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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON THE WIDTHS OF THE ARNOL'D TONGUES

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Abstract

Let $F : \mathbb{R} \to \mathbb{R}$ be a real analytic increasing diffeomorphism with $F - \mathrm{Id}$ being 1 periodic. Consider the translated family of maps $(F_t : \mathbb{R} \to \mathbb{R})_{t \in \mathbb{R}}$ defined as $F_t(x) = F(x) + t$. Let $\mathrm{Trans}(F_t)$ be the translation number of F_t defined by

$$\mathrm{Trans}(F_t) := \lim_{n \to +\infty} \frac{F_t^{\circ n} - \mathrm{Id}}{n}.$$

Assume that there is a Herman ring of modulus 2τ associated to F and let p_n/q_n be the n-th convergent of $\mathrm{Trans}(F) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$. Denoting ℓ_θ as the length of the interval $\{t \in \mathbb{R} \mid \mathrm{Trans}(F_t) = \theta\}$, we prove that the sequence (ℓ_{p_n/q_n}) decreases exponentially fast with respect to q_n. More precisely

$$\limsup_{n \to +\infty} \frac{1}{q_n} \log \ell_{p_n/q_n} \leq -2\pi\tau.$$

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1. Introduction

In the whole article, $F : \mathbb{R} \to \mathbb{R}$ is an increasing analytic diffeomorphism such that $F - \mathrm{Id}$ is 1 periodic. We identify the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ and $\mathbb{S}^1$ via $[x] \simeq e^{2i\pi x}$, where $[x]$ denotes the class of real number x modulo 1. The map F induces an orientation preserving analytic circle diffeomorphism $f : \mathbb{T} \to \mathbb{T}$ given by $[x] \mapsto [F(x)]$. The definitions of translation number of F and the rotation number of f are based upon the following result of Poincaré.

Proposition 1.1 (Poincaré [B]). *The sequence of maps* $\dfrac{F^{\circ n} - \mathrm{Id}}{n}$ *converges uniformly on* $\mathbb{R}$ *to a constant.*

Definition 1.2. *The translation number of* F *is defined as*

$$\mathrm{Trans}(F) := \lim_{n \to +\infty} \frac{F^{\circ n} - \mathrm{Id}}{n}.$$

The rotation number of f *is the quantity* $\rho(f) := \mathrm{Trans}(F) \mod 1$.

Let us now consider the translated family of maps $F_t : \mathbb{R} \to \mathbb{R}$ defined by $F_t(x) = F(x) + t$ for all $x \in \mathbb{R}$ and for every $t \in \mathbb{R}$. This family induces a family f_t of analytic circle diffeomorphisms. The function

$$\mathcal{H} : t \mapsto \mathrm{Trans}(F_t)$$

is continuous and non decreasing. The graph of $\mathcal{H}$ is like a devil's staircase generally. The preimage of an irrational translation number is just a point under this map. And the preimage of a rational translation number is a closed interval, generally not reduced to a point. Thus it is interesting to study the lengths of these intervals

$$I_\theta := \{t \in \mathbb{R} \mid \mathrm{Trans}(F_t) = \theta\}.$$

Let us denote the length of the interval I_θ as ℓ_θ. The function ℓ_θ is discontinuous at all the rational points. Our aim is to estimate these lengths under certain conditions.

These lengths could be connected with the widths of the Arnol'd tongues in the following way. Let us define a two parameter family of maps of the real line

$$F_{t,a}(x) = x + t + a \sin(2\pi x).$$

This gives a family of increasing diffeomorphisms of $\mathbb{R}$ when $t \in \mathbb{R}$ and $a \in (-1/2\pi, 1/2\pi)$ and for each (t, a) in this domain it induces an orientation preserving analytic circle diffeomorphism. This family is often called the Arnol'd family or the standard family after Arnol'd [A].

Definition 1.3. *The Arnol'd tongue* $\mathcal{T}_\theta$ *of translation number* θ *is defined as the following set*

$$\mathcal{T}_\theta := \{(t, a) \in \mathbb{R} \times (-1/2\pi, 1/2\pi) \mid \mathrm{Trans}(F_{t,a}) = \theta\}.$$

If we fix $a \in (-1/2\pi, 1/2\pi)$ and set $F_t = F_{t,a}$, then the length $\ell_{p/q}$ is the width of the Arnol'd tongue $\mathcal{T}_{p/q}$ sliced at the height a.

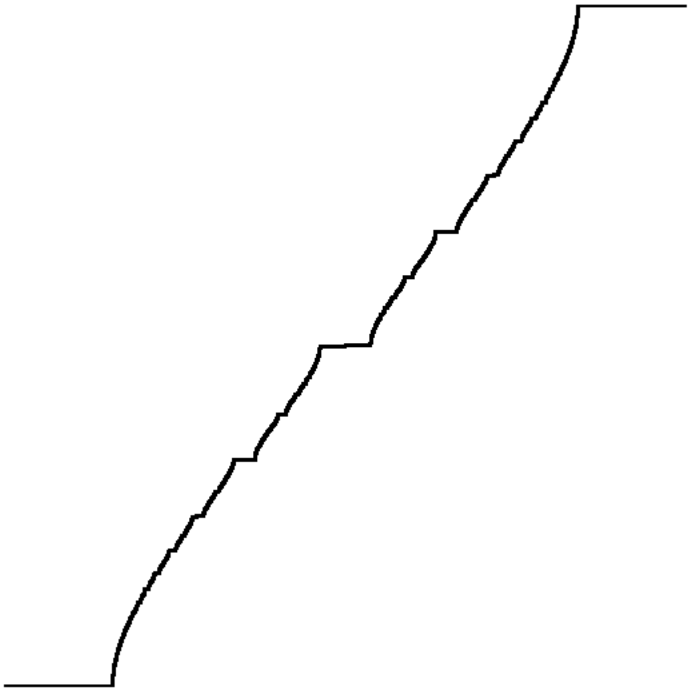

FIGURE 1. The graph of $\mathcal{H} : t \mapsto \mathrm{Trans}(F_t)$ for $0 \le t \le 1$, where $F_t(x) = x + t + \dfrac{1}{4\pi}\sin(2\pi x)$.

