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HYBRID APPROXIMATION OF SOLUTIONS OF NONLINEAR OPERATOR EQUATIONS AND APPLICATION TO EQUATION OF HAMMERSTEIN-TYPE

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Abstract

In this paper we study the hybrid iterative scheme to find a common element of a set of solutions of generalized mixed equilibrium problem, a set of common fixed points of finite family of weak relatively nonexpansive mapping, and null spaces of finite family of γ-inverse strongly monotone mappings in a 2-uniformly convex and uniformly smooth real Banach space. Our results extend, improve and generalize the results of several authors which were announced recently. An application of our theorem to the solution of equations of Hammerstein-type is of independent interest.

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1. Introduction

Let $(E, \|.\|)$ be a real normed space with dual E^* and let $\langle\ .\ ,\ .\ \rangle$ be the duality pairing between members of E and E^*. The mapping $J : E \rightarrow 2^{E^*}$ defined by

$$J(x) = \{f^* \in E^* : \langle x, f^* \rangle = ||x||^2; ||f^*|| = ||x||\}, x \in E,$$

is called the normalized duality mapping. We note that in a Hilbert space H, J is the identity operator. It is well known that J is single valued, if the dual space E^* of E is strictly convex. Furthermore, it is also well known that if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is singled-valued, one-to-one, surjective, and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see [25]).

Let C be a closed convex nonempty subset of a real Banach space E with dual space E^*. Let $f : C \times C \rightarrow \mathbb{R}$ be a bifunction, $\Phi : C \rightarrow \bar{\mathbb{R}}$ be a proper extended real-valued function, where $\bar{\mathbb{R}}$ denotes the extended real numbers and let $B : C \rightarrow E^*$ be a nonlinear mapping. The generalized mixed equilibrium problem (abbreviated $GMEP$) for f, Φ, B is to find $u^* \in C$ such that

$$f(u^*, y) + \Phi(y) - \Phi(x^*) + \langle y - u^*, Bu^* \rangle \geq 0\ \forall\ y \in C.$$

The set of solutions for $GMEP$ is denoted by

$$GMEP(f, \Phi, B) = \{x \in C : f(x, y) + \Phi(y) - \Phi(x) + \langle y - x, Bx \rangle \geq 0\ \forall\ y \in C\}. \tag{1}$$

Special Remarks

(a) If $\Phi \equiv 0 \equiv B$ in (1), then (1) reduces to the classical equilibrium problem (abbreviated EP) and $GMEP(f, 0, 0)$ is denoted by $EP(f)$, where

$$EP(f) = \{x \in C : f(x, y) \geq 0\ \forall\ y \in C\}.$$

(b) If $f \equiv 0 \equiv \Phi$ in (1), then (1) reduces to the classical variational inequality problem and $GMEP(0, 0, B)$ is denoted by $VI(B, C)$, where

$$VI(B, C) = \{x \in C : \langle y - x, Bx \rangle \geq 0\ \forall\ y \in C\}.$$

(c) If $f \equiv 0 \equiv B$, then (1) reduces to the following minimization problem: find $u^* \in C$ such that $\Phi(y) \geq \Phi(u^*)\ \forall\ y \in C$; and $GMEP(0, \Phi, 0)$ is denoted by $argmin(\Phi)$, where

$$argmin(\Phi) = \{x \in C : \Phi(y) \geq \Phi(x)\ \forall\ y\ \in C\}.$$

(d) If $B \equiv 0$, then (1) reduces to the mixed equilibrium problem (MEP) and $GMEP(F, \Phi, 0)$ is denoted by $MEP(f, \Phi)$, where

$$MEP(f, \Phi, 0) = \{x \in C : f(x, y) + \Phi(y) - \Phi(x) \geq 0\ \forall\ y \in C\}.$$

(e) If $\Phi \equiv 0$, then (1) becomes the generalized equilibrium problem (GEP) and $GMEP(f, 0, B)$ is denoted by $GEP(f, B)$, where

$$GEP(f, B) = \{x \in C : f(x, y) + \langle y - x, Bx \rangle \geq 0\ \forall\ y \in C\}.$$

(f) If $f \equiv 0$, then (1) reduces to the generalized variational inequality problem (GVIP) and $GMEP((0, \Phi, B)$ is denoted by $GVIP(\Phi, B, C)$, where

$$GVIP(\Phi, B, C) = \{x \in C : \Phi(y) - \Phi(x) + \langle y - x, Bx \rangle \geq 0 \ \forall \ y \in C\}.$$

Problem (1) is very general in the sense that it includes, as special cases optimization problems, variational inequality problems, Min-Max problems and countless others (see e.g., [3, 21]).

