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QUADRATIC PBW-ALGEBRAS, YANG-BAXTER EQUATION AND ARTIN-SCHELTER REGULARITY

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Abstract

We study quadratic algebras over a field $\mathbf{k}$. We show that an n-generated PBW-algebra A has finite global dimension and polynomial growth *iff* its Hilbert series is $H_A(z) = 1/(1-z)^n$. A surprising amount can be said when the algebra A has *quantum binomial relations*, that is the defining relations are binomials $xy - c_{xy}zt$, $c_{xy} \in \mathbf{k}^{\times}$, which are square-free and nondegenerate. We prove that in this case various good algebraic and homological properties are closely related. The main result shows that for an n-generated quantum binomial algebra A the following conditions are equivalent: (i) A is a PBW-algebra with finite global dimension; (ii) A is PBW and has polynomial growth; (iii) A is an Artin-Schelter regular PBW-algebra; (iv) A is a Yang-Baxter algebra; (v) $H_A(z) = 1/(1-z)^n$; (vi) The dual $A^!$ is a quantum Grassman algebra; (vii) A is a binomial skew polynomial ring. This implies that the problem of classification of Artin-Schelter regular PBW-algebras of global dimension n is equivalent to the classification of square-free set-theoretic solutions of the Yang-Baxter equation (X, r), on sets X of order n.

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1. Introduction

In the paper we work with quadratic algebras A over a ground field $\mathbf{k}$. Following a classical tradition (and recent trend), we take a combinatorial approach to study A. The properties of A will be read off a presentation $A = \mathbf{k}\langle X\rangle/(\Re)$, where X is a finite set of generators of degree 1, $|X| = n$, $\mathbf{k}\langle X\rangle$ is the unitary free associative algebra generated by X, and $(\Re)$ is the two-sided ideal of relations, generated by a *finite* set $\Re$ of homogeneous polynomials of degree two. Clearly A is a connected graded $\mathbf{k}$-algebra (naturally graded by length) $A = \bigoplus_{i\geq 0} A_i$, where $A_0 = \mathbf{k}$, A is generated by $A_1 = Span_{\mathbf{k}} X$, so each A_i is finite dimensional.

A quadratic algebra A is *a PBW-algebra* if there exists an enumeration of X, $X = \{x_1, \cdots x_n\}$ such that the quadratic relations $\Re$ form a (noncommutative) Gröbner basis with respect to the degree-lexicographic ordering on $\langle X\rangle$ induced from $x_1 < x_2 < \cdots < x_n$. In this case the set of normal monomials (mod $\Re$) forms a $\mathbf{k}$-basis of A called a *PBW-basis* and $x_1, \cdots, x_n$ (taken exactly with this enumeration) are called *PBW-generators of A*. The notion of a PBW-algebra was introduced by Priddy, [40], his *PBW-basis* is a generalization of the classical Poincaré-Birkhoff-Witt basis for the universal enveloping of a finite dimensional Lie algebra. PBW-algebras form an important class of Koszul algebras. The interested reader can find information on PBW-algebras and more references in [39]. One of the central problems that we consider is

the classification of Artin-Schelter regular PBW-algebras.

It is far from its final resolution. The first question to be asked is

> *What can be said about PBW-algebras with polynomial growth and finite global dimension?*

We find that, surprisingly, the class $\mathfrak{C}_n$ of n-generated PBW-algebras with polynomial growth and finite global dimension is determined uniquely by its Hilbert series, this is in section 3.

Theorem 1.1. *Let $A = \mathbf{k}\langle X\rangle/(\Re)$ be a quadratic PBW-algebra, where $X = \{x_1, x_2, \cdots, x_n\}$ is a set of PBW-generators. The following are equivalent*

(1) *A has polynomial growth and finite global dimension.*
(2) *A has exactly $\binom{n}{2}$ relations and finite global dimension.*
(3) *The Hilbert series of A is*
$$H_A(z) = \frac{1}{(1-z)^n}.$$
(4) *There exists a permutation $y_1, \cdots, y_n$ of $x_1 \cdots x_n$, such that the set*
$$\mathcal{N} = \{y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \mid \alpha_i \geq 0 \text{ for } 1 \leq i \leq n\} \tag{1.1}$$
is a $\mathbf{k}$-basis of A.

Furthermore, the class $\mathfrak{C}_n$ of all n-generated PBW-algebras with polynomial growth and finite global dimension contains a unique (up to isomorphism) monomial algebra:
$$A^0 = \langle x_1, \cdots, x_n\rangle/(x_j x_i \mid 1 \leq i < j \leq n).$$

Note that $y_1, y_2, \cdots, y_n$ is possibly a "new" enumeration of X, which induces a degree-lexicographic ordering $\prec$ on $\langle X\rangle$ with $y_1 \prec y_2 \prec \cdots \prec y_n$ different from the original ordering. The defining relations remain the same, but their leading terms w.r.t. $\prec$ may be different from the original ones, and $y_1, y_2, \cdots, y_n$ are not necessarily PBW-generators of A. In the terminology of Gröbner bases, $\mathcal{N}$ is not necessarily *a normal basis* of A w.r.t. $\prec$.

A class of PBW-Artin-Schelter regular rings of arbitrarily high global dimension n, were introduced and studied in [14], [28], [20], [19]. These are *the binomial skew-polynomial rings.* It was shown in [28] that they are also closely related to the set-theoretic solutions of the Yang-Baxter equation. So we consider the so-called *quantum binomial algebras* introduced and studied in [19], [24]. These are quadratic algebras (not necessarily PBW) with square-free non-degenerate binomial relations, see Definition 2.7. The second question that we ask in the paper is

Which are the PBW-Artin-Schelter regular algebras in the class of quantum binomial algebras?

We prove that each quantum binomial PBW algebra with finite global dimension is a Yang-Baxter algebra, and therefore a binomial skew-polynomial ring. This implies that *in the class of quantum binomial algebras* the three notions: a PBW-Artin-Schelter regular algebra, a binomial skew-polynomial ring, and a Yang-Baxter algebra (in the sense of Manin) are equivalent. The following result is proven in Section 5.

Theorem 1.2. *Let $A = \boldsymbol{k}\langle X\rangle/(\Re)$ be a quantum binomial algebra. The following conditions are equivalent.*

(1) *A is a PBW-algebra with finite global dimension.*
(2) *A is a PBW-algebra with polynomial growth.*
(3) *A is a PBW-Artin-Schelter regular algebra.*
(4) *A is a Yang-Baxter algebra, that is the set of relations $\Re$ defines canonically a solution of the Yang-Baxter equation.*
(5) *A is a binomial skew polynomial ring, with respect to some appropriate enumeration of X.*
(6)
$$\dim_{\boldsymbol{k}} A_3 = \binom{n+2}{3}, \quad \textit{or equivalently} \quad \dim_{\boldsymbol{k}} A^!_3 = \binom{n}{3}.$$
(7)
$$H_A(z) = \frac{1}{(1-z)^n}$$
(8) *The Koszul dual $A^!$ is a quantum Grassman algebra.*

Each of these conditions implies that A is Koszul and a Noetherian domain.

The problem of classification of Artin-Schelter regular PBW-algebras with quantum binomial relations and global dimension n is equivalent to the classification of square-free set-theoretic solutions of YBE, (X, r), on sets X of order n.

Even under these strong restrictions on the shape of the relations, the problem is highly non-trivial. However, for reasonably small n (say $n \leq 10$) the square-free solutions of YBE (X, r) are known. A possible classification for general n can be based on the so-called *multipermutation level* of the solutions, see [26].

The paper is organized as follows.

In section 2 are recalled basic definitions, and some facts used throughout the paper. In section 3 we study the general case of PBW-algebras with finite global dimension and polynomial growth and prove Theorem 1.1. The approach is combinatorial. To each PBW-algebra A we associate two finite oriented graphs. The first is *the graph of normal words* $\Gamma_{\mathbf{N}}$ (this is a particular case of the Ufnarovski graph [45]), it determines the growth and the Hilbert series of A. The second is *the graph of obstructions,* $\Gamma_{\mathbf{W}}$, dual to $\Gamma_{\mathbf{N}}$. We define it via the set obstructions (in the sense of Anick, [1], [2]), it gives a precise information about *the global dimension* of the algebra A. We prove that all algebras in $\mathfrak{C}_n$ determine a unique (up to isomorphism) graph of obstructions Γ_W, which is the complete oriented graph K_n with no cycles. The two graphs play an important role in the proof of Theorem 1.1. They can be used whenever PBW-algebras are studied.

In section 4 we find some interesting combinatorial results on quantum binomial sets (X, r) and the corresponding quadratic algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. We study the action of the infinite Dihedral group, $\mathcal{D} = \mathcal{D}(r)$, associated with r on X^3 and find some *counting formulae* for the $\mathcal{D}$-orbits. These are used to show that (X, r) is a set-theoretic solution of the Yang-Baxter equation *iff* $\dim_{\mathbf{k}} \mathcal{A}_3 = \binom{n+2}{3}$.

In section 5 we prove Theorem 1.2. The proof involves the results of sections 3, 4, and results on binomial skew polynomial rings and set-theoretic solutions of YBE from [28], [20], [19], [24].

2. Preliminaries some definitions and facts

In this section we recall basic notions and results which will be used in the paper. This paper is a natural continuation of [19]. We shall use the terminology, notation and results from [14, 28, 19, 20, 24]. The reader acquainted with these can proceed to the next section.

A connected graded algebra is called *Artin-Schelter regular* (or *AS regular*) if

(i) A has *finite global dimension* d, that is, each graded A-module has a free resolution of length at most d.

(ii) A has *finite Gelfand-Kirillov dimension*, meaning that the integer-valued function $i \mapsto \dim_{\mathbf{k}} A_i$ is bounded by a polynomial in i.

(iii) A is *Gorenstein*, that is, $Ext^i_A(\mathbf{k}, A) = 0$ for $i \neq d$ and $Ext^d_A(\mathbf{k}, A) \cong \mathbf{k}$.

AS regular algebras were introduced and studied first in [3], [4], [5]. When $d \leq 3$ all regular algebras are classified. Since then AS regular algebras and their geometry are intensively studied. The problem of classification of regular rings is difficult and remains open even for regular rings of global dimension 4. The study of Artin-Schelter regular rings, their classification, and finding new classes of such rings is one of the basic problems for noncommutative geometry. Numerous works on this topic appeared during the last two decades, see for example [4, 5], [46], [32], et al.

