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HYPERCYCLIC ABELIAN SEMIGROUPS OF MATRICES ON $\mathbb{C}^n$

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Abstract

We give a complete characterization of existence of dense orbit for any abelian semigroup of matrices on $\mathbb{C}^n$. For finitely generated semigroups, this characterization is explicit and is used to determine the minimal number of matrices in normal form over $\mathbb{C}$ which forms a hypercyclic abelian semigroup on $\mathbb{C}^n$. In particular, we show that no abelian semigroup generated by n matrices on $\mathbb{C}^n$ can be hypercyclic.

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1. Introduction

Let $M_n(\mathbb{C})$ be the set of all square matrices over $\mathbb{C}$ of order $n \geq 1$ and by $\mathrm{GL}(n, \mathbb{C})$ the group of invertible matrices of $M_n(\mathbb{C})$. Let G be an abelian sub-semigroup of $M_n(\mathbb{C})$. For a vector $v \in \mathbb{C}^n$, we consider the orbit of G through v: $G(v) = \{Av : \ A \in G\} \subset \mathbb{C}^n$. A subset $E \subset \mathbb{C}^n$ is called *G-invariant* if $A(E) \subset E$ for any $A \in G$. The orbit $G(v) \subset \mathbb{C}^n$ is *dense* (resp. *locally dense*) in $\mathbb{C}^n$ if $\overline{G(v)} = \mathbb{C}^n$ (resp. $\overset{\circ}{\overline{G(v)}} \neq \emptyset$), where $\overline{E}$ (resp. $\overset{\circ}{E}$) denotes the closure (resp. the interior) of a subset $E \subset \mathbb{C}^n$. The semigroup G is called *hypercyclic* if there exists a vector $v \in \mathbb{C}^n$ such that $G(v)$ is dense in $\mathbb{C}^n$. Hypercyclic is also called topologically transitive. We refer the reader to the recent book [3] and [7] for a thorough account on hypercyclicity.

So, the question to investigate is the following: When can an abelian sub-semigroup of $M_n(\mathbb{C})$ be hypercyclic?

The main purpose of this paper is twofold: firstly, we give a general result answering the above question for any abelian *sub-semigroup* of $M_n(\mathbb{C})$. Notice that in [1], the authors answer this question for any abelian *subgroup* of $\mathrm{GL}(n, \mathbb{C})$, so this paper can be viewed as a continuation of that work. Secondly, we prove that the minimal number of matrices in normal form in $\mathcal{K}_{\eta,r}(\mathbb{C})$ (see definition below) required to form a hypercyclic abelian semigroup in $\mathbb{C}^n$ is $2n - r + 1$ (see Corollary 1.6). In particular, $n+1$ is the minimal number of matrices on $\mathbb{C}^n$ required to form a hypercyclic abelian semigroup on $\mathbb{C}^n$, this was recently showed in [2] answering a question raised by Feldman in ([5], Section 6). Notice that in [5], Feldman showed that in $\mathbb{C}^n$ there exist a hypercyclic semigroup generated by $(n+1)$-tuple of diagonal matrices on $\mathbb{C}^n$ and that no semigroup generated by n-tuple of diagonalizable matrices on $\mathbb{C}^n$ can be hypercyclic.

To state our main results, we need to introduce the following notations and definitions for the sequel.

Write $\mathbb{N}_0 = \mathbb{N}\backslash\{0\}$. Let $n \in \mathbb{N}_0$ fixed. For each $m = 1, 2, \dots, n$, denote by:

- $\mathbb{T}_m(\mathbb{C})$ the set of matrices over $\mathbb{C}$ of the form:

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix} \qquad (1)$$

- $\mathbb{T}_m^*(\mathbb{C}) = \mathbb{T}_m(\mathbb{C}) \cap \mathrm{GL}(m, \mathbb{C})$ the group of matrices of the form (1) with $\mu \neq 0$.

Let $r \in \mathbb{N}_0$ and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$, $\sum\limits_{i=1}^{r} n_i = n$. Denote by:

- $\mathcal{K}_{\eta,r}(\mathbb{C}) := \mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \mathbb{T}_{n_r}(\mathbb{C})$.
- $\mathcal{K}_{\eta,r}^*(\mathbb{C}) := \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \mathrm{GL}(n, \mathbb{C})$.

In particular:

If $r = 1$, $\mathcal{K}_{\eta,r}(\mathbb{C}) = \mathbb{T}_n(\mathbb{C})$ and $\mathcal{K}_{\eta,r}^*(\mathbb{C}) = \mathbb{T}_n^*(\mathbb{C})$. If $r = n$, $\mathcal{K}_{\eta,r}(\mathbb{C})$ is the group of diagonal matrices on $\mathbb{C}^n$.

• $G^* = G \cap \mathrm{GL}(n, \mathbb{C})$.
Consider the matrix exponential map
$\exp: \ M_n(\mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$, set $\exp(M) = e^M$.
It is proved (see Proposition 2.2) that for every abelian sub-semigroup G of $M_n(\mathbb{C})$ there exist a $P \in \mathrm{GL}(n, \mathbb{C})$ and a partition $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$ of n for some $1 \leq r \leq n$ such that $P^{-1}GP$ is an abelian sub-semigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$. We call $P^{-1}GP$ the *normal form* of G. For such a choice of matrix P, we let:
• $\mathrm{g} = \exp^{-1}(G) \cap \left(P\left(\mathcal{K}_{\eta,r}(\mathbb{C})\right)P^{-1}\right)$.
• $\mathrm{g}_u = \{Bu : \ B \in \mathrm{g}\}, \ u \in \mathbb{C}^n$.
Throughout the paper, for a vector $v \in \mathbb{C}^n$, we will be denoting by v^T the transpose of v.
• $\mathcal{B}_0 = (e_1, \dots, e_n)$ the canonical basis of $\mathbb{C}^n$ and I_n the identity matrix on $\mathbb{C}^n$.
• $u_0 = [e_{1,1}, \dots, e_{r,1}]^T \in \mathbb{C}^n$ where $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$, $k = 1, \dots, r$.
• $e^{(k)} = [e_1^{(k)}, \dots, e_r^{(k)}] \in \mathbb{C}^n$ where for every $j = 1, \dots, r$,

$$e_j^{(k)} = \begin{cases} 0 \in \mathbb{C}^{n_j}, & \text{if } j \neq k \\ e_{k,1}, & \text{if } j = k \end{cases}$$

An equivalent formulation is
$e^{(1)} = e_1, \dots, e^{(k)} = e_{\ell_k}$ where $\ell_k := \sum_{j=1}^{k-1} n_j + 1, \quad k = 2, \dots, r.$
• Finally, take $v_0 = Pu_0$.

