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ON THE L-INFINITY DESCRIPTION OF THE HITCHIN MAP

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Abstract

Recently, E. Martinengo obtained results on obstructions to deformations of Higgs pairs by describing an L-infinity morphism inducing the Hitchin map. In this note we show that analogous results hold for principal G-Higgs bundles. Moreover, we show that the L-infinity morphism is a “global analogue” of a Lie-algebraic construction.

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1 Introduction

One of the routes to deformation theory in characteristic zero – as laid down by Deligne, Drinfeld, Kontsevich, Quillen and others – is via differential graded Lie algebras. Moreover, one expects, after Kontsevich, that locally the formal geometry of moduli problems is in fact controlled by a richer structure: an L_∞ -algebra (strongly homotopy Lie algebra). According to this paradigm, natural morphisms of moduli spaces (deformation functors) are induced by L_∞ -morphisms of the corresponding L_∞ -algebras.

Let G be a complex reductive Lie group. Consider a Higgs pair (\mathbf{P}, θ) on a compact, Kähler manifold X . Here \mathbf{P} is a principal G -bundle on X and $\theta \in H^0(X, \text{ad}\mathbf{P} \otimes \Omega_X^1)$, satisfying $\theta \wedge \theta = 0$. We can associate to such a pair the spectral invariants of θ . For instance, if G is a classical group, θ is given locally by a matrix of holomorphic 1-forms, and one takes the coefficients of its characteristic polynomial.

There exist (coarse) moduli spaces of (semi-stable) Higgs pairs: these were constructed by Hitchin in dimension one and by Simpson in higher dimensions. The “characteristic polynomial map” (the Hitchin map) is in fact a proper map from the moduli space to a vector space. It has many wonderful properties, especially if $\dim_{\mathbb{C}} X = 1$, when the Hitchin map turns the moduli space into an algebraic completely integrable Hamiltonian system.

In this simple note we exhibit an L_∞ -morphism inducing the Hitchin map. We obtain it as a “global version” of a simple Lie-algebraic problem. If we choose homogeneous generators for $\mathbb{C}[\mathfrak{g}]^G$, we can write the adjoint quotient map as $\chi : \mathfrak{g} \rightarrow \mathbb{C}^N$. Fixing also an element of \mathfrak{g} allows one to interpret \mathfrak{g} and \mathbb{C}^N as Maurer-Cartan functors of differential graded Lie algebras and χ as a natural transformation between these, induced by an L_∞ -morphism. The induced natural transformation of deformation functors is the isomorphism $\mathfrak{g} // G \simeq \mathbb{C}^N$. The L_∞ -morphism can be written in elementary and explicit terms, using the polarisation of G -invariant polynomials. This construction “globalises” easily, and gives rise to an L_∞ -description of the Hitchin map. This is the natural generalisation of the calculations in [Mar10], where the author treats the case of $G = GL(n, \mathbb{C})$ and bundle-valued Higgs pairs, using heavily the properties of traces and powers of matrices.

The existence of the L_∞ -morphism gives additional information on obstructions, generalising the corresponding result in [Mar10].

The content of this note is organised as follows. In the second section, we set up notation and recall basic facts about L_∞ -algebras. In section three we prove two simple lemmas about invariant polynomials and formulate the Lie-algebraic version of our result. Namely, let $C^\bullet := \mathfrak{g} \otimes \mathbb{C}[\epsilon]/\epsilon^2$, with differential $\epsilon \text{ad}v$, where $v \in \mathfrak{g}$. We describe in **Propositions** 3.1, 3.2 an L_∞ -morphism $h_\infty : C^\bullet \rightarrow \mathbb{C}^N[-1]$, with the property that $\text{MC}(h_\infty) = \chi : \mathfrak{g} \rightarrow \mathbb{C}^N$ and, consequently, $\text{Def}(h_\infty)$ is the Chevalley isomorphism. We prove the global version in section four. Given a

Higgs pair (\mathbf{P}, θ) , we show in **Propositions** 4.1, 4.2 that h_∞ induces an L_∞ -morphism

$$h_\infty : \bigoplus_{p+q=\bullet} A^{0,p}(\mathrm{ad}\mathbf{P} \otimes \Omega_X^q) \longrightarrow \bigoplus_{m_i \in E} A^{0,\bullet}(S^{m_i} \Omega_X^1).$$

and the natural transformation of deformation functors induced by h_∞ is the Hitchin map: $\mathrm{Def}(h_\infty) = H$.

Finally, we show in **Corollary** 4.1 that obstructions to deforming a Higgs pair (\mathbf{P}, θ) are contained in $\ker H^2(h_1)$.

