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EXACT SOLUTION OF A GENERALIZED ANNNI MODEL ON A CAYLEY TREE

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Abstract

We consider the Ising model on a Cayley tree of order two with nearest neighbor interactions and competing next nearest neighbor interactions restricted to spins belonging to the same branch of the tree. This model was studied by Vannimenus who found a new modulated phase, in addition to the paramagnetic, ferromagnetic, antiferromagnetic phases and a (+ + - -) periodic phase. Vannimenus' results are based on the recurrence equations (relating the partition function of an $n-$ generation tree to the partition function of its subsystems containing $(n-1)$ generations) and most results are obtained numerically. In this paper we analytically study the recurrence equations and obtain some exact results: critical temperatures and curves, a number of several phases, and partition function.

1. Introduction

The model considered by Vannimenus [20] consists of Ising spins ($\sigma = \pm 1$) on a Cayley tree of branching ratio 2, so that every spin has three nearest-neighbor (NN). Two kinds of bonds are present: NN interactions of strength J_1 and next-nearest-neighbor (NNN) interactions J_2, these being restricted to spins belonging to the same branch of the tree. The phase diagram described by Vannimenus contains a modulated phase, as found for similar models on periodic lattices, but the multicritical Lifshitz point is at zero temperature. The variation of the wavevector with temperature in the modulated phase is studied in detail and narrow commensurate steps between incommensurate regions are shown. The behavior of the coherence length near the different transitions is also analyzed.

Vannimenus' model was then generalized in many directions:

In [8] a model with the competing NN and NNN interactions Ising model on a Cayley tree was considered, but in their case it allowed for all interbranch NNN interactions on the coordination number three, which was discussed earlier in [9]. In addition to the expected paramagnetic, ferromagnetic and antiferromagnetic phases, an intermediate range of $J_2/J_1 < 0$ values, where the local magnetization has chaotic oscillatory glass-like behavior, were obtained.

Another generalization is due to Mariz et al [10]. These authors studied the phase diagram for the Ising model on a Cayley tree with competing NN interactions J_1 and NNN interactions J_2 and J_3 in the presence of an external magnetic field. At vanishing temperature, the phase diagram is fully determined for all values and signs of J_2/J_1 and J_3/J_2; in particular, it was verified that values of J_3/J_2 high enough favor the paramagnetic phase. At finite temperatures, several interesting features (evolution of re-entrances, separation of the modulated region into two disconnected pieces, etc.) are exhibited for typical values of J_2/J_1 and J_3/J_2.

The next generalization is considered in [11], where the lattice spin model with Q-component discrete spin variables restricted to having orientations orthogonal to the faces of Q-dimensional hypercube is considered on the Cayley tree (Bethe lattice). The partition function of the model with dipole-dipole and quadrupole-quadrupole interaction is presented in terms of double graph expansions. By analyzing the regions of stability of different types of fixed points of the system of recurrent relations (which is generalization of the Vannimenus' equations), the phase diagrams of the model are plotted. For $Q \leq 2$ the phase diagram of the model is found to have three tricritical points.

The next generalizations are considered in [2], [3] and [4]. These authors have studied the phase diagram for Potts model on a Cayley tree with competing NN interactions J_1, prolonged NNN interactions J_p and one level NNN interactions J_o. In [4] the Potts model with $J_o \neq 0$ is considered and it is shown that for some values of J_o the multicritical Lifshitz point is at non-zero temperature and it is proven that as soon as the same-level interaction J_o is nonzero, the paramagnetic phase found at high temperatures for $J_o = 0$ disappears, while the Ising model does not obtain such property.

But most results of the above mentioned works are obtained numerically. Thus it is natural to try to get some of these results in an analytical way.

In this paper we consider the same model which was considered by Vannimenus (not its generalization) and study its phases by an analytical argument. Here we shall combine analytical arguments of papers [5], [14], [15], [19].

