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ZERO-SUM FLOWS IN DESIGNS

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Abstract

Let D be a t-(v,k,λ) design and let $N_i(D)$, for $1 \leq i \leq t$, be the higher incidence matrix of D, a $(0,1)$-matrix of size $\binom{v}{i} \times b$, where b is the number of blocks of D. A *zero-sum flow* of D is a nowhere-zero real vector in the null space of $N_1(D)$. A *zero-sum k-flow* of D is a zero-sum flow with values in $\{\pm 1, \ldots, \pm(k-1)\}$. In this paper we show that every non-symmetric design admits an integral zero-sum flow, and consequently we conjecture that every non-symmetric design admits a zero-sum 5-flow. Similarly, the definition of zero-sum flow can be extended to $N_i(D)$, $1 \leq i \leq t$. Let $D = t$-$(v,k,\binom{v-t}{k-t})$ be the complete design. We conjecture that $N_t(D)$ admits a zero-sum 3-flow and prove this conjecture for $t = 2$.

1 Introduction

Let G be a directed graph. A *k-flow* of G is an assignment of integers with maximum absolute value $k-1$ to each edge of G such that for any vertex of G, the sum of the labels on incoming edges is equal to that of the labels on outgoing edges. A *nowhere-zero k-flow* is a k-flow with no zeros. A celebrated conjecture of Tutte is:

Conjecture(Tutte's 5-flow Conjecture [8]). Every bridgeless graph has a nowhere-zero 5-flow.

Clearly, in the language of linear algebra a nowhere-zero flow for a directed graph G is a vector of the null space of the incidence matrix of G with no zero entry. Motivated with this concept in [1, 2], we defined the zero-sum flow for the simple graphs. For an undirected graph G, a *zero-sum flow* is a nowhere-zero element in the null space of the incidence matrix of G. In this paper we would like to extend this concept to hypergraphs and in particular to t-designs.

A *hypergraph* H is a pair $H = (X, \mathcal{E})$, where X is a set of elements called *vertices* and $\mathcal{E}$ is a set of nonempty subsets of X called *hyperedges*.

Let $X = \{x_1, \ldots, x_n\}$ and $\mathcal{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$. Then, the incidence matrix of the hypergraph $H = (X, \mathcal{E})$ is an $n \times m$ $(0, 1)$-matrix $N = [n_{ij}]$, where n_{ij} is 1 if $x_i \in \mathcal{E}_j$ and 0 otherwise.

A *zero-sum flow* for a hypergraph H is a nowhere-zero real vector in the null space of the incidence matrix of H. An *integral zero-sum flow* is a flow with integer entries.

A *t-(v, k, λ) design D* (briefly, t-design), is a pair $(X, \mathcal{B})$, where X is a v-set of points and $\mathcal{B}$ is a collection of k-subsets of X (blocks) with the property that every t-subset of X is contained in exactly λ blocks. Traditionally, in the case of 2-designs, the number of blocks and the frequency of occurrences of points in blocks are denoted by b and r, respectively. A 2-design is called *symmetric* if $b = v$. A 2-design is called *resolvable* if there exists a partition of the set of blocks into parallel classes, each of which in turn partitions X. A 2-$(v, 3, 1)$ design is called a *Steiner Triple System* of order v, STS(v). A t-(v, k, λ) design is called *complete* if it contains all the k-subsets of X.

Let D be a t-(v, k, λ) design with blocks $\mathcal{B}$ and let I and J be an i-subset and a j-subset of $\{1, \ldots, v\}$, respectively ($i, j < t$). Then the *derive design* of D with respect to I, D_I, is defined to be all the blocks of $\mathcal{B}$ containing I and then removing I from all those blocks. Then D_I is a $(t-i)$-$(v-i, k-i, \lambda')$ design. Also the *residual design* of D with respect to J, D^J, consists of all blocks $B \in \mathcal{B}$ such that $B \cap J = \varnothing$. Then D^J is a $(t-j)$-$(v-j, k, \lambda'')$ design. For $u \in \mathrm{null}(N_t(D))$ and D' any subdesign of D, let $u|_{D'}$ be a vector obtained from u by restricting u to D'.