2 Preliminaries

2.1 Notation and Conventions

The ground field is \mathbb{C} . We denote by $\mathrm{Art}_{\mathbb{C}}$ the category of local Artin \mathbb{C} -algebras with residue field \mathbb{C} , and \mathfrak{m}_A denotes the maximal ideal of $A \in \mathrm{Art}_{\mathbb{C}}$. For a vector space V , we use $T(V)$ to denote its tensor algebra and $S(V) = \bigoplus_{k \geq 0} S^k(V)$ to denote its symmetric algebra. The same notation is used for the tensor or symmetric algebra of a *graded* vector space, as well as for the underlying vector space of the corresponding coalgebras. When we want to emphasise the coalgebra structure, we use $S_c(V)$, etc. The *reduced* symmetric (co)algebra is denoted $\overline{S(V)} = \bigoplus_{k \geq 1} S^k(V)$, and similarly for the tensor (co)algebra. By $S(k, n-k)$ we denote the $(k, n-k)$ shuffles, that is, permutations of n elements, for which $\sigma(i) < \sigma(i+1)$ for all $i \neq k$. We use the standard acronym “dgla” for a “differential graded Lie algebra”.

Next, G is a complex reductive Lie group, and $\mathfrak{g} = \mathrm{Lie}(G)$. We use fixed homogeneous generators, $\{p_i\}$ of the ring of G -invariant polynomials on \mathfrak{g} . The adjoint quotient map is always given in terms of this basis, i.e., $\chi : \mathfrak{g} \rightarrow \mathbb{C}^N$. We denote by E the set of exponents of G , and $N = |E|$. The exponents themselves are denoted by m_i and the degrees of the invariant polynomials by $d_i = m_i + 1 = \deg p_i$. We denote by \mathcal{P}_{dk} various polarisation maps, introduced in section 3.1.

The base manifold is assumed to be compact and Kähler, denoted by X . For a principal bundle, \mathbf{P} , we use $\mathrm{ad}\mathbf{P}$ for the associated bundle of Lie algebras, $\mathrm{ad}\mathbf{P} = \mathbf{P} \times_{\mathrm{ad}} \mathfrak{g}$. We denote by Ω_X^p the sheaf of holomorphic p -forms on X . We use $A^{p,q}$ to denote the global sections of the sheaf $\mathcal{A}^{p,q}$ of complex differential forms of type (p, q) .

2.2 Differential Graded Lie Algebras

Two great references for this material are [GM88] and [Man04], as well as [Man99], which is a shorter version of the latter. We shall need the bare minimum of the technology developed there. Below we give only some of the basic definitions, without attempting to motivate them in any way. A *differential graded Lie algebra* (dgla) is a triple $(\mathcal{C}^\bullet, d, [\ , \])$. Here $\mathcal{C}^\bullet = \bigoplus_{i \in \mathbb{N}} \mathcal{C}^i$ is a graded vector space, endowed with a bracket $[\ , \] : \mathcal{C}^i \times \mathcal{C}^j \rightarrow \mathcal{C}^{i+j}$. The bracket is graded skew-symmetric and satisfies a graded Jacobi identity. Finally, $d : \mathcal{C}^\bullet \rightarrow \mathcal{C}^{\bullet+1}$ is a

differential ($d^2 = 0$), which is a graded derivation of the bracket. To a dgla \mathcal{C} one associates a Maurer-Cartan functor $\text{MC}_{\mathcal{C}^\bullet} : \text{Art}_{\mathbb{C}} \rightarrow \text{Set}$, defined as

$$\text{MC}_{\mathcal{C}^\bullet}(A) = \left\{ u \in \mathcal{C}^1 \otimes \mathfrak{m}_A \mid du + \frac{1}{2}[u, u] = 0 \right\}$$

and a deformation functor $\text{Def}_{\mathcal{C}} : \text{Art}_{\mathbb{C}} \rightarrow \text{Set}$

$$\text{Def}_{\mathcal{C}^\bullet(A)} = \text{MC}_{\mathcal{C}^\bullet}(A) / \{\text{gauge equivalence}\}.$$

We are not giving the gauge action explicitly as we are not going to use it. It is described by the BCH-formula and can be found in [GM88], [Man04] or [Man99]. On the other hand, deformation problems come with a deformation functor $\text{Art}_{\mathbb{C}} \rightarrow \text{Set}$, and we say that the problem is governed (controlled) by a dgla, if its deformation functor is isomorphic to the deformation functor arising from some dgla \mathcal{C} .

As stated in the introduction, for us a Higgs bundle (Higgs pair) is a pair (\mathbf{P}, θ) , $\theta \in H^0(X, \text{ad}\mathbf{P} \otimes \Omega_X^1)$, $\theta \wedge \theta = 0$. There exist other variants of this construction (e.g., one can replace Ω_X^1 by a ‘‘coefficient bundle’’ L) but we are not going to use any of these. The construction of the coarse moduli spaces (of semi-stable Higgs bundles) was treated by various authors, among whom Hitchin ([Hit87a], [Hit87b]) and Simpson ([Sim92], [Sim94]). These spaces have amazingly rich geometry and are an object of current research. The corresponding stacks are described, e.g., in [DG02], in a more general setup. The infinitesimal deformations of Higgs bundles were studied by many people; in particular, Biswas and Ramanan ([BR94]) identified a deformation complex for this problem when X is a curve. In [Bis94] a deformation complex was written for the case of higher dimensional X and for varying base manifold. In general, i.e., for arbitrary dimension and arbitrary reductive G , the differential graded Lie algebra controlling deformations of a Higgs bundle (\mathbf{P}, θ) on a fixed (compact, complex) manifold, X , turns out to be ([Sim97], [Sim94], [Sim92])

