'_i + \mathfrak{m} = R$. Suppose that $x \in I_i$. Thus there are two elements $a \in I'_i$ and $b \in \mathfrak{m}$ such that $a + b = x$. So we have $b = x - a \in I_i \cap \mathfrak{m} = 0$. This implies that $I_i = I'_i$. Therefore there are infinitely many minimal left ideals $\{I_i\}_{i=1}^{\infty}$ such that $I_i + \mathfrak{m} = R$ and $I_i \cap \mathfrak{m} = 0$, for every i, $i \geq 1$. The argument shows that if I is a non-trivial left ideal and $I \cap \mathfrak{m} = 0$, then I is a minimal left ideal of R. This implies that the number of

left ideals of R which are not minimal is finite. Now, since the intersection of every two distinct minimal left ideals of R is zero, one can color all minimal left ideals of R by a color and color each other vertex with a new color to obtain a proper vertex coloring of $G(R)$. This completes the proof of Part (ii).

To prove the Part (iii), consider $soc(R)$. By [13, Exercise 19, p.69], $soc(R)$ is a two sided ideal containing all minimal left ideals of R. If $soc(R) = R$, then since every minimal left ideal is a simple left module, we conclude that R is a semisimple left Artinian ring. Since $deg(\mathfrak{m}) < \infty$, by Wedderburn-Artin Theorem and Lemma 7, we conclude that $R \cong D_1 \times \cdots \times D_k$, where D_i is a division ring for every i, $1 \leq i \leq k$. This yields that $G(R)$ is a finite graph, a contradiction. Therefore $soc(R) \neq R$. Now, since the number of minimal left ideals of R is infinite, we conclude that $deg(soc(R))$ is infinite and Part (iii) is proved. The proof is complete. □

Remark 9. Let R be a ring. If for every maximal left ideal $\mathfrak{m}$ of R, $deg(\mathfrak{m}) < \infty$, then $G(R)$ is null or a finite graph. To see this as the previous proof shows, if $G(R)$ is infinite, then the number of minimal left ideals of R is infinite. Now, by Part (i) of Theorem 8, there exists a maximal left ideal $\mathfrak{m}_j$ which contains infinitely many minimal left ideals. So $deg(\mathfrak{m}_j) = \infty$, a contradiction.

Now, we propose a question: If R is a ring and $deg(\mathfrak{m}) < \infty$, for a maximal left ideal $\mathfrak{m}$ of R, then is it true that $G(R)$ is null or finite?

In the next theorem we will show that every intersection graph of ideals which is regular, is a complete graph or a null graphs.

Theorem 10. *Suppose that R is a ring and $G(R)$ is an r-regular graph , for some non-negative integer r. Then either $G(R)$ is a complete graph or a null graph.*

Proof. Suppose that $G(R)$ is not null. By Lemma 5, R is a left Artinian ring. Toward a contradiction suppose that $G(R)$ is not a complete graph. It follows from [10, Theorem 2.11] that R has at least two minimal left ideals, say I_1 and I_2. Since I_1 and I_2 are not adjacent and $diam(G(R)) \leq 2$, there exists an ideal J of R such that $I_1 — J — I_2$. So both I_1 and I_2 are contained in J. Thus each vertex adjacent to I_1 is adjacent to J too. This argument shows that $deg(J) > deg(I_1)$, a contradiction. Therefore, $G(R)$ is complete. □

In the sequel of this section we study some rings whose intersection graph of ideals are complete, i.e., $diam(G(R)) = 1$.

Theorem 11. *Let R be a commutative ring. Then R is an integral domain if and only if R is a reduced ring and $G(R)$ is a complete graph.*

Proof. One side is clear. For the other side suppose that R is a reduced ring and $G(R)$ is a complete graph. We claim that if $I,\ J \in I(R)^*$ and $IJ = 0$, then $I \cap J = 0$. By contrary, suppose that $I \cap J \neq 0$. Then there exists a non-zero element x in $I \cap J$. So $(x) \subseteq I$ and $(x) \subseteq J$ and hence $(x)^2 \subseteq IJ = 0$. Therefore, $x = 0$, a contradiction. Now, if R is not a domain, then there exist non-zero elements $x, y \in R$ such that $xy = 0$. If $(x) = (y)$, then $x^2 = 0$, a contradiction. Thus $(x) \neq (y)$, and by the claim $(x)(y) = (x) \cap (y) = 0$, a contradiction. The proof is complete. $\square$

The condition of R to be a reduced ring in the previous theorem is necessary. To see this let $R = \mathbb{Z}_{p^3}$, where p is a prime number. Then R is not a reduced ring, $G(R)$ is a complete graph but R is not an integral domain.