2. Preliminaries

The Cayley tree (Bethe lattice [1]) Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that from each vertex of which issues exactly $k+1$ edges. Let $\Gamma^k = (V, L)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k. Two vertices x and y are called *nearest neighbors* (NN) if there exists an edge $l \in L$ connecting them, which is denoted by $l = \langle x, y\rangle$. A collection of the pairs $\langle x, x_1\rangle, ..., \langle x_{d-1}, y\rangle$ is called a *path* from x to y. Then the distance $d(x, y), x, y \in V$, on the Cayley tree, is the number of edges in the shortest path from x to y.

For a fixed $x^0 \in V$ we set

$$W_n = \{x \in V | d(x, x^0) = n\}, \qquad V_n = \bigcup_{k=1}^{n} W_k.$$

Denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

this set is called a set of *direct successors* of x.

The vertices x and y are called *next-nearest-neighbor* (NNN) which is denoted by $\rangle x, y\langle$, if there exists a vertex $z \in V$ such that x, z and y, z are NN. We will consider only *prolonged* NNN $\rangle x, y\langle$, for which there exist n such that $x \in W_n$ and $y \in W_{n+2}$, this kind of NNN is considered in [20] and [3].

We consider the Ising model with competing NN and prolonged NNN interactions on a Cayley tree where the spin takes values in the set $\Phi := \{-1, 1\}$, and is assigned to the vertices of the tree [20]. A configuration σ on V is then defined as a function $x \in V \mapsto \sigma(x) \in \Phi$; the set of all configurations is Φ^V. The (formal) Hamiltonian of the model is

$$H(\sigma) = -J_1 \sum_{\langle x,y\rangle \in L} \sigma(x)\sigma(y) - J_2 \sum_{\rangle x,y\langle} \sigma(x)\sigma(y), \tag{2.1}$$

where $J_1, J_2 \in R$ are coupling constants and $\langle x, y\rangle$ stands for NN vertices and $\rangle x, y\langle$ stands for prolonged NNN.

As usual, one can introduce the notions of Gibbs measure (phase) of the Ising model with a competing interactions on the Cayley tree [7], [18], [13].

The standard approach consists in writing the recurrence equation relating the partition function

$$Z_n = \sum_{\sigma_n \in \Phi^{V_n}} \exp\{-\beta H(\sigma_n)\},$$

of n-generation tree to the partition function Z_{n-1} of its subsystems containing $(n-1)$ generations. In [20] the partition function Z_n of the Hamiltonian (2.1) is given by

$$Z_n = \left(u_1^{(n)} + u_2^{(n)}\right)^2 + \left(u_3^{(n)} + u_4^{(n)}\right)^2, \ n \geq 1. \tag{2.2}$$

Here $u^{(n)} = \left(u_1^{(n)}, u_2^{(n)}, u_3^{(n)}, u_4^{(n)}\right)$ satisfies the following recurrent equation

$$\begin{aligned} u_1^{(n+1)} &= a\left(bu_1^{(n)} + b^{-1}u_2^{(n)}\right)^2 \\ u_2^{(n+1)} &= a^{-1}\left(bu_3^{(n)} + b^{-1}u_4^{(n)}\right)^2 \\ u_3^{(n+1)} &= a^{-1}\left(b^{-1}u_1^{(n)} + bu_2^{(n)}\right)^2 \\ u_4^{(n+1)} &= a\left(b^{-1}u_3^{(n)} + bu_4^{(n)}\right)^2, \end{aligned} \tag{2.3}$$

where $a = \exp(J_1\beta)$, $b = \exp(J_2\beta)$.