Since every t-design is a hypergraph, therefore the definition of zero-sum flows makes sense for designs too. Moreover, for every i, $2 \le i \le t$, if $N_i(D)$ denotes the higher incidence matrix of D (i.e., the inclusion matrix of i-subsets versus blocks), then one can extend the definition of zero-sum flow to $N_i(D)$, namely every zero-sum flow is an element of the $\mathrm{null}(N_i(D))$ (i.e. null space of $N_i(D)$). In the case of complete design D, $N_t(D)$ is denoted by $W_{t,k}(v)$ (following R. Wilson). If there is no risk of ambiguity, we drop v. In the case of complete design we always assume that $v-t \geq k \geq t$. In the literature of combinatorial designs [7], the elements of the null space of $W_{tk}(v)$ is called $T[N](t,k,v)$ *trades*. Therefore, the zero-sum flows are trades with non-zero components.

Now, more definitions from design theory. The *complement* of a t-(v,k,λ) design D, D^c is a design, where its blocks are complement of blocks of D with respect to the point set X. A *large set* of t-(v,k,λ) designs of size N, denoted by $\mathrm{LS}[N](t,k,v)$, is a partition of the complete design into N disjoint t-$(v,k,\binom{v-t}{k-t}/N)$ designs.

2 Some preliminary results

In this section we state some results which will be used in the proof of our theorems.

Lemma 1 *Let D be a t-(v,k,λ) design with b blocks. Then* $\mathrm{null}(N_t(D)) \underset{\neq}{\subset} \mathrm{null}(N_{t-1}(D))$.

Proof. By Proposition 2 of [11], we have

$$W_{(t-1)t}N_t(D) = (k-t+1)N_{t-1}(D).$$

On the other hand, since $N_t(D)$ is a full rank matrix [9, p.789], hence

$$\dim(\mathrm{null}(N_t(D))) < \dim(\mathrm{null}(N_{t-1}(D))),$$

and the proof is complete. □

Before stating the next lemma we need a notation. Let $v_i = (v_{i1},\ldots,v_{in})$, $1 \leq i \leq r$, be a set of real vectors. Then we have

$$\max_{1\leq i\leq r} v_i := (\max_{1\leq i\leq r} v_{i1},\ldots,\max_{1\leq i\leq r} v_{in}).$$

Lemma 2 *Let $Y \subseteq \{1,\ldots,v\}$ and $N_Y(D)$ be the row corresponding to the subset Y of the matrix $N_{|Y|}(D)$. Then*

$$N_Y(D^c) = j - max_{a\in Y}(N_{\{a\}}(D)) = j + \sum_{L\subseteq Y, 1\leq |L|\leq |Y|} (-1)^{|L|}N_L(D),$$

where j is the all 1 vector.

Proof. The proof follows by a simple use of Inclusion-Exclusion Principle. □

Corollary 1 *For every i, $1 \le i \le t$,* $\mathrm{null}(N_i(D)) = \mathrm{null}(N_i(D^c))$.

Proof. Let $u \in \mathrm{null}(N_i(D))$ and Y be an i-subset of $\{1, \ldots, v\}$. Then by Lemma 2, we have

$$N_Y(D^c)u = ju + \sum_{L \subseteq Y, 1 \le |L| \le |Y|} (-1)^{|L|} N_L(D)u \ .$$

Now, by Lemma 1 it follows that $u \in \mathrm{null}(N_i(D^c))$. Note that since the sum of all the rows of $N_i(D)$ is a multiple of j, therefore $N_i(D)u = 0$ implies that $ju = 0$. □

Corollary 2 *Let D be a t-(v, k, λ) design and let I and J be an i-subset and a j-subset of $\{1, \ldots, v\}$, respectively. If $u \in \mathrm{null}(N_t(D))$, then the following hold:*

(i) $u|_{D_I} \in \mathrm{null}(N_{t-i}(D_I))$

(ii) $u|_{D^J} \in \mathrm{null}(N_{t-j}(D^J))$.

Proof. Clearly, $u|_{D_I} \in \mathrm{null}(N_{t-i}(D_I))$. On the other hand $N_{t-j}(D^J) = N_{t-j}((D^c)_J)$. Now, by Corollary 1, $u \in \mathrm{null}(N_t(D))$ implies that $u \in \mathrm{null}(N_t(D^c))$. By Part (i) we have

$$u|_{(D^c)_J} \in \mathrm{null}(N_{t-j}(D^c)_J) = \mathrm{null}(N_{t-j}(D^J)).$$

But $u|_{(D^c)_J} = u|_{D^J}$ and the proof is complete. □

3 Zero-sum flows for non-symmetric designs

In this section we would like to prove that the incidence matrix of every non-symmetric 2-design admits an integral zero-sum flow and then we propose a conjecture similar to Tutte's 5-flow Conjecture for 2-designs.