Theorem 12. *Suppose that R is a commutative Noetherian ring and $G(R)$ is a complete graph. Then* $\mathrm{Ass}(R) = \mathrm{Min}(R)$ *has just one element.*

Proof. Suppose that $0 = \cap_{i=1}^{n} Q_i$ is a minimal primary decomposition of the ideal 0, see [15, Corollary 4.35]. Since $G(R)$ is complete and $\cap_{i=1}^{n} Q_i$ is a minimal primary decomposition, the ideal 0 is primary and so by [15, Remarks 9.33], $\mathrm{Ass}(R) = \{\sqrt{0}\}$. Therefore by [15, Corollary 9.36], $Z(R) = \mathrm{Ass}(R) = \{\sqrt{0}\} = \mathrm{Nil}(R)$. Since R is a Noetherian ring and 0 is a primary ideal, $\sqrt{0} = \mathfrak{p}$ is a prime ideal and so by [7, Proposition 1.8], $\mathrm{Min}(R) = \{\mathfrak{p}\}$. $\square$

The following example shows that the converse of the previous result is not true.

Example 13. Let $R = \frac{k[x,y]}{(x,y)^2}$, where k is an infinite field and x, y are indeterminates. Clearly, R is an Artinian local ring with the unique maximal ideal $\mathfrak{m} = \overline{(x,y)}$. Since $\mathfrak{m}^2 = 0$, by [15, Theorem 4.9], 0 is a primary ideal of R. Thus $\mathrm{Ass}(R) = \mathrm{Min}(R) = \{\sqrt{0}\}$, but $\overline{(x)} \cap \overline{(y)} = 0$.

We close this section with the following theorem.

Theorem 14. *Let $(R, \mathfrak{m})$ be a commutative Artinian local ring. Then the following statements are equivalent.*
(i) *R is a Gorenstein ring.*
(ii) *$I = AnnAnnI$ for all ideals I of R.*
(iii) *$G(R)$ is a complete graph.*
(iv) *R has a unique minimal ideal.*

Proof. By [9, Exercise 3.2.15] and [10, Theorem 2.11] the proof is complete. $\square$

3. The Clique Number and the Chromatic Number of the Intersection Graph of Ideals

In this section we characterize all rings whose intersection graph of ideals are triangle-free. There are many graphs whose clique numbers are finite whereas chromatic numbers are infinite. We show that if the clique number of an intersection graph of ideals is finite, then its chromatic number is also finite and they are equal when R is a reduced ring. Finally, it is proved that if R is a commutative reduced ring with $|\mathrm{Min}(R)| < \infty$, then $\alpha(G(R)) = |\mathrm{Min}(R)|$.

Lemma 15. *Let R be a ring such that $\omega(G(R)) < \infty$. Then R is a left Artinian ring.*

Proof. Suppose to the contrary that R is not a left Artinian ring. Then there exists a descending chain $I_1 \supset I_2 \supset \cdots \supset I_n \supset \cdots$ of left ideals of R. Hence $\{I_t\}_{t=1}^{\infty}$ is an infinite clique of $G(R)$, a contradiction. □

Note that the converse of the above lemma is not true. In fact, the following example is an Artinian ring such that $\omega(G(R)) = \infty$.

Example 16. Let $R = \frac{k[[x,y]]}{(x^2,y^2)}$, where k is a field and x, y are indeterminates. It is a routine exercise in commutative algebra that R is an Artinian Gorenestien ring with infinitely many non-trivial ideals. Therefore, by Theorem 14, $G(R)$ is a complete graph and so $\omega(G(R)) = \infty$.