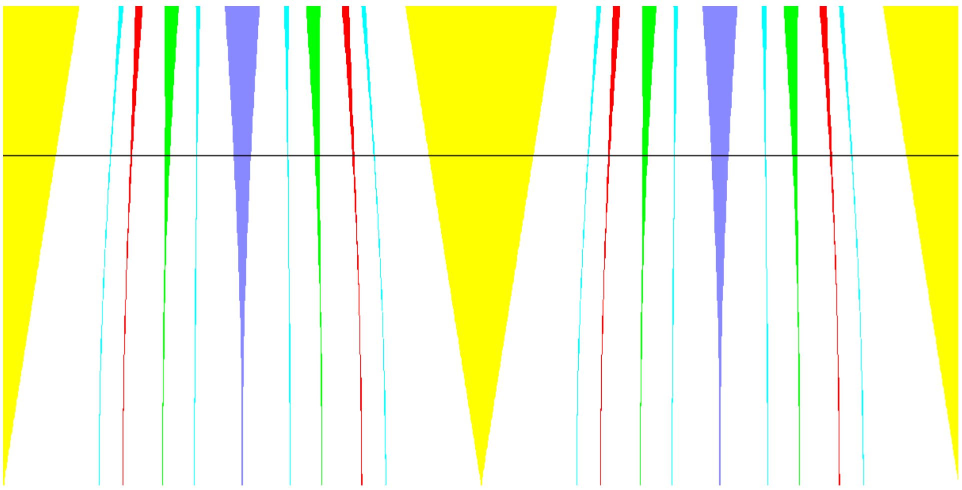

FIGURE 2. Arnol'd tongues of the standard family sliced at a fixed positive height.

2. PRELIMINARIES

Before we proceed further we would recall some basic facts about translation and rotation numbers. Every time we write p/q for a rational number, we implicitely assume that p and q are coprime.

Proposition 2.1 (Poincaré [B]). *If $\rho(f) \in \mathbb{Q}/\mathbb{Z}$ then $f : \mathbb{T} \to \mathbb{T}$ has a periodic point. More precisely, if* $\mathrm{Trans}(F) = p/q \in \mathbb{Q}$ *then there is a point $a \in \mathbb{R}$ such that $F^{\circ q}(a) = a + p$.*

Note that $G := F^{\circ q} - \mathrm{Id} - p$ vanishes on the whole F-orbit of a, in particular on the q-set $\{a, F(a), \ldots, F^{\circ(q-1)}(a)\}$ whose image in $\mathbb{T}$ is a cycle of f. We shall say that such a cycle has rotation number p/q. The derivative of G is constant along the orbit of a under iteration of F. As G is analytic, either it has a double root, or it vanishes at least once with positive derivative and once with negative derivative. This shows that counting multiplicities, f has at least 2 cycles with rotation number p/q.

Proposition 2.2 (Poincaré [B]). *If $\rho(f) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$, then $f : \mathbb{T} \to \mathbb{T}$ is semi-conjugate to the rotation $\mathbb{T} \ni [x] \mapsto [x + \alpha] \in \mathbb{T}$.*

In fact, the semiconjugacy may be obtained as follows. The sequence of maps Φ_N defined as

$$\Phi_N(x) := \frac{1}{N} \sum_{k=0}^{N-1} \Big(F^{\circ k}(x) - F^{\circ k}(0) \Big)$$

converges, as $N \to +\infty$, to a non-decreasing continuous surjective map $\Phi_F : \mathbb{R} \to \mathbb{R}$, which satisfies

$$\Phi_F(x+1) = \Phi_F(x) + 1 \quad \text{and} \quad \Phi_F \circ F(x) = T_\alpha \circ \Phi_F(x)$$

where $T_\alpha : x \mapsto x + \alpha$ is the translation by α.

The following result of Denjoy implies that when F is an analytic diffeomorphism, then the semiconjugacy is in fact an actual conjugacy. In other words $\Phi_F : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism.

Theorem 2.3 (Denjoy [B]). *If $\rho(f) = \alpha \in \mathbb{R}\setminus\mathbb{Q}$ and if f is a $\mathcal{C}^2$ diffeomorphism, then $f : \mathbb{T} \to \mathbb{T}$ is conjugate to the rotation of angle α.*

3. Herman Ring

From now on, we assume that $\mathrm{Trans}(F) \in \mathbb{R}\setminus\mathbb{Q}$ and so, that $\Phi_F : \mathbb{R} \to \mathbb{R}$ is a homeomorphism. We will now be interested in the regularity of Φ_F. It is known that when $\mathrm{Trans}(F)$ satisfies an appropriate arithmetic condition, then the conjugacy Φ_F is itself an analytic diffeomorphism.