Consider mapping $F : u = (u_1, u_2, u_3, u_4) \in R_+^4 \to F(u) = (u_1', u_2', u_3', u_4') \in R_+^4$ defined by

$$\begin{aligned} u_1' &= a\left(bu_1 + b^{-1}u_2\right)^2 \\ u_2' &= a^{-1}\left(bu_3 + b^{-1}u_4\right)^2 \\ u_3' &= a^{-1}\left(b^{-1}u_1 + bu_2\right)^2 \\ u_4' &= a\left(b^{-1}u_3 + bu_4\right)^2. \end{aligned} \tag{2.4}$$

Then the recurrent equation (2.3) can be written as $u^{(n+1)} = F(u^{(n)})$, $n \geq 0$ which in the theory of dynamical systems is called *trajectory* of the initial point $u^{(0)}$ under action of the mapping F. Thus the asymptotic behavior of Z_n for $n \to \infty$ can be determined by values of $\lim u^{(n)}$ i.e., trajectory of $u^{(0)}$ under action of F. In this paper we study the trajectory (dynamical system) for a given initial point $u^{(0)} \in R^4$.

3. Dynamics of F

3.1. **Fixed points.** In this subsection we are going to define fixed points, i.e., solutions to $F(u) = u$.

Denote $\mathrm{Fix}(F) = \{u : F(u) = u\}$.

We introduce the new variables $\alpha = \sqrt{a}$, $v_i = \sqrt{u_i}$, $i = 1, 2, 3, 4$. Then the equation $F(u) = u$ becomes as

$$\begin{aligned} v_1 &= \alpha\left(bv_1^2 + b^{-1}v_2^2\right) \\ v_2 &= \alpha^{-1}\left(bv_3^2 + b^{-1}v_4^2\right) \\ v_3 &= \alpha^{-1}\left(b^{-1}v_1^2 + bv_2^2\right) \\ v_4 &= \alpha\left(b^{-1}v_3^2 + bv_4^2\right). \end{aligned} \tag{3.1}$$

Lemma 3.1. *If a vector u is a fixed point of F then $u \in M_1 = \{u = (u_1, u_2, u_3, u_4) \in R_+^4 : u_1 = u_4,\ u_2 = u_3\}$ or $u \in M_2 = \{u = (u_1, u_2, u_3, u_4) \in R_+^4 :\ \sqrt{u_1} + \sqrt{u_4} = \varphi(\sqrt{u_2} + \sqrt{u_3})\}$, where $\varphi(x) = \frac{1+a^{-1}bx}{ab+(b^2-b^{-2})x}$.*

Proof. From (3.1) we get

$$\begin{cases} (v_1 - v_4)[\alpha b(v_1 + v_4) - 1] + (v_2 - v_3)[\alpha b^{-1}(v_2 + v_3)] = 0 \\ (v_1 - v_4)[(\alpha b)^{-1}(v_1 + v_4)] + (v_2 - v_3)[1 + \alpha^{-1} b(v_2 + v_3)] = 0. \end{cases} \tag{3.2}$$

If $v_1 = v_4$ (resp. $v_2 = v_3$) from the second equation of (3.2) we get $v_2 = v_3$ (resp. $v_1 = v_4$). Thus $v_1 = v_4$ if and only if $v_2 = v_3$. Assume now $v_1 \neq v_4$ and $v_2 \neq v_3$ then system (3.2) can be reduced to the equation

$$(b^2 - b^{-2})(v_1 + v_4)(v_2 + v_3) + \alpha b(v_1 + v_4) - \alpha^{-1} b(v_2 + v_3) - 1 = 0. \tag{3.3}$$

Equation (3.3) gives $v_1 + v_4 = \varphi(v_2 + v_3)$. □

Let us first study the fixed points of F which belong in M_1: the condition $u_1 = u_4,\ u_2 = u_3$ reduces the equation $F(u) = u$ to the following equation

$$x = f(x) \equiv a^2 \left(\frac{1 + b^2 x}{b^2 + x} \right)^2, \tag{3.4}$$

where $x = \frac{u_1}{u_2}$. Denote

$$\tilde{a} = a^{-2} b^{-6}, \quad \tilde{b} = b^4, \quad y = b^2 x.$$

The following lemma gives a full description of solutions to (3.4).