Lemma 3 *Let A be an $m \times n$ rational matrix. If there exists a nowhere-zero real vector in* $\mathrm{null}(A)$*, then there exists a nowhere-zero integer vector in* $\mathrm{null}(A)$.

Proof. Let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ be an n-vector where 1 is in the ith position. First we claim that if F is an infinite field and B is an $m \times n$ matrix over F, then B has a nowhere-zero vector in its null space if and only if no e_i, $1 \le i \le m$, is contained in the row space of B.

One side is clear. For the other side suppose that no $e_i, 1 \le i \le m$, is contained in the row space of B. Thus there exists a vector $X_i \in \mathrm{null}(B)$ such that e_i and X_i are not orthogonal. This implies that the ith component of X_i is non-zero. Let W_i be the set of all vectors in $\mathrm{null}(B)$ whose ith components are zero. If $\mathrm{null}(B) \subseteq \cup_{i=1}^{n} W_i$, then by [5, p.283], $\mathrm{null}(B) \subseteq W_j$, for some j, which is a contradiction. Thus there is a vector in $\mathrm{null}(B)\setminus \cup_{i=1}^{n} W_i$, which is clearly nowhere-zero and the claim is proved.

Now, by assumption there exists a nowhere-zero real vector in $\mathrm{null}(A)$ and so no e_i, $1 \le i \le m$, is a linear combination of the rows of A over that rational number. Thus by the claim, there is a rational nowhere-zero vector in $\mathrm{null}(A)$ and the proof is complete. $\square$

Theorem 1. *The incidence matrix of every non-symmetric* 2-(v, k, λ) *design admits an integral zero-sum flow.*

Proof. Let D be a non-symmetric 2-(v, k, λ) design with the incidence matrix N. First we show that every column of N is a linear combination of the other columns of N. With no loss of generality we prove that the last column of N is a linear combination of the other columns. By a suitable ordering of the elements of $\{1, \dots, v\}$, we can assume that the last column of N is $[1, \dots, 1, 0, \dots, 0]^T$, in which the number of 1 is k. Now, we remove the last column of N and call the remaining matrix by M. We have the following equality:

$$L = MM^T = \begin{bmatrix} (r-\lambda)I_k + (\lambda-1)J_k & \lambda J_{k,v-k} \\ \lambda J_{v-k,k} & (r-\lambda)I_{v-k} + \lambda J_{v-k} \end{bmatrix},$$

where $J_{p,q}$ is the $p \times q$ all 1 matrix. For simplicity we denote $J_{p,p}$ by J_p.

Let L_i be the ith row of L, $1 \le i \le v$. Now, by some elementary row operations, we replace every L_j, $k+1 \le j \le v$, with $L_j + a\sum_{i=1}^{k} L_i$ where $a = \frac{-\lambda}{(r-1)+(k-1)(\lambda-1)}$. Consequently the resulting matrix, L' will be of the following form:

$$L' = \begin{bmatrix} (r-\lambda)I_k + (\lambda-1)J_k & \lambda J_{k,v-k} \\ 0 & (r-\lambda)I_{v-k} + \lambda(1+ak)J_{v-k} \end{bmatrix}.$$

Thus

$$\begin{aligned}\det(L') &= \det((r-\lambda)I_k + (\lambda-1)J_k)\det((r-\lambda)I_{v-k} + \lambda(1+ak)J_{v-k}) \\ &= (r-\lambda)^{(k-1)}((r-\lambda)+(\lambda-1)k)((r-\lambda)^{(v-k-1)}[(r-\lambda)+\lambda(1+ak)(v-k)])\end{aligned}$$