$$\mathcal{C}^\bullet = \bigoplus_{p+r=\bullet} A^{0,p}(X, \text{ad}\mathbf{P} \otimes \Omega_X^r),$$

with differential $\bar{\partial}_{\mathbf{P}} + \text{ad}\theta$. This is the Dolbeault resolution of the complex of Biswas and Ramanan. For the case of $G = GL(n, \mathbb{C})$ and L -valued pairs, one has $\Lambda^q(L)$ instead of Ω_X^q , and this setup is discussed in great detail in [Mar10], where the author also treats deformations with a varying base manifold.

Let $M_{Dol}(G)$ be the moduli space of isomorphism classes of semi-stable Higgs pairs. The Hitchin map

$$H : M_{Dol}(G) \longrightarrow \mathcal{B} = \bigoplus_{m_i \in E} H^0(X, S^{m_i} \Omega_X^1)$$

is given by $[(\mathbf{P}, \theta)] \mapsto \{p_i(\theta)\}_i$. For details, see [Hit87a], [Hit87b], [Sim92].

The Hitchin base \mathcal{B} is a deformation functor in many ways. To any $\xi \in \mathcal{B}$ we associate the functor of deformations of ξ , Def_ξ . An element $s \in \mathcal{B} \otimes \mathfrak{m}_A$ gives an A -deformation $\xi + s$ of ξ over

A. This functor is isomorphic to the deformation functor of the abelian dgla $\oplus_i A^{0,\bullet}(S^{m_i}\Omega_X^1)[-1]$. The isomorphism is tautological, induced by $H^0(X, S^{m_i}\Omega_X^1) \subset A^{0,0}(X, S^{m_i}\Omega_X^1) = A^{0,\bullet}(S^{m_i}\Omega_X^1)[-1]^1$.

For a given Higgs bundle (\mathbf{P}, θ) , the Hitchin map induces a morphism of deformation functors

$$H(A) : \text{Def}_{(\mathbf{P}, \theta)}(A) \rightarrow \text{Def}_{\chi(\mathbf{P}, \theta)}(A), \quad A \in \text{Art}_{\mathbb{C}}.$$

As we saw, there are dgla's associated to both deformation functors, but H is not a dgla morphism – it is not even linear, unless $\dim X = 1$ and $G = \mathbb{C}^\times$.

2.3 L-infinity Algebras

The notion of L_∞ -algebra (strongly homotopy Lie algebra) generalises the notion of a dgla. The main idea is to relax the Jacoby identity, allowing that it be satisfied only “up to homotopy” (“BRST-exact term” in physics parlance), determined by “higher brackets”. This structure encompasses the notion of a dgla as a special case. Moreover, L_∞ -morphisms of dglas (see below) are richer than dgla morphisms: one allows morphisms of complexes, preserving the bracket only up to homotopy.

We do not attempt to give an overview of this immense subject or its relations to topology, Poisson geometry, quantization, mirror symmetry, non-commutative geometry, etc. We simply review the relevant definitions in a very brief manner. Our conventions are as in [Man04].

Recall that the space of homomorphisms between two graded vector spaces is naturally graded. Let V be a graded vector space. Suppose given a collection $\{q_k\}$, $q_k \in \text{Hom}^1(S^k(V[1]), V[1])$, $k \geq 1$. Then ([Man04], chapter 8) $q = \sum_k q_k$ extends to a degree 1 coderivation Q from $\overline{S_c(V[1])}$ to itself. We say that (V, q) is an L_∞ -algebra if Q is actually a codifferential, i.e. $Q^2 = 0$. The coderivation Q is expressed in terms of q as ([Man04])

$$Q = \pi \sum_{n=1}^{\infty} \frac{1}{n!} (q \otimes id^n) \circ \Delta^n.$$

Here π is the canonical projection $\overline{T_c(V)} \rightarrow \overline{S_c(V)}$, and Δ is the comultiplication on $T_c(V)$. For the very special case at hand, we shall write the formula explicitly.

By décalage, one can replace the q_k 's by maps $\mu_k \in \text{Hom}^{2-k}(\Lambda^k V, V)$: these are the higher brackets mentioned above. To an L_∞ -algebra (V, q) one associates a Maurer-Cartan functor $\text{MC}_V : \text{Art}_{\mathbb{C}} \rightarrow \text{Set}$ defined by

$$\text{MC}_V(A) = \left\{ u \in V^1 \otimes \mathfrak{m}_A \mid \sum_{k \geq 1} \frac{q_k(u^k)}{k!} = 0 \right\}$$

and a deformation functor Def_V , defined by $\text{MC}(A)/\text{homotopy equivalence}$, $A \in \text{Art}_{\mathbb{C}}$. We are not giving explicitly the definition of homotopy equivalence, partially in view of the next two sentences. A dgla becomes an L_∞ -algebra if we take $q_1(a) = -da$, $q_2(a \cdot b) = (-1)^{\deg a}[a, b]$, $q_k = 0, k \geq 3$. Then the Maurer-Cartan equations clearly coincide, and, moreover, gauge equivalence coincides with homotopy equivalence ([MF07]).