Lemma 3.2. *(Cf. Lemma 10.7 in* [13]*). Equation (3.4) has a unique positive, stable solution if $\tilde{b} \leq 9$. If $\tilde{b} > 9$, then there exist $\nu_1(\tilde{b})$ and $\nu_2(\tilde{b})$ such that the conditions $0 < \nu_1(\tilde{b}) < \nu_2(\tilde{b})$ are satisfied and equation (3.4) has three solutions, $x_1^* < x_2^* < x_3^*$, x_1^* and x_3^* are stable and x_2^* is unstable, if $\nu_1(\tilde{b}) < \tilde{a} < \nu_2(\tilde{b})$ and has two solutions, x_1^*, x_2^*, x_1^* is unstable (saddle) and x_2^* is stable, if $\tilde{a} = \nu_1(\tilde{b})$ or $\tilde{a} = \nu_2(\tilde{b})$. In this case, we have*

$$\nu_i(\tilde{b}) = \frac{1}{y_i} \left(\frac{1 + y_i}{\tilde{b} + y_i} \right)^2,$$

where y_1 and y_2 are the solutions of the equation $y^2 + (3 - \tilde{b})y + \tilde{b} = 0$.

Now we shall give some arguments to find fixed points of F which belong in M_2. Again using variables v_i, $i = 1, 2, 3, 4$, assume $v_2 + v_3 = C$, with $C > \max\{0, \frac{\alpha b}{b^{-2} - b^2}\}$. Using Lemma 3.1 we get $v_3 = C - v_2$ and $v_4 = \varphi(C) - v_1$. Then from the first equation of (3.1) we obtain $v_2 = \sqrt{b(\alpha^{-1} v_1 - b v_1^2)}$ and from the second equation of (3.1) we have $P_4(v_1) = 0$ with a polynomial P_4 of degree 4, coefficients of which depend on parameters α, b and C. Thus a quartic equation can be obtained. Such an equation can be solved using known formulas (see http://mathworld.wolfram.com/QuarticEquation.html), since we will have some complicated formulas for the coefficients and the solutions, we do not present the solution here.

3.2. **Periodic points.** A point u in R_+^4 is called a *periodic point* of F if there exists p so that $F^p(u) = u$ where F^p is the pth iterate of F. The smallest positive integer p satisfying the above is called the *prime period* or least period of the point u. Denote by $\mathrm{Per}_p(F)$ the set of periodic points with prime period p.

Note that the set M_1 is invariant with respect to F, i.e., $F(M_1) \subset M_1$. In this subsection we shall describe some periodic points of F which belong in M_1.

Let us first describe periodic points with $p = 2$ on M_1. In this case the equation $F(F(u)) = u$ can be reduced to the description of 2-periodic points of the function f defined in (3.4) i.e., to the solution of the equation

$$f(f(x)) = x. \tag{3.5}$$

Note that the fixed points of f are solutions to (3.5). To find other solutions we consider the equation

$$\frac{f(f(x)) - x}{f(x) - x} = 0.$$

Simple calculations show that the last equation is equivalent to the following

$$b^4(1 + a^2b^2)^2x^2 + a^2\{b^8 + 2(a^{-2} + a^2)b^6 + 4b^4 - 1\}x + b^4(a^2 + b^2)^2 = 0. \tag{3.6}$$

The equation has two positive solutions iff $B < 0$ and $D > 0$ where

$$B = a^2\{b^8 + 2(a^{-2} + a^2)b^6 + 4b^4 - 1\}, \quad D = B^2 - (2b^4(a^2 + b^2)(1 + a^2b^2))^2.$$