Theorem 17. *Let R be a ring and $G(R)$ be a triangle-free graph which is not null. Then R is a local ring and one of the following holds:*
(i) *The maximal left ideal of R is principle and moreover $G(R) = K_1$ or $G(R) = K_2$.*
(ii) *The minimal generating set of $\mathfrak{m}$ has size 2, $\mathfrak{m}^2 = 0$ and $G(R)$ is a star graph.*

Proof. Toward a contradiction suppose that R has two maximal left ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. If $\mathfrak{m}_1 \cap \mathfrak{m}_2$ is non-zero, then $\mathfrak{m}_1$, $\mathfrak{m}_2$, $\mathfrak{m}_1 \cap \mathfrak{m}_2$ form a triangle, a contradiction. Thus assume that $\mathfrak{m}_1 \cap \mathfrak{m}_2 = 0$. By Theorem 4, there exists an ideal J which is adjacent to both $\mathfrak{m}_1$ and $\mathfrak{m}_2$. If $J \cap \mathfrak{m}_1 = J$, then $J \subset \mathfrak{m}_1$ and so $\mathfrak{m}_1 \cap \mathfrak{m}_2$ is non-zero, a contradiction. Thus J, $J \cap \mathfrak{m}_1$, $J \cap \mathfrak{m}_2$, form a triangle, a contradiction. Therefore R is a local ring. Let $\mathfrak{m}$ be the unique maximal left ideal of R. Since R is triangle-free, by Lemma 15, R is a left Artinian ring. Thus by Hopkins Theorem (see [11, Theorem 4.15]), R is a left Noetherian ring. So $\mathfrak{m}$ is a finitely generated left ideal. If a minimal generating set of $\mathfrak{m}$ has at least size 3 and contains, $x, y, z, \ldots$, then $\mathfrak{m}, (x, y), (y, z)$ form a triangle, a contradiction. Thus $\mathfrak{m}$ is generated with at most two elements. First assume that $\mathfrak{m} = (x)$ is a principle left ideal. Since R is a left Artinian ring, $J(R) = \mathfrak{m}$ is nilpotent. If $\mathfrak{m}^3 \neq 0$, then by Nakayama's Lemma (see [13, 4.22]), $\mathfrak{m}$, $\mathfrak{m}^2$, $\mathfrak{m}^3$ are distinct and so form a triangle, a contradiction. Thus $\mathfrak{m}^3 = 0$. Let I be a non-trivial left ideal of R. We show that $I = \mathfrak{m}$ or $I = (x^2)$. Let $a \in I$ be a non-zero element. Thus $a = bx$, for some $b \in R$. If $b \notin \mathfrak{m}$, then b is a unit and so $I = \mathfrak{m}$. Otherwise b is contained in $\mathfrak{m}$ and so $a = cx^2$, for some $c \in R$. Again if c is not in $\mathfrak{m}$, then $I = (x^2)$, otherwise $I = 0$, a contradiction. If $\mathfrak{m}^2$ is non-zero, then we find $G(R) = K_2$. If $\mathfrak{m}^2 = 0$, then we have $G(R) = K_1$.
Now, suppose that $\mathfrak{m}$ is not principle and $\mathfrak{m} = (x, y)$. We show that $\mathfrak{m}^2 = 0$. Clearly, if $I, J \neq \mathfrak{m}$ be two non-trivial left ideals of R, then they are minimal left ideals and so $IJ = 0$ or $IJ = J$. If $IJ = J$, then since $I \subseteq J(R)$, by Nakayama's Lemma we conclude that $J = 0$, a contradiction. Therefore $IJ = 0$ and this implies that $\mathfrak{m}^2 = 0$. The proof is complete. □

Theorem 18. *Let R be a ring and $\omega(G(R)) < \infty$. Then $\chi(G(R)) < \infty$.*

Proof. By Lemma 15, R is a left Artinian ring. Thus the length of every left ideal of R is finite. Note that $l(R) \leq \omega(G(R)) + 1$. If $G(R)$ is finite, then we are done. Thus assume that $G(R)$ is infinite. Since R is a left Artianian ring, every left ideal of R contains a minimal left ideal. So if $G(R)$ is infinite and the number of minimal left ideals of R is finite, then $G(R)$ has an infinite clique, a contradiction. Thus R has an infinite number of minimal left ideals.