Our principal results can now be stated as follows:

Theorem 1.1. *Let G be an abelian sub-semigroup of $M_n(\mathbb{C})$. The following are equivalent:*

(i) *G is hypercyclic*
(ii) *The orbit $G(v_0)$ is dense in $\mathbb{C}^n$*
(iii) *g_{v_0} is an additif sub-semigroup dense in $\mathbb{C}^n$*

Remark 1. If all matrices of $G \backslash I_n$ are non invertible (i.e. $G^* = \{I_n\}$) then G is not hypercyclic by Proposition 4.1.

Theorem 1.2. *Let G be an abelian sub-semigroup of $M_n(\mathbb{C})$ with normal form $P^{-1}GP$ in $\mathcal{K}_{\eta,r}(\mathbb{C})$ for some $r \in \mathbb{N}_0$ and let $B_1, \dots, B_p \in \mathrm{g}$ such that $e^{B_1}, \dots, e^{B_p}$ generate G^*. The following are equivalent:*

(i) *G is hypercyclic*
(ii) *the orbit $G(v_0)$ is dense in $\mathbb{C}^n$*
(iii) $\mathrm{g}_{v_0} = \sum_{k=1}^{p} \mathbb{N}B_k v_0 + \sum_{k=1}^{r} 2i\pi\mathbb{Z}Pe^{(k)}$ *is an additif sub-semigroup dense in $\mathbb{C}^n$.*

Corollary 1.3. *If G is an abelian semigroup (with normal form in $\mathcal{K}_{\eta,r}(\mathbb{C})$) generated by $(2n-r)$ matrices of $M_n(\mathbb{C})$, it has no dense orbit.*

Corollary 1.4. ([2]). *If G is an abelian semigroup generated by n matrices of $M_n(\mathbb{C})$, it has no dense orbit.*

Theorem 1.5. *For every $n \in \mathbb{N}_0$, $r = 1, \dots, n$, and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$, there exist $(2n-r+1)$ matrices in $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ that generate an hypercyclic abelian semigroup.*

As a consequence, from Theorem 1.5 and Corollary 1.3, we obtain the following corollary.

Corollary 1.6. *For every $n \in \mathbb{N}_0$, $r = 1, \dots, n$, and $\eta = (n_1, \dots, n_r) \in \mathbb{N}_0^r$, the minimum number of matrices of $M_n(\mathbb{C})$ with normal form in $\mathcal{K}_{\eta,r}(\mathbb{C})$ that generate an hypercyclic abelian semigroup is $2n - r + 1$.*

In particular:
For $r = n$, we obtain Feldman's theorem:

Corollary 1.7. ([5]) *The minimum number of diagonalizable matrices of $M_n(\mathbb{C})$ that generate an hypercyclic abelian semigroup is $n + 1$.*

For $r = 1$, we obtain the following corollary:

Corollary 1.8. *The minimum number of matrices of $\mathbb{T}_n(\mathbb{C})$ that generate an hypercyclic abelian semigroup is $2n$.*

This paper is organized as follows: In Section 2 we introduce the normal form of an abelian sub-semigroup of $M_n(\mathbb{C})$. Section 3 is devoted to the characterization of abelian sub-semigroups of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ with a dense orbit. The proof of Theorem 1.1 is done in Section 4. In Section 5, we prove Theorem 1.2, Corollaries 1.3 and 1.4. Theorem 1.5 is proved in Section 6. In Section 7, we give some examples in the cases $n = 1, 2$.

2. Normal form of abelian sub-semigroups of $M_n(\mathbb{C})$

Recall first the following proposition.

Proposition 2.1 ([1], Proposition 6.1). *Let G be an abelian* subgroup *of $GL(n, \mathbb{C})$. Then there exists a $P \in GL(n, \mathbb{C})$ such that $P^{-1}GP$ is an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$, for some $\eta \in \mathbb{N}_0^r$ and $r \in \{1, \dots, n\}$.*

The analogous of Proposition 2.1 for sub-semigroup is the following:

Proposition 2.2. *Let G be an abelian sub-semigroup of $M_n(\mathbb{C})$. Then there exists a $P \in GL(n, \mathbb{C})$ such that $P^{-1}GP$ is an abelian subsemigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$, for some $\eta \in \mathbb{N}_0^r$ and $r \in \{1, \dots, n\}$.*

Proof. For every $A \in G$, there exists $\lambda_A \in \mathbb{C}$ so that $(A - \lambda_A I_n) \in \mathrm{GL}(n, \mathbb{C})$ (it suffices to take λ_A not to be an eigenvalue of A). Write $\widehat{L}$ to be the group generated by $L := \{A - \lambda_A I_n : \ A \in G\}$. Then $\widehat{L}$ is an abelian subgroup of $\mathrm{GL}(n, \mathbb{C})$ and by Proposition 2.1, there exists a $P \in \mathrm{GL}(n, \mathbb{C})$ such that $P^{-1}\widehat{L}P \subset \mathcal{K}^*_{\eta,r}(\mathbb{C})$, for some $\eta \in \mathbb{N}_0^r$ and $r \in \{1, \dots, n\}$. As

$$P^{-1}LP = \left\{P^{-1}AP - \lambda_A I_n : \ A \in G\right\}$$

then $P^{-1}GP \subset \mathcal{K}_{\eta,r}(\mathbb{C})$, this proves the proposition. □

3. Abelian sub-semigroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ with a dense orbit

The aim of this section is to give results for an abelian sub-semigroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ analogous to those in ([1], Section 6). Let G be an abelian sub-semigroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. Denote by

- $\mathcal{C}(G) := \{A \in \mathcal{K}_{\eta,r}(\mathbb{C}) : \ AB = BA, \ \forall \ B \in G\}$.
- $\widehat{G}$ the group generated by G.
- $\widehat{\mathrm{g}} := \exp^{-1}(\widehat{G}) \cap (\mathcal{K}_{\eta,r}(\mathbb{C}))$

Since G is abelian, $G \subset \mathcal{C}(G)$.