A class of PBW AS regular algebras of global dimension n was introduced and studied by the author, [14], [28] [20], [19]. These are *the binomial skew-polynomial rings.*

Definition 2.1. [14] A *binomial skew polynomial ring* is a quadratic algebra $A = \mathbf{k}\langle x_1, \cdots, x_n\rangle/(\Re)$ with precisely $\binom{n}{2}$ defining relations $\Re = \{x_j x_i - c_{ij} x_{i'} x_{j'}\}_{1 \leq i < j \leq n}$ such that

(a) $c_{ij} \in \mathbf{k}^{\times}; = \mathbf{k} \setminus \{0\}$;

(b) For every pair i, j $1 \leq i < j \leq n$, the relation $x_j x_i - c_{ij} x_{i'} x_{j'} \in \Re$, satisfies $j > i'$, $i' \leq j'$;

(c) Every ordered monomial $x_i x_j$, with $1 \leq i < j \leq n$ occurs on the right-hand side of some relation in $\Re$;

(d) $\Re$ is the *reduced Gröbner basis* of the two-sided ideal $(\Re)$,(with respect to the order $\prec$ on $\langle X\rangle$) or equivalently the ambiguities $x_k x_j x_i$, with $k > j > i$ do not give rise to new relations in A.

We say that $\Re$ are *relations of skew-polynomial type* if conditions 2.1 (a), (b) and (c) are satisfied (we do not assume (d)).

By [6] condition 2.1 (d) may be rephrased by saying that *the set of ordered monomials*

$$\mathcal{N}_0 = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_n \geq 0 \text{ for } 1 \leq i \leq n\}$$

is a $\mathbf{k}$-basis of A.

Remark 2.2. In the terminology of this paper *a binomial skew polynomial ring* is a quadratic PBW algebra A with PBW generators $x_1 \cdots, x_n$ and relations of skew-polynomial type.

More generally, we will consider a class of quadratic algebras with binomial relations, called *quantum binomial algebras*, these are not necessarily PBW-algebras.

We need to recall first the notions of a quadratic set and the associated quadratic algebras.

Definition 2.3. Let X be a nonempty set and let $r : X \times X \longrightarrow X \times X$ be a bijective map. In this case we shall use notation (X, r) and refer to it as a *quadratic set*. We present the image of (x, y) under r as

$$r(x,y) = ({}^x y, x^y). \tag{2.1}$$

The formula (2.1) defines a "left action" $\mathcal{L} : X \times X \longrightarrow X$, and a "right action" $\mathcal{R} : X \times X \longrightarrow X$, on X as:

$$\mathcal{L}_x(y) = {}^x y, \quad \mathcal{R}_y(x) = x^y, \tag{2.2}$$

for all $x, y \in X$. r is *nondegenerate* if the maps $\mathcal{L}_x$ and $\mathcal{R}_x$) are bijective for each $x \in X$. r is *involutive* if $r^2 = id_{X \times X}$.

As a notational tool, we shall often identify the sets $X^{\times k}$ of ordered k-tuples, $k \geq 2$, and X^k, the set of all monomials of length k in the free monoid $\langle X\rangle$.

As in [20, 23, 24, 25, 26], to each quadratic set (X, r) we associate canonically algebraic objects (see Definition 2.4) generated by X and with quadratic defining relations $\Re_0 = \Re_0(r)$ naturally determined as

$$\begin{array}{lll} xy = y'x' \in \Re_0(r), & \text{whenever} & \\ r(x,y) = (y', x') & \text{and} \quad (x,y) \neq (y',x') & \text{hold in } X \times X. \end{array} \tag{2.3}$$

We can tell the precise number of defining relations, whenever (X, r) is nondegenerate and involutive, with $\mid X \mid = n$. In this case the set of defining relations $\Re(r)$, contains precisely $\binom{n}{2}$ quadratic relations (see [22] Proposition 2.3).

Definition 2.4. [20, 24] Assume that $r : X^2 \longrightarrow X^2$ is a bijective map.

(i) The monoid

$$S = S(X, r) = \langle X;\ \Re_0(r)\rangle,$$

with a set of generators X and a set of defining relations $\Re(r)$, is called *the monoid associated with* (X, r). The *group* $G = G(X, r)$ *associated with* (X, r) is defined analogously.

(ii) For arbitrary fixed field $\mathbf{k}$, *the* $\mathbf{k}$*-algebra associated with* (X, r) is defined as

$$\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) = \mathbf{k}\langle X;\ \Re_0(r)\rangle \simeq \mathbf{k}\langle X\rangle/(\Re), \tag{2.4}$$

where $\Re = \Re(r)$ is the set of quadratic *binomial relations*

$$\Re = \{xy - y'x' \mid xy = y'x' \in \Re_0(r)\}. \tag{2.5}$$

Clearly $\mathcal{A}$ is *a quadratic algebra*, generated by X and with defining relations $\Re(r)$. Furthermore, $\mathcal{A}$ is isomorphic to the monoid algebra $\mathbf{k}S(X, r)$. In many cases the associated algebra will be *standard finitely presented* with respect to the degree-lexicographic ordering induced by an appropriate enumeration of X, that is a PBW-algebra. It is known in particular, that the algebra $\mathcal{A}(\mathbf{k}, X, r)$ has remarkable algebraic and homological properties when r is involutive, nondegenerate and obeys the braid or Yang-Baxter equation in $X \times X \times X$. Set-theoretic solutions were introduced in [8, 47] and have been under intensive study during the last decade. There are many works on set-theoretic solutions and related structures, of which a relevant selection for the interested reader is [47, 28, 41, 11, 31, 20, 19, 43, 48, 24, 25, 26, 7].

Definition 2.5. Let (X, r) be a quadratic set.

(1) (X, r) is said to be *square-free* if $r(x, x) = (x, x)$ for all $x \in X$.
(2) (X, r) is called a *quantum binomial set* if it is nondegenerate, involutive and square-free.
(3) (X, r) is a *set-theoretic solution of the Yang-Baxter equation* (YBE) if the braid relation

$$r^{12}r^{23}r^{12} = r^{23}r^{12}r^{23}$$

holds in $X \times X \times X$. In this case (X, r) is also called a *braided set.* If in addition r is involutive (X, r) is called *a symmetric set.*

Example 2.6. $X = \{x_1, x_2, x_3, x_4, x_5\}$ and r is defined via the actions $\mathcal{L}, \mathcal{R} = (\mathcal{L})^{-1} \in \mathrm{Sym}(X)$ as

$$\begin{array}{l} r(x, y) = (\mathcal{L}_x(y), (\mathcal{L}_y)^{-1}(x)) \quad \text{where} \\ \mathcal{L}_{x_1} = \mathcal{L}_{x_3} = (x_2x_4) \quad \mathcal{L}_{x_2} = \mathcal{L}_{x_4} = (x_1x_3) \\ \mathcal{L}_{x_5} = (x_1x_2x_3x_4). \end{array} \tag{2.6}$$

This is a (square-free) symmetric set. (It has multipermutation level 3, see [26]). Presenting the solution (X, r) via the left and the right actions is an elegant and convenient way to express the corresponding $\binom{n}{2}$ quadratic relations $\Re(r)$ of the algebra $\mathcal{A}(\mathbf{k}, X, r)$, especially when n is large. We recommend the reader to write down explicitly the ten quadratic relations encoded in (2.6). Enumerated this way, $x_1, \cdots, x_5$ are not PBW-generators. If we reorder the generators as

$$y_1 = x_1 \prec y_2 = x_3 \prec y_3 = x_2 \prec y_4 = x_4 \prec y_5 = x_5,$$

then $y_1, \cdots, y_5$ are PBW-generators, and $\mathcal{A}$ is a binomial skew-polynomial ring w.r.t this new enumeration.

Definition 2.7. A quadratic algebra $A(\mathbf{k}, X, \Re) = \mathbf{k}\langle X\rangle/(\Re)$ is *a quantum binomial algebra* if the relations $\Re$ satisfy the following conditions

(a) Each relation in $\Re$ is of the shape

$$xy - c_{yx}y'x', \quad \text{where} \quad x, y, x', y' \in X, \quad \text{and} \quad c_{xy} \in \mathbf{k}^{\times} \tag{2.7}$$

(this is what we call *a binomial relation*).

(b) Each $xy, x \neq y$ of length 2 occurs at most once in $\Re$.
(c) Each relation is *square-free*, i.e. it does not contain a monomial of the shape xx, $x \in X$.
(d) The relations $\Re$ are *non degenerate*, i.e. the canonical bijection $r = r(\Re) : X \times X \longrightarrow X \times X$, associated with $\Re$, is non degenerate.

Relations satisfying conditions (a)-(d) are called *quantum binomial relations.*

Clearly, each binomial skew-polynomial ring is a PBW-quantum binomial algebra. The algebra $\mathcal{A}(\mathbf{k}, X, r)$ from Example 2.6 is a concrete quantum binomial algebra. See more examples at the end of the section. We recall that (although this is not part of the definition) every n-generated quantum binomial algebra has exactly $\binom{n}{2}$ relations.

With each quantum binomial algebra we associate two maps determined canonically via its relations.

Definition 2.8. Let $\Re \subset \mathbf{k}\langle X\rangle$ be a set of quadratic binomial relations, satisfying conditions (a) and (b). Let $V = Span_{\mathbf{k}}X$.

The canonically *associated quadratic set* (X, r), with $r = r(\Re) : X \times X \longrightarrow X \times X$ is defined as

$$r(x, y) = (y', x'), \text{ and } r(y', x') = (x, y) \text{ if } xy - c_{xy}y'x' \in \Re.$$
$$\text{If } xy \text{ does not occur in any relation we set } r(x, y) = (x, y).$$

$(X, r(\Re)) = (X, r)$ is *the quadratic set (canonically) associated with* $\Re$.

The automorphism associated with $\Re$, $R = R(\Re) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$, is defined as follows:

$$\text{If } xy - c_{xy}y'x' \in \Re, \text{ we set } R(x \otimes y) = c_{xy}y' \otimes x', \text{and } R(y' \otimes x') = (c_{xy})^{-1}x \otimes y.$$
$$\text{If } xy \text{ does not occur in any relation } (x = y \text{ is also possible}), \text{ we set } R(x \otimes y) = x \otimes y.$$

R is called *non-degenerate* if r is non-degenerate. In this case we shall also say that the defining relations $\Re$ are *non degenerate binomial relations.*

Let V be a $\mathbf{k}$-vector space. A linear automorphism R of $V \otimes V$ is *a solution of the Yang-Baxter equation*, (YBE) if the equality

$$R^{12}R^{23}R^{12} = R^{23}R^{12}R^{23} \tag{2.8}$$

holds in the automorphism group of $V \otimes V \otimes V$, where R^{ij} means R acting on the i-th and j-th component.