Since $r > \lambda$, we have

$$(r-\lambda)^{k-1}((r-\lambda)+(\lambda-1)k)(r-\lambda)^{v-k-1} > 0.$$

Now, we claim that $(r-\lambda)+\lambda(1+ak)(v-k) \neq 0$. To justify the claim, replace a with $\frac{-\lambda}{(r-1)+(k-1)(\lambda-1)}$ and obtain the following expression:

$$\mathbf{h} := (v-k)[\lambda(r-1)+(\lambda^2-\lambda)(k-1)-\lambda^2 k] + r(r-1)$$

$$+r(\lambda-1)(k-1) - \lambda(r-1) - \lambda(\lambda-1)(k-1)$$

$$= ((v-1)\lambda + r)(r-k-\lambda) + \lambda k^2.$$

It suffices to show that $\mathbf{h}$ is positive. To show this, we replace $\lambda(v-1)$ by $r(k-1)$ and we obtain

$$\mathbf{h} = k(r-k)(r-\lambda).$$

Since $r > k$ (by Fisher's inequality, [10, p.222]), we have $\mathbf{h} > 0$. Therefore,

$$\mathrm{rank}(L) = \mathrm{rank}(M) = \mathrm{rank}(N) = v.$$

This implies that every column of N is a linear combination of the remaining columns of N. Hence for every i, $1 \le i \le b$, there is a vector in the null(N) whose ith component is nonzero. Now assume that

$$\mathrm{U}_i = \{\, x \in \mathrm{null}(N) \,|\, x_i = 0 \,\} \qquad 1 \le i \le b.$$

If $\mathrm{null}(N) \subseteq \bigcup_{i=1}^{b} \mathrm{U}_i$, then by [5, p.283], $\mathrm{null}(N) \subseteq \mathrm{U}_j$, for some $j, 1 \le j \le b$ which is a contradiction. Consequently, there exists an element $\alpha \in \mathrm{null}(N) \setminus \bigcup_{i=1}^{b} \mathrm{U}_i$ and by Lemma 3 we are done. □

Now, we are in a position to state a conjecture similar to Tutte's 5-flow Conjecture (1954, [8]) for 2-designs.

Conjecture 1. *The incidence matrix of every non-symmetric design admits a zero-sum* 5*-flow.*

Remark 1. The incidence matrix of every resolvable 2-design admits a zero-sum 3-flow. Moreover, if a complete design D has a large set, then $N_i(D)$, $1 \le i \le t$, admits a zero-sum 3-flow. A computer search shows that the incidence matrix of every STS(v), $7 < v \le 15$, admits a zero-sum 3-flow.

Conjecture 2. *The incidence matrix of every* STS(v)*,* $v > 7$*, admits a zero-sum* 3*-flow.*

4 Zero-sum flows for $W_{t,k}(v)$

First we recall that for a t-(v, k, λ) design D, by a zero-sum flow of $N_i(D)$ $(1 \le i \le t)$, we mean an element of the null$(N_i(D))$. Also recall that if D is a complete design with v points and block size k, $N_t(D)$ is denoted by $W_{t,k}(v)$. In this section we focus our attention to zero-sum flows for $W_{t,k}(v)$.

A celebrated theorem due to Baranyai [4] states that if $k|v$, then the complete design 1-$(v, k, \binom{v-1}{k-1})$ is resolvable. Along this line the following result due to Hartman [6] is of great interest.

Lemma 4 *If N is a natural number and N divides both $\binom{v}{k}$ and $\binom{v-1}{k-1}$, then* $\mathrm{LS}[N](1, k, v)$ *exists.*

We note that in this lemma by letting $N = \binom{v-1}{k-1}$, Baranyai's Result follows.

Now, we propose the following conjecture.

Conjecture 3 *The matrix $W_{t,k}(v), v \neq t + k$, admits a zero-sum 3-flow.*

Remark 2. A celebrated conjecture due to A. Hartman (Halving Conjecture) states that:

$$LS[2](t, k, v) \text{ exists if and only if } 2 \,|\, \binom{v-i}{k-i}, \text{ for every } i,\ 0 \leq i \leq t.$$

In the language of zero-sum flow the Hartman's Conjecture is equivalent to the existence of zero-sum 2-flow for the matrix $W_{t,k}(v)$.