A mapping $A : D(A) \subset E \to E^*$ is called a monotone operator if for all $x, \, y \in D(A)$, we have that $\langle x - y, Ax - Ay \rangle \geq 0$. The operator A is called γ-inverse strongly monotone if there exists a positive real number γ such that for all $x, \, y \in D(A)$, $\langle x - y, Ax - Ay \rangle \geq \gamma ||Ax - Ay||^2$. If A is a γ-inverse strongly monotone mapping then it is Lipschitz continuous with Lipschitz constant $\frac{1}{\gamma}$, i.e., $||Ax - Ay|| \leq \frac{1}{\gamma}||x - y|| \, \forall \, x, y \in D(A)$.

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a non-empty closed convex subset of E. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy \rangle + ||y||^2 \tag{2}$$

for $x, y \in E$, was studied by Alber [1], Kamimula and Takahashi [16] and Riech [23]. It follows easily from the definition of ϕ that

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2 \tag{3}$$

for $x, y \in E$. The generalized projection mapping introduced by Alber [1], is a mapping $\Pi_C : E \to C$ defined by $\Pi_C(y) = arg \min_{x \in C} \phi(x, y) \ \forall \ y \in E$. That is, if $x_0 \in C$ is such that $\phi(x_0, y) = \min_{x \in C} \phi(x, y)$ for all $y \in E$, then $\Pi_C(y) = x_0$. In fact we have the following results.

Lemma 1. *([1]). Let C be a nonempty closed and convex subset of a reflexive, strictly convex and smooth real Banach space E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = min\{\phi(z, x) : z \in C\}$.*

If E is a Hilbert space, then $\phi(x, y) = ||x - y||^2$ and Π_C is the metric projection of E onto C.

Let C be a closed convex nonempty subset of E and let $T : C \to C$ be a mapping. We denote by $F(T)$, the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. A point $p \in C$ is said to be *an asymptotic fixed point* of T (see e.g., [16]) if C contains a sequence $\{x_n\}_{n \geq 1}$ which converges weakly to p and such that $\lim_{n \to \infty} ||Tx_n - x_n|| = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping $T : C \to C$ is called nonexpansive if $||Tx - Ty|| \leq ||x - y|| \ \forall \ x, y \in C$. A mapping $T : C \to C$ is said to be relatively nonexpansive if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $\phi(p, Tx) \leq \phi(p, x), \forall\ p \in F(T), x \in C$;
(3) $\hat{F}(T) = F(T)$.

A point $p \in C$ is said to be *a strong asymptotic fixed point* of T (see e.g., [28]) if C contains a sequence $\{x_n\}_{n\geq 1}$ which converges strongly to p and such that $\lim_{n\to\infty} ||Tx_n - x_n|| = 0$. The set of strong asymptotic fixed points of T will be denoted by $\tilde{F}(T)$. A mapping $T : C \to C$ is called *weak relatively nonexpansive* if the following conditions are satisfied:

(1) $F(T)$ is nonempty;
(2) $\phi(p, Tx) \leq \phi(p, x), \forall\ p \in F(T), x \in C$;
(3) $\tilde{F}(T) = F(T)$.

If E is strictly convex and reflexive real Banach space and $A : E \to E^*$ is a continuous monotone mapping with $A^{-1}(0) \neq \varnothing$ then it is proved in [18, 20] that $J_r := (J + rA)^{-1}J$, for $r > 0$ is relatively nonexpansive. It is obvious that every relatively nonexpansive mapping is weak relatively nonexpansive. In fact, for any mapping $T : C \to C$ we have $F(T) \subset \tilde{F}(T) \subset \hat{F}(T)$. Therefore, if T is relatively nonexpansive mapping then $F(T) = \tilde{F}(T) = \hat{F}(T)$.

The following is an example of a weak relatively nonexpansive mapping, which is not a relatively nonexpansive mapping.