The first result obtained in this direction is a result of Herman. Recall that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is a Diophantine number if there is a small constant $C > 0$ and there is a large constant $\tau \geq 2$ such that for all rational number p/q

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C}{q^\tau}.$$

Theorem 3.1 (Herman [He1]). *If* $\mathrm{Trans}(F)$ *is a Diophantine number, then $\Phi_F : \mathbb{R} \to \mathbb{R}$ is an analytic diffeomorphism.*

The optimal arithmetic condition which guaranties that the conjugacy is an analytic diffeomorphism, has been obtained by Yoccoz [Y] but is too complicated to be recalled here.

Now we can introduce the definition of the Herman ring.

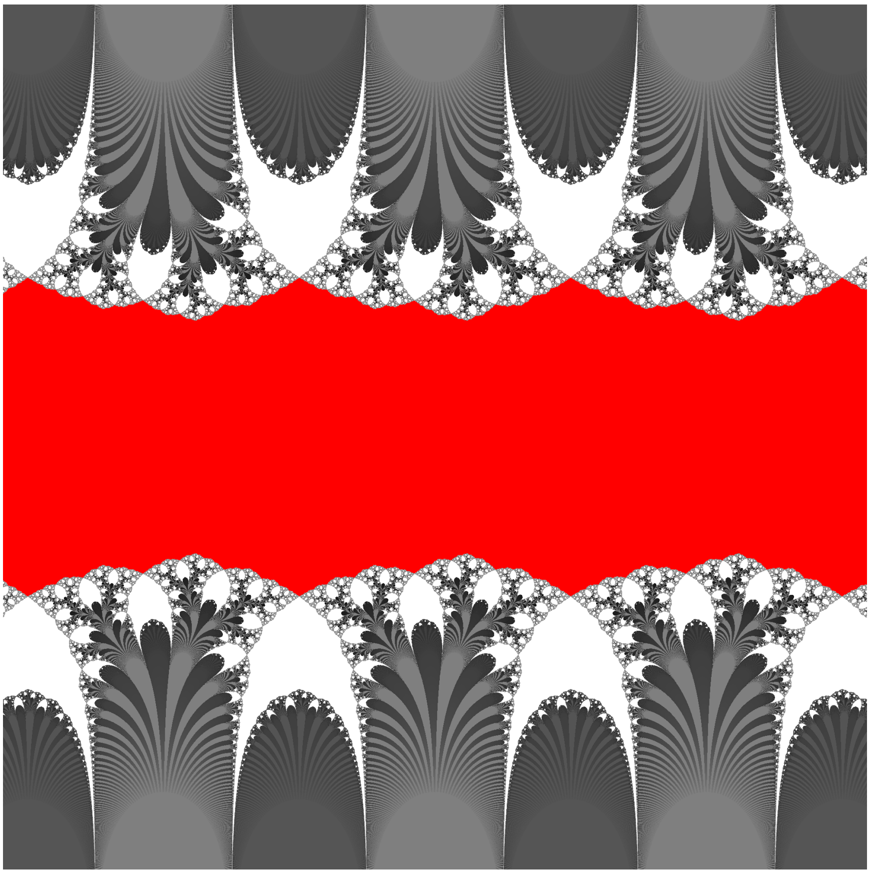

FIGURE 3. A Herman strip in the family $F_t(x) = x + t + \dfrac{1}{4\pi}\sin(2\pi x)$. The translation number is the golden mean $(\sqrt{5}-1)/2$.

Definition 3.2. *Assume that* $\mathrm{Trans}(F) \in \mathbb{R} \setminus \mathbb{Q}$ *and* $\Phi_F : \mathbb{R} \to \mathbb{R}$ *is analytic. Let* τ *be the largest number such that* $\Psi_F := \Phi_F^{-1}$ *extends univalently to* $S(\tau) := \{z \in \mathbb{C} \mid -\tau < \mathrm{Im}(z) < \tau\}$. *The map* F *extends analytically to* $\mathcal{HS}(F) := \Psi_F\big(S(\tau)\big)$. *We call* $\mathcal{HS}(F)$ *the* Herman strip *of* F. *The image of* $\mathcal{HS}(F)$ *in* $\mathbb{C}/\mathbb{Z}$ *is called a* Herman ring *of* f *associated to* F. *The modulus of the Herman ring is* 2τ.

From now on, we assume that $\alpha := \mathrm{Trans}(F)$ is irrational and that Φ_F is an analytic diffeomorphism, i.e., F has a Herman Strip. Then, we would study the length ℓ_{p_n/q_n} where p_n/q_n is the n-th convergent of the continued fraction expansion of α.

4. Main result and its comparison to earlier works

Theorem 4.1. *Suppose that* $F : \mathbb{R} \to \mathbb{R}$ *is an increasing* $\mathbb{R}$*-analytic diffeomorphism. For any* $t \in \mathbb{R}$, *define the translated family of maps* $F_t(x) = F(x) + t$ *for* $x \in \mathbb{R}$. *Assume that*

- $\mathrm{Trans}(F) = \alpha \in \mathbb{R} \setminus \mathbb{Q}$ *and*
- *there is a Herman ring associated to* F *with modulus* 2τ.

Let p_n/q_n be the n-th continued fraction convergent of α. Then we have the following inequality

$$\limsup_{n\to+\infty} \frac{1}{q_n} \log \ell_{p_n/q_n} \leq -2\pi\tau.$$

In our set up, when we approach a map which has a Herman strip then the corresponding length ℓ_{p_n/q_n} decreases exponentially with respect to q_n. In particular, when we take a horizontal slice of the Arnol'd tongues, and when we approach a parameter whose associated map has a Herman ring, the width of the tongue ℓ_{p_n/q_n} decreases exponentially with respect to q_n.