A quantum binomial algebra $A = \mathbf{k}\ \langle X; \Re\rangle$, is *a Yang-Baxter algebra* (in the sense of Manin [37]) if the associated map $R = R(\Re)$ is a solution of the Yang-Baxter equation.

It was shown in [28] that each binomial skew polynomial ring is a Yang-Baxter algebra.

The results below can be extracted from [28], [14], and [19], Theorem B.

Fact 2.9. *Let $A = \mathbf{k}\langle X\rangle/(\Re)$ be a quantum binomial algebra. Then the following two conditions are equivalent.*

(1) *A is a binomial skew polynomial ring, with respect to some appropriate enumeration of X.*
(2) *The automorphism $R = R(\Re) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$ is a solution of the Yang-Baxter equation, so A is a Yang-Baxter algebra.*

Each of these conditions implies that A is an Artin-Schelter regular PBW-algebra. Furthermore, A is a left and right Noetherian domain.

We shall prove in Section 5 that conversely, in the class of quantum binomial algebras each Artin-Schelter regular PBW-algebra defines canonically a solution of the YBE, and therefore is a Yang-Baxter algebra and a binomial skew-polynomial ring.

We end up the section with two concrete examples of quantum binomial algebras with 4 generators.

Example 2.10. Let $A = \mathbf{k}\langle x, y, z, t\rangle/(\Re)$, where $X = \{x, y, z, t\}$, and

$$\Re \quad = \quad \{xy - zt, \quad ty - zx, \quad xz - yx, \quad tz - yt, \quad xt - tx, \quad yz - zy\}.$$

Clearly, the relations are square-free, a direct verification shows that they are nondegenerate. So A is a quantum binomial algebra. More sophisticated proof shows that the set of relations $\Re$ is not a Gröbner basis w.r.t. deg-lex ordering coming from any order (enumeration) of the set X. This example is studied with details in Section 4.

The second is an example of a quantum binomial algebra which is a *binomial skew-polynomial ring*, and therefore a Yang-Baxter algebra. It is a PBW-algebra, w.r.t various enumerations of X.

Example 2.11. Let $A = \mathbf{k}\langle X\rangle/(\Re)$, where $X = \{x, y, z, t\}$, and

$$\Re \quad = \quad \{xy - zt, \quad ty - zx, \quad xz - yt, \quad tz - yx, \quad xt - tx, \quad yz - zy\}.$$

We fix $t > x > z > y$, and take the corresponding deg-lex ordering on $\langle X\rangle$. Then direct verification shows that $\Re$ is a Gröbner basis. To do this one has to show that the ambiguities txz, txy, tzy, xzy are solvable. In this case the set

$$\mathcal{N} = \{y^\alpha z^\beta x^\gamma t^\delta \mid \alpha, \beta, \gamma, \delta \geq 0\}$$

is the normal basis of A, (mod $\Re$).

Note that any order in which $\{t, x, \} > \{z, y\}$, or $\{z, y\} > \{t, x, \}$, makes A a PBW-algebra, there are exactly eight such enumerations of X.

Furthermore, A is a Yang-Baxter algebra, and an AS-regular domain of global dimension 4.

3. PBW algebras with polynomial growth and finite global dimension

Let $X = \{x_1, \cdots x_n\}$. As usual, we fix the deg-lex ordering $< \langle X\rangle$. Each element $g \in \mathbf{k}\langle X\rangle$ has the shape $g = cu + h$, where $u \in \langle X\rangle$, $c \in \mathbf{k}^\times$, and either $h = 0$, or $h = \sum_\alpha c_\alpha u_\alpha$ is a linear combination of monomials $u_\alpha \in< u$. u is called the leading monomial of g (w.r.t. $<$) and denoted $LM(g)$. Every finitely presented graded algebra, A $A = \mathbf{k}\langle X\rangle/I$, where I is an ideal of $\mathbf{k}\langle X\rangle$ has a uniquely determined reduced Gröbner basis $\mathbf{G}$. In general, G is infinite. Anick introduces *the set of obstructions* ***W*** *for a connected graded algebra*, see [2]. It is easy to deduce from his definition that the set of obstructions $\mathbf{W}$ is exactly *the set of leading monomials* $\mathbf{W} = \{LM(g) \mid g \in G\}$. The obstructions are used to construct a free resolution of the field k considered as an A-module, [2].

Consider now a PBW-algebra $A = \mathbf{k}\langle X\rangle/(\Re)$ with PBW generators $X = \{x_1, \cdots x_n\}$. In this case *the set of obstructions* $\mathbf{W}$ is simply the set of leading monomials of the defining relations.

$$\mathbf{W} \ = \{LM(f) \mid f \in \Re\}.$$

Then $\mathbf{N} = X^2 \backslash \mathbf{W}$ is the set of *normal monomials* (mod $\mathbf{W}$) of length 2.

Notation 3.1. We set

$$\mathbf{N}^{(0)} = \{1\}, \quad N^{(1)} = X$$

$$\mathbf{N}^{(m)} = \{x_{i_1}x_{i_2}\cdots x_{i_m} \mid x_{i_k}x_{i_{k+1}} \in \mathbf{N},\ 1 \le k \le m-1\}, m = 2, 3, \cdots$$

$$\mathbf{N}^{\infty} = \bigcup_{m \geq 0} N^{(m)}$$

Note the set $\mathbf{N}^{(m)}$, $m \geq 0$ is a $\mathbf{k}$-basis of A_m, $\mathbf{N}^{\infty}$ is the set of all normal (mod $\mathbf{W}$) words in $\langle X \rangle$. It is well-known that the set $\mathbf{N}^{\infty}$ project to a basis of A. More precisely, the free associative algebra $\mathbf{k}\langle X \rangle$ splits into a direct sum of subspaces

$$\mathbf{k}\langle X \rangle \simeq Span_{\mathbf{k}}\mathbf{N}^{\infty} \bigoplus I$$

So there are isomorphisms of vector spaces

$$A \simeq Span_{\mathbf{k}}\mathbf{N}^{\infty}$$

$$A_m \simeq Span_{\mathbf{k}}\mathbf{N}^{(m)},\ \dim A_m = |\mathbf{N}^{(m)}|, m = 0, 1, 2, 3, \cdots$$

For a PBW algebra A there is a canonically associated monomial algebra $A^0 = \mathbf{k}\langle X \rangle/(\mathbf{W})$. As a monomial algebra, A^0 is also PBW. Both algebras A and A^0 have the same set of obstructions $\mathbf{W}$ and therefore they have the same *normal basis* $\mathbf{N}^{\infty}$, the same Hilbert series and the same growth. It follows from the results of Anick that $gl.\dim A = gl.\dim A^0$. More generally, the sets of obstructions $\mathbf{W}$ determine uniquely the Hilbert series, growth and the global dimension for the whole family of PBW algebras A sharing the same W. The binomial skew polynomial rings are a well-known example of PBW-algebras with polynomial growth and finite global dimension, moreover they are ***AS-regular Noetherian domains, see [28]. We recall the original definition.

We will introduce now two oriented graphs which are used effectively to measure the growth and the global dimension of A.

Definition 3.2. Let $M \subset X^2$ be a set of monomials of length 2. We introduce *the graph* Γ_M *corresponding to* M. By definition Γ_M is *a directed graph* with a set of vertices $V(\Gamma_M) = X$ and a set of directed edges (arrows) $E = E(\Gamma_M)$ defined as follows

$$x \longrightarrow y \in E \text{ iff } xy \in M.$$

We recall that *the order* of a graph Γ is the number of its vertices, i.e. $|V(\Gamma)|$. *A cycle (of length k)* in Γ is an (oriented) path of the shape $v_1 \longrightarrow v_2 \longrightarrow \cdots v_k \longrightarrow v_1$ where $v_1, \cdots, v_k$ are distinct vertices. *A loop* is a cycle of length 0, $x \longrightarrow x$. So the graph Γ_M contains a loop $x \longrightarrow x$ whenever $xx \in M$, and a cycle of length two $x \longrightarrow y \longrightarrow x$, whenever $xy, yx \in M$. In this case $x \longrightarrow y, y \longleftarrow x$ are called bidirected edges. Note that, following the terminology in graph theory, *we make a difference between directed and oriented graphs.* A directed graph having no symmetric pair of directed edges (i.e., no bidirected edges, pairs $x \longrightarrow y$ and $y \longrightarrow x$) is known *as an oriented graph.* An oriented graph with no cycles is called *acyclic oriented* graph. In particular, such a graph has no loops.

Denote by $\overline{M}$ the complement $X^2 \backslash M$. Then the graph $\Gamma_{\overline{M}}$ is *dual to* Γ_M in the sense that

$$x \longrightarrow y \in E(\Gamma_{\overline{M}}) \text{ iff } x \longrightarrow y \text{ is not an edge of } \Gamma_M$$

Let A be a PBW algebra, let $\mathbf{W}$ and $\mathbf{N}$ be the set of obstructions, and the set of normal monomials of length 2, respectively. Then the graph $\Gamma_{\mathbf{N}}$ gives complete information about the growth of A, while the global dimension of A, can be read of $\Gamma_{\mathbf{W}}$.

The graph of normal words of A, $\Gamma_{\mathbf{N}}$ was introduced in a more general context by Ufnarovski [45].

Note that, in general, $\Gamma_{\mathbf{N}}$ is a directed graph (but *not necessarily an oriented graph*) which may contain pairs of edges, $x \longrightarrow y, y \longrightarrow x$ or loops $x \longrightarrow x$.

The following is a particular case of a more general result of Ufnarovski.

Fact 3.3. [45] *For every $m \geq 1$ there is a one-to-one correspondence between the set $\boldsymbol{N}^{(m)}$ of normal words of length m and the set of paths of length $m-1$ in the graph $\Gamma_{\boldsymbol{N}}$. The path $a_1 \longrightarrow a_2 \longrightarrow a_2 \longrightarrow \cdots \longrightarrow a_m$ (these are not necessarily distinct vertices) corresponds to the word $a_1 a_2 \cdots a_m \in \boldsymbol{N}^{(m)}$. The algebra A has polynomial growth of degree m* iff

(i) *the graph $\Gamma_{\boldsymbol{N}}$ has no intersecting cycles, and*

(ii) *m is the largest number of (oriented) cycles occurring in a path of $\Gamma_{\boldsymbol{N}}$.*

Example 3.4. Each binomial skew-polynomial algebra A with 5 generators has graph $\Gamma_{\mathbf{N}}$ as in Figure 1. The graph of obstruction $\Gamma_{\mathbf{W}}$ for A can be seen in Figure 2. The Koszul dual $A^!$ has a corresponding graph of normal words $\Gamma_{\mathbf{N}^!}$ represented in Figure 3. The graphs in Figure 2 and Figure 3 are acyclic tournaments, see Definition 3.6.