We have

$$B = \begin{cases} > 0, \ \text{ if } \ b \geq \sqrt{\sqrt{2} - 1} \ \text{ or } \ b \leq \sqrt{\sqrt{2} - 1}, \ a^2 \in (0, \mathbf{b}^-) \cup (\mathbf{b}^+, +\infty) \\ 0, \ \text{ if } \ b \leq \sqrt{\sqrt{2} - 1}, \ \ a^2 = \mathbf{b}^-, \mathbf{b}^+ \\ < 0, \ \text{ if } \ b \leq \sqrt{\sqrt{2} - 1}, \ a^2 \in (\mathbf{b}^-, \mathbf{b}^+) \end{cases}$$

where

$$\mathbf{b}^{\pm} = \frac{1 - 4b^4 - b^8 \pm (1 - b^4)\sqrt{(b^4 - 1)^2 - 4b^4}}{4b^6};$$

$$D = -a^2(b^4 - 1)^2(4b^6a^4 + (3b^8 + 6b^4 - 1)a^2 + 4b^6) =$$

$$\begin{cases} < 0, \ \text{ if } \ \sqrt{3^{-1}} < b, \ b \neq 1 \ \text{ or } \ b \leq \sqrt{3^{-1}}, a^2 \in (0, b_*^-) \cup (b_*^+, +\infty) \\ 0, \ \text{ if } \ b = 1 \ \text{ or } \ b \leq \sqrt{3^{-1}}, \ a^2 = b_*^-, b_*^+ \\ > 0, \ \text{ if } \ b \leq \sqrt{3^{-1}}, a^2 \in (b_*^-, b_*^+) \end{cases} \tag{3.7}$$

where

$$b_*^{\pm} = \frac{1 - 3b^8 - 6b^4 \pm \sqrt{(b^4 - 1)^3(9b^4 - 1)}}{8b^6}.$$

One can check that $\sqrt{3^{-1}} < \sqrt{\sqrt{2} - 1}$, and for $b \leq \sqrt{3^{-1}}$ one has $\mathbf{b}^- \leq b_*^-$ and $b_*^+ \leq \mathbf{b}^+$. Thus we have proved the following

Lemma 3.3. *The solutions to (3.5), which are different from fixed points of f, vary as follows:*

1) If $\sqrt{3^{-1}} < b$, $b \neq 1$ or $b \leq \sqrt{3^{-1}}$, $a^2 \in (0, b_*^-) \cup (b_*^+, +\infty)$ then equation (3.6) has no positive solution.

2) If $b = 1$ or $b \leq \sqrt{3^{-1}}$ and $a^2 = b_*^-, b_*^+$ then equation (3.6) has a unique positive solution $x_1 = \frac{-B}{2b^4(1+a^2b^2)^2}$.

3) If $b \leq \sqrt{3^{-1}}$, $a^2 \in (b_*^-, b_*^+)$ then there are two positive solutions $x_{\pm} = \frac{-B \pm \sqrt{D}}{2b^4(1+a^2b^2)^2}$ to (3.6).

The following lemma gives useful properties of the function f.

Lemma 3.4. *1) If $b > 1$ then the sequence $x_n = f(x_{n-1})$, $n = 1, 2, \dots$ converges for any initial point $x_0 > 0$, where f is defined in (3.4).*

2) If $b < 1$ then the sequence $y_n = g(y_{n-1})$, $n = 1, 2, \dots$ converges for any initial point $y_0 > 0$, where $g(x) = f(f(x))$.