Example 2. Consider the sequence space $\ell_2(\mathbb{R})$ and let $\{x_n\}_{n\geq 1} \subset \ell_2(\mathbb{R})$ be generated as $x_0 = (1, 0, 0, 0, ...)$, $x_1 = (1, 1, 0, 0, 0, ...)$, $x_2 = (1, 0, 1, 0, 0, 0, ...)$, $x_3 = (1, 0, 0, 1, 0, 0, 0, ...)$, ..., $x_n = (1, 0, 0, 0, ..., 0, 0, 1, 0, ...)$, ...

It is obvious that $\{x_n\}_{n\geq 1}$ converges weakly to x_0. On the other hand, we have $||x_n - x_m|| = \sqrt{2}$ for any $n \neq m$, n, m sufficiently large. Define a mapping $T : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ by

$$T(x) = \begin{cases} \frac{n}{n+1} x_n, & \text{if } x = x_n; \\ -x, & \text{if } x \neq x_n. \end{cases}$$

Clearly $F(T) = \{0\}$ and $||Tx - 0|| = ||Tx|| \leq ||x|| = ||x - 0||, \forall\ x \in \ell_2(\mathbb{R})$.

Since $\ell_2(\mathbb{R})$ is a real Hilbert space, we have that

$$\phi(0, Tx) = ||0 - Tx||^2 = ||Tx||^2 \leq ||x||^2 = ||0 - x||^2 = \phi(0, x)\ \forall\ x \in \ell_2(\mathbb{R}).$$

The operator T is a weak relatively nonexpansive mapping. In fact, for any strongly convergent sequence $\{z_m\}_{m\geq 1} \subset \ell_2(\mathbb{R})$ such that $z_m \to z_0 \in E$ and $||z_m - Tz_m|| \to 0$ as $m \to \infty$, there exists $N_0 \in \mathbb{N}$ such that $z_m \neq x_n$ for all $n, m \geq N_0$. Thus, $Tz_m = -z_m$ for $m \geq N_0$; and it follows from $||z_m - Tz_m|| \to 0$ that $2z_m \to 0$ as $m \to \infty$ and hence $z_m \to z_0 = 0$ as $m \to \infty$. This implies that $\tilde{F}(T) = \{0\} = F(T)$, so that T is a weak relatively nonexpansive mapping.

The operator T is, however, not a relatively nonexpansive mapping. This follows from the fact that, though $x_n \rightharpoonup x_0$ and $||Tx_n - x_n|| = ||\frac{n}{n+1}x_n - x_n|| = \frac{1}{n+1}||x_n|| \to 0$ as $n \to \infty$, we have that $x_0 \notin F(T)$. This means that $\hat{F}(T) \neq F(T)$.

Iterative approximation of fixed points and zeros of nonlinear operators has been studied extensively by many authors to solve nonlinear operator equations as well as variational inequality problems

Zegeye and Shahzad [29] studied the following iterative scheme: $x_0 \in C$,

$$(4)\qquad \begin{aligned} y_n &= \Pi_C J^{-1}(Jx_n - \alpha_n Ax_n)\\ z_n &= Ty_n\\ H_0 &= \{v \in C : \phi(v, z_0) \leq \phi(v, y_0) \leq \phi(v, x_0)\}\\ H_n &= \{v \in H_{n-1} \cap W_{n-1} : \phi(v, z_n) \leq \phi(v, y_n) \leq \phi(v, x_n)\}\\ W_0 &= C\\ W_n &= \{v \in H_{n-1} \cap W_{n-1} : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\}\\ x_{n+1} &= \Pi_{H_n \cap W_n}(x_0), n \geq 0. \end{aligned}$$

They proved that the sequence $\{x_n\}_{n\geq 0}$ generated by (4) converges strongly to a common element of the set of fixed points of weak relatively nonexpansive mapping T and the set of solutions of a certain variational inequality problem under appropriate conditions.

In [24], Takahashi and Zembayashi introduced the following iterative scheme in a uniformly smooth and uniformly convex Banach space: $x_0 \in C$,

$$(5)\qquad \begin{aligned} y_n &= (J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n))\\ u_n &\in C : F(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \ \forall\ y \in C\\ H_n &= \{v \in C : \phi(v, u_n) \leq \phi(v, x_n)\}\\ W_n &= \{v \in C : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\}\\ x_{n+1} &= \Pi_{H_n \cap W_n}(x_0), n \geq 0, \end{aligned}$$

for all $n \geq 0$, where T is relatively nonexpansive mapping, $\{\alpha_n\}_{n\geq 1} \subset [0,1]$ and $\{r_n\}_{n\geq 1} \subset [a, \infty)$ satisfy some appropriate conditions. They proved that $\{x_n\}_{n\geq 0}$ generated by (5) converges strongly to $z = \Pi_{F(T)\bigcap EP(F)}(x_0)$.