The graph $\Gamma_{\mathbf{W}}$ is dual to $\Gamma_{\mathbf{N}}$, i.e. $x \longrightarrow y \in E(\mathbf{W})$ *iff* $x \longrightarrow y$ is not an edge in $\Gamma_{\mathbf{N}}$. Similar to $\Gamma_{\mathbf{N}}$, $\Gamma_{\mathbf{W}}$ is a directed graph which, in general, may contain pairs of edges, $x \longrightarrow y, y \longrightarrow x$ or loops $x \longrightarrow x$.

It is easy to see that for every $m \geq 1$ there is a one-to-one correspondence between the set of m-chains, in the sense of Anick, and the set of paths of length m in $\Gamma_{\mathbf{W}}$. The m- chain $y_{m+1}y_m \cdots y_1$, where $y_{i+1}y_i \in \mathbf{W},\ 1 \leq i \leq m$, corresponds to the path $y_{m+1} \longrightarrow y_m \longrightarrow \cdots \longrightarrow y_1$ of length m in $\Gamma_{\mathbf{W}}$. For completeness, the 0-*chains* are the elements of X, by definition.

Note that Anick's resolution [2] [1] is minimal for PBW algebras and therefore A has finite global dimension $d < \infty$ *iff* there is a $d-1$-chain, and there are no d-chains in A. The following lemma is a "translation" of this in terms of the properties of $\Gamma_{\mathbf{W}}$.

Lemma 3.5. $\operatorname{gl.dim} A = d < \infty$ iff $\Gamma_{\boldsymbol{W}}$ *is an acyclic oriented graph, and $d-1$ is the maximal length of a path occurring in $\Gamma_{\boldsymbol{W}}$.*

All PBW algebras with the same set of PBW generators $x_1, \cdots, x_n$ and the same sets of obstructions $\mathbf{W}$, share the same graphs $\Gamma_{\mathbf{N}}$ and $\Gamma_{\mathbf{W}}$. In some cases it is convenient to study the corresponding monomial algebra A^0 instead of A.

Definition 3.6. A complete oriented graph Γ (i.e., a directed graph in which each pair of vertices is joined by a single edge having a unique direction) is called *a tournament or tour.* Clearly a complete directed graph with no cycles (of any length) is *an acyclic tournament.*

The following is straightforward.

Remark 3.7. An acyclic oriented graph with n vertices is a tournament *iff* it has exactly $\binom{n}{2}$ (directed) edges.

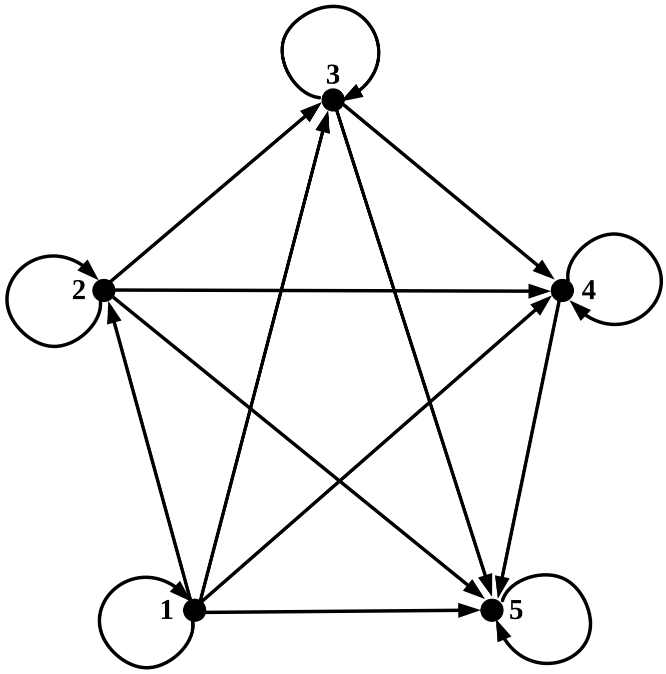

FIGURE 1. This is the graph of normal words $\Gamma_{\mathbf{N}}$ for a PBW algebra A with 5 generators, polynomial growth and finite global dimension.

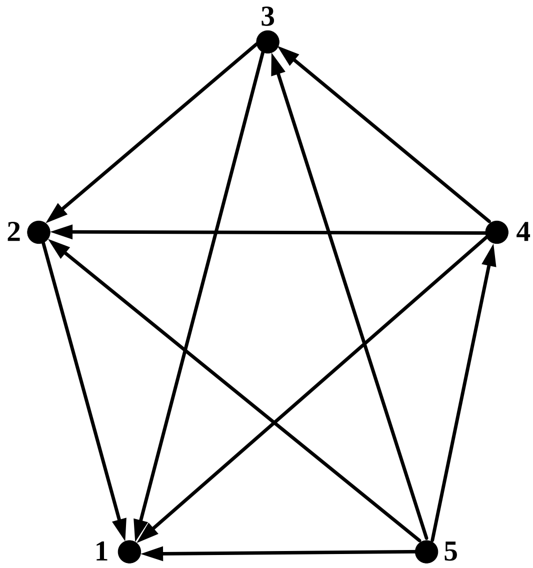

FIGURE 2. This is the graph of obstructions $\Gamma_{\mathbf{W}}$, dual to $\Gamma_{\mathbf{N}}$. It is **an acyclic tournament of order 5**, labeled "properly", as in Proposition 3.8.

We shall need the following easy proposition about oriented graphs.

Proposition 3.8. *Let Γ be an acyclic tournament, of order n. Then the set of its vertices $V = V(\Gamma)$ can be labeled $V = \{y_1, y_2, \cdots, y_n\}$, so that*

$$E(\Gamma) = \{y_j \longrightarrow y_i \mid 1 \le i < j \le n\}. \tag{3.1}$$

Analogously, the vertices can be labeled $V = \{z_1, z_2, \cdots, z_n\}$, so that $E(\Gamma) = \{z_i \longrightarrow z_j \mid 1 \le i < j \le n\}$

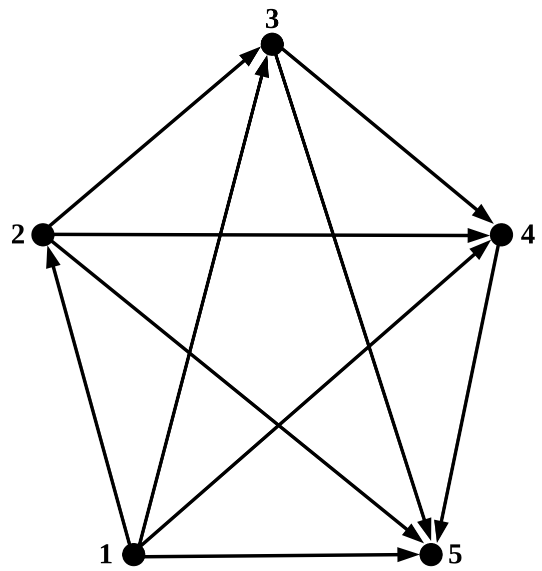

FIGURE 3. This is the graph of normal words $\Gamma^!$ for the Koszul dual $A^!$. It is **an acyclic tournament of order 5**.

Proof. We prove this by induction on the order of Γ.

The statement is obvious for $n = 2$. Assume the statement of the proposition is true for graphs with $n-1$ vertices. Let Γ be an acyclic tournament, of order n with vertices labeled $\{1, \cdots n\}$ and set of edges $E = E(\Gamma)$. Let Γ_{n-1} be the subgraph of Γ with a set of vertices $V' = V(\Gamma_{n-1}) = \{1, \cdots n-1\}$ and set of edges $E' = E(\Gamma_{n-1})$, for $1 \leq i, j \leq n-1$, $i \longrightarrow j$ is an edge of Γ_{n-1} *iff* $i \longrightarrow j \in E(\Gamma)$. By the inductive assumption the set of vertices V' can be relabeled $V' = \{v_1, \cdots v_{n-1}\}$, s.t.

$$E(\Gamma_{n-1}) = \{v_j \longrightarrow v_i \mid 1 \leq i < j \leq n-1\}.$$

Denote by v the n-th vertex of Γ. Two cases are possible.

(a) $v_j \longrightarrow v \in E(\Gamma)$, $\forall\, 1 \leq j \leq n-1$. In this case the relabeling is clear, we set $y_1 = v$, and $y_{j+1} = v_j, 1 \leq j \leq n-1$. Then the labeling $V = \{y_1 \cdots y_n\}$ agrees with 3.1.

(b) There exists a $j, 1 \leq j \leq n-1$, such that $v \longrightarrow v_j \in E(\Gamma)$. Let j, $1 \leq j \leq n-1$, be the maximal index with the property $v \longrightarrow v_j \in E(\Gamma)$.

Assume $j > 1$, and let $1 \leq i < j$ We claim that $v \longrightarrow v_i \in E(\Gamma)$. Assume the contrary. By assumption the vertices v, v_i are connected with a directed edge, so $v_i \longrightarrow v \in E(\Gamma)$. Note that by the inductive assumption $v_j \longrightarrow v_i \in E(\Gamma)$. So the graph Γ contains the cycle

$$v \longrightarrow v_j \longrightarrow v_i \longrightarrow v,$$

which contradicts the hypothesis. Thus we have

$$\begin{aligned} &v \longrightarrow v_i \in E(\Gamma), \qquad \forall\, i,\ 1 \leq i \leq j \\ &v_k \longrightarrow v \in E(\Gamma), \qquad \forall\, j,\ j < k \leq n-1 \quad (\text{if } j < n-1). \end{aligned}$$

Three cases are possible

A. $j = 1$. In this case we set

$$y_1 = v_1,\ y_2 = v, y_{k+1} = v_k,\ 2 \leq k \leq n-1.$$

B. $1 < j < n-1$. In this case we set

$$y_k = v_k,\ 1 \le k \le j-1,\ y_j = v,\ y_{k+1} = v_k,\ j \le k \le n-1.$$

C. $j = n-1$. In this case we set

$$y_k = v_k,\ 1 \le k \le n-1,\ y_n = v, .$$

□

A is *a quadratic monomial algebra* if it has a presentation $A = \mathbf{k}\langle X\rangle/(W)$, where W is a set of monomials of length 2. A quadratic monomial algebra is always a PBW-algebra.