Proof. 1) For $b > 1$ we have $f'(x) = 2a^2(b^4 - 1)\frac{1+b^2x}{(b^2+x)^3} > 0$ i.e., f is an increasing function. Here we consider the case when the function f has three fixed points x_i^*, $i = 1, 2, 3$ (see Lemma 3.2. This proof is more simple for cases when f has one or two fixed points). We have that the point x_2^* is a repeller i.e., $f'(x_2^*) > 1$ and the points x_1^*, x_3^* are attractive i.e., $f'(x_1^*) < 1$, $f'(x_3^*) < 1$. Now we shall take arbitrary $x_0 > 0$ and prove that $x_n = f(x_{n-1})$, $n \geq 1$ converges as $n \to \infty$. Consider the following partition $(0, +\infty) = (0, x_1^*) \cup \{x_1^*\} \cup (x_1^*, x_2^*) \cup \{x_2^*\} \cup (x_2^*, x_3^*) \cup \{x_3^*\} \cup (x_3^*, +\infty)$. For any $x \in (0, x_1^*)$ we have $x < f(x) < x_1^*$, since f is an increasing function, from the last inequalities we get $x < f(x) < f^2(x) < f(x_1^*) = x_1^*$. Iterating this argument we obtain $f^{n-1}(x) < f^n(x) < x_1^*$, which for any $x_0 \in (0, x_1^*)$ gives $x_{n-1} < x_n < x_1^*$ i.e., x_n converges and its limit is a fixed point of f, since f has a unique fixed point x_1^* in $(0, x_1^*]$ we conclude that the limit is x_1^*. For $x \in (x_1^*, x_2^*)$ we have $x_2^* > x > f(x) > x_1^*$, consequently $x_n > x_{n+1}$ i.e., x_n converges and its limit is again x_1^*. Similarly, one can show that if $x_0 > x_2^*$ then $x_n \to x_3^*$ as $n \to \infty$.

2) For $b < 1$ we have f is decreasing and has unique fixed point x_1 which is repelling, but g is increasing since $g'(x) = f'(f(x))f'(x) > 0$. By Lemma 3.3 we have that g has at most three fixed points (including x_1). The point x_1 is repelling for g too, since $g'(x_1) = f'(f(x_1))f'(x_1) = (f'(x_1))^2 > 1$. But fixed points x_-, x_+ (see Lemma 3.3) of g are attractive. Hence one can repeat the same argument of the proof of part 1) for the increasing function g and complete the proof. □

Lemma 3.3 shows that if $b > 1$ i.e., $J_2 > 0$ then there is no any 2-periodic trajectory of F on M_1. Moreover, the following lemma says that if $J_2 > 0$ then there is no any periodic trajectory on M_1.

Lemma 3.5. *1) If $J_2 > 0$ then for any $p \geq 2$ the equation $F^p(u) = u$ has no solution $u \in M_1 \setminus \mathrm{Fix}(F)$.*

2) If $J_2 < 0$ then for any $p \geq 3$ the equation $F^p(u) = u$ has no solution $u \in M_1 \setminus (\mathrm{Fix}(F) \cup \mathrm{Per}_2(F))$.

Proof. 1) Assume there is a solution $u^{(0)} \in M_1 \setminus \mathrm{Fix}(F)$ then we get $p-$periodic trajectories $u_i^{(n+p)} = u_i^{(n)}$, $i = 1, 2$; $n = 0, 1, 2, \dots$. Since the set M_1 is invariant with respect to F, we obtain

$$x_{n+p} = \frac{u_1^{(n+p)}}{u_2^{(n+p)}} = \frac{u_1^{(n)}}{u_2^{(n)}} = x_n = f^n(x_0).$$

This is a contradiction, since by Lemma 3.4 we have x_n is not periodic.

2) Assume there is a solution $u^{(0)} \in M_1 \setminus (\mathrm{Fix}(F) \cup \mathrm{Per}_2(F))$ then we have

$$y_{n+p} = x_{2n+2p} = \frac{u_1^{(2n+2p)}}{u_2^{(2n+2p)}} = \frac{u_1^{(2n)}}{u_2^{(2n)}} = y_n = f^{2n}(y_0).$$

This is a contradiction, since by Lemma 3.4 we have y_n is not periodic. □

4. Exact values

Starting from a random initial condition (with $u_1^{(0)} \neq u_4^{(0)}$ and $u_2^{(0)} \neq u_3^{(0)}$), one iterates the recurrence equations (2.3) and observes their behavior after a large number of iterations. In the simplest situation a fixed point $u^* = (u_1^*, u_2^*, u_3^*, u_4^*) \in R_+^4$ is reached. It corresponds to (see [20]):

a paramagnetic phase if $u^* \in M_1$;

a ferromagnetic phase if $u^* \in M_2$.