Herman studied the function $\mathcal{H} : t \mapsto \mathrm{Trans}(F_t)$ in his paper [He2]. From his works one can have an estimate on the behaviour of these lengths using the result $\left|\alpha - \dfrac{p_n}{q_n}\right| = \mathcal{O}\Big(\dfrac{1}{q_n^2}\Big)$ [HW].

Theorem 4.2 (Herman [B]). *Suppose that F is conjugate to the translation by* $\mathrm{Trans}(F)$ *by a $\mathcal{C}^1$ diffeomorphism Φ_F, then the function $\mathcal{H} : t \mapsto \mathrm{Trans}(F_t)$ is differentiable at $t = 0$ and it has a positive derivative.*

Corollary 4.3. If Φ_F is $\mathcal{C}^1$, then

$$\ell_{p_n/q_n} = o\left(\alpha - \frac{p_n}{q_n}\right) = o\left(\frac{1}{q_n^2}\right).$$

Proof. Suppose that

$$I_{p_n/q_n} = [t_n^-, t_n^+].$$

According to Theorem 4.2,

$$\begin{aligned}
&\frac{p_n/q_n - \alpha}{t_n^\pm - 0} \to \mathcal{H}'(0) \quad (\neq 0) \Rightarrow \frac{t_n^\pm}{p_n/q_n - \alpha} \to \frac{1}{\mathcal{H}'(0)};\\
&\Rightarrow t_n^\pm = \frac{1}{\mathcal{H}'(0)}\left(\frac{p_n}{q_n} - \alpha\right) + o\left(\frac{p_n}{q_n} - \alpha\right).\\
&\therefore \ell_{p_n/q_n} = t_n^+ - t_n^- = o\left(\frac{p_n}{q_n} - \alpha\right) = o\left(\frac{1}{q_n^2}\right).
\end{aligned}$$

□

Thus according to Herman ℓ_{p_n/q_n} decreases faster than $1/q_n^2$. Our theorem states that the decay is much faster, it is an exponential decay. More precisely

$$\ell_{p_n/q_n} \leq e^{-2\pi\tau q_n + o(q_n)}.$$

This is a better estimate and our bound involves the modulus of the Herman ring. There is automatically a Herman ring if the rotation number satisfies the generalized Herman condition [Y], in particular if the rotation number is of bounded type (all the continued fraction entries are bounded) or Diophatine. There is also a Herman ring if the rotation number is Brjuno and F is univalent on a sufficiently large strip. The set of Diophantine numbers is of full measure and it is a subset of the Brjuno numbers. Thus the bound holds true on a full measure subset of the translation numbers. It follows from [He2] that this is true for a set of parameters t of positive measure.

5. Proof of the main result

We shall start with estimating the length $\ell_{p/q}$ under some conditions. Remember that $\ell_{p/q}$ is the length of the interval $I_{p/q}$, which consists of the values of t such that $\mathrm{Trans}(F_t) = p/q$. In other words we are interested in finding the parameters t, for which the graph $y = F_t^{\circ q}(x) - p$ intersects the line $y = x$. By the choice of the family, the graph $y = F_t^{\circ q}(x) - p$ moves upward as we increase t. Assuming the graph moves upward with a certain speed, it can be shown that the interval $I_{p/q}$ can not be too big when the amount $|F_t^{\circ q}(x) - x - p|$ is bounded for all $t \in I_{p/q}$ and for all x. Along this path we obtain the following upper bound of $\ell_{p/q}$ in the next lemma. This estimate is valid for more general families which may not be analytic.

Lemma 5.1. *Let $(\mathcal{F}_t)_{t\in\mathbb{R}}$ be a family of continuous increasing homeomorphisms of $\mathbb{R}$ with $\mathcal{F}_t - \mathrm{Id}$ being periodic of period 1 and the family is differentiable with respect to the parameter t. We assume that this is an increasing family in the sense that $\mathrm{Trans}(\mathcal{F}_t)$ increases with t. Also assume that there are $\varepsilon_0 > 0$ and $v_0 > 0$ such that for all $t \in I_{p/q}$ and for all $x \in \mathbb{R}$,*

- $m_t \leq \mathcal{F}_t^{\circ q}(x) - x - p \leq M_t$, *with* $M_t - m_t \leq \varepsilon_0$ *and*
- $\dfrac{\partial \mathcal{F}_t^{\circ q}(x)}{\partial t} \geq v_0$.

Then,

$$\ell_{p/q} \leq \frac{\varepsilon_0}{v_0}.$$

Proof. Since $t \mapsto \mathrm{Trans}(\mathcal{F}_t)$ is non decreasing the set $I_{p/q} = \{t \in \mathbb{R} \mid \mathrm{Trans}(\mathcal{F}_t) = p/q\}$ is an interval say $[t^-_{p/q}, t^+_{p/q}]$. Also with an increasing family, we have $M_{t^-_{p/q}} = 0$ and $m_{t^+_{p/q}} = 0$. The assumption that $\dfrac{\partial \mathcal{F}_t^{\circ q}(x)}{\partial t} \geq v_0$ implies that

$$\begin{aligned}
\mathcal{F}^{\circ q}_{t^+_{p/q}}(x) - x - p &\geq \mathcal{F}^{\circ q}_{t^-_{p/q}}(x) - x - p + v_0(t^+_{p/q} - t^-_{p/q}) \\
\Rightarrow m_{t^+_{p/q}} &\geq m_{t^-_{p/q}} + v_0(t^+_{p/q} - t^-_{p/q}) \\
\Rightarrow M_{t^-_{p/q}} - m_{t^-_{p/q}} &\geq v_0(t^+_{p/q} - t^-_{p/q}) \quad (\because M_{t^-_{p/q}} = m_{t^+_{p/q}} = 0) \\
\Rightarrow \frac{\varepsilon_0}{v_0} &\geq (t^+_{p/q} - t^-_{p/q}).
\end{aligned}$$