Theorem 3.9. *Let $A^0 = \boldsymbol{k}\langle X\rangle/(\boldsymbol{W})$ be a quadratic monomial algebra. The following conditions are equivalent*

(1) *A^0 has finite global dimension, and polynomial growth.*

(2) *A^0 has finite global dimension, and $|\boldsymbol{W}| = \binom{n}{2}$.*

(3) *A^0 has polynomial growth, $\boldsymbol{W}\bigcap \operatorname{diag} X^2 = \emptyset$, and $|\boldsymbol{W}| = \binom{n}{2}$.*

(4) *The graph $\Gamma_{\mathbf{W}}$ is an acyclic tournament.*

(5)

$$H_{A^0}(z) = \frac{1}{(1-z)^n}.$$

(6) *There is a permutation $y_1, \cdots, y_n$ of $x_1 \cdots, x_n$, such that*

$$\boldsymbol{N}^{\infty} = \{y_1^{\alpha_1} \cdots y_n^{\alpha_n} \mid \alpha_i \ge 0,\ 1 \le i \le n\}. \tag{3.2}$$

(7) *There is a permutation $y_1, \cdots, y_n$ of $x_1 \cdots, x_n$, such that*

$$\boldsymbol{W} = \{y_j y_i \mid\ 1 \le i < j \le n\}.$$

Furthermore, in this case

$$\operatorname{gl.dim} A^0 = n = \text{ the degree of polynomial growth of } A.$$

Proof. Condition (4) is central for our proof.

A. We will start with several easy implications.

Suppose (4) holds, so $\Gamma_{\mathbf{W}}$ is an acyclic tournament. By Proposition 3.8 the set of its vertices $V = V(\Gamma_{\mathbf{W}})$ can be relabeled $V = \{y_1, y_2, \cdots, y_n\}$, so that

$$E(\Gamma_{\mathbf{W}}) = \{y_j \longrightarrow y_i \mid 1 \le i < j \le n\}. \tag{3.3}$$

This clearly implies condition (7). The inverse implication is also clear. So (4) $\iff$ (7).

The following implications are straightforward

$$(6) \iff (7) \implies (5)$$
$$(7) \implies (3)$$

As an acyclic tournament $\Gamma_{\mathbf{W}}$ contains exactly $\binom{n}{2}$ edges, and therefore $|\mathbf{W}| = \binom{n}{2}$. (3.3) implies also that the graph $\Gamma_{\mathbf{W}}$ has a path $y_n \longrightarrow y_{n-1} \longrightarrow \cdots \longrightarrow y_1$ of length $n-1$, and there are no longer paths, thus $gl.\dim A^0 = n$. Hence, (4) $\implies$ (2). It is also clear that (4) $\implies$ (1). (2) $\implies$ (4). First by $|\mathbf{W}| = \binom{n}{2}$ the graph has exactly $\binom{n}{2}$ edges, and next $gl.\dim A^0 < \infty$ implies that $\Gamma_{\mathbf{W}}$ is an acyclic oriented graph, (see Lemma 3.5) so by Remark 3.7 $\Gamma_{\mathbf{W}}$ is an acyclic tournament.

(3) $\implies$ (4). Assume (3) holds. Then $\Gamma_{\mathbf{W}}$ has exactly $\binom{n}{2}$ edges $x \longrightarrow y$, with $x \ne y$. Its dual graph $\Gamma_{\mathbf{N}}$ has a loop $x \longrightarrow x$ at every vertex, and exactly $\binom{n}{2}$ edges $x \longrightarrow y$, with $x \ne y$. The

polynomial growth of A^0 implies that $\Gamma_{\mathbf{N}}$ has no cycles of length ≥ 2, and therefore every two vertices in $\Gamma_{\mathbf{N}}$ are connected with a single directed edge, so $\Gamma_{\mathbf{N}}$ is an oriented graph. It follows then that $\Gamma_{\mathbf{W}}$ is an acyclic oriented tournament, which verifies the implication. The inverse implication is clear. We have shown

$$(2) \Longleftrightarrow (4) \Longleftrightarrow (3).$$

It remains to show (1) $\Longrightarrow$ (4), and (5) $\Longrightarrow$ (4). The two implications are given by similar argument.

B. (5) $\Longrightarrow$ (4). Note first that

$$H_{A^0}(z) = \frac{1}{(1-z)^n} = 1 + nz + \binom{n+1}{2} z^2 + \binom{n+2}{3} z^3 + \cdots . \tag{3.4}$$

So

$$\dim A_2 = |\mathbf{N}| = \binom{n+1}{2}, \quad \text{which implies} \quad |\mathbf{W}| = \binom{n}{2}.$$

Secondly, the special shape of Hilbert series $H_{A^0}(z)$ implies that A^0 has polynomial growth of degree n. Therefore by Fact 3.3 the graph $\Gamma_{\mathbf{N}}$ contains a path with n cycles. The only possibility for such a path is

$$\begin{tikzcd} \underset{a_1}{\bullet} \arrow[loop above] \arrow[r] & \underset{a_2}{\bullet} \arrow[loop above] \arrow[r] & \underset{a_3}{\bullet} \arrow[loop above] \arrow[r] & \cdots \arrow[r] & \underset{a_n}{\bullet} \arrow[loop above] \end{tikzcd} \tag{3.5}$$

Indeed $\Gamma_{\mathbf{N}}$ has exactly n vertices, and has no intersecting cycles. Each loop $x \longrightarrow x$ in $\Gamma_{\mathbf{N}}$ implies $xx \in N$, so $\Delta_2 \subset \mathbf{N}$ ($\Delta_2 = \operatorname{diag}(X^2)$)***. Then the complement $\mathbf{N} \backslash \Delta_2$ contains exactly $\binom{n}{2}$ monomials of the shape $xy, x \neq y$, or equivalently $\Gamma_{\mathbf{N}}$ has $\binom{n}{2}$ edges of the shape $x \longrightarrow y, x \neq y$. Clearly no pair $x \longrightarrow y, y \longrightarrow \in E(\Gamma_{\mathbf{N}})$, otherwise $\Gamma_{\mathbf{N}}$ will have two intersecting cycles $x \longrightarrow x$ and $x \longrightarrow y \longrightarrow x$, so it is an oriented graph.

Consider now the dual graph $\Gamma_{\mathbf{W}}$. The properties of $\Gamma_{\mathbf{N}}$ imply that (a) $\Gamma_{\mathbf{W}}$ has no loops. (b) $\Gamma_{\mathbf{W}}$ has no cycles of length ≥ 2. Each edge $x \longrightarrow y$ in $E(\Gamma_{\mathbf{N}})$ has a corresponding edge $x \longleftarrow y \in E(\Gamma_{\mathbf{W}})$. So $\Gamma_{\mathbf{W}}$ is an acyclic oriented graph with $\binom{n}{2}$ edges, and Remark 3.7 again implies that it is an acyclic tournament. This proves (5) $\Longrightarrow$ (4).

C. Finally we show (1) $\Longrightarrow$ (4).

Assume that A has a polynomial growth and finite global dimension. We shall use once more the nice balance between the dual graphs $\Gamma_{\mathbf{W}}$ and $\Gamma_{\mathbf{N}}$. Note first that $\Gamma_{\mathbf{N}}$ has no intersecting cycles, since otherwise A would have exponential growth. On the other hand $\Gamma_{\mathbf{W}}$ is acyclic, therefore it is an acyclic oriented graph. In particular, $\Gamma_{\mathbf{W}}$ has no loops, or equivalently $\mathbf{W}$ does not contain monomials of the type xx). It follows then that the dual graph $\Gamma_{\mathbf{N}}$ has loops $x \longrightarrow x$ at every vertex. Secondly, for each pair $x \neq y$ of vertices, there is exactly one edge $x \longrightarrow y$, or $y \longrightarrow x$, in $\Gamma_{\mathbf{N}}$. Indeed, $x \longrightarrow y, y \longrightarrow x \in E(\Gamma_{\mathbf{N}})$ would imply that $\Gamma_{\mathbf{N}}$ has intersecting cycles, which is impossible. Moreover, if there is no edge connecting x and y in $\Gamma_{\mathbf{N}}$ this would imply that both $x \longrightarrow y$, $y \longrightarrow x$ are edges of $\Gamma_{\mathbf{W}}$, hence $\Gamma_{\mathbf{W}}$ has a cycle $x \longrightarrow y \longrightarrow x$, which is impossible.

We have shown that So $\Gamma_{\mathbf{W}}$ is an acyclic oriented graph with $\binom{n}{2}$ edges, and Remark 3.7 again implies that it is an acyclic tournament. $\square$

Remark 3.10. The implication (1) $\Longrightarrow$ (5) follows straightforwardly from a result of Anick, see [1] Theorem 6.

Proof of Theorem 1.1. Assume now that $A = \mathbf{k}\langle X\rangle/(\Re)$ is a quadratic PBW-algebra, with PBW generators $X = \{x_1, \cdots x_n\}$. Let $\mathbf{W}$ be the set of obstructions, and let $A^0 = \mathbf{k}\langle X\rangle/(\mathbf{W})$ be the corresponding monomial algebra. The set $\mathbf{N}^\infty$, (Notation 3.1) is a $\mathbf{k}$-basis for both algebras A and A^0. As we have noticed before, the two algebras have the same Hilbert series, equal degrees of growth, and by Lemma 3.5, there is an equality $gl.\dim A = gl.\dim A^0$.

(1) $\Longrightarrow$ (4). Suppose A has finite global dimension and polynomial growth. Then the same is valid for A^0. By Theorem 3.9.6 there is a permutation $y_1, \cdots, y_n$ of $x_1 \cdots, x_n$, such that

$$\mathbf{N}^\infty = \{y_1^{\alpha_1} \cdots y_n^{\alpha_n} \mid \alpha_i \geq 0,\ 1 \leq i \leq n\},$$

so A has a $\mathbf{k}$-basis of the desired form. (In general, it is not true that $\mathbf{N}^\infty$ is *a normal basis* for A w.r.t. the deg-lex ordering $\prec$ defined via $y_1 \prec \cdots \prec y_n$.)

(4) $\Longrightarrow$ (3) is clear.

(3) $\Longrightarrow$ (1).