To find a common element of $F^* = F(T) \cap F(S) \cap GEP(f, A)$, Chang et al. [8] introduced the following iterative scheme on a uniformly smooth and uniformly convex Banach space: $x_0 \in C$,

$$
\begin{array}{rcl}
z_n &=& J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\
y_n &=& J^{-1}(\beta_n Jx_n + (1-\beta_n)JSx_n), \\
u_n &\in& C : f(u_n, y) + \langle Au_n, y - u_n\rangle + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0 \ \forall\ y \in C, \\
H_n &=& \{v \in C : \phi(v, u_n) \leq \beta_n\phi(v, x_n) + (1-\beta_n)\phi(v, z_n) \leq \phi(v, x_n)\}; \\
W_n &=& \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\} \\
x_{n+1} &=& \Pi_{H_n \cap W_n}(x_0), n \geq 0,
\end{array} \tag{6}
$$

where $J : E \to E^*$ is the normalized duality mapping, T and S are relatively nonexpansive mappings; $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ are sequences in [0, 1] and $\{r_n\}_{n\geq 0} \subset [a, \infty)$, satisfying appropriate conditions. Furthermore, they proved that $\{x_n\}_{n\geq 0}$ generated by (6) converges strongly to $\Pi_{F^*}(x_0)$, where Π_{F^*} is the generalized projection of E onto F^*.

Recently, for finding a common element of $F' = F(S) \cap VI(A, C) \cap EP(f)$, Kang, et al. [17] introduced the following scheme in a 2-uniformly convex and uniformly smooth Banach space E: $x_0 \in C$,

$$
\begin{array}{rcl}
y_n &=& \Pi_C J^{-1}(Jx_n - \beta_n Ax_n) \\
z_n &=& J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSy_n) \\
u_n &\in& C : F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jz_n\rangle \geq 0 \ \forall\ y \in C \\
H_0 &=& \Big\{v \in C : \phi(v, u_0) \leq \phi(v, z_0) \\
&& \leq \alpha_0\phi(v, x_0) + (1-\alpha_0)\phi(v, y_0) \leq \phi(v, x_0)\Big\} \\
H_n &=& \Big\{v \in H_{n-1} \cap W_{n-1} : \phi(v, u_n) \leq \phi(v, z_n) \\
&& \leq \alpha_n\phi(v, x_n) + (1-\alpha_n)\phi(v, y_n) \leq \phi(v, x_n)\Big\} \\
W_0 &=& C \\
W_n &=& \{v \in H_{n-1} \cap W_{n-1} : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\} \\
x_{n+1} &=& \Pi_{H_n \cap W_n}(x_0), n \geq 0,
\end{array}
$$

where S is weak relatively nonexpansive, $\{\alpha_n\}_{n\geq 0} \subset [0, 1]$ and $\{r_n\}_{n\geq 0} \subset [a, \infty)$ satisfy certain conditions. Furthermore, they proved that $\{x_n\}_{n\geq 0}$ and $\{u_n\}_{n\geq 0}$ converges strongly to $z \in F$, where $z = \Pi_F(x_0)$.

Remark 3. Though all the iteration processes, as introduced by the authors mentioned above, worked it is easy to see that these processes seem cumbersome and complicated in the sense that at each stage of iteration, two different sets H_n and W_n are computed and the next iterate taken as the generalized projection of x_0 on the intersection of H_n and W_n. This seems difficult to do in application. Furthermore, apart from the scheme studied by Chang et al.[8] in which they considered two weak relatively nonexpansive mappings, we observe that the other authors mentioned above considered only a single weak relatively nonexpansive operator T.

Motivated and inspired by the results of these authors, it is our aim in this paper, to introduce a refined hybrid iterative scheme for finding a common element of null spaces of finite family of inverse strongly monotone mappings, a set of common fixed points of a finite family of weak relatively nonexpansive mappings and the set of solutions of generalized mixed equilibrium problem; and which involves computation of only one set C_n at each stage of iteration. Our iteration scheme seems simpler and better than those studied in [29, 24, 8, 17]. Our theorems improve and generalize the results of Chang *et al.* [8], Kang, et al. [17], Takahashi and Zambayeshi [24], Zegeye [29] and a host of other authors. The application of our theorem to iterative approximation of solution of equation of Hammerstein-type is of indepeendent interest.