Lemma 3.1. *Under the notation above, we have: $\mathcal{C}(\widehat{G}) = \mathcal{C}(G)$.*

Proof. If $B \in \mathcal{C}(G)$ and $A \in G$ then $A^{-1}B = BA^{-1}$ (since $AB = BA$). We conclude that $B \in \mathcal{C}(\widehat{G})$. □

Recall the following results proved in [1].

Lemma 3.2 ([1], Corollary 3.2)**.** *Let G be an abelian* subgroup *of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. Then* $\mathrm{g} \subset \mathcal{C}(G)$ *and all matrix of* g *commute.*

Proposition 3.3 ([1], Corollaries 6.4 and 5.4)**.** *Let G be an abelian* subgroup *of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. If $\overline{\mathrm{g}_{u_0}} = \mathbb{C}^n$ then there exists an isomorphism Ψ from $\mathbb{C}^n$ to $\mathcal{C}(G)$ satisfying $\Psi(Bu_0) = B$ for every $B \in \mathcal{C}(G)$. Moreover, $h := \Psi^{-1} \circ exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})} \circ \Psi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is well defined, continuous and satisfies $h(Bu_0) = e^B u_0$, for every $B \in \mathcal{C}(G)$. In particular, $h(\mathrm{g}_{u_0}) = G(u_0)$.*

Proposition 3.4 ([1], Corollaries 6.5 and 5.4)**.** *Let G be an abelian* subgroup *of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. If $\overline{G(u_0)} = \mathbb{C}^n$ then there exists an isomorphism Φ from $\mathbb{C}^n$ to $\mathcal{C}(G)$ satisfying $\Phi(Bu_0) = B$ for every $B \in \mathcal{C}(G)$. Moreover, $f = \Phi^{-1} \circ exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})} \circ \Phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is well defined and satisfies*

(i) f *is an open map*

(ii) $f^{-1}(G(u_0)) = \mathrm{g}_{u_0}$.

Proposition 3.5. *Let G be an abelian sub-semigroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$.*

(1) *If $\overline{\mathrm{g}_{u_0}} = \mathbb{C}^n$ then there exists an isomorphism Ψ from $\mathbb{C}^n$ to $\mathcal{C}(G)$ satisfying $\Psi(Bu_0) = B$ for every $B \in \mathcal{C}(G)$. Moreover, $h := \Psi^{-1} \circ exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})} \circ \Psi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is well defined, continuous and satisfies $h(Bu_0) = e^B u_0$, for every $B \in \mathcal{C}(G)$. In particular, $h(\mathrm{g}_{u_0}) = G(u_0)$.*

(2) *If $\overline{G(u_0)} = \mathbb{C}^n$ then there exists an isomorphism Φ from $\mathbb{C}^n$ to $\mathcal{C}(G)$. Moreover, $f := \Phi^{-1} \circ exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})} \circ \Phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is well defined and satisfies*
   (i) f *is an open map*
   (ii) $f^{-1}(G(u_0)) = \mathrm{g}_{u_0}$

Proof. (1): Suppose that $\overline{\mathrm{g}_{u_0}} = \mathbb{C}^n$. Then $\overline{\widehat{\mathrm{g}}_{u_0}} = \mathbb{C}^n$. Applying Proposition 3.3 to $\widehat{G}$, so there exists an isomorphism Ψ from $\mathbb{C}^n$ to $\mathcal{C}(\widehat{G})$ such that $h := \Psi^{-1} \circ \exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})} \circ \Psi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is well defined and satisfies $h(Bu_0) = e^B u_0$, for every $B \in \mathcal{C}(\widehat{G}) = \mathcal{C}(G)$ (Lemma 3.1). By Lemma 3.2, $\widehat{\mathrm{g}} \subset \mathcal{C}(G)$ and since $\mathrm{g} \subset \widehat{\mathrm{g}}$, it follows that $\mathrm{g} \subset \mathcal{C}(G)$ and therefore $h(\mathrm{g}_{u_0}) = G(u_0)$.

Similar arguments as before apply to the proof of (2) using Proposition 3.4. □

Proposition 3.6. *Let G be an abelian sub-semigroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. The following assertions are equivalent:*

(i) the orbit $G(u_0)$ is dense in $\mathbb{C}^n$

(ii) g_{u_0} is an additif sub-semigroup dense in $\mathbb{C}^n$.

Proof. *(ii)* $\Longrightarrow$ *(i)* : Suppose that $\overline{\mathrm{g}_{u_0}} = \mathbb{C}^n$. So by Proposition 3.5, (1), the map $h := \Psi^{-1} \circ exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})} \circ \Psi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is continuous and satisfies $h(\mathrm{g}_{u_0}) = G(u_0)$. Hence $h(\mathbb{C}^n) = h(\overline{\mathrm{g}_{u_0}}) \subset \overline{G(u_0)}$ and so $\overline{G(u_0)} = \mathbb{C}^n$.