Next we define morphisms between L_∞ -algebras. Suppose given two L_∞ -algebras (V, q) and (W, \hat{q}) and a sequence of linear maps $h_k \in \text{Hom}^0(S_c^k V[1], W[1])$, $k \geq 1$. These induce ([Man04]) a morphism $H : \overline{S_c V[1]} \rightarrow \overline{S_c W[1]}$. We say that we have a morphism of L_∞ -algebras $h_\infty : (V, q) \rightarrow (W, \hat{q})$ if H intertwines Q and \hat{Q} , i.e., $H \circ Q = \hat{Q} \circ H$. Write Q_k^n for the component of Q mapping $S_c^k(V)$ to $S_c^n(V)$. The morphism condition is equivalent to

$$\sum_{a=1}^{\infty} h_a \circ Q_k^a = \sum_{a=1}^{\infty} \hat{q}_a \circ H_k^a, \quad k \in \mathbb{N}.$$

The dgla subcategory is not a full subcategory of the category of L_∞ -algebras. In the last section, we are going to exhibit the Hitchin map as an honest L_∞ -morphism between two differential graded Lie algebras.

For a dgla one has that $Q_k^a = 0$, unless $a = k$ and $a = k - 1 \neq 0$. If, moreover, the dgla is *abelian*, then $q_2 = 0 = Q_k^{k-1}$. We are interested in the special case of an L_∞ -morphism $h : (V, q) \rightarrow (W, \hat{q})$, where V, W are dgla's and W is abelian. Let us spell out the condition for an L_∞ -morphism in this situation. First off, we have

$$Q_k^k(s_1 \cdots s_k) = \sum_{\sigma \in S(1, k-1)} \varepsilon(\sigma) q_1(s_{\sigma_1}) \cdot s_{\sigma_2} \cdots s_{\sigma_k} \quad (1)$$

$$Q_k^{k-1}(s_1 \cdots s_k) = \sum_{\sigma \in S(2, k-2)} \varepsilon(\sigma) q_2(s_{\sigma_1} \cdot s_{\sigma_2}) \cdot s_{\sigma_3} \cdots s_{\sigma_k}. \quad (2)$$

The h_k 's determine an L_∞ -morphism if

$$h_1 \circ q_1 = \hat{q}_1 \circ h_1, \quad (3)$$

which says that h_1 is a morphism of complexes, and

$$h_k \circ Q_k^k + h_{k-1} \circ Q_k^{k-1} = \hat{q}_1 \circ h_k, \quad k \geq 2. \quad (4)$$

The last condition, when evaluated on homogeneous elements s_1, \dots, s_k reads

$$\begin{aligned} & h_k \left(- \sum_{\sigma \in S(1, k-1)} \varepsilon(\sigma) d(s_{\sigma_1}) \cdot s_{\sigma_2} \cdots s_{\sigma_k} \right) + \\ & h_{k-1} \left(\sum_{\sigma \in S(2, k-2)} \varepsilon(\sigma) (-1)^{\deg s_{\sigma_1}} [s_{\sigma_1}, s_{\sigma_2}] \cdot s_{\sigma_3} \cdots s_{\sigma_k} \right) = \\ & -dh_k(s_1 \cdots s_k). \end{aligned} \quad (5)$$

It expresses the failure of h_{k-1} to preserve the bracket in terms of a homotopy given by h_k (to make this literally correct, one has to apply the décalage isomorphism first).

Finally, let us mention that MC is a functor $L_\infty \rightarrow \text{Fun}(\text{Art}_{\mathbb{C}}, \text{Set})$ and the action on morphisms is given by sending an L_∞ -morphism $h_\infty \in \text{Hom}_{L_\infty}(V, W)$ to a natural transformation $\text{MC}(h_\infty)$, acting, for each $A \in \text{Art}_{\mathbb{C}}$, as

$$\text{MC}_V(A) \ni x \mapsto \sum_{k=1}^{\infty} \frac{1}{k!} h_k(x^k) \in \text{MC}_W(A). \quad (6)$$

This descends to a natural transformation $\text{Def}_V \rightarrow \text{Def}_W$.

3 The Adjoint Quotient in L_∞ -terms

If we think of the Hitchin map H as a “global analogue” of the adjoint quotient map $\chi : \mathfrak{g} \rightarrow \mathbb{C}^N$, it is natural to look for a Lie-algebraic version of the expected L_∞ -morphism inducing H .