2. Preliminaries

A real normed space $(E, ||\cdot||)$ is said to be strictly convex if $\frac{||x+y||}{2} < 1$ for all $x, y \in E$, with $||x|| = 1, ||y|| = 1$ and $x \neq y$. E is said to be uniformly convex if for each $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in E$ with $||x|| \leq 1, ||y|| \leq 1$ and $||x - y|| \geq \epsilon$, $||x + y|| \leq 2(1 - \delta)$ holds. The modulus of uniform convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - ||\tfrac{1}{2}(x + y)|| : ||x|| \leq 1, ||y|| \leq 1, ||x - y|| \geq \epsilon\},$$

for all $\epsilon \in [0, 2]$. It is known that E is uniformly convex if and only if $\delta_E(\epsilon) > 0 \ \forall \ \epsilon \in (0, 2]$. It is also well known (see e.g., [14, 27]) that $\delta_E(\epsilon)$ is a strictly increasing continuous function with $\delta_E(0) = 0$; furthermore $\frac{\delta_E(\epsilon)}{\epsilon}$ is nondecreasing for all $\epsilon \in (0, 2]$. Let $p > 1$, then E is said to be *p-uniformly convex* if there exists a constant $c > 0$ such that $\delta_E(\epsilon) \geq c\epsilon^p$, $\forall \ \epsilon \in (0, 2]$. It is easy to see that every p-uniformly convex space is uniformly convex. It is well known (see e.g., [26]) that

$$L^p(l^p) \text{ or } W^p_m \text{ is} \begin{cases} p - \text{uniformly convex}, & \text{if } p \geq 2; \\ 2 - \text{uniformly convex}, & \text{if } 1 < p \leq 2, \end{cases}$$

and that Hilbert spaces are 2-uniformly convex Banach spaces. A real normed space $(E, ||\cdot||)$ (with dual space E^*) is said to be smooth if and only if for all $x \in E$ with $||x|| = 1$, there exists a unique $f^* \in E^*$ such that $||f^*|| = 1$ and $\langle x, f^* \rangle = ||x||$. The space E is said to be uniformly smooth if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $||x|| = 1$ and $||y|| \leq \delta$, the inequality $\dfrac{||x + y|| + ||x - y||}{2} - 1 \leq \epsilon||y||$ holds. It is well known that every uniformly smooth real Banach space is smooth and reflexive.

In the sequel, we shall make use of the following lemmas.

Lemma 4. *([2, 26]). Let E be a 2-uniformly convex real Banach space. Then for all $x, y \in E$, the inequality $||x-y|| \leq \frac{2}{c^2}||Jx - Jy||$ holds, where J is the normalized duality mapping on E and $0 < c \leq 1$.*

Lemma 5. *([1, 16]). Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then the following conclusions hold:*

(1) $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$ $x \in E$ *and* $\forall\ y \in C$,

(2) *Let* $x \in E$ *and* $z \in C$, *then* $z = \Pi_C(x) \Leftrightarrow \langle y - z, Jx - Jz\rangle \leq 0\ \ \forall\ y \in C$.

Lemma 6. *([16]). Let E be smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$ then* $\lim\limits_{n\to\infty} ||x_n - y_n|| = 0$.

Lemma 7. *([1]). Let E be uniformly convex real Banach space, $\lambda \in [0, 1]$ and $x, y \in E$. Then*

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - 2\lambda(1-\lambda)C^2\delta_E\Big(\frac{||x-y||}{2C}\Big),$$

where $C = \sqrt{\frac{||x||^2+||y||^2}{2}}$.

Remark 8. (See [1]). If $||x|| \leq R$ and $||y|| \leq R$, for some positive number R, then in Lemma 7 we have that $C \leq R$ and $2C^2\delta_E\Big(\frac{||x-y||}{2C}\Big) \geq R^2\frac{\delta_E\Big(\frac{||x-y||}{2R}\Big)}{2L^*}$, where L^* is a constant called the *Figiel constant* and is such that $1 < L^* < 1.7$.

For solving the equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, we assume that f satisfies the following conditions:

(A1): $f(x, x) = 0$ for all $x \in C$;

(A2): f is monotone, i.e., $f(x, y) + f(y, x) \leq 0\ \forall\ x, y \in C$;

(A3): for each $x, y, z \in C, \limsup_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;

(A4): The map $y \mapsto f(x, y)$ is convex and lower semi-continuous.