(i) $\Longrightarrow$ *(ii)* : Suppose that $\overline{G(u_0)} = \mathbb{C}^n$. By Proposition 3.5, (2), (ii), the map $f := \Phi^{-1} \circ \exp_{/\mathcal{K}_{\eta,r}(\mathbb{C})} \circ \Phi : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is open and satisfies $f^{-1}(G(u_0)) = \mathrm{g}_{u_0}$. It follows that:

$$\mathbb{C}^n = f^{-1}(\mathbb{C}^n) = f^{-1}\left(\overline{G(u_0)}\right) \subset \overline{f^{-1}(G(u_0))} = \overline{\mathrm{g}_{u_0}}$$

and so $\overline{\mathrm{g}_{u_0}} = \mathbb{C}^n$. □

4. Proof of Theorem 1.1

The proof of Theorem 1.1 needs the following propositions and lemmas.

Proposition 4.1. *Let G be an abelian sub-semigroup of $M_n(\mathbb{C})$ and $u \in \mathbb{C}^n$. Then $G(u)$ is locally dense (resp. dense) if and only if so is $G^*(u)$.*

Proof. Suppose that $\overset{\circ}{\overline{G^*(u)}} \neq \emptyset$ (resp. $\overline{G^*(u)} = \mathbb{C}^n$), for some $u \in \mathbb{C}^n$. Since $G^*(u) \subset G(u)$, so $\overset{\circ}{\overline{G(u)}} \neq \emptyset$ (resp. $\overline{G(u)} = \mathbb{C}^n$). Conversely, suppose that $\overset{\circ}{\overline{G(u)}} \neq \emptyset$ (resp. $\overline{G(u)} = \mathbb{C}^n$), for some $u \in \mathbb{C}^n$. We can assume using Proposition 2.2 that $G \subset \mathcal{K}_{\eta,r}(\mathbb{C})$. We let $G' := G \setminus G^*$.
- If $G' = \emptyset$ then $G = G^*$ and so G^* is locally hypercyclic (resp. hypercyclic).
- If $G' \neq \emptyset$ then

$$G(u) \subset \left(\bigcup_{A \in G'} Im(A) \right) \cup G^*(u).$$

As every $A \in G'$ is non invertible, then $Im(A) \subset \bigcup\limits_{k=1}^{r} H_k$ where

$$H_k := \left\{ u = [u_1, \dots, u_r]^T \in \mathbb{C}^n : \ u_j \in \mathbb{C}^{n_j}, \ u_k \in \{0\} \times \mathbb{C}^{n_k - 1}, \ 1 \leq j \neq k \leq r \right\}.$$

It follows that

$$G(u) \subset \left(\bigcup_{k=1}^{r} H_k \right) \cup G^*(u),$$

and so

$$\overline{G(u)} \subset \left(\bigcup_{k=1}^{r} H_k \right) \cup \overline{G^*(u)}.$$

Since H_k has dimension $n-1$, $\overset{\circ}{H_k} = \emptyset$, for every $1 \leq k \leq r$ and therefore $\overset{\circ}{\overline{G^*(u)}} \neq \emptyset$ (resp. $\overline{G^*(u)} = \mathbb{C}^n$). □

Lemma 4.2. *Let G be an abelian sub-semigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$ and $\mathrm{g}^* = exp^{-1}(G^*) \cap (\mathcal{K}_{\eta,r}(\mathbb{C}))$. Then $\mathrm{g} = \mathrm{g}^*$.*

Proof. Since $G^* \subset G$, we see that $\mathrm{g}^* \subset \mathrm{g}$. Conversely, if $B \in \mathrm{g}$ then $e^B \in G^*$, so $B \in \exp^{-1}(G^*) \cap (\mathcal{K}_{\eta,r}(\mathbb{C})) = \mathrm{g}^*$, hence $\mathrm{g} \subset \mathrm{g}^*$. □

For an abelian subgroup G of $\mathrm{GL}(n, \mathbb{C})$, denote by
- $E(u) := \mathrm{Vect}(G(u))$ the vector subspace of $\mathbb{C}^n$ generated by $G(u)$, $u \in \mathbb{C}^n$.
- $\mathrm{Vect}(G)$ the vector subspace of $M_n(\mathbb{C})$ generated by G. One can easily check that $\mathrm{Vect}(G)$ is the algebra generated by G. In particular, if G is an abelian subgroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$ then $\mathrm{Vect}(G)$ is a vector subspace of $\mathcal{K}_{\eta,r}(\mathbb{C})$.

Lemma 4.3. ([4], *Proposition 3.1*) *Let G be an abelian subgroup of $GL(n,\mathbb{C})$. If $u\in\mathbb{C}^n$ and $v\in E(u)$, then there exists $B\in$ Vect(G) such that $Bu=v$.*

Lemma 4.4. ([1], *Proposition 6.6*) *If G is an abelian subgroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$, then $U:=\prod_{k=1}^{r}\mathbb{C}^*\times\mathbb{C}^{n_k-1}$ is G-invariant, and all orbits of U are minimal in U.*

Proposition 4.5. *Let G be an abelian sub-semigroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$. If $\overline{G(v)}=\mathbb{C}^n$, for some $v\in\mathbb{C}^n$, then there exists $B\in$ Vect$(G)\cap GL(n,\mathbb{C})$ such that $Bu_0=v$, and hence $\overline{G(u_0)}=\mathbb{C}^n$.*