Hence $\ell_{p/q} \leq \dfrac{\varepsilon_0}{v_0}$. □

Remark 5.2. One can prove by induction on k that in our family $(F_t)_{t\in\mathbb{R}}$ where $F_t(x) = F(x) + t$, for any $t \in \mathbb{R}$, for all $x \in \mathbb{R}$ and for all $k \geq 1$,

$$\frac{\partial F_t^{\circ k}(x)}{\partial t} \geq 1.$$

This lemma gives an estimate of $I_{p/q}$ using an upper bound of $|M_t - m_t|$ and a lower bound $\dfrac{\partial \mathcal{F}_t^{\circ q}(x)}{\partial t} \geq v_0$. According to the remark v_0 can be taken as 1 in our family. Now we are interested in finding a bound of $|M_t - m_t|$ in our family. From now onwards we are in the set up of the given analytic family $(F_t)_{t\in\mathbb{R}}$ as in the theorem.

Choose a sequence $(t_n \in I_{p_n/q_n})_{n\geq 1}$. Define

$$G_n(x) := F_{t_n}^{\circ q_n}(x) - x - p_n.$$

The function G_n is of period 1 and vanishes along at least two sets of q_n points corresponding to two cycles of period q_n for f_{t_n}.

It is enough to show that for all $\tau' < \tau$, we have

$$\sup_{x\in\mathbb{R}} \big|G_n(x)\big| = \mathcal{O}\big(e^{-2\pi\tau' q_n}\big).$$

Choose $\tau' < \tau$ and set $S' := \Phi_F^{-1}\big(S(\tau')\big) \subset \mathcal{HS}(F)$.

Proposition 5.3. *If n is large enough, then for all $k \leq q_n$, $F_{t_n}^{\circ k}$ is defined in S' with values in $\mathcal{HS}(F)$. The sequence of maps (G_n) converges uniformly to 0 on S'.*

To prove this proposition let us prove parts of it in the following lemmas. Choose τ'' and τ''' so that $\tau' < \tau'' < \tau''' < \tau$. Set $S''' := \Phi_F^{-1}\big(S(\tau''')\big) \subset \mathcal{HS}(F)$ and $S'' := \Phi_F^{-1}\big(S(\tau'')\big) \subset \mathcal{HS}(F)$. Note that $S' \subset S'' \subset S''' \subset \mathcal{HS}(F)$ and $S(\tau') \subset S(\tau'') \subset S(\tau''') \subset S(\tau)$. We shall use the fact that the function Φ_F is analytic on $\overline{S'''}$ and $\Phi_F - \mathrm{Id}$ is of period 1. Consequently Φ_F' is of period 1 and analytic on $\overline{S'''}$ too, thus bounded. The same holds for the derivative of its inverse on $\overline{S(\tau''')}$ and the analytic function F on $\overline{S'''}$. Since F is conjugate to a translation, observe that the iterates of a point $z \in S'$ under F are inside S', and the same is true for points of S'' or S'''. Also it is easy to work with the conjugate of F, which is the translation T_α. We shall compare F_{t_n} (and its iterates) with the translation T_α (and its iterates) after conjugating with Φ_F and transport back the estimates on S'.

In this process we need the Mean Value Inequality (abbreviated MVI), which says that for an analytic function λ defined on a convex compact set and with z_1, z_2 in the domain of analyticity,

$$|\lambda(z_1) - \lambda(z_2)| \leq \sup_z |\lambda'(z)||z_1 - z_2|.$$

The set $\overline{S'''}$ may not be convex, we use an inequality similar to MVI in the following manner. As Φ_F is univalent, for z_1 and z_2 in $\overline{S'''}$ we can define a continuous non vanishing function

$$(z_1, z_2) \mapsto \frac{\Phi_F(z_1) - \Phi_F(z_2)}{z_1 - z_2}$$

on $\overline{S'''} \times \overline{S'''} \setminus \Delta(\overline{S'''})$; where $\Delta(\overline{S'''})$ is the diagonal of $\overline{S'''} \times \overline{S'''}$. We can extend this function on the diagonal continuously by setting the non zero value $\Phi_F'(z_1)$ for $z_1 = z_2$. The map $\Phi_F - \mathrm{Id}$ is of period 1 and this implies that the constructed function is of period 1. Thus there is a constant $C(\Phi_F|_{\overline{S'''}})$ such that

$$|\Phi_F(z_1) - \Phi_F(z_2)| \leq C(\Phi_F|_{\overline{S'''}})|z_1 - z_2|.$$

We call this the Modified Mean Value Inequality (abbreviated MMVI) for Φ_F taken on $\overline{S'''}$.

During these computations we also use the fact that (p_n/q_n) are convergents of α and q_n grows very fast, in fact exponentially.

Lemma 5.4. $\sup_{S'''}|F^{\circ q_n}(z) - z - p_n| \to 0$ *as* $n \to +\infty$.