Lemma 9. *([3]) Let C be a closed convex subset of a smooth, strictly convex and reflexive real Banach space E and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). Let $r > 0$ and $x \in E$, then there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0\ \forall\ y \in C.$$

By similar argument of the proofs of Lemmas 2.8 and 2.9 of [24], we have the following two lemmas.

Lemma 10. *Let C be a closed convex subset of a smooth, strictly convex and reflexive real Banach space E. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4), $B : C \to E^*$ be a monotone mapping and let $\Phi : C \to \mathbb{R}$ be a lower semi-continuous convex function. For*

$r > 0$ and $x \in E$, define a mapping $T_r : C \to C$ as follows: $T_r(x) = \Big\{ z \in C : f(z, y) + \Phi(y) - \Phi(z) + \langle y - z, Bx \rangle + \frac{1}{r} \langle y - z, Jz - Jx \rangle \geq 0, \ \forall \ y \in C \Big\}$ for all $x \in E$. Then , the following hold:

(1): *T_r is single-valued;*

(2): *T_r is a firmly nonexpansive-type mapping [19], i.e., for all $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y \rangle \leq \langle T_r x - T_r y, Jx - Jy \rangle;$$

(3): *$F(T_r) = GMEP(f, \Phi, B)$;*

(4): *$GMEP(f, \Phi, B)$ is closed and convex.*

Lemma 11. *Let C be a closed convex subset of a smooth, strictly convex and reflexive real Banach space E and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). For $r > 0$, $x \in E$ and $p \in F(T_r)$, we have*

$$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x).$$

We make use of the function $V : E \times E^* \to \Re$ defined by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^* \rangle + ||x^*||^2,$$

for all $x \in E$ and $x^* \in E^*$, which was studied by Alber in [1]. Observe that, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. The following lemma is well known.

Lemma 12. *([1]). Let E be a reflexive strictly convex and smooth Banach space with E^* as its dual. Then $V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*)$, for all $x \in E$ and $x^*, y^* \in E^*$.*

3. Main results

Let C be a closed convex nonempty subset of a 2-uniformly convex and uniformly smooth real Banach space E with dual space E^*. we shall study the strong convergence of sequence of iteration $\{x_n\}_{n \geq 0}$ which is generated as follows:

$$\begin{aligned}
&x_0 \in C_0 = C; \ y_n = \Pi_C(J^{-1}(Jx_n - \beta A_{n+1} x_n));\\
&z_n = J^{-1}(\alpha J x_n + (1 - \alpha) J T_{n+1} y_n);\\
&u_n \in C \text{ such that } G(u_n, z_n, y) \geq 0 \ \forall \ \ y \in C;\\
&C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\};\\
&x_{n+1} = \Pi_{C_{n+1}}(x_0), \ n \geq 0,
\end{aligned} \tag{7}$$

where $G(u_n, z_n, y) = f(u_n, y) + \Phi(y) - \Phi(u_n) + \langle y - u_n, Bu_n \rangle + \frac{1}{r_n} \langle y - u_n, Ju_n - Jz_n \rangle$, $A_n = A_{n \ (mod \ m)}$, $T_n = T_{n \ (mod \ \omega)}$, with the mod functions taking values in the sets $\{1, 2, ..., m\}$ and $\{1, 2, ..., \omega\}$ respectively; $r_n \in [d, \infty)$ for some $d > 0$; β is such the $0 < \beta < \frac{c^2 \gamma}{2}$, where c is the 2-uniformly convex constant of E and $\alpha \in (0, 1)$ is fixed.

Theorem 13. *Let C be a closed convex nonempty subset of a 2-uniformly convex and uniformly smooth real Banach space E with dual space E^*. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4) and let $T_j : C \to E, j = 1, 2, 3, ..., \omega$ be a finite family of weak relatively nonexpansive mappings. Let $A_i : C \to E^*, i = 1, 2, 3, ..., m$ be a finite family of γ_i-inverse strongly monotone mappings and let $\gamma = min\{\gamma_i, i = 1, 2, 3, ..., m\}$. Let $\Phi : C \to \mathbb{R}$ be a lower semi-continuous convex function and let $B : C \to E^*$ be a continuous monotone function. Suppose $F := \bigcap_{j=1}^{\omega} F(T_j) \cap \bigcap_{i=1}^{m} A_i^{-1}(0) \cap GMEP(f, \Phi, B) \neq \varnothing$ and the sequence $\{x_n\}_{n\geq 0}$ is defined iteratively by (7), then $\{x_n\}_{n\geq 0}$ converges strongly to some element of F.*

Proof. The proof is divided into five steps.
Step 1: We show that the sequence $\{x_n\}_{n\geq 0}$ is well defined.