Proof. Since $\overline{G(v)}=\mathbb{C}^n$, we see that $\overline{\widehat{G}(v)}=\mathbb{C}^n$ where $\widehat{G}\subset\mathcal{K}^*_{\eta,r}(\mathbb{C})$ is the group generated by G. So $E(v):=\mathrm{Vect}(\widehat{G}(v))=\mathbb{C}^n$. By Lemma 4.4, $U=\prod_{k=1}^{r}\mathbb{C}^*\times\mathbb{C}^{n_k-1}$ is $\widehat{G}$-invariant, dense and open subset of $\mathbb{C}^n$, hence $v\in U$. Write $v=[v_1,\dots,v_r]^T$ with $v_k=[x_{k,1},\dots,x_{k,n_k}]^T\in\mathbb{C}^*\times\mathbb{C}^{n_k-1}$. Applying Lemma 4.3 to $\widehat{G}$, there exists $B\in\mathrm{Vect}(\widehat{G})$ such that $Bu_0=v$. Since $\mathcal{K}_{\eta,r}(\mathbb{C})$ is a vector space, $\mathrm{Vect}(\widehat{G})\subset\mathcal{K}_{\eta,r}(\mathbb{C})$ and one can write $B=\mathrm{diag}(B_1,\dots,B_r)$ with $B_k\in\mathbb{T}_{n_k}(\mathbb{C})$, $k=1,\dots,r$. Let μ_k be the eigenvalue of B_k. From $Bu_0=v$, we see that $\mu_k=x_{k,1}\neq0$ for every $k=1,\dots,r$, hence $B\in\mathrm{GL}(n,\mathbb{C})$. As $\mathrm{Vect}(\widehat{G})\subset\mathcal{C}(\widehat{G})=\mathcal{C}(G)$ (Lemma 3.1), then $B(G(u_0))=G(v)$ and so $\overline{G(u_0)}=\mathbb{C}^n$. □

Proof of Theorem 1.1. One can assume that G is an abelian sub-semigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$ by Proposition 2.2. So $(ii)\Longleftrightarrow(iii)$ is equivalent to: $\overline{G(u_0)}=\mathbb{C}^n$ if and only if $\overline{\mathrm{g}_{u_0}}=\mathbb{C}^n$. Write $\mathrm{g}^*=exp^{-1}(G^*)\cap\mathcal{K}_{\eta,r}(\mathbb{C})$ and $\mathrm{g}^*_{u_0}=\{Bu_0:\ \ B\in\mathrm{g}^*\}$. Applying Proposition 3.6 to G^*, we see that $\overline{G^*(u_0)}=\mathbb{C}^n$ if and only if $\overline{\mathrm{g}^*_{u_0}}=\mathbb{C}^n$. Hence the equivalence follows from Proposition 4.1 and Lemma 4.2 since $\mathrm{g}^*_{u_0}=\mathrm{g}_{u_0}$.
The equivalence $(i)\Longleftrightarrow(ii)$ results directly from Proposition 4.5. □

5. Proof of Theorem 1.2 and Corollaries 1.3 and 1.4

Lemma 5.1 ([1], Proposition 3.5). *Let $A,B\in\mathbb{T}_n(\mathbb{C})$ such that $AB=BA$. If $e^A=e^B$, then $A=B+2ik\pi I_n$ for some $k\in\mathbb{Z}$.*

Proposition 5.2. *Let G be an abelian sub-semigroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ and let $B_1,\dots,B_p\in\mathcal{K}_{\eta,r}(\mathbb{C})$ $(p\geq1)$ such that $e^{B_1},\dots,e^{B_p}$ generate G. We have*

$$\mathrm{g}_{u_0}=\sum_{k=1}^{p}\mathbb{N}B_ku_0+\sum_{k=1}^{r}2i\pi\mathbb{Z}e^{(k)}.$$

Proof. • First we determine g. Let $C \in \mathrm{g}$. Then $C = \mathrm{diag}(C_1, \dots, C_r) \in \mathcal{K}_{\eta,r}(\mathbb{C})$ and $e^C \in G$. So $e^C = \mathrm{diag}(e^{C_1}, \dots, e^{C_r}) = e^{m_1 B_1} \dots e^{m_p B_p}$ for some $m_1, \dots, m_p \in \mathbb{N}$. Since $B_1, \dots, B_p \in \mathrm{g}$, they pairwise commute (Lemma 3.2). Therefore, $e^C = e^{m_1 B_1 + \cdots + m_p B_p}$. Write $B_j = \mathrm{diag}(B_{j,1}, \dots, B_{j,r})$, then $e^{C_k} = e^{m_1 B_{1,k} + \cdots + m_p B_{p,k}}$, $k = 1, \dots, r$. As $C \in \mathrm{g}$, we also have $CB_j = B_j C$ and so $C_k B_{j,k} = B_{j,k} C_k$, $j = 1, \dots, p$. It follows that: $C_k = m_1 B_{1,k} + \cdots + m_p B_{p,k} + 2i\pi s_k I_{n_k}$ for some $s_k \in \mathbb{Z}$ (Lemma 5.1). Therefore

$$\begin{aligned} C &= \mathrm{diag}\left(\sum_{j=1}^{p} m_j B_{j,1} + 2i\pi s_1 I_{n_1}; \dots, \dots; \sum_{j=1}^{p} m_j B_{j,r} + 2i\pi s_r I_{n_r} \right) \\ &= \sum_{j=1}^{p} m_j B_j + \mathrm{diag}(2i\pi s_1 I_{n_1}, \dots, 2i\pi s_r I_{n_r}) \end{aligned}$$

Write $J_k := \mathrm{diag}(J_{k,1}, \dots, J_{k,r})$ where $J_{k,i} = \begin{cases} 0 \in \mathbb{T}_{n_i}(\mathbb{C}) & \text{if} \quad i \neq k \\ I_{n_k} & \text{if} \quad i = k \end{cases}$. We have

$$\mathrm{diag}(2i\pi s_1 I_{n_1}, \dots, 2i\pi s_r I_{n_r}) = \sum_{k=1}^{r} 2i\pi s_k J_k$$

and therefore

$$C = \sum_{j=1}^{p} m_j B_j + \sum_{k=1}^{r} 2i\pi s_k J_k.$$

We conclude that

$$\mathrm{g} = \sum_{j=1}^{p} \mathbb{N} B_j + \sum_{k=1}^{r} 2i\pi \mathbb{Z} J_k.$$