The question then is the following: both \mathfrak{g} and \mathbb{C}^N are vector spaces, but χ is not linear. Can we identify an L_∞ -morphism between suitable dgla’s which will capture this non-linearity? Let us fix an element, $v \in \mathfrak{g}$. There are two deformation functors associated with this data.

On one hand, we have the categorical quotient $\mathfrak{g} // G$. On the other hand, we have $\text{Def}_{\chi(v)}$, the functor of deformations of $\chi(v) \in \mathbb{C}^N$ (which is \mathbb{C}^N itself). The two functors are isomorphic. Next, we identify dglas controlling them. Of course, \mathfrak{g} itself is a dgla with trivial differential. Take $C^\bullet = \mathfrak{g} \otimes \mathbb{C}[\varepsilon]/\varepsilon^2$ with the “twisted” differential εadv . Thus we have the two-term complex, $\mathfrak{g} \xrightarrow{\text{adv}} \mathfrak{g}$, with bracket $[(a_0, a_1), (b_0, b_1)] = ([a_0, b_0], [a_0, b_1] + [a_1, b_0])$ and differential $d_1 = \text{adv}$. Clearly $\text{MC}_{C^\bullet} = \mathfrak{g}$. Next, we take $B^\bullet = \mathbb{C}^N[-1]$, with trivial bracket and differential. Now, for a given $A \in \text{Art}_{\mathbb{C}}$ we identify deformations of $\chi(v)$ with $\mathbb{C}^N \otimes_{\mathfrak{m}_A}$ via $\chi(v+b) - \chi(v)$. But χ is given by homogeneous polynomials, and for these Taylor’s formula can be expressed conveniently by polarisation. Since we have actually G -invariant polynomials on \mathfrak{g} , they satisfy extra relations. These two facts are sufficient to identify an L_∞ -morphism between C^\bullet and B^\bullet with the desired property.

3.1 Polarisation and Invariant Polynomials

Suppose V is a (finite dimensional) vector space concentrated in degree zero (i.e., without grading). Then we have, for each $d, k \in \mathbb{N}$, a homomorphism

$$\mathcal{P}_{dk} : S^d(V^\vee) \rightarrow T^k(V^\vee)^{\Sigma_k} \otimes S^{d-k}(V^\vee)$$

defined by

$$\mathcal{P}_{dk}(p)(X_1, \dots, X_k; v) = \mathcal{L}_{X_1} \dots \mathcal{L}_{X_k}(p)(v),$$

where $\mathcal{L}_X(p)$ is the Lie derivative of p . We shall occasionally denote by $\mathcal{P}_{dk}(p)_v$ the corresponding map $T^k(V) \rightarrow \mathbb{C}$.

In lay terms, $\mathcal{P}_{dk}(p)(X_1, \dots, X_k; v)$ is the coefficient in front of $t_1 \dots t_k$ in the Taylor expansion of $p(v + \sum t_i X_i)$.

For example, if $V = \mathfrak{gl}(r, \mathbb{C})$ and $p(A) = \text{tr} A^d$, then $\mathcal{P}_{dk}(p)(X_1, \dots, X_k; v) = \frac{d!}{(d-k)!} \text{tr}(X_1 \cdots X_k v^{d-k})$.

Notice that $\mathcal{P}_{dk} = 0$ for $k > d$ and that \mathcal{P}_{dd} is the usual polarisation map, identifying invariants and coinvariants for the symmetric group. In particular, we have $p(X) = \frac{1}{d!} \mathcal{P}_{dd}(p)(X, \dots, X)$ and, more generally

$$\mathcal{P}_{dk}(p)(X_1, \dots, X_k; v) = \frac{1}{(d-k)!} \mathcal{P}_{dd}(p)(v, \dots, v, X_1, \dots, X_k).$$

We also have

$$p(v + X) - p(v) = \sum_{k=1}^{\infty} \frac{1}{k!} \mathcal{P}_{dk}(p)(X, \dots, X; v), \quad (7)$$

which follows from Taylor's formula.

Lemma 3.1 *Let $p \in \mathbb{C}[\mathfrak{g}]^G$ be a homogeneous G -invariant polynomial of degree d . Then $\forall v, X \in \mathfrak{g}$, $\mathcal{P}_{d1}(p)(\text{ad}_X(v); v) \equiv \mathcal{L}_{[X,v]}p(v) = 0$.*

Proof:

We are claiming that $\frac{d}{dt}p(v + t\text{ad}_X(v))|_{t=0} = 0$, which is an infinitesimal form of the G -invariance of p : its Lie derivative vanishes in any direction tangent to a G -orbit. Alternatively, one can write the above expression as $\frac{1}{(d-1)!}$ times

$$\mathcal{P}_{dd}(p)(v, \dots, v, \text{ad}_X(v)) = \frac{1}{d} \frac{d}{dt} \mathcal{P}_{dd}(p)(\text{Ad}(e^{tX})v, \dots, \text{Ad}(e^{tX})v)|_{t=0} = 0.$$