Proof. Suppose that $w = \Phi_F(z)$ for $z \in S'''$, then $\Phi_F \circ F^{\circ q_n}(z) = w + q_n\alpha$. By choice $(\Phi_F^{-1})'$ is bounded on $\overline{S(\tau''')}$. Assuming $C_1 = \max_{w\in\overline{S(\tau''')}} |(\Phi_F^{-1})'(w)|$, we have

$$
\begin{aligned}
|F^{\circ q_n}(z) - z - p_n| &= |\Phi_F^{-1}(w + q_n\alpha - p_n) - \Phi_F^{-1}(w)| \\
&\le C_1|w + q_n\alpha - p_n - w| \quad \text{(by MVI)} \\
&= C_1|q_n\alpha - p_n| \le \frac{C_1}{q_n}.
\end{aligned}
$$

The last inequality uses the bound $|\alpha - p_n/q_n| \le 1/q_n^2$ [HW], which holds since p_n/q_n is the n-th continued fraction convergent of α. Hence

$$\sup_{S'''}|F^{\circ q_n}(z) - z - p_n| \underset{n\to+\infty}{\longrightarrow} 0. \qquad \square$$

In the following lemma we try to locate the iterates of F_{t_n} up to order q_n when the starting point is taken on S''. But before doing that we shall work with the conjugate of F_{t_n} by Φ_F inside the strip $S(\tau)$. Note that for n large enough $F_{t_n}(S'') \subset S'''$ and so the map $H_n : w \mapsto \Phi_F \circ F_{t_n} \circ \Phi_F^{-1}$ is defined on $S(\tau'')$ with its values in $S(\tau''')$ when n is large.

Lemma 5.5. *There exists a constant* C_2 *such that for any* $w \in S(\tau'')$ *and* n *large* $H_n(w) \in S(\tau''')$ *and*

$$|H_n(w) - T_\alpha(w)| \le \frac{C_2}{q_n^2}.$$

Proof. We assume that $w \in S(\tau'')$ and $w = \Phi_F(z)$ for $z \in S''$. Thus

$$
\begin{aligned}
|H_n(w) - T_\alpha(w)| &= |H_n \circ \Phi_F(z) - T_\alpha \circ \Phi_F(z)| \\
&= |\Phi_F \circ F_{t_n}(z) - \Phi_F \circ F(z)| \\
&\le C(\Phi_F|_{\overline{S'''}})|F_{t_n}(z) - F(z)| \text{ (By MMVI)} \\
&= C(\Phi_F|_{\overline{S'''}})|t_n|
\end{aligned}
$$

Since $\mathcal{H} : t \mapsto \mathrm{Trans}(F_t)$ has a positive derivative at 0 by Theorem 4.2, we have

$$\left|\alpha - \frac{p_n}{q_n}\right| \ge \frac{\mathcal{H}'(0)}{2}|t_n|.$$

Consequently,

$$|H_n(w) - T_\alpha(w)| \le \frac{2C(\Phi_F|_{\overline{S'''}})}{\mathcal{H}'(0)}\left|\alpha - \frac{p_n}{q_n}\right| \le \frac{C_2}{q_n^2}$$

for some constant $C_2 = \dfrac{2C(\Phi_F|_{\overline{S'''}})}{\mathcal{H}'(0)}$, which does not depend on n and w. This calculation implies that if we choose n large enough then H_n maps $S(\tau'')$ inside $S(\tau''')$. $\square$

In the next step we work with the iterates of H_n up to order q_n. We extend our previous result with the help of the following observation which is also proved next. For an increasing continuous map $K : \mathbb{R} \to \mathbb{R}$ satisfying the condition that $K - \mathrm{Id}$ is of period 1, we use the fact

$$|K^{\circ l}(x) - T_\alpha^{\circ l}(x)| \le l \sup_{x\in\mathbb{R}}|K(x) - T_\alpha(x)|.$$

Lemma 5.6. *For n large and for all $k \le q_n$, $H_n^{\circ k}$ maps $S(\tau')$ inside $S(\tau)$ and*

$$|H_n^{\circ k}(w) - T_\alpha^{\circ k}(w)| \le \frac{kC_2}{q_n^2}.$$

Proof. To show that the iterates of a point in $S(\tau')$ up to order q_n under the map H_n fall inside $S(\tau)$, we proceed by induction. The base case $k=1$ has been proved in Lemma 5.5. We assume that for all integer l, $1 \le l < k \le q_n$, and n sufficiently large the image of $S(\tau')$ under the map $H_n^{\circ l}$ is inside $S(\tau'')$. Starting with $w \in S(\tau')$ we see that

$$\begin{aligned} |H_n^{\circ k}(w) - T_\alpha^{\circ k}(w)| &\le |H_n \circ H_n^{\circ k-1}(w) - T_\alpha \circ H_n^{\circ k-1}(w)| \\ &\quad + |T_\alpha \circ H_n^{\circ k-1}(w) - T_\alpha \circ T_\alpha^{\circ k-1}(w)| \\ &\le \frac{C_2}{q_n^2} + |H_n^{\circ k-1}(w) - T_\alpha^{\circ k-1}(w)| \text{ (by 5.5)} \end{aligned}$$

Continuing this inductive calculation we obtain

$$|H_n^{\circ k}(w) - T_\alpha^{\circ k}(w)| \le \frac{kC_2}{q_n^2}.$$

Therefore choosing n large enough we can ensure that $H_n^{\circ k}(w) \in S(\tau''')$ for any $w \in S(\tau')$. $\square$

Now we are in a position to look at the iterates of F_{t_n} on S' up to order q_n.

Lemma 5.7. *For n large and for all $k \le q_n$, $F_{t_n}^{\circ k}$ maps S' inside $\mathcal{HS}(F)$ and*

$$|F_{t_n}^{\circ k}(z) - F^{\circ k}(z)| \le \frac{C_1 k C_2}{q_n^2}$$

for any $z \in S'$.