• Second, we determine g_{u_0}. Let $B \in \mathrm{g}$. We have $B = \sum_{j=1}^{p} m_j B_j + \sum_{k=1}^{r} 2i\pi s_k J_k$ for some $m_1, \dots, m_p \in \mathbb{N}$, and $s_1, \dots, s_r \in \mathbb{Z}$. We also have

$$\begin{aligned} J_k u_0 &= \mathrm{diag}(J_{k,1}, \dots, J_{k,r})[e_{1,1}, \dots, e_{r,1}]^T \\ &= [e_1^{(k)}, \dots, e_r^{(k)}]^T \\ &= e^{(k)}. \end{aligned}$$

Hence

$$Bu_0 = \sum_{j=1}^{p} m_j B_j u_0 + \sum_{k=1}^{r} 2i\pi s_k e^{(k)}$$

and therefore $\mathrm{g}_{u_0} = \sum_{j=1}^{p} \mathbb{N} B_j u_0 + \sum_{k=1}^{r} 2i\pi \mathbb{Z} e^{(k)}$. This proves the proposition. □

Proof of Theorem 1.2. By Proposition 2.2, there exists a $P \in \mathrm{GL}(n, \mathbb{C})$ such that $P^{-1}GP$ is an abelian sub-semigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$. We let $G' := P^{-1}GP$, $\mathrm{g}' := \exp^{-1}(G') \cap \mathcal{K}_{\eta,r}(\mathbb{C})$, $G'* := G' \cap \mathrm{GL}(n, \mathbb{C})$, $\mathrm{g}'* := \exp^{-1}(G'*) \cap \mathcal{K}_{\eta,r}(\mathbb{C})$ and $\mathrm{g}'*_{u_0} := \{Bu_0 : \quad B \in \mathrm{g}'*\}$. By Lemma

4.2, $g' = g'^{*}$. Write $B'_1 = P^{-1}B_1P, \dots, B'_p = P^{-1}B_pP$. Then $e^{B'_1}, \dots, e^{B'_p}$ generate $G'*$. Now we apply Proposition 5.2 to G'^{*}, and obtain

$$g'_{u_0} = g'^{*}_{u_0} \ = \ \sum_{j=1}^{p}\mathbb{N}B'_j u_0 \ + \ \sum_{k=1}^{r}2i\pi\mathbb{Z}e^{(k)}.$$

As $g = Pg'P^{-1}$ and $Pu_0 = v_0$, then

$$g_{v_0} \ = P(g'_{u_0}) = \ \sum_{j=1}^{p}\mathbb{N}B_j v_0 \ + \ \sum_{k=1}^{r}2i\pi\mathbb{Z}Pe^{(k)}.$$

By applying Theorem 1.1, the proof is over. □

Lemma 5.3. ([8], *page 35*). *Let* $H = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_m$ *with* $u_k = (u_{k,1}, \dots, u_{k,n}) \in \mathbb{C}^n$ *and* $u_{k,j} = \mathrm{Re}(u_{k,j}) + i\mathrm{Im}(u_{k,j})$, $k = 1, \dots, m$, $j = 1, \dots, n$. *Then* H *is dense in* $\mathbb{C}^n$ *if and only if for every* $(s_1, \dots, s_m) \in \mathbb{Z}^m \backslash \{0\}$:

$$rank\left(\begin{bmatrix} \mathrm{Re}(u_{1,1}) & \dots & \dots & \mathrm{Re}(u_{m,1}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{Re}(u_{1,n}) & \dots & \dots & \mathrm{Re}(u_{m,n}) \\ \mathrm{Im}(u_{1,1}) & \dots & \dots & \mathrm{Im}(u_{m,1}) \\ \vdots & \vdots & \vdots & \vdots \\ \mathrm{Im}(u_{1,n}) & \dots & \dots & \mathrm{Im}(u_{m,n}) \\ s_1 & \dots & \dots & s_m \end{bmatrix}\right) = \ 2n + 1.$$

Proof of Corollary 1.3. • We show first that if $H = \mathbb{Z}u_1 + \dots + \mathbb{Z}u_m$, $u_k \in \mathbb{C}^n$ with $m \leq 2n$, then H cannot be dense: Write $u_k = [u_{k,1}, \dots, u_{k,n}]^T \in \mathbb{C}^n$, $u_{k,j} = \mathrm{Re}(u_{k,j}) + i\mathrm{Im}(u_{k,j})$ and $v_k = [\mathrm{Re}(u_{k,j}), \mathrm{Im}(u_{k,j}), s_j]^T \in \mathbb{R}^{2n+1}$, $1 \leq j \leq n$, $1 \leq k \leq m$. Since $m \leq 2n$, it follows that $\mathrm{rank}(v_1, \dots, v_m) \leq 2n$, and so H is not dense in $\mathbb{C}^n$ by Lemma 5.3.
• Now, by applying Theorem 1.2 and the fact that $m = p+r \leq 2n$, the Corollary 1.3 follows. □

Proof of Corollary 1.4. This follows from the fact that $n \leq 2n-r$ since $r \leq n$, and by applying Corollary 1.3. □

6. Proof of Theorem 1.5

We will construct for every $r = 1, \dots, n$ and for every partition $\eta \in \mathbb{N}_0^r$ of n, $(2n - r + 1)$ matrices $A_1, \dots, A_{2n-r+1} \in \mathcal{K}^*_{\eta,r}(\mathbb{C})$ that they generate an hypercyclic abelian semigroup.

We used repeatedly the following multidimensional version of Kronecker's Theorem stated below. (See for example [6], Theorem 442):

Kronecker's Theorem. *Let* $\alpha_1, \dots, \alpha_n$ *be negative real numbers such that the numbers* $1, \alpha_1, \dots, \alpha_n$ *are linearly independent over* $\mathbb{Q}$. *Then the set*

$$\mathbb{N}^n + \mathbb{N}[\alpha_1, \dots, \alpha_n]^T := \left\{[s_1, \dots, s_n]^T + k[\alpha_1, \dots, \alpha_n]^T : \ k, s_1, \dots, s_n \in \mathbb{N}\right\}$$

is dense in $\mathbb{R}^n$.