□

Lemma 3.2 *Let $V = \bigoplus_{i=0}^{k-1} V_i$, let $F \in T^d(V^\vee)^\Sigma$, and let $L \in \prod_i GL(V_i)$. The decomposition of V induces a decomposition of $S^d(V^\vee)$, indexed by ordered partitions of d of length k . The projection of $F \circ (L \otimes 1^{\otimes d-1})$ onto the subspace corresponding to $(d-k+1, 1, \dots, 1)$ maps $v \otimes X_1 \otimes \dots \otimes X_{k-1}$ to*

$$\begin{aligned} & \frac{d!}{(d-k)!} F(L(v), X_1, \dots, X_{k-1}, v, \dots, v) + \\ & \sum_{\sigma \in S(1, n-2)} \frac{d!}{(d-k+1)!} F(L(X_{\sigma_1}), X_{\sigma_2}, \dots, X_{\sigma_k}, v, \dots, v). \end{aligned}$$

Proof:

Count the terms containing exactly one of each X_i in $F(L(v + \sum_i X_i), v + \sum_i X_i, \dots, v + \sum_i X_i)$.

□

Corollary 3.1 *Let $p \in \mathbb{C}[\mathfrak{g}]^G$ be a homogeneous G -invariant polynomial of degree d . Let $2 \leq k \leq d$, and let $v, Y, X_1, \dots, X_{k-1} \in \mathfrak{g}$. Then*

$$\begin{aligned} & \mathcal{P}_{dk}(p)([Y, v], X_1, \dots, X_{k-1}; v) + \\ & \sum_{\sigma \in S(1, k-2)} \mathcal{P}_{d, k-1}(p)([Y, X_{\sigma_1}], \dots, X_{\sigma_{k-1}}; v) = 0. \end{aligned}$$

Proof:

We apply Lemma 3.2 to $F = \mathcal{P}_{dd}(p)$ and $L = \text{ad}Y$ and use Lemma 3.1 to argue that $F \circ (L \otimes 1)$ is zero. \square

This formula is the key to the L_∞ -description of the Chevalley (and Hitchin) map, and it clearly has an operadic origin.

We can evaluate polynomials in $\mathbb{C}[\mathfrak{g}]^G$ on sections of $\text{ad}\mathbf{P} \otimes \Omega_X^1$ (or forms with coefficients therein) obtaining values in $H^0(S^\bullet \Omega_X^1)$. We can also apply the various operators \mathcal{P}_{dk} as above. We shall keep the notation the same. For example, if $s_i = \alpha_i \otimes X_i \in A^{0,\bullet}(\text{ad}\mathbf{P} \otimes \Omega_X^1)$ are homogeneous, and $v \in H^0(X, \text{ad}P \otimes \Omega_X^1)$, we write $\mathcal{P}_{dk}(p)(s_1, \dots, s_k; v) = \alpha_1 \wedge \dots \wedge \alpha_k \mathcal{P}_{dk}(X_1, \dots, X_k; v)$.

3.2 L-infinity Morphism

Proposition 3.1 *Let m be an exponent of G and $p \in \mathbb{C}[\mathfrak{g}]^G$ homogeneous, $\deg p = d = m + 1$. Let $pr_0^{\otimes k} : S^k(C^\bullet[1]) \rightarrow S^k(C^1)$, $k \geq 1$, be the natural projection. The maps*

$$h_k^m = \mathcal{P}_{dk}(p)_v \circ pr_0^{\otimes k} : S^k(C^\bullet[1]) \longrightarrow \mathbb{C}$$

that is,

$$(a_1, b_1) \cdot \dots \cdot (a_k, b_k) \mapsto \mathcal{P}_{dk}(p)(b_1, \dots, b_k; v)$$

induce an L_∞ -morphism

$$h_\infty^m : C^\bullet \longrightarrow \mathbb{C}[-1].$$

Proof:

We check 3.4. First, the linear part, h_1^m , must be a morphism of complexes. The differentials are, respectively, $\text{ad}v$ and 0, and we have to show that, for $s = (a, b) \in \mathfrak{g}^{\oplus 2}$, $h_1^m([v, s]) = 0$, i.e., $\mathcal{P}_{d,1}(p)([v, b]; v) = 0$. But this is Lemma 3.1. Next, consider $k \geq 2$. Since B^\bullet has trivial differential, the right side in 4 is identically zero. The left side is zero on $S^k(C^1)$, since $[C^1, C^1] = 0$. Moreover, it is zero on $S^r(C^0) \cdot S^{k-r}(C^1)$, $r \geq 2$, since the maps h_k^m factor through $pr_0^{\otimes k}$. So we only have to verify the formula on $C^0 \cdot S^{k-1}(C^1)$: in this case Q_k^{k-1} contributes via the bracket and Q_k^k via $\text{ad}v$. Take homogeneous elements $s_j = (0, b_j)$, $j \geq 2$ and $s_1 = (a, 0)$. In the first summand of 5, shuffles with $\sigma_1 \neq 1$ give zero, and $\sigma_1 = 1$ means $\sigma = \text{id}$, so we have $h_k^m([v, s_1] \cdot s_2 \cdot \dots \cdot s_k) = (-1) \mathcal{P}_{dk}(p_i)([a, v], b_2, \dots, b_k; v)$. The second summand in 5 is $h_{k-1}^m \circ Q_k^{k-1}(s_1 \cdot \dots \cdot s_k)$ and the non-vanishing terms correspond to $(2, k-2)$ shuffles for which $\sigma_1 = 1$, so the summation is in fact over $(1, k-2)$ shuffles and we have

$$h_{k-1}^m \left(\sum_{\sigma \in S(1, k-2)} (-1)^\epsilon(\sigma) [s_1, s_{\sigma_1}] \cdot \dots \cdot s_{\sigma_{k-1}} \right).$$