If the iterations give a cyclic (periodic), say with period p sequence then the corresponding phase is called a (p-)*commensurate phase*. Finally, the system may remain aperiodic, which corresponds to an *incommensurate phase*.

The condition $\tilde{b} > 9$ of Lemma 3.2 requires that $J_2 > 0$. Denote

$$T_c = \frac{2J_2}{\ln 3}, \quad J_2 > 0.$$

Lemma 3.2 gives the following

Theorem 4.1. *If $T \geq T_c$ then the model (2.1) has a unique paramagnetic phase; If $T < T_c$ then there are exactly three (resp. two) paramagnetic phases if (J_1, J_2) is such that $b^3\sqrt{\nu_1} < a^{-1} < b^3\sqrt{\nu_2}$ (resp. $a^{-1} = b^3\sqrt{\nu_1}$ or $a^{-1} = b^3\sqrt{\nu_2}$).*

For the condition $b < \sqrt{3^{-1}}$ of Lemma 3.3 we need to condition $J_2 < 0$. In this case we have $T_c = \frac{-2J_2}{\ln 3}, \quad J_2 < 0$.

From Lemma 3.3 we get the following

Theorem 4.2. *If $T \geq T_c$ then the model (2.1) (on M_1) has a unique 2-commensurate phase. If $T < T_c$ then there are exactly two (resp. one) 2-commensurate phases if $a^2 \in (b_*^-, b_*^+)$ (resp. $a^2 = b_*^-$ or $a^2 = b_*^+$).*

For a fixed temperature $T = \beta^{-1} < T_c$ we have two critical curves $a^2 = b_*^{\pm}$ i.e., on terms of J_1 and $J_2 < 0$ they are given by the following explicit relations

$$J_1 = \frac{1}{2\beta} \ln \left(8^{-1}\{1 - 3e^{8J_2\beta} - 6e^{4J_2\beta} \pm \sqrt{(e^{4J_2\beta} - 1)^3(9e^{4J_2\beta} - 1)}\} \right) - 3J_2.$$

Using Lemma 3.3 and formula (2.2) we can get explicit formulas for the sequence of periodic partition functions:

$$Z_n = Z_n(y) = 2a^{-2/3} \times$$

$$\begin{cases} \left(\left(ab(b+\frac{1}{by})^2+\frac{1}{ab}(\frac{1}{b}+\frac{b}{y})^2\right)^{-\frac{2}{3}}+a^{\frac{2}{3}}\left(\frac{a}{b}(by+\frac{1}{b})^2+\frac{b}{a}(\frac{y}{b}+b)^2\right)^{-\frac{2}{3}}\right)^2, & n=2m \\ \left(\left(ab(b+\frac{1}{bf(y)})^2+\frac{1}{ab}(\frac{1}{b}+\frac{b}{f(y)})^2\right)^{-\frac{2}{3}}+a^{\frac{2}{3}}\left(\frac{a}{b}(bf(y)+\frac{1}{b})^2+\frac{b}{a}(\frac{f(y)}{b}+b)^2\right)^{-\frac{2}{3}}\right)^2, & n=2m+1 \end{cases}$$

where $m=0,1,2,\dots$; y is one of x_1, x_-, x_+ defined in Lemma 3.3 and function f is given in (3.4).