We deduce the complex version as follows.

Corollary 6.1. *Let* $\alpha_1, \dots, \alpha_n;\ \beta_1, \dots, \beta_n$ *be negative real numbers such that the numbers* $1, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ *are linearly independent over* $\mathbb{Q}$. *Then*

$$\mathbb{N}^n + i\mathbb{N}^n + \mathbb{N}[\alpha_1 + i\beta_1, \dots, \alpha_n + i\beta_n]^T$$

is dense in $\mathbb{C}^n$.

Proof. Let $f : \mathbb{R}^{2n} \longrightarrow \mathbb{C}^n$ denote the isomorphism defined by $f(u, v) = u + iv$ where $\mathbb{C}^n$ is considered as an $\mathbb{R}$-vector space of dimension $2n$ with basis $(e_1, \dots, e_n; ie_1, \dots, ie_n)$. Write $H := \mathbb{N}^n + i\mathbb{N}^n + \mathbb{N}[\alpha_1 + i\beta_1, \dots, \alpha_n + i\beta_n]^T$. Then, $f^{-1}(H) = \mathbb{N}^{2n} + \mathbb{N}[\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n]^T$ is dense in $\mathbb{R}^{2n}$ by Kronecker's Theorem, and therefore H is dense in $\mathbb{C}^n$. $\square$

Proposition 6.2. *Let* $n \in \mathbb{N}_0$ *and* $r = 1, \dots, n$. *Then there exist* $2n-r+1$ *vectors* $u_1, \dots, u_{2n-r+1}$ *of* $\mathbb{C}^n$ *such that*

$$\sum_{k=1}^{2n-r+1} \mathbb{N}u_k + \sum_{k=1}^{r} 2i\pi\mathbb{Z}e^{(k)}$$

is dense in $\mathbb{C}^n$.

Proof. Let $\alpha_1, \dots, \alpha_n;\ \beta_1, \dots, \beta_n$ be negative real numbers such that the numbers $1, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ are linearly independent over $\mathbb{Q}$. Recall that $e^{(k)} = e_{\ell_k}$ where $\ell_1 = 1$ and $\ell_k = \sum_{j=1}^{k-1} n_j + 1$, for $k = 2, \dots, r$. Denote by $(e_{i_{r+1}}, \dots, e_{i_n}) := \mathcal{B}_0 \backslash (e_{\ell_1}, \dots, e_{\ell_r})$ and define the matrix S by

$$Se_k = \begin{cases} 2i\pi e^{(k)}, \text{ if } 1 \leq k \leq r \\ e_{i_k}, \text{ if } r+1 \leq k \leq n \end{cases}$$

We see that $S \in \mathrm{GL}(n, \mathbb{C})$.

We let $u = [\alpha_1 + i\beta_1, \dots, \alpha_n + i\beta_n]^T$ and define

$$u_k := \begin{cases} Se_{r+k}, & \text{if} \quad 1 \leq k \leq n-r \\ iSe_{r-n+k}, & \text{if} \quad n-r+1 \leq k \leq 2n-r \\ Su, & \text{if} \quad k = 2n-r+1 \end{cases}$$

Write

$$H := \sum_{k=1}^{2n-r+1} \mathbb{N}u_k + \sum_{k=1}^{r} 2i\pi\mathbb{Z}e^{(k)}$$

and

$$H' := \sum_{k=1}^{n-r} \mathbb{N}e_{r+k} + \sum_{k=1}^{n} i\mathbb{N}e_k + \mathbb{N}u + \sum_{k=1}^{r} \mathbb{Z}e_k.$$

Then we have

$$\begin{aligned} S(H') &= \sum_{k=1}^{n-r}\mathbb{N}Se_{r+k} + \sum_{k=1}^{n} i\mathbb{N}Se_k + \mathbb{N}Su + \sum_{k=1}^{r}\mathbb{Z}Se_k \\ &= \sum_{k=1}^{n-r}\mathbb{N}u_k + \sum_{k=1}^{n}\mathbb{N}iSe_k + \mathbb{N}u_{2n-r+1} + \sum_{k=1}^{r}\mathbb{Z}2i\pi e^{(k)} \\ &= \sum_{k=1}^{n-r}\mathbb{N}u_k + \sum_{k=n-r+1}^{2n-r}\mathbb{N}u_k + \mathbb{N}u_{2n-r+1} + \sum_{k=1}^{r}2i\pi\mathbb{Z}e^{(k)}. \end{aligned}$$

Therefore $S(H') = H$. Since $\mathbb{N}^n + i\mathbb{N}^n + \mathbb{N}u \subset H'$, we see that H' is dense in $\mathbb{C}^n$ by Corollary 6.1, and so is H. This proves the proposition. □

Proof of Theorem 1.5. By Proposition 6.2, there exist $u_1, \dots, u_{2n-r+1} \in \mathbb{C}^n$ so that

$$H := \sum_{k=1}^{2n-r+1}\mathbb{N}u_k + \sum_{k=1}^{r}2i\pi\mathbb{Z}e^{(k)}$$

is dense in $\mathbb{C}^n$. Write $u_k = [u_{k,1}, \dots, u_{k,r}]^T$ with $u_{k,j} = [x^{(k)}_{j,1}, \dots, x^{(k)}_{j,n_j}]^T$. We will construct $(2n - r + 1)$ matrices B_k, $1 \leq k \leq 2n - r + 1$ satisfying $B_k u_0 = u_k$. For this, we let $B_k = \mathrm{diag}(B_{k,1}, \dots, B_{k,r})$ where

$$B_{k,j} = \begin{bmatrix} x^{(k)}_{j,1} & & & & 0 \\ \vdots & \ddots & & & \\ \vdots & 0 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ x^{(k)}_{j,n_j} & 0 & \dots & 0 & x^{(k)}_{j,1} \end{bmatrix}, \quad 1 \leq j \leq r;\ 1 \leq k \leq 2n - r + 1.$$

Let G be the sub-semigroup of $\mathcal{K}^*_{\eta,r}(\mathbb{C})$ generated by $e^{B_1}, \dots, e^{B_{2n-r+1}}$ and write $\mathrm{g} := \exp^{-1}(G) \cap \mathcal{K}_{\eta,r}(\mathbb{C})$.