Theorem 2. *Let t be a positive integer. If the Conjecture 3 is true for $k = t + 1$, then it is true for every v, k, t, with $v \geq k + t + 1$ and $k \geq t + 1$.*

Proof. For $t = 1$ we have $k \geq 2$, and so $\binom{v}{k} \not| v$. Therefore, the equality $v\binom{v-1}{k-1} = k\binom{v}{k}$ implies that $\binom{v}{k}$ and $\binom{v-1}{k-1}$ are not coprime. Now, by Lemma 4 we find a zero-sum 3-flow. Indeed if the number of partitions is even, then it has a zero-sum 2-flow and otherwise, a zero-sum 3-flow. By assumption we may assume that $t \geq 2$ and $k > t + 1$. We prove the theorem by induction on v. If $v = k + t + 1$, then by Corollary 1, we have

$$\mathrm{null}(W_{t,k}(k + t + 1)) = \mathrm{null}(W_{t,t+1}(k + t + 1)),$$

and by assumption $W_{t,k}(k + t + 1)$ admits a zero-sum 3-flow. Thus suppose that $v \geq k + t + 2$. We have $k - 1 \geq t + 1$ and $v - 1 \geq (k - 1) + t + 1$. So by induction hypothesis $W_{t,k-1}(v - 1)$ admits a zero-sum 3-flow, say u. By Lemma 1, $u \in \mathrm{null}(W_{t-1,k-1}(v - 1))$. On the other hand $v - 1 \geq k + t + 1$. Hence by induction hypothesis, there exists a zero-sum flow, say y, for $W_{t,k}(v - 1)$. By a suitable ordering of the rows and the columns we obtain the following form of $W_{t,k}(v)$ as follows:

$$W_{t,k}(v) = \begin{bmatrix} W_{t-1,k-1}(v-1) & 0 \\ W_{t,k-1}(v-1) & W_{t,k}(v-1) \end{bmatrix}.$$

Clearly, the vector $\begin{bmatrix} u \\ y \end{bmatrix}$ is a zero-sum 3-flow for $W_{t,k}(v)$ and the proof is complete. □

Corollary 3 *Conjecture 3, is true for $t = 2$.*

Proof. By Lemma 2, it is sufficient to prove that for every v, $v \geq 6$, $W_{2,3}(v)$ admits a zero-sum 3-flow. For $v \geq 9$ every 2-$(v, 3, v-2)$, see [7, p.98] complete design has a large set and therefore admits a zero-sum 3-flow. In the case of $v = 7$, $k = 3$, the complete design is partitioned into two Fano planes and a 2-$(7, 3, 3)$ design [3]. Thus it admits a zero-sum 3-flow. For the case $v = 8$ we provide a zero-sum 3-flow explicitly as follows:

123	124	125	126	127	128	134	135	136	137	138	145	146	147
-2	2	2	-2	2	-2	-2	1	2	-1	2	-2	-1	2

148	156	157	158	167	168	178	234	235	236	237	238	245	345
1	1	-1	-1	-1	1	-1	2	-1	-2	1	2	-2	1

247	248	256	257	258	267	268	278	246	346	347	348	356	357
-2	-1	1	-1	1	1	1	-1	1	1	-1	-1	-1	1

358	367	368	378	456	457	458	467	468	478	567	568	578	678
-1	1	-1	-1	1	1	1	-1	-1	1	-1	-1	1	1

Remark 3. We would like to close the paper with a report on the existence of zero-sum flows for small t-designs with $k = t + 1$ and small t.

• For $t = 2$, the 2-$(6, 3, 4)$ design is partitioned into two copies of 2-$(6, 3, 2)$ designs [7], therefore $W_{2,3}(6)$ admits a zero-sum 2-flows.

• For $t = 3$, the 3-$(8, 4, 5)$ design is partitioned into two copies of 3-$(8, 4, 1)$ designs and a copy of 3-$(8, 5, 3)$ design, [3]. Thus $W_{3,4}(8)$ admits a nowhere-zero 3-flow.

• For $t = 4$, no result is known.

• For $t = 5$, the 5-$(12, 6, 7)$ design is partitioned into 2 copies of 5-$(12, 6, 1)$ designs (the famous small Witt design) and a copy of 5-$(12, 6, 5)$ design [3]. Therefore, there is a zero-sum 4-flow for $W_{5,6}(12)$.

• For $t = 6$, the 6-$(14, 7, 8)$ design is partitioned into two copies of 6-$(14, 7, 4)$ designs [7] and hence admits a zero-sum 2-flow.

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