It is easy to see that if x is a fixed point of f then the corresponding fixed point of F has the form $u^*(x)=(u_1^*(x),u_2^*(x),u_2^*(x),u_1^*(x))$ with $u_1^*(x)=a^{-1}(b+(bx)^{-1})^{-2}$ and $u_2^*(x)=a(b+b^{-1}x)^{-2}$. If y is a fixed point of g then the corresponding 2-periodic point of F has the form $u^{\rm per}(y)=(u_1^{\rm per}(y),u_2^{\rm per}(y),u_2^{\rm per}(y),u_1^{\rm per}(y))$ with

$$u_1^{\rm per}(y)=a^{-1/3}\left(ab(b+(by)^{-1})^2+(ab)^{-1}(by^{-1}+b^{-1})^2\right)^{-2/3},$$

$$u_2^{\rm per}(y)=a^{1/3}\left(a^{-1}b(b+b^{-1}y)^2+ab^{-1}(by+b^{-1})^2\right)^{-2/3}.$$

Lemma 3.5 gives

Theorem 4.3. *The model (2.1) (on M_1) has an uncountable set S of incommensurate phases μ_u, where $u\in M_1\setminus(\mathrm{Fix}(F)\cup\mathrm{Per}_2(F))$. Moreover, the set of incommensurate phases can be classified to (uncountable) subsets*

$$S_x=\{\mu_u:\quad u\in M_1\setminus(\mathrm{Fix}(F)\cup\mathrm{Per}_2(F))\quad \textit{with}\quad \lim_{n\to\infty}F^n(u)=u^*(x)\},$$

where x is an attractive fixed point of f and

$$S_y^{\rm per}=\{\mu_u:\quad u\in M_1\setminus(\mathrm{Fix}(F)\cup\mathrm{Per}_2(F))\quad \textit{with}\quad \lim_{n\to\infty}F^{2n}(u)=u^{\rm per}(y)\},$$

where y is an attractive fixed point of g.

5. Concluding remarks

Usually, to describe phases (Gibbs measures) of a given Hamiltonian on a Cayley tree one has correspondence between Gibbs measures and a collection of vectors (real numbers in some particular cases) $\{h_x,\, x\in V\}$, which satisfy a non-linear equation (see for example, [5], [6], [12]–[16], [19]). The recurrent equation (2.3) considered in this paper (which was obtained in [20]) describes a vector function $\{u^{(n)}, n\in N\}$ which is a particular case of the above mentioned function h_x obtained as $h_x=u^{(n)}$ if $x\in W_n$ i.e., depends only on the number of the generation set where x belongs, but not on x itself. Thus the solutions to the recurrent equation (2.3) do not fully describe phases of the model (2.1). But deriving the functional equation for h_x corresponding to the Hamiltonian (2.1) is also difficult, since there is prolonged NNN interaction. Such a model can also be studied by a contour argument (see [17] and references therein), but this argument does not give exact solutions, in general.

By a process of iteration, for the model (2.1) Vannimenus found a new modulated phase, in addition to the expected paramagnetic and ferromagnetic (fixed point) phases and a $(++--)$ periodic (four cycle antiferromagnetic phase, which consisted of commensurate (periodic) and

incommensurate (aperiodic) regions corresponding to so-called "devil's staircase". In this paper, using the theory of dynamical systems we have analytically proved many of the above mentioned results, i.e., the following exact results are obtained:

Paramagnetic phase: The exact critical temperature and exact critical curves are found. It is proven that the number of the paramagnetic phases can be at most three. (Theorem 4.1).

Ferromagnetic phase: We reduced the description of such phases to a quartic equation (i.e., solution of the equations on M_2). But we were not able to study the periodic solutions on M_2.

Commensurate phase: The exact critical temperature (which is obtained from the critical temperature of the paramagnetic phase by replacing J_2 with $-J_2$) and exact critical curves are found. On the set M_1 it is proven that the number of the 2-commensurate phases can be at most two and there is not p-commensurate phases if $p \geq 3$ (Lemma 3.5, Theorem 4.2). We also described exact values of periodic partition functions.

Incommensurate phase: We proved that the model has uncountably many such phases. Moreover, we classified them in two classes: the first class contains the phases which are "asymptotically fixed" (set S_x); the second class contains the phases which are "asymptotically periodic" (set S_y^{per}). Note that for the usual Ising model with external field on Cayley trees such infinitely many phases are known (see [7], p.250).

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