Proof. From Lemma 5.6, one observes that if n is sufficiently large then $H_n^{\circ k}\big(S(\tau')\big) \subset S(\tau''')$ for any $k \le q_n$. Using the fact that H_n is conjugate to F_{t_n} via Φ_F^{-1} we see that $F_{t_n}^{\circ k}$ maps S' inside S''' for large n. Now choosing $z \in S'$ and assuming $w = \Phi_F(z) \in S(\tau')$, we have

$$\begin{aligned} |F_{t_n}^{\circ k}(z) - F^{\circ k}(z)| &= |\Phi_F^{-1} \circ \Phi_F \circ F_{t_n}^{\circ k}(z) - \Phi_F^{-1} \circ \Phi_F \circ F(z)| \\ &= |\Phi_F^{-1} \circ H_n^{\circ k}(w) - \Phi_F^{-1} \circ T_\alpha^{\circ k}(w)| \\ &\le \frac{C_1 k C_2}{q_n^2} \text{ (by MVI and 5.6).} \end{aligned}$$

$\square$

Proof of Proposition 5.3. In Lemma 5.7 we have already seen that for large n and for all $k \le q_n$, $F_{t_n}^{\circ k}$ is defined on S' with values in $\mathcal{HS}(F)$. And for $z \in S'$,

$$\begin{aligned} \big|G_n(z)\big| = \big|F_{t_n}^{\circ q_n}(z) - z - p_n\big| &= \big|F_{t_n}^{\circ q_n}(z) - F^{\circ q_n}(z) + F^{\circ q_n}(z) - z - p_n\big| \\ &\le \big|F_{t_n}^{\circ q_n}(z) - F^{\circ q_n}(z)\big| + \big|F^{\circ q_n}(z) - z - p_n\big|. \end{aligned}$$

From Lemma 5.4 we know that $\sup_{S'''}\big|F^{\circ q_n}(z) - z - p_n\big| \to 0$ as $n \to +\infty$. And by Lemma 5.7,

$$|F_{t_n}^{\circ q_n}(z) - F^{\circ q_n}(z)| \le \frac{C_1 k C_2}{q_n^2}.$$

Since $k \le q_n$, the fraction $\frac{C_1 k C_2}{q_n^2} \underset{n \to +\infty}{\longrightarrow} 0$. This completes the proof. $\square$

In our next step we work with the conjugating map Φ_F and give another sequence of maps converging to it. We set

$$\Phi_n := \frac{1}{q_n}\sum_{k=0}^{q_n-1}\big(F^{\circ k} - F^{\circ k}(0)\big) \quad \text{and} \quad \widehat{\Phi}_n := \frac{1}{q_n}\sum_{k=0}^{q_n-1}\big(F_{t_n}^{\circ k} - F^{\circ k}(0)\big).$$

According to Proposition 2.2,

$$\Phi_n \underset{n\to+\infty}{\longrightarrow} \Phi_F.$$

We shall see that this is also true for the sequence $\widehat{\Phi}_n$.

Lemma 5.8. *If n is large enough, the domain of $\widehat{\Phi}_n$ contains S'. As $n \to +\infty$, the sequence $\widehat{\Phi}_n$ converges to the linearizing map Φ_F and thus $\widehat{\Phi}_n$ has a univalent inverse $\widehat{\Psi}_n : S(\tau') \to \mathcal{HS}(F)$ for large n.*

Proof. Taking $z \in S'$ we note that for any $k \le q_n$ the point $F_{t_n}^{\circ k}(z)$ is inside S''' for large n. Moreover

$$\begin{aligned}\left|\widehat{\Phi}_n(z) - \Phi_n(z)\right| &= \frac{1}{q_n}\left|\sum_{k=0}^{q_n-1}\big(F_{t_n}^{\circ k}(z) - F^{\circ k}(z)\big)\right| \\ &\le \frac{1}{q_n}\sum_{k=1}^{q_n-1}\frac{C_1C_2}{q_n^2}\cdot k \text{ (by 5.7)} \\ &= \frac{C_1C_2}{q_n^3}\cdot\frac{(q_n-1)(q_n-2)}{2}.\end{aligned}$$

This implies that as $n \to +\infty$, the sequence $\widehat{\Phi}_n$ converges to the linearizing map Φ_F. Moreover $\widehat{\Phi}_n : S' \to S(\tau)$ is defined and it is univalent for large n. Thus it has a univalent inverse $\widehat{\Psi}_n : S(\tau') \to \mathcal{HS}(F)$ when n is large enough. □

Lemma 5.9. *Counting multiplicities, the map $G_n \circ \widehat{\Psi}_n$ vanishes at least along two sets of the form $a_n + kp_n/q_n + l$ and $b_n + kp_n/q_n + l$ with $a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$ and $(k,l) \in \mathbb{Z}\times\mathbb{Z}$.*

Proof. The map f_{t_n} has at least two q_n-cycles in $\mathbb{T}$, counting multiplicities. The map G_n vanishes on the set of points which are the lifts of the q_n-cycles from $\mathbb{T}$ to $\mathbb{R}$. Let us assume that G_n vanishes on the sets $\{F_{t_n}^{\circ j}(a) + l \mid l \in \mathbb{Z}\}$ and $\{F_{t_n}^{\circ j}(b) + l \mid l \in \mathbb{Z}\}$ for $j = 0, \cdots, q_n - 1$ and for some $a \in \mathbb{R}$, $b \in \mathbb{R}$. We have

$$\widehat{\Phi}_n(a) = \frac{1}{q_n}\sum_{k=0}^{q_n-1}\left(F_{t_n}^{\circ k}(a) - F^{\circ k}(0)\right).$$