Since $C^1 = C^\bullet[1]^0$, there is no Koszul sign and we have

$$(-1) \sum_{\sigma \in S(1, k-2)} \mathcal{P}_{d, k-1}(p_i)([a, b_{\sigma_1}], \dots, b_{\sigma_{k-1}}; v).$$

The sum of the two terms is zero by Corollary 3.1. \square

Next, we put the maps for the various exponents together.

Proposition 3.2 *The maps $h_\infty^{m_i}$, $m_i \in E$ induce an L_∞ -morphism $h_\infty : C^\bullet \rightarrow \mathbb{C}^N[-1]$. It has the property that $MC(h_\infty) = \chi$ and, consequently, $Def(h_\infty)$ is the Chevalley isomorphism.*

Proof:

The fact that we have an L_∞ -morphism is clear from the previous proposition. Next, let $A \in \text{Art}_{\mathbb{C}}$, and suppose $s = (0, b) \in MC(A) \simeq \mathfrak{g} \otimes A \subset \mathfrak{g}^{\oplus 2} \otimes A$. Then, by 6 $MC(h_\infty)(s) = \sum_{d=1}^{\infty} \frac{1}{d!} h_\infty(s^d)$, but by ‘‘Taylor’s’’ formula 7 this is $\{p_i(v+b) - p_i(v)\}_i$, which is – by definition – the same as $\chi(v+b) - \chi(v)$. \square

4 The Hitchin Map

Here we identify the L_∞ -morphism giving rise to the Hitchin map. In view of the previous section, it is now clear how to define the morphism – the difference is that one has to take care of differential forms. Naturally, the structure of the proof is similar to that in [Mar10], though the context is slightly different, as discussed.

What we are looking for is an L_∞ -morphism

$$h_\infty : \mathcal{C}^\bullet = \bigoplus_{p+r=\bullet} A^{0,p}(\text{ad}\mathbf{P} \otimes \Omega^r) \longrightarrow \bigoplus_{m_i \in E} A^{0,\bullet}(S^{m_i} \Omega_X^1)$$

such that $Def(h_\infty) = H$.

Proposition 4.1 *Let m be an exponent of G and $p \in \mathbb{C}[\mathfrak{g}]^G$ homogeneous with $\deg p = d = m + 1$. Let pr_0 be the projection $pr_0 : S^k(\mathcal{C}^\bullet) \rightarrow S^k(A^{0,\bullet}(\text{ad}\mathbf{P} \otimes \Omega_X^1))$, $k \geq 1$. Then the maps defined by*

$$h_k^m = \mathcal{P}_{dk}(p)_\theta \circ pr_0 : S^k(\mathcal{C}^\bullet[1]) \longrightarrow A^{0,\bullet}(S^m \Omega),$$

that is,

$$s_1^{p_1,1} \cdot s_2^{p_2,1} \cdot \dots \cdot s_k^{p_k,1} \mapsto \mathcal{P}_{dk}(p)(s_1^{p_1,1}, \dots, s_k^{p_k,1}; \theta)$$

and zero otherwise, induce an L_∞ -morphism

$$h_\infty^m : \bigoplus_{p+r=\bullet} A^{0,p}(\text{ad}\mathbf{P} \otimes \Omega^r) \rightarrow A^{0,\bullet}(S^m \Omega).$$

Proof:

We check the conditions 3,4. The differentials are $\bar{\partial} + \text{ad}\theta$ and $\bar{\partial}$, so 3 is equivalent to $\mathcal{P}_{d,1}(p)([\theta, s]; \theta) = 0$, which is guaranteed by Lemma 3.1. Next assume $k \geq 2$. Since by definition h_k^m factors through pr_0 , both sides of 4 are identically zero, except possibly for *two* cases. Case 1: when evaluated on $S^k(A^{0,\bullet}(\text{ad}P \otimes \Omega^1))$ and Case 2: when evaluated on $A^{0,\bullet}(\text{ad}\mathbf{P}) \cdot S^{k-1}(A^{0,\bullet}(\text{ad}\mathbf{P} \otimes \Omega^1))$. Notice that we are working with $\mathcal{C}^\bullet[1]$, so the degree of a homogeneous element in $A^{0,\bullet}(\text{ad}P \otimes \Omega_X^1)$ is \bullet ! We start with Case 1, evaluating on decomposable homogeneous elements $s_i = \alpha_i \otimes X_i$, $i = 1 \dots k$. Since $[s_{\sigma_1}, s_{\sigma_2}]$ and $\text{ad}\theta(s_{\sigma_1})$ belong to $A^{0,\bullet}(\text{ad}P \otimes \Omega^2)$, they do not contribute to the left side of 5. And since $\sum_{\sigma \in S(1, k-1)} \epsilon(\sigma) \bar{\partial}(\alpha_{\sigma_1}) \wedge \dots \wedge \alpha_{\sigma_k} = \bar{\partial}(\alpha_1 \wedge \dots \wedge \alpha_k)$, the left side of 5 gives