Assume (3) holds. Then obviously A has polynomial growth of degree n. The equalities

$$H_{A^0}(z) = H_A(z) = \frac{1}{(1-z)^n}$$

and Theorem 3.9 imply that the monomial algebra A^0 has $gl.\dim A^0 = n$, and $\mid W \mid = \binom{n}{2}$. Clearly, then A has $\binom{n}{2}$ relations and global dimension n. This gives the implications (3) $\Longrightarrow$ (1), and (3) $\Longrightarrow$ (2)

Similarly, condition (2) is satisfied simultaneously by A and A^0, and therefore, by Theorem 3.9, (3) holds.

The theorem has been proved.

4. Combinatorics in quantum binomial sets

In this section (X, r) is a finite quantum binomial set.

When we study the monoid $S = S(X, r)$, or the algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r) \simeq \mathbf{k}\,[S]$ associated with (X, r) (see Definition 2.4), it is convenient to use the action of the infinite groups, $\mathcal{D}_k(r)$, generated by maps associated with the quadratic relations, as follows. We consider the bijective maps

$$r^{ii+1} : X^m \longrightarrow X^k,\ 1 \leq i \leq m-1, \quad \text{where} \quad r^{ii+1} = Id_{X^{i-1}} \times r \times Id_{X^{m-i-1}}.$$

Note that these maps are elements of the symmetric group $\mathrm{Sym}(X^m)$. Then the group $\mathcal{D}_k(r)$ generated by r^{ii+1}, $1 \leq i \leq m-1$, acts on X^m. r is involutive, so the bijective maps r^{ii+1} are involutive, as well, and $\mathcal{D}_m(r)$ is the infinite group

$$\mathcal{D}_k(r) = \ _{\mathrm{gr}}\langle r^{ii+1} \mid \quad (r^{ii+1})^2 = e, \quad 1 \leq i \leq k-1\rangle. \tag{4.1}$$

When $m = 3$, we use notation $\mathcal{D} = \mathcal{D}_3(r)$. In this case $\mathcal{D} = \ _{\mathrm{gr}}\langle r^{ii+1} \mid \quad (r^{ii+1})^2 = e, \quad 1 \leq i \leq 2\rangle$ is simply *the infinite dihedral group.*

Note that for S and $\mathcal{A}$ *the problem of equality of words* is solvable. Two elements $\omega, \omega' \in \langle X\rangle$ are equal in S *iff* they have the same length, $|\omega| = |\omega'| = m$ and belong to the same orbit of $\mathcal{D}_m(r)$ in X^m

Assuming that (X, r) is a quantum binomial set we will find some counting formulae and inequalities involving the orders of the $\mathcal{D}$-orbits in X^3, and their number, see Proposition 4.4. These are used to find a necessary and sufficient condition for (X, r) to be a symmetric set, Proposition 4.8, and to give upper bounds for $\dim A_3$ and $\dim A_3^!$ in the general case of quantum binomial algebra A, Corollary 4.10.

As usual, the orbit of a monomial $\omega \in X^3$ under the action of $\mathcal{D}$ will be denoted by $\mathcal{O} = \mathcal{O}(\omega)$.

Denote by Δ_i the diagonal of $X^{\times i}, 2 \leq i \leq 3$. One has $\Delta_3 = \Delta_2 \times X \bigcap X \times \Delta_2$.

Definition 4.1. We call a $\mathcal{D}$-orbit $\mathcal{O}$ *square-free* if

$$\mathcal{O} \bigcap (\Delta_3 \bigcup (\Delta_2 \times X) \bigcup (X \times \Delta_2)) = \emptyset.$$

A monomial $\omega \in X^3$ is *square-free* in S if its orbit $\mathcal{O}_{\mathcal{D}}(\omega)$ is square-free.

Remark 4.2. We recall that whenever (X, r) is a quadratic set, the left and the right "actions"

$${}^{z}\bullet : X \times X \longrightarrow X \quad \text{and} \quad \bullet^{z} : X \times X \longrightarrow X$$

induced by r reflect each property of r, see [24], and [22], Remark 2.1. We will use the following simple properties of the actions when r is square-free and nondegenerate.

$$(4.2) \qquad \begin{array}{lllll} {}^{z}t = {}^{z}u & \Longrightarrow & t = u & \Longleftarrow & t^{z} = u^{z} \qquad \text{(by the nondegeneracy of } r) \\ {}^{z}t = z & \Longleftrightarrow & t = z & \Longleftrightarrow & t^{z} = z \qquad (r \text{ is square-free and nondegenerate}). \end{array}$$

Lemma 4.3. *Let (X, r) be a quantum binomial set, and let $\mathcal{O}$ be a square-free $\mathcal{D}$-orbit in X^3. Then $|\mathcal{O}| \geq 6$.*

Proof. Suppose $\mathcal{O} = \mathcal{O}(xyz)$ is a square-free orbit. Consider the set

$$O_1 = \{v_i \mid 1 \leq i \leq 6\} \subseteq \mathcal{O}$$

consisting of the first six elements of the "Yang-Baxter" diagram

$$(4.3) \qquad \begin{array}{ccc} v_1 = xyz & \xrightarrow{r^{12}} & ({}^{x}yx^{y})z = v_2 \\ r^{23}\big\downarrow & & \big\downarrow r^{23} \\ v_3 = x({}^{y}zy^{z}) & & ({}^{x}y)({}^{x^{y}}z)(x^{y})^{z} = v_5 \\ r^{12}\big\downarrow & & \big\downarrow r^{12} \\ v_4 = {}^{x}({}^{y}z)(x^{{}^{y}z})(y^{z}) & & [{}^{{}^{x}y}({}^{x^{y}}z)][({}^{x}y)^{({}^{x^{y}}z)}][(x^{y})^{z}] = v_6. \end{array}$$

Clearly,

$$O_1 = U_1 \bigcup U_3 \bigcup U_5, \quad \text{where} \quad U_j = \{v_j,\ r^{12}(v_j) = v_{j+1}\}, \quad j = 1, 3, 5.$$

We claim that U_1, U_3, U_5 are pairwise disjoint sets, and each of them has order 2. Note first that since v_j is a square-free monomial, for each $j = 1, 3, 5$, one has $v_j \neq r_{12}(v_j) = v_{j+1}$, therefore

$$|U_j| = 2, \quad j = 1, 3, 5.$$

The monomials in each U_j have the same "tail". More precisely, $v_1 = (xy)z, v_2 = r(xy)z$, have a "tail" z, the tail of v_3, and v_4 is y^z, and the tail of v_5, and v_6 is $(x^y)^z$. It will be enough to show that the three elements $z, y^z, (x^y)^z \in X$ are pairwise distinct. But $\mathcal{O}(xyz)$ is square-free, so $y \neq z$ and by (4.2) $y^z \neq z$. Furthermore, $v_2 = ({}^x y)(x^y)z \in \mathcal{O}(xyz)$ and therefore, $x^y \neq y$ and $x^y \neq z$. Now by (4.2) one has

$$\begin{array}{lcl} x^y \neq z & \Longrightarrow & (x^y)^z \neq z \\ x^y \neq y & \Longrightarrow & (x^y)^z \neq y^z. \end{array}$$

We have shown that the three elements $z, y^z, (x^y)^z \in X$ occurring as tails in U_1, U_3, U_5, respectively, are pairwise distinct, so the three sets are pairwise disjoint. This implies $|O_1| = 6$, and therefore $|\mathcal{O}| \geq 6$. $\square$

Proposition 4.4. *Suppose (X, r) is a finite quantum binomial set. Let $\mathcal{O}$ be a $\mathcal{D}$-orbit in X^3, denote $\mathbf{E}(\mathcal{O}) = \mathcal{O} \bigcap ((\Delta_2 \times X \bigcup X \times \Delta_2) \backslash \Delta_3)$.*

(1) *The following implications hold.*

$$\begin{array}{llcl} \textbf{(i)} & \mathcal{O} \bigcap \Delta_3 \neq \emptyset & \Longrightarrow & |\mathcal{O}| = 1. \\ \textbf{(ii)} & \mathbf{E}(\mathcal{O}) \neq \emptyset & \Longrightarrow & |\mathcal{O}| \geq 3 \quad \textit{and} \quad |\mathbf{E}(\mathcal{O})| = 2. \\ \textbf{(iii)} & \mathcal{O} \bigcap (\Delta_2 \times X \bigcup X \times \Delta_2) = \emptyset & \Longrightarrow & |\mathcal{O}| \geq 6. \end{array}$$

(2) *There are exactly $n(n-1)$ orbits $\mathcal{O}$ in X^3 of type* **(ii)**.

(3) *(X, r) satisfies the cyclic condition* iff *each orbit $\mathcal{O}$ of type* **(ii)** *has order $|\mathcal{O}| = 3$. In this case*

$$^{x^y}y = {}^x y \quad x^{^x y} = x^y, \forall x, y \in X.$$

An orbit $\mathcal{O}$ which satisfies $\mathbf{E}(\mathcal{O}) \neq \emptyset$ will be called *an orbit of type* **(ii)**

Proof. Clearly, the "fixed" points under the action of $\mathcal{D}$ on X^3 are exactly the monomials xxx, $x \in X$. This gives (i).

Assume now that $\mathcal{O}$ is of type **(ii)**. Then it contains an element of the shape $\omega = xxy$, or $\omega = xyy, x, y \in X, x \neq y$. Without loss of generality we can assume $\omega = xxy \in \mathcal{O}$.

The orbit $\mathcal{O}(\omega)$ can be obtained as follows. We fix as an initial element of the orbit $\omega = xxy$. Then there is *a unique finite sequence* $r^{23}, r^{12}, r^{23}, \cdots$ that exhausts the whole orbit, and produces at every step a "new" element. r is involutiove and square-free, thus in order to produce new elements at every step, the sequence must start with r^{23} and at every next step we have to alternate the actions r^{23}, and r^{12}.

We look at the "Yang-Baxter" diagram starting with ω and exhausting the whole orbit (without repetitions).

$$\omega = \omega_1 = xxy \longleftrightarrow^{r^{23}} \omega_2 = x({}^x y)(x^y) \longleftrightarrow^{r^{12}} \omega_3 = ({}^{x^2}y)(x^{^x y})(x^y) \longleftrightarrow \cdots \longleftrightarrow \omega_m. \tag{4.4}$$

Note first that the first three elements $\omega_1, \omega_2, \omega_3$ are distinct monomials in X^3. Indeed, $x \neq y$ implies $r(xy) \neq xy$ in X^2, so $\omega_2 \neq \omega_1$. Note (X, r) is square-free, so ${}^x x = x$, but by the nondegeneracy $y \neq x$, also implies ${}^x y \neq x$. So $r(x^x y) \neq x^x y$, and therefore $\omega_3 \neq \omega_2$. Furthermore, $\omega_3 \neq \omega_1$. Indeed, if we assume $x = {}^{x^2}y = {}^x({}^x y)$ so by (4.2) one has ${}^x y = x$, and therefore $y = x$, a contradiction. We have obtained that $\mid \mathcal{O} \mid \geq 3$.