• Firstly, let us check that G is abelian. For this, it suffices to show that $B_k B_{k'} = B_{k'} B_k$, for every $k,\ k' = 1, \dots, 2n - r + 1$:

Write $B_{k,j} := N_{k,j} + x^{(k)}_{j,1} I_{n_j}$ where

$$N_{k,j} = \begin{bmatrix} 0 & 0 \\ T_{k,j} & 0 \end{bmatrix} \in \mathbb{T}_{n_j}(\mathbb{C}), \text{ with } T_{k,j} = [x^{(k)}_{j,2}, \dots, x^{(k)}_{j,n_j}]^T,\ j = 1, \dots, r.$$

We see that $N_{k,j}N_{k',j} = N_{k',j}N_{k,j} = 0$, for every $j = 1, \dots, r$. Hence $B_{k,j}B_{k',j} = B_{k',j}B_{k,j}$ and so $B_k B_{k'} = B_{k'} B_k$.

• Secondly, by Proposition 5.2 we have

$$\begin{aligned}
\mathrm{g}_{u_0} &= \sum_{k=1}^{2n-r+1} \mathbb{N}B_k u_0 + \sum_{k=1}^{r} 2i\pi\mathbb{Z}e^{(k)} \\
&= \sum_{k=1}^{2n-r+1} \mathbb{N}u_k + \sum_{k=1}^{r} 2i\pi\mathbb{Z}e^{(k)} \\
&= H
\end{aligned}$$

Therefore $\overline{\mathrm{g}_{u_0}} = \mathbb{C}^n$ and by Theorem 1.1, $\overline{G(u_0)} = \mathbb{C}^n$. □

7. Examples

Example 7.1. Let G be the sub-semigroup of $\mathbb{C}^*$ generated by $a_1 = e^{2\pi}$, $a_2 = e^{-2(\sqrt{2}+i\sqrt{3})\pi}$. Then G is hypercyclic.

Proof. In this case, we have $u_0 = 1$ and $\mathrm{g} = \exp^{-1}(G)$. By Proposition 5.2, $\mathrm{g}_1 = 2\pi\mathbb{N} - 2(\sqrt{2} + i\sqrt{3})\pi\mathbb{N} + 2i\pi\mathbb{Z} = 2\pi H$ where $H := \mathbb{N} - (\sqrt{2} + i\sqrt{3})\mathbb{N} + i\mathbb{Z}$. As $1, \sqrt{2}$ and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$, then by Corollary 6.1, $\mathbb{N} - (\sqrt{2} + i\sqrt{3})\mathbb{N} + i\mathbb{N} \subset H$ is dense in $\mathbb{C}$ and so is H. Therefore, $\overline{\mathrm{g}_1} = \mathbb{C}$ and by Theorem 1.1, $\overline{\mathrm{G}(1)} = \mathbb{C}$. □

Example 7.2. Let G be the semigroup generated by $A_1 = \mathrm{diag}(e^{2\pi},\ e^{2\pi})$, $A_2 = \begin{bmatrix} 1 & 0 \\ 2\pi & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 \\ 2i\pi & 1 \end{bmatrix}$ and $A_4 = e^{-2\pi(\sqrt{2}+i\sqrt{3})} \begin{bmatrix} 1 & 0 \\ 2\pi(1-i\sqrt{5}) & 1 \end{bmatrix}$. Then G is abelian and hypercyclic.

Proof. By construction, G is an abelian sub-semigroup of $\mathbb{T}_2^*(\mathbb{C})$ and we have $u_0 = e_1$ and $A_k = e^{B_k}$, $k = 1, \dots, 4$ where $B_1 = \mathrm{diag}(2\pi,\ 2\pi)$, $B_2 = \begin{bmatrix} 0 & 0 \\ 2\pi & 0 \end{bmatrix}$, $B_3 = \begin{bmatrix} 0 & 0 \\ 2i\pi & 0 \end{bmatrix}$ and $B_4 = \begin{bmatrix} -2\pi(\sqrt{2}+i\sqrt{3}) & 0 \\ 2\pi(1-i\sqrt{5}) & -2\pi(\sqrt{2}+i\sqrt{3}) \end{bmatrix}$. By Proposition 5.2,

$$\begin{aligned}
\mathrm{g}_{e_1} &= \sum_{k=1}^{4} \mathbb{N}B_k e_1 + 2i\pi\mathbb{Z}e_1 \\
&= 2\pi H
\end{aligned}$$

where

$$H := \mathbb{N}e_1 + \mathbb{N}e_2 + i\mathbb{N}e_2 + \mathbb{N}[-\sqrt{2} - i\sqrt{3},\ 1 - i\sqrt{5}]^T + i\mathbb{Z}e_1.$$

We let

$$K := \mathbb{N}e_1 + \mathbb{N}e_2 + i\mathbb{N}e_2 + \mathbb{N}[-\sqrt{2} - i\sqrt{3},\ 1 - i\sqrt{5}]^T + i\mathbb{N}e_1.$$

Then

$$K = \mathbb{N}^2 + i\mathbb{N}^2 + \mathbb{N}[-\sqrt{2} - i\sqrt{3},\ 1 - i\sqrt{5}]^T \subset H.$$

By Corollary 6.1, K is dense in $\mathbb{C}^2$ since $1, -\sqrt{2}, -\sqrt{3}$ and $-\sqrt{5}$ are linearly independent over $\mathbb{Q}$, and so is H. We conclude by Theorem 1.1 that $G(e_1)$ is dense in $\mathbb{C}^2$. □

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