For every r, $1 \leq r \leq \omega(G(R)) + 1$, let

$$\mathcal{S}_r = \{\, I \in I(R)^* \,|\, l(I) = r \,\}.$$

Let t be the maximum natural number such that $\mathcal{S}_t$ is infinite. Note that since $\mathcal{S}_1$ is infinite, t exists and $t \geq 1$. By the definition of t, the number of left ideals of R of length $t+1$ is finite. By Schreier Refinement Theorem (see [11, Theorem 4.10]) every left ideal of length t is contained in at least one left ideal of length $t+1$. So there is a left ideal J of length $t+1$ such that J contains infinitely many left ideals of length t. Since $\omega(G(R)) < \infty$, there exist left ideals $L, K \in \mathcal{S}_t$ such that $L, K \subseteq J$ and $L \cap K = 0$. Now, by [11, Proposition 4.12], we have

$$t + 1 = l(J) \geq l(L + K) = l(L \oplus K) = l(L) + l(K) = 2t.$$

Therefore $t = 1$. This implies that the set of all left ideals of R which are not minimal is finite. Since the intersection of two distinct minimal left ideals of R is zero, so one can color all minimal left ideals of R with the same color and color every non-minimal left ideal with a new color. Thus $\chi(G(R))$ is finite and the proof is complete. □

Theorem 19. *Let R be a ring and $\omega(G(R)) < \infty$. Then the following hold:*
(i) *If R is not local and it is commutative, then $G(R)$ is finite.*
(ii) *If $(R, \mathfrak{m})$ is a local ring, then either $G(R)$ is finite or the size of every minimal generating set of $\mathfrak{m}$ is 2.*

Proof. (i) Since $\omega(G(R))$ is finite, by Lemma 15, R is an Artinian ring and so by [7, Theorem 8.7], $R \cong R_1 \times \cdots \times R_n$, where R_i is a local Artinian ring for $i = 1, \ldots, n$. If $n > 1$ and at least one of the R_i has infinite number of ideals, then clearly, $\omega(G(R))$ is infinite, a contradiction. So if R is not a local ring, then $G(R)$ is finite.
(ii) If $(R, \mathfrak{m})$ is a local ring, then $R/\mathfrak{m}$ is a division ring (see [5, Proposition 15.15]) and $\mathfrak{m}/\mathfrak{m}^2$ is a left $(R/\mathfrak{m})$-module. If its dimension is more than 2 and $R/\mathfrak{m}$ is infinite, then every 1-dimensional subspace is contained in infinitely many 2-dimensional subspaces and so $\mathfrak{m}$ has infinitely many left ideals whose intersection is non-zero. Thus $\omega(G(R))$ is infinite, a contradiction. Hence suppose that $R/\mathfrak{m}$ is infinite and the dimension of $\mathfrak{m}/\mathfrak{m}^2$ over $R/\mathfrak{m}$ is at most 2.
First assume that the dimension of $\mathfrak{m}/\mathfrak{m}^2$ over $R/\mathfrak{m}$ is 2. We show that every minimal generating set of $\mathfrak{m}$ has two elements. Suppose that $\{a + \mathfrak{m}^2, b + \mathfrak{m}^2\}$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$ over $R/\mathfrak{m}$. Then

$\mathfrak{m} = (a,b) + \mathfrak{m}^2$. By Nakayama's Lemma we have $\mathfrak{m} = (a,b)$. Now, if $\{a,b\}$ is not a minimal generating set for $\mathfrak{m}$, then $\mathfrak{m}$ is principle and so the dimension of $\mathfrak{m}/\mathfrak{m}^2$ over $R/\mathfrak{m}$ is 1, a contradiction.