We claim now that the intersection $\mathbf{E} = \mathbf{E}(\mathcal{O})$ contains exactly two elements. We analyze the diagram (4.4) looking from left to right.

Suppose we have made $k-1$ "steps" to the right obtaining new elements, so we have obtained

$$\omega = \omega_1 = xxy \longleftrightarrow^{r^{23}} \omega_2 = x^x y x^y \longleftrightarrow^{r^{12}} \omega_3 = {}^{x^2}y x^{^x y} x^y \longleftrightarrow \cdots \omega_{k-1} \longleftrightarrow^{r^{ii+1}} \omega_k,$$

where $\omega_1, \omega_2, \cdots, \omega_k$ are pairwise distinct. Note that all elements $\omega_s, 2 \leq s \leq k-1$ have the shape $\omega_s = a_s b_s c_s$, with $a_s \neq b_s$ and $b_s \neq c_s$. Two cases are possible.

(a) $\omega_k = a_k b_k c_k$, with $a_k \neq b_k$ and $b_k \neq c_k$, then applying r^{jj+1} (where $j = 2$ if $i = 1$ and $j = 1$ if $i = 2$), we obtain a new member of the orbit.

(b) $\omega_k = aac$, or $\omega_k = acc$, $a \neq c$. In this case r^{jj+1} with $j \neq i$ keeps ω_k fixed, so the process of obtaining new elements of the orbit stops at this step and the diagram is complete.

But our diagram is finite, so as a final step on the right it has to "reach" some $\omega_m = aac$, or $\omega_m = acc$, $a \neq c$ (we have already shown that $m \geq 3$). Note that $\omega_m \neq \omega_1$. Hence, the intersection $\mathbf{E} = \mathbf{E}(\mathcal{O})$ contains exactly two elements.

We claim that there exists exactly $n(n-1)$ orbits of type **(ii)**. Indeed, let $\mathcal{O}_1, \cdots \mathcal{O}_p$ be all orbits of type **(ii)**. The intersections $E_i = E(\mathcal{O}_i), 1 \leq i \leq p$, are disjoint sets and each of them contains two elements. Now the equalities

$$\begin{array}{lcl} \bigcup_{1\leq i\leq p} E_i & = & (\Delta_2 \times X \bigcup X \times \Delta_2)\backslash\Delta_3 \\ \mid E_i \mid = 2, & & \mid \Delta_2 \times X \bigcup X \times \Delta_2)\backslash\Delta_3 \mid = 2n(n-1) \end{array}$$

imply $p = n(n-1)$

Condition (1)(**iii**) follows from Lemma 4.3.

Condition (3) follows straightforwardly from (4.4). $\square$

Example 4.5. Consider the quantum binomial algebra A given in Example 2.10. Let (X, r) be the associated quadratic set, $S = S(X, r)$ the corresponding monoid. The relations are semigroup relations, so $A = \mathcal{A}(\mathbf{k}, X, r) \simeq \mathbf{k}S$. We will find the corresponding $\mathcal{D}$ orbits in X^3. There are 12 orbits of type **(ii)**. This agrees with Proposition 4.8.

$$\mathcal{O}_1 = xxy \xrightarrow{r^{23}} xzt \xrightarrow{r^{12}} yxt \xrightarrow{r^{23}} ytx \xrightarrow{r^{12}} tzx \xrightarrow{r^{23}} tty$$

$$\mathcal{O}_2 = xxz \xrightarrow{r^{23}} xyx \xrightarrow{r^{12}} ztx \xrightarrow{r^{23}} zxt \xrightarrow{r^{12}} tyt \xrightarrow{r^{23}} ttz$$

$$\mathcal{O}_3 = yxx \xrightarrow{r^{12}} xzx \xrightarrow{r^{23}} xty \xrightarrow{r^{12}} txy \xrightarrow{r^{23}} tzt \xrightarrow{r^{12}} ytt$$

$$\mathcal{O}_4 = zxx \xrightarrow{r^{12}} tyx \xrightarrow{r^{23}} txz \xrightarrow{r^{12}} xtz \xrightarrow{r^{23}} xyt \xrightarrow{r^{12}} ztt$$

$$\mathcal{O}_5 = xyy \xrightarrow{r^{12}} zty \xrightarrow{r^{23}} zxz; \quad \mathcal{O}_6 = xzz \xrightarrow{r^{12}} yxz \xrightarrow{r^{23}} yyx$$

$$\mathcal{O}_7 = tyy \xrightarrow{r^{12}} zxy \xrightarrow{r^{23}} zzt; \quad \mathcal{O}_8 = tzz \xrightarrow{r^{12}} ytz \xrightarrow{r^{23}} yyt$$

$$\mathcal{O}_9 = txx \xrightarrow{r^{12}} xtx \xrightarrow{r^{23}} xxt; \quad \mathcal{O}_{10} = xtt \xrightarrow{r^{12}} txt \xrightarrow{r^{23}} ttx$$

$$\mathcal{O}_{11} = yzz \xrightarrow{r^{12}} zyz \xrightarrow{r^{23}} zzy; \quad \mathcal{O}_{12} = zyy \xrightarrow{r^{12}} yzy \xrightarrow{r^{23}} yyz$$

There are only two square-free orbits, $\mathcal{O}^{(1)} = \mathcal{O}(xyz)$, and $\mathcal{O}^{(2)} = \mathcal{O}(tyz)$. Each of them has order 6.

$$
\begin{array}{ccccccc}
xyz & \xrightarrow{r^{12}} & ztz & \qquad\qquad & tyz & \xrightarrow{r^{12}} & zxz \\
{\scriptstyle r^{23}}\downarrow & & \downarrow{\scriptstyle r^{23}} & & {\scriptstyle r^{23}}\downarrow & & \downarrow{\scriptstyle r^{23}} \\
xzy & & zyt & & tzy & & zyx \\
{\scriptstyle r^{12}}\downarrow & & \downarrow{\scriptstyle r^{12}} & & {\scriptstyle r^{12}}\downarrow & & \downarrow{\scriptstyle r^{12}} \\
yxy & \xrightarrow{r^{12}} & yzt & & yty & \xrightarrow{r^{12}} & yzx
\end{array}
\tag{4.5}
$$

The one element orbits are $\{xxx\}, \{yyy\}, \{zzz\}, \{ttt\}$.

A more detailed study of the orbits shows that A is *not PBW w.r.t. any enumeration of* X. Clearly, r does not satisfy the braid relation, so (X, r) is not a symmetric set.

Lemma 4.6. *A quantum binomial set (X, r) is symmetric* iff *the orders of $\mathcal{D}$-orbits $\mathcal{O}$ in X^3 satisfy the following two conditions.*

$$
\begin{array}{lll}
\text{(a)} & \mathcal{O}\bigcap(\Delta_2 \times X \bigcup X \times \Delta_2)\backslash\Delta_3)) \neq \emptyset & \Longleftrightarrow |\mathcal{O}| = 3. \\
\text{(b)} & \mathcal{O}\bigcap(\Delta_2 \times X \bigcup X \times \Delta_2) = \emptyset & \Longleftrightarrow |\mathcal{O}| = 6.
\end{array}
$$

Proof. Look at the corresponding YBE diagrams. □

Let (X, r) be a quantum binomial set, let $\mathcal{D}$ be the infinite dihedral group acting on X^3. We fix the following notation for the $\mathcal{D}$-orbits in X^3.

Notation 4.7. We denote by $\mathcal{O}_i, 1 \leq i \leq n(n-1)$ the orbits of type **(ii)**, and by $\mathcal{O}^{(j)}, 1 \leq j \leq q$ all square-free orbits in X^3. The remaining $\mathcal{D}$-orbits in X^3 are the one-element orbits $\{xxx\}, x \in X$, their union is Δ_3.

Proposition 4.8. *Let (X, r) be a finite quantum binomial set. Let $\mathcal{O}^{(j)}, 1 \leq j \leq q$, be the set of all (distinct) square-free $\mathcal{D}$-orbits in X^3. Then*

(1) $q \leq \binom{n}{3}$.
(2) *(X, r) is a symmetric set* iff $q = \binom{n}{3}$.

Proof. Clearly, X^3 is a disjoint union: of its $\mathcal{D}$-orbits, so

$$X^3 = \Delta_3 \bigcup_{1 \leq i \leq n(n-1)} \mathcal{O}_i \bigcup_{1 \leq j \leq q} \mathcal{O}^{(j)}.$$

Thus

$$| X^3 |=| \Delta_3 | + \sum_{1 \leq i \leq n(n-1)} | \mathcal{O}_i | + \sum_{1 \leq j \leq q} | \mathcal{O}^{(j)} | \tag{4.6}$$

Denote $m_i = |\mathcal{O}_i|,\ 1 \leq i \leq n(n-1),\ \ n_j = |\mathcal{O}^{(j)}|,\ 1 \leq j \leq q$. By Proposition 4.4 one has

$$m_i \geq 3,\ 1 \leq i \leq n(n-1), \quad \text{and} \quad n_j \geq 6,\ 1 \leq j \leq q.$$

We replace these inequalities in (4.6) and obtain

$$n^3 = n + \sum_{1 \leq i \leq n(n-1)} m_i + \sum_{1 \leq j \leq q} n_j \geq\ n + 3n(n-1) + 6q. \tag{4.7}$$

So

$$q \leq \frac{n^3 - 3n^2 + 2n}{6} = \binom{n}{3},$$

which verifies (1). Assume now $q = \binom{n}{3}$. Then (4.7) implies

$$n^3 = n + \sum_{1 \leq i \leq n(n-1)} m_i + \sum_{1 \leq j \leq \binom{n}{3}} n_j \geq \; n + 3n(n-1) + 6\binom{n}{3} = n^3.$$

This is possible **iff** the following equalities hold

$$\begin{aligned} m_i &= |\mathcal{O}_i| = 3, \; 1 \leq i \leq n(n-1), \\ n_j &= |\mathcal{O}^{(j)}| = 6, \; 1 \leq j \leq q. \end{aligned} \tag{4.8}$$

By Lemma 4.6, the equalities (4.7) hold *iff* (X, r) is a symmetric set. □

Corollary 4.9. *Let (X, r) be a finite quantum binomial set, $S = S(X, r)$, $\mathcal{A} = \mathcal{A}(\boldsymbol{k}, X, r)$ the associated monoid and monoidal algebra. (X, r) is a symmetric set* iff $\dim \mathcal{A}_3 = \binom{n+2}{3}$.