It is enough to show that C_n is closed, convex and nonempty for each $n \geq 0$. That C_n is equal to its closure follows easily from its definition for all $n \geq 0$. Convexity of C_n follows from the fact that, for $z \in C_n$, $\phi(z, u_{n-1}) \leq \phi(z, x_{n-1})$ is equivalent to $2\langle z, Jx_{n-1} - Ju_{n-1}\rangle - ||x_{n-1}||^2 + ||u_{n-1}||^2 \leq 0$. Next, we show that $F \subset C_n$ for each $n \geq 0$. We do this by induction on n. Clearly, $F \subset C_0 = C$. Suppose $F \subset C_k$ for some $k \geq 1$. Let $p \in F$, then by Lemma 11 and using the fact that $|| \cdot ||^2$ is convex and T_j is weak relatively nonexpansive for each $j = 1, 2, 3, ..., \omega$, we have that

$$\begin{aligned}
\phi(p, u_k) &= \phi(p, T_{r_k} z_k) \\
&\leq \phi(p, z_k) \\
&= \phi\Big(p, J^{-1}(\alpha J x_k + (1-\alpha) J T_{k+1} y_k)\Big) \\
&= ||p||^2 - 2\langle p, \alpha J x_k + (1-\alpha) J T_{k+1} y_k\rangle \\
&\quad + ||\alpha J x_k + (1-\alpha) J T_{k+1} y_k||^2 \\
&\leq ||p||^2 - 2\alpha\langle p, J x_k\rangle - 2(1-\alpha)\langle p, J T_{k+1} y_k\rangle \\
&\quad + \alpha ||x_k||^2 + (1-\alpha)||T_{k+1} y_k||^2 \\
&= \alpha\phi(p, x_k) + (1-\alpha)\phi(p, T_{k+1} y_k) \\
&\leq \alpha\phi(p, x_k) + (1-\alpha)\phi(p, y_k). \qquad (8)
\end{aligned}$$

Moreover, by Lemmas 4, 5 and 12 we have that,

$$
\begin{aligned}
\phi(p, y_k) &= \phi(p, \Pi_C J^{-1}(Jx_k - \beta A_{k+1} x_k)) \\
&\leq \phi(p, J^{-1}(Jx_k - \beta A_{k+1} x_k)) \\
&= V(p, Jx_k - \beta A_{k+1} x_k) \\
&\leq V\Big(p, (Jx_k - \beta A_{k+1} x_k) + \beta A_{k+1} x_k\Big) \\
&\quad -2\big\langle J^{-1}(Jx_k - \beta A_{k+1} x_k) - p, \beta A_{k+1} x_k\big\rangle. \\
&= V(p, Jx_k) - 2\beta\big\langle J^{-1}(Jx_k - \beta A_{k+1} x_k) - p, A_{k+1} x_k\big\rangle \\
&= \phi(p, x_k) - 2\beta\big\langle x_k - p, A_{k+1} x_k\big\rangle \\
&\quad -2\beta\big\langle J^{-1}(Jx_k - \beta A_{k+1} x_k) - x_k, A_{k+1} x_k\big\rangle \\
&\leq \phi(p, x_k) - 2\beta\gamma ||A_{k+1} x_k||^2 \\
&\quad +2\beta ||J^{-1}(Jx_k - \beta A_{k+1} x_k) - J^{-1} Jx_k|| \times ||A_{k+1} x_k|| \\
&\leq \phi(p, x_k) - 2\beta\gamma ||A_{k+1} x_k||^2 + \frac{4\beta^2}{c^2} ||A_{k+1} x_k||^2 \\
&\leq \phi(p, x_k) + 2\beta\Big(\frac{2}{c^2}\beta - \gamma\Big) ||A_{k+1} x_k||^2. \qquad (9)
\end{aligned}
$$

Thus, using the fact that $\beta \leq \frac{c^2}{2}\gamma$, we have that

$$\phi(p, y_k) \leq \phi(p, x_k). \qquad (10)$$

Using (10) in (8) gives

$$\phi(p, u_k) \leq \phi(p, x_k). \qquad (11)$$

That is, $p \in C_{k+1}$. Then, by induction $F \subset C_n$ for each $n \geq 0$, hence the sequence $\{x_n\}_{n\geq 0}$ is well defined.