Now, assume that the dimension of $\mathfrak{m}/\mathfrak{m}^2$ over $R/\mathfrak{m}$ is 1. Thus $\mathfrak{m} = (x)$ is principle. We show that every proper left ideal of R is a power of $\mathfrak{m}$. Let I be a non-zero left ideal of R. Since $\mathfrak{m}$ is nilpotent, there exists a natural number i such that $I \subseteq \mathfrak{m}^i$ but $I \nsubseteq \mathfrak{m}^{i+1}$. Let $a \in I \setminus \mathfrak{m}^{i+1}$. We have $a = bx^i$, for some $b \in R$. If $b \in \mathfrak{m}$, then $a \in \mathfrak{m}^{i+1}$, a contradiction. Thus b is unit. Hence $x^i \in I$. This implies that $I = \mathfrak{m}^i$. By Hopkins Theorem (see [11, Theorem 4.15]), $\mathfrak{m}$ is a nilpotent ideal and so $G(R)$ is a finite graph. If $\mathfrak{m} = 0$, then R is a division ring and we are done.

Now, we prove that if $(R,\mathfrak{m})$ is a local left Artinian ring, and $R/\mathfrak{m}$ is finite, then R is a finite ring. By Wedderburn's "Little" Theorem (see [13, 13.1]), $F := R/\mathfrak{m}$ is a field. Since R is a left Artinian ring by Hopkins Theorem (see [11, Theorem 4.15]), $\mathfrak{m}$ is a nilpotent left ideal and R is a left Noetherian ring. Suppose that $\mathfrak{m}^r = 0$. Consider the F-vector space $\mathfrak{m}^{i-1}/\mathfrak{m}^i$, $2 \le i \le r$. Since R is a left Noetherian ring, for each i, $\mathfrak{m}^i$ is a finitely generated R-module. This implies that for every i, $2 \le i \le r$, $\mathfrak{m}^{i-1}/\mathfrak{m}^i$ is a finite dimensional F-vector space. Since F is finite, we conclude that R is a finite ring. The proof is complete. $\square$

Corollary 20. *Let R be a Gorenstein ring. Then $\omega(G(R)) = n$ if and only if R has n non-trivial ideals.*

Proof. Let R be a Gorenstein ring and $\omega(G(R)) = n$. By Lemma 15, R is an Artinian ring. It turns out by Theorem 14, $G(R)$ is a complete graph, and so R has exactly n non-trivial ideals. Conversely, suppose that R has n non-trivial ideals. Then R is an Artinian ring. Again Theorem 14 implies that $G(R)$ is a complete graph, and hence $\omega(G(R)) = n$. $\square$

Theorem 21. *Let R be a reduced ring with $\omega(G(R)) < \infty$. Then $\omega(G(R)) = \chi(G(R))$.*

Proof. By Lemma 15, R is a left Artinian ring. By [11, Theorem 4.15], since $J(R)$ is nilpotent, we conclude that $J(R) = 0$ and so R is a semisimple ring. Since R is reduced by Wedderburn-Artin Theorem, $R \cong D_1 \times \cdots \times D_n$, where D_i is a division ring, for each i, $1 \le i \le n$. Therefore, R has $2^n - 2$ non-trivial left ideals. Now, consider the ideal $I = D_1 \times 0 \times \cdots \times 0$. Clearly, every ideal of the form $D_1 \times I_2 \times \cdots \times I_n$ contains I, where I_i is a left ideal of D_i, for $i = 2, \ldots, n$. But the set of these left ideals forms a clique $\mathcal{C}$ of $G(R)$ and so $\omega(G(R)) \ge 2^{n-1} - 1$. Now, we show that $\chi(G(R)) = 2^{n-1} - 1$. First we color the vertices of the clique $\mathcal{C}$ by $2^{n-1} - 1$ different colors. Let $J = (0 = J_1) \times J_2 \times \cdots \times J_n$ be a vertex not contained in $\mathcal{C}$. We color J with the color of the vertex $L = D_1 \times L_2 \times \cdots \times L_n$ of $\mathcal{C}$ in which $L_i = 0$ if $J_i = D_i$ and $L_i = D_i$ if $J_i = 0$. Thus we obtain a proper vertex coloring for $G(R)$ using $2^{n-1} - 1$ colors, as desired. $\square$

It follows from Theorem 21, if R is a reduced ring and $\omega(G(R)) < \infty$, then R has finitely many ideals. This is not true necessarily in the case non-reduced. See the following example.