And

$$
\begin{aligned}
\widehat{\Phi}_n\big(F_{t_n}(a)\big) &= \frac{1}{q_n}\sum_{k=0}^{q_n-1}\Big(F_{t_n}^{\circ k+1}(a) - F^{\circ k}(0)\Big)\\
&= \frac{1}{q_n}\sum_{k=0}^{q_n-1}\Big(F_{t_n}^{\circ k}(a) - F^{\circ k}(0)\Big) + \frac{F_{t_n}^{\circ q_n}(a) - a}{q_n}\\
&= \widehat{\Phi}_n(a) + \frac{p_n}{q_n}
\end{aligned}
$$

since $F_{t_n}^{\circ q_n}(a) = a + p_n$. In a similar way we obtain

$$
\begin{aligned}
&\widehat{\Phi}_n\big(F_{t_n}^{\circ j+1}(a)\big) - \widehat{\Phi}_n\big(F_{t_n}^{\circ j}(a)\big)\\
&= \widehat{\Phi}_n\big(F_{t_n}^{\circ j+1}(b)\big) - \widehat{\Phi}_n\big(F_{t_n}^{\circ j}(b)\big) = \frac{p_n}{q_n}
\end{aligned}
$$

for all $0 \le j \le q_n - 2$. This proves that $G_n \circ \widehat{\Psi}_n$ vanishes at least along two sets of the form $a_n + kp_n/q_n + l$ and $b_n + kp_n/q_n + l$ with $a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$ and $(k,l) \in \mathbb{Z}\times\mathbb{Z}$.

$\square$

Now note that the map G_n and $\widehat{\Psi}_n$ are of period 1. For n large enough, the map $G_n \circ \widehat{\Psi}_n$ is holomorphic on $S(\tau')$. And as $n \to +\infty$, the sequence $(G_n \circ \widehat{\Psi}_n)$ converges uniformly to 0 on $S(\tau')$. The required estimate is a consequence of the following lemma.

Lemma 5.10. *Assume that G is holomorphic on $S(\tau')$, periodic of period 1 and vanishes on two sets of the form $a + kp/q + l$ and $b + kp/q + l$ where $(k,l) \in \mathbb{Z}\times\mathbb{Z}$. Then*

$$
\max_{x\in\mathbb{R}}\big|G(x)\big| \le \frac{\sup\limits_{z\in S(\tau')}\big|G(z)\big|}{\big(\sinh(\pi q\tau')\big)^2} \underset{q\to+\infty}{=} \mathcal{O}\left(\sup_{z\in S(\tau')}\big|G(z)\big|\cdot e^{-2\pi q\tau'}\right).
$$

Proof. Suppose that $G_0(z) = \sin\big(\pi q(z-a)\big)\sin\big(\pi q(z-b)\big)$. Then G_0 vanishes exactly on the sets of the form $a + kp/q + l$ and $b + kp/q + l$ with $(k,l) \in \mathbb{Z}\times\mathbb{Z}$. Hence the function $\dfrac{G(z)}{G_0(z)}$ does not have a pole and thus it is holomorphic on the strip $S(\tau')$. By Maximum Modulus Principle, for $z \in S(\tau')$ we have

$$
\left|\frac{G(z)}{G_0(z)}\right| \le \frac{\sup\limits_{z\in S(\tau')}|G(z)|}{\inf\limits_{z\in\partial S(\tau')}|G_0(z)|}.
$$

Since G and G_0 are non constant and periodic, these supremum and infimum values actually occur on the boundary of $S(\tau')$.

Assuming $z = x \pm i\tau'$ we see that

$$
\begin{aligned}
|\sin\big(\pi q(z-a)\big)| &= \left|\frac{e^{i\pi q(z-a)} - e^{-i\pi q(z-a)}}{2i}\right|\\
&= \left|\frac{e^{i\beta}}{i}\right|\left|\frac{e^{\mp\pi q\tau'} - e^{\pm\pi q\tau'}e^{-2i\beta}}{2}\right| \quad (\text{where } \ \beta = \pi q x - \pi q a)\\
&\ge \frac{e^{\pi q\tau'}}{2} - \frac{e^{-\pi q\tau'}}{2}\\
&= \sinh(\pi q\tau').
\end{aligned}
$$

For q large enough, such that $\pi q\tau' > \log\sqrt{2}$, we see that

$$|\sin\big(\pi q(z-a)\big)| \geq \sinh(\pi q\tau') > \frac{e^{\pi q\tau'}}{4}.$$

This proves the lemma. □

Proof of the Theorem 4.1. By Lemma 5.10 we obtain

$$\max_{y\in\mathbb{R}}\big|G_n\circ\widehat{\Psi}_n(y)\big| \leq \frac{\sup\limits_{w\in S(\tau')}\big|G_n\circ\widehat{\Psi}_n(w)\big|}{\big(\sinh(\pi q_n\tau')\big)^2}$$

$$\underset{q_n\to+\infty}{=} \mathcal{O}\left(\sup_{w\in S(\tau')}\big|G_n\circ\widehat{\Psi}_n(w)\big|\cdot e^{-2\pi q_n\tau'}\right).$$

Since $\widehat{\Psi}_n : \mathbb{R}\to\mathbb{R}$ is surjective and since Proposition 5.3 asserts that $\sup\limits_{z\in S'}|G_n(z)| \to 0$ as $n\to+\infty$, we see that

$$\sup_{x\in\mathbb{R}}|G_n(x)| = \mathcal{O}\big(e^{-2\pi\tau' q_n}\big)$$

for any $\tau' < \tau$. Hence the theorem is proved. □

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