$$-\bar{\partial}(\alpha_1 \wedge \dots \wedge \alpha_k) \otimes \mathcal{P}_{dk}(p)(X_1, \dots, X_k; \theta) = \hat{q}_1 \circ h_k^m(s_1 \cdot \dots \cdot s_k).$$

Next we proceed with Case 2. We take decomposable homogeneous elements $s_i = \alpha_i \otimes X_i$, $s_1 \in A^{0,\bullet}(\text{ad}P)$, $s_2, \dots, s_k \in A^{0,\bullet}(\text{ad}P \otimes \Omega)$ and evaluate 5 on their product. The right hand side is zero, so we just compute the left side. The terms with $\sigma_1 \neq 1$ are identically zero, and $\sigma_1 = 1$ means $\sigma = \text{id}$, so we obtain

$$h_k^m([\theta, s_1] \cdot s_2 \cdot \dots \cdot s_k) = (-1)^{\deg s_1} \alpha_1 \wedge \dots \wedge \alpha_k \mathcal{P}_{dk}(p)([X_1, \theta], X_2, \dots, X_k; \theta).$$

The non-vanishing contributions from $h_{k-1}^m \circ Q_k^{k-1}$ in 5 correspond to $(2, k-2)$ shuffles for which $\sigma_1 = 1$, so the summation is in fact over $(1, k-2)$ shuffles and we have

$$h_{k-1}^m \left(\sum_{\sigma \in S(1, k-2)} (-1)^{\deg s_1} \epsilon(\sigma) [s_1, s_{\sigma_1}] \cdot \dots \cdot s_{\sigma_{k-1}} \right).$$

Due to the shift, the Koszul sign is traded for reordering the forms and we are left with

$$(-1)^{\deg s_1} \alpha_1 \wedge \dots \wedge \alpha_k \sum_{\sigma \in S(1, k-1)} \mathcal{P}_{d, k-1}(p)([X_1, X_{\sigma_1}], \dots, X_{\sigma_{k-1}}; \theta).$$

Then the sum of the two terms is zero by Corollary 3.1, as before. \square

Remark: We did not use anywhere the fact that our Higgs bundles are Ω_X^1 -valued: exactly the same calculation works for any coefficient vector bundle L .

Proposition 4.2 *The maps $h_\infty^{m_i}$, $m_i \in E$ induce an L_∞ -morphism*

$$h_\infty : \bigoplus_{p+q=\bullet} A^{0,p}(\text{ad}\mathbf{P} \otimes \Omega_X^q) \longrightarrow \bigoplus_{m_i \in E} A^{0,\bullet}(S^{m_i} \Omega_X^1).$$

The natural transformation of deformation functors induced by h_∞ is the Hitchin map: $\text{Def}(h_\infty) = H$.

Proof:

Suppose $s = (s', s'') \in \text{MC}(A)$, $A \in \text{Art}_{\mathbb{C}}$. Then, by 6 $\text{Def}(h_\infty)(s) = \sum_{d=1}^{\infty} \frac{1}{d!} h_\infty(s^d)$, but by the

formula 7 this is $\oplus_i p_i(\theta + s') - p_i(\theta)$. On the other hand, the latter is exactly $\chi(P_A, \theta_A) - \chi(P, \theta)$. \square

The existence of an L_∞ -morphism inducing H allows us to generalise the result on obstructions from [Mar10].

Corollary 4.1 *The obstructions to deforming a Higgs bundle (\mathbf{P}, θ) are contained in $\ker H^2(h_1)$, where*

$$H^2(h_1) : \mathbb{H}^2(\mathcal{C}^\bullet) \longrightarrow \bigoplus_{m_i \in E} H^1(X, S^{m_i} \Omega_X^1)$$

is the map on \mathbb{H}^2 induced by the linear part, h_1 , of h_∞ . Explicitly, in terms of the Dolbeault resolution:

$$H^2(h_1) : \{s^{j, 2-j}\} \mapsto \{\mathcal{P}_{d1}(p_i)([s^{1,1}, \theta]; \theta)\}_i.$$

Proof:

The “linear part” of an L_∞ -morphism is a morphism of obstruction theories. But since $\bigoplus A^{0, \bullet}(S^{m_i} \Omega_X^1)$ is abelian, its obstruction space is trivial (this is immediate from the definition of obstruction space). \square

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