Example 22. Let $R = \frac{k[x,y]}{(x,y)^2}$, where k is an infinite field and x, y are indeterminates. Then $\overline{(x,y)}, \overline{(x)}, \overline{(y)}$ and $\{\overline{(x+ay)} |\ a \in k\}$ are all non-trivial ideals of R. Therefore, $G(R)$ is an infinite star graph.

Theorem 23. *Let R be a commutative reduced ring and $|\mathrm{Min}(R)| < \infty$. Then $\alpha(G(R)) = |\mathrm{Min}(R)|$.*

Proof. Suppose that $\mathrm{Min}(R) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Let $\widehat{\mathfrak{p}}_j = \bigcap_{i=1, i\neq j}^{n} \mathfrak{p}_i$, for $j = 1, \ldots, n$. Since R is reduced, the intersection of all minimal prime ideals is zero and so it is easily checked that $\{\widehat{\mathfrak{p}}_1, \ldots, \widehat{\mathfrak{p}}_n\}$ is an independent set of $G(R)$. Hence $|\mathrm{Min}(R)| \leq \alpha(G(R))$. Obviously, $\alpha(G(R)) = \omega(\overline{G(R)})$. Thus it suffices to show that $\chi(\overline{G(R)}) \leq \alpha(G(R))$. Define a coloring f on $V(\overline{G(R)})$ by $f(I) = \min\{i, I \nsubseteq \mathfrak{p}_i\}$. We show that f is a proper vertex coloring of $\overline{G(R)}$. First note that, since R is a reduced ring, for each non-zero ideal I, there exists a minimal prime ideal $\mathfrak{p}$ such that $I \nsubseteq \mathfrak{p}$. Now, suppose that I and J are adjacent vertices in $\overline{G(R)}$ and $f(I) = f(J) = i$, $1 \leq i \leq n$. Since R is a commutative ring, we deduce that $IJ = 0$. Therefore $IJ \subseteq \mathfrak{p}_i$ and so either $I \subseteq \mathfrak{p}_i$ or $J \subseteq \mathfrak{p}_i$, a contradiction. Then f is a proper vertex coloring of $\overline{G(R)}$ and hence $\chi(\overline{G(R)}) \leq n$. Therefore, $\alpha(G(R)) = |\mathrm{Min}(R)|$. $\square$

We end this paper with with following result about the clique number and the chromatic number of the intersection graph of a direct product of rings.
Obviously, if R is a ring and $R \cong R_1 \times \cdots \times R_k$, where R_i is a ring, then $\omega(G(R)) < \infty$ if and only if $\omega(G(R_i)) < \infty$, for every i, $1 \leq i \leq k$. Now, we prove this fact for the chromatic number instead of the clique number.

Theorem 24. *Let $R_1, \ldots, R_k$ be rings and $R \cong R_1 \times \cdots \times R_k$. Then $\chi(G(R)) < \infty$ if and only if $\chi(G(R_i)) < \infty$, for every i, $1 \leq i \leq k$.*

Proof. First suppose that $R \cong R_1 \times R_2$. Let $c_i : V(G(R_i)) \longrightarrow \{1, \ldots, \chi(G(R_i))\}$ be a proper vertex coloring of $G(R_i)$, $i = 1, 2$. We extend the map c_i to $I(R_i)$ by defining $c_i(R_i) = -1$ and $c_i(0) = 0$, for $i = 1, 2$. Define a map c on $I(R)^*$ by

$$c(I \times J) = (c_1(I), c_2(J)), \quad \text{for every} \quad I \in I(R_1) \quad \text{and} \quad J \in I(R_2).$$

It is not hard to check that c is a proper coloring of $G(R_1 \times R_2)$. Conversely, let c be a proper vertex coloring of $G(R)$, then the restriction of c to $I(R_1 \times 0)^*$ and $I(0 \times R_2)^*$ provides a proper coloring of $G(R_1 \times 0)$ and $G(0 \times R_2)$, respectively. If $k > 2$, then the assertion follows from the case $k = 2$ and induction. $\square$

Acknowledgments. This work was completed while the first author was visiting the Abdus Salam International Centre for Theoretical Physics (ICTP) under the auspices of Combinatorics Program 2010. He would like to express his gratitude for the support.

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