Proof. The distinct elements of S form a $\mathbf{k}$-basis of the monoidal algebra $\mathbf{k}S \simeq \mathcal{A}(\mathbf{k}, X, r)$. In particular $\dim \mathcal{A}_3$ equals the number of $\mathcal{D}$-orbits in X^3. □

Assume now that A is a quantum binomial algebra. We want to estimate the dimension $\dim A_3$. Let (X, r) be the corresponding quantum binomial set, $S = S(X, r)$, $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. We use Proposition 4.8 to find an upper bound for the number of distinct $\mathcal{D}$-orbits in X^3, or equivalently, the order of the set S_3 of (distinct) elements of length 3 in S. One has

$$|S_3| = n + n(n-1) + q \leq n + n(n-1) + \binom{n}{3} = \binom{n+2}{3}.$$

Hence

$$\dim \mathcal{A}_3 = |S_3| \leq \binom{n+2}{3}.$$

In the general case, a quantum binomial algebra satisfies $\dim A_3 \leq \dim \mathcal{A}_3$, due to the coefficients c_{xy} appearing in the set of relations. We have proven the following corollary.

Corollary 4.10. *If A is a quantum binomial algebra, then*

$$\dim A_3 \leq \binom{n+2}{3}, \quad \dim A^!_3 \leq \binom{n}{3}.$$

5. Quantum binomial algebras. Yang-Baxter equation and Artin-Schelter regularity

Definition 5.1. [37], [38] A graded algebra $A = \bigoplus_{i \geq 0} A_i$ is called *a Frobenius algebra of dimension n*, (or *a Frobenius quantum space of dimension n*) if

(a) *$dim(A_n) = 1$, $A_i = 0$,* for *$i > n$.*
(b) For all $n \geq j \geq 0$) the multiplicative map $m : A_j \otimes A_{n-j} \to A_n$ is a perfect duality (nondegenerate pairing).

A Frobenius algebra A is called *a quantum Grassmann algebra* if in addition

(c) *$dim_{\mathbf{k}} A_i = \binom{n}{i}$,* for $1 \leq i \leq n$.

Lemma 5.2. *Let $A = \boldsymbol{k}\langle X; \Re\rangle$ be a quantum binomial algebra, $|X| = n$, $A^!$ its Koszul dual. Let $(X, r), r = r(\Re)$ be the associated quantum binomial set, $S = S(X, r)$. Then each of the following three conditions implies that (X, r) is a symmetric set.*

(1) $\dim A_3 = \binom{n+2}{3}$.
(2) $\dim A^!_3 = \binom{n}{3}$.
(3) *X can be enumerated $X = \{x_1 \cdots , x_n\}$, so that the set of ordered monomials of length 3*

$$N_3 = \{x_{i_1}x_{i_2}x_{i_3} \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq n\} \tag{5.1}$$

projects to a $\boldsymbol{k}$-basis of A_3.

Proof. Consider the relations

$$\binom{n+2}{3} = \dim A_3 \leq \dim \mathcal{A}_3 \leq \binom{n+2}{3}.$$

This implies $\mathcal{A}_3 = \binom{n+2}{3}$, (equivalently $|S_3| = \binom{n+2}{3}$ and therefore there are exactly $q = \binom{n}{3}$ square-free $\mathcal{D}$-orbits in X^3. Then Proposition 4.8 (2) implies that (X, r) is a symmetric set, which verifies (1)$\Longrightarrow$ (3). The converse (3) $\Longrightarrow$ (1) is straightforward. Finally, one has

$$\dim A_3 = \binom{n+2}{3} \Longleftrightarrow \dim A^!_3 = \binom{n}{3}.$$

This can be proved directly using the $\mathcal{D}$-orbits in X^3. It is also straightforward from the following formula for quadratic algebras, see [39], p 85.

$$\dim A^!_3 = (\dim A_1)^3 - 2(\dim A_1)(\dim A_2) + \dim A_3.$$

□

Lemma 5.3. *Let A be a quadratic algebra with relations of skew-polynomal type, let $A^!$ be its Koszul dual. Then the following conditions are equivallent.*

(1) $\dim A_3 = \binom{n+2}{3}$
(2) $\dim A^!_3 = \binom{n}{3}$.
(3) *The set of defining relations $\Re$ for A is a Gröbner basis, so A is a skew polynomial ring, and therefore a PBW algebra.*
(4) *The set of defining relations $\Re^\perp$ for $A^!$ is a Gröbner basis, so $A^!$ is a PBW algebra with PBW generators $x_1, \cdots , x_n$.*

Sketch of the proof. As we already mentioned (1) $\Longleftrightarrow$ 2).

(1) $\Longleftrightarrow$ 3). Assume condition (1) holds. The set of monomials of length 3, which are normal mod the ideal $(\Re)$ form a $\mathbf{k}$-basis of A_3. The skew-polynomial shape of the relations implies that each monomial u which is normal mod the ideal $(\Re)$ is in the set of ordered monomials

$$N_3 = \{x_{i_1}x_{i_2}x_{i_3} \mid 1 \leq i_1 \leq i_2 \leq i_3 \leq n\}.$$

There are equalities

$$\mid N_3 \mid = \binom{n+2}{3} = \dim A_3,$$

hence all ordered monomials of length 3 are normal mod $(\Re)$. It follows then that all ambiguities $x_kx_jx_i, 1 \leq i < j < k \leq n$ are resolvable, and by Bergman's Diamond lemma [6], $\Re$ is a Gröbner basis of the ideal $(\Re)$. Thus A is a PBW algebra, more precisely, A is a binomial skew-polynomial ring. This gives (1) $\Longrightarrow$ (3). The converse implication (1) $\Longleftarrow$ (3) is clear.

(2) $\Longleftarrow$ (4) is clear.

(2) $\Longrightarrow$ (4) is analogous to (1) $\Longrightarrow$ 3).

Theorem 5.4. *Let $A = \boldsymbol{k}\langle X\rangle/(\Re)$ be a quantum binomial algebra, $|X| = n$, and let $\Re$ be the associated automorphism $R = R(\Re) : V^{\otimes 2} \longrightarrow V^{\otimes 2}$, see Definition 2.8. Then the following three conditions are equivalent*

(1) *$R = R(\Re)$ is a solution of the Yang-Baxter equation*

(2) *A is a binomial skew-polynomal ring*

(3)
$$\dim_{\boldsymbol{k}} A_3 = \binom{n+2}{3}.$$

Proof. We start with the implication (3) $\Longrightarrow$ (2).

Assume that $\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}$. Consider the corresponding quadratic set (X, r) and monoidal algebra $\mathcal{A} = \mathcal{A}(\mathbf{k}, X, r)$. As before we conclude that $\dim_{\mathbf{k}} \mathcal{A}_3 = \binom{n+2}{3}$ and therefore, by Corollary 4.9 (X, r) is a symmetric set.

Then by [20] Theorem 2.26, there exists an ordering on X, $X = \{x_1, x_2, \cdots, x_n\}$ so that the algebra $\mathcal{A}$ is a skew-polynomial ring and therefore a PBW-algebra, with PBW generators $x_1, x_2, \cdots, x_n$. It follows then that the relations $\Re$ of A are relations of skew-polynomial type, that is, conditions (a), (b) and (c) of Definition 2.1 are satisfied.

By assumption $\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}$ so Lemma 5.3 implies that the set of relations $\Re$ is a Gröbner basis (A is PBW) and therefore A is a binomial skew polynomial ring. This verifies (3) $\Longrightarrow$ (2).

It follows from Definition 2.1 that if A is a binomial skew-polynomial ring then the set of monomials
$$\mathcal{N} = \{x_{i_1}x_{i_2}x_{i_3} \mid i_1 \leq x_{i_2} \leq x_{i_3}\}$$
is a $\mathbf{k}$-basis of A_3, so $\dim_{\mathbf{k}} A_3 = \binom{n+2}{3}$, hence (2) $\Longrightarrow$ (2).

The equivalence (1) $\Longleftrightarrow$ (2) is proven in [19], Theorem B. □

Proof of Theorem 1.2. The equivalence of conditions (4), (5) and (6) follows from Theorem 5.4.

The implication (7) $\Longrightarrow$ (6) is clear. The converse follows from (6) $\Longrightarrow$ (5) $\Longrightarrow$ (7).

It is straightforward that each binomial skew polynomial ring A is Koszul and satisfies (1) and (2). It is proven in [19], that the Koszul dual $A^!$ of a binomial skew polynomial ring is a quantum Grassman algebra. Thus (5) $\Longrightarrow$ (8).

Clearly, (8) $\Longrightarrow$ (6). So, (5) $\Longleftrightarrow$ (8).

It is also known that a Koszul algebra A is Gorenstein *iff* its dual $A^!$ is Frobenius. It follows then that (5) $\Longrightarrow$ (3).

The result that every binomial skew polynomial ring is AS regular follows also from the earlier work [28].

Finally we will show (1) $\Longrightarrow$ (5) and (2) $\Longrightarrow$ (5).

We know that a quantum binomial algebra A has exactly $\binom{n}{2}$ relations. Assume now that A PBW-algebra. Then its set of obstructions $\mathbf{W}$ has order $|\mathbf{W}| = \binom{n}{2}$. Consider now the corresponding monomial algebra $A^0 = \mathbf{k}\langle X\rangle/(\mathbf{W})$. Each of the conditions (1) and (2) is satisfied by A *iff* it is true for A^0.

Assume first that A satisfies (1). Then the monomial algebra A^0 satisfies condition (2) of Theorem 3.9 and by the same theorem the Hilbert series of A^0 satisfies (7). The algebras A and A^0 have the same Hibert series, and therefore (1)$\Longrightarrow 7 \Longrightarrow$ (5).

Similarly, if A satisfies (2), then the monomial algebra A^0 satisfies condition (3) of Theorem 3.9. (Note that the relations are square free, so $\mathbf{W} \bigcap \operatorname{diag} X^2 = \emptyset$.) Analogously to the previous implication we conclude (2)$\Longrightarrow 7 \Longrightarrow$ (5). The equivalence of the conditions (1) $\cdots$ (8) has been verified.

Each of these conditions imply that A is a binomial skew-polynomial ring and therefore it is a Noetherian domain, see [28], or Fact 2.9.

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