Step 2: We show that the sequence $\{x_n\}_{n\geq 0}$ is bounded and Cauchy.

From $x_n = \Pi_{C_n}(x_0)$ and by Lemma 5, we have

$$\phi(x_n, x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0)$$

for each $p \in F \subset C_n$ and for $n \geq 0$. This implies that the sequence $\{\phi(x_n, x_0)\}_{n\geq 0}$ is bounded. Since $x_n = \Pi_{C_n}(x_0)$ and $x_{n+1} = \Pi_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$. We also have from Lemma 5 that

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \qquad (12)$$

which implies that $\phi(x_n, x_0)$ is nondecreasing and by inequality (3) bounded below by 0. So, $\lim\limits_{n\to\infty} \phi(x_n, x_0)$ exists. Next, for any positive integer m, and using Lemma 5 again we have that

$$\phi(x_{n+m}, x_n) = \phi(x_{n+m}, \Pi_{C_n} x_0) \leq \phi(x_{n+m}, x_0) - \phi(x_n, x_0) \qquad (13)$$

for all $n \geq 0$. Since $\lim\limits_{n\to\infty} \phi(x_n, x_0)$ exists, we obtain from (13) that

$$\phi(x_{n+m}, x_n) \to 0 \text{ as } n \to \infty. \tag{14}$$

Thus, by Lemma 6, we have that $||x_{n+m} - x_n|| \to 0$ as $n \to \infty$. This implies that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence in C, and since C is complete being a closed nonempty subset of real Banach space E, there exists an x^* in C such that

$$x_n \to x^* \text{ as } n \to \infty. \tag{15}$$

Step 3: We show that $x^* \in \bigcap\limits_{j=1}^{\omega} F(T_j)$.

Since $x_{n+1} \in C_{n+1}$, we have that,

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \tag{16}$$

and by (14) and Lemma 6, we get that $\lim\limits_{n\to\infty} ||x_{n+1} - u_n|| = 0$. But

$$||x_n - u_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - u_n||.$$

Thus,

$$\lim_{n\to\infty} ||x_n - u_n|| = 0. \tag{17}$$

This implies that $u_n \to x^*$ as $n \to \infty$, and since J is norm-to-norm uniformly continuous on bounded subsets of E we have from (17) that

$$\lim_{n\to\infty} ||Jx_n - Ju_n|| = 0. \tag{18}$$

Now, since E is a uniformly smooth real Banach space we have that E^* is uniformly convex and so for $p \in F$ and the fact that $\{x_n\}_{n\geq 0}$, $\{T_j x_n\}_{n\geq 0}, j = 1, 2, ..., \omega$ are all bounded, we obtain by Lemma 7, Remark 8, Lemma 11 and using (8) and (9) that

$$
\begin{aligned}
\phi(p,u_n) &= \phi(p,T_{r_n}z_n)\\
&\leq \phi(p,z_n)\\
&= \phi\Big(p, J^{-1}(\alpha Jx_n+(1-\alpha)JT_{n+1}y_n)\Big)\\
&= ||p||^2-2\big\langle p,\alpha Jx_n+(1-\alpha)JT_{n+1}y_n\big\rangle+||\alpha Jx_n+(1-\alpha)JT_{n+1}y_n||^2\\
&\leq ||p||^2-2\alpha\big\langle p,Jx_n\big\rangle-2(1-\alpha)\big\langle p,JT_{n+1}y_n\big\rangle\\
&\quad +\alpha||x_n||^2+(1-\alpha)||T_{n+1}y_n||^2-\alpha(1-\alpha)M_0^2\frac{\delta_E\Big(\frac{||Jx_n-JT_{n+1}y_n||}{2M_0}\Big)}{2L^*}\\
&= \alpha\phi(p,x_n)+(1-\alpha)\phi(p,T_{n+1}y_n)-\alpha(1-\alpha)M_0^2\frac{\delta_E\Big(\frac{||Jx_n-JT_{n+1}y_n||}{2M_0}\Big)}{2L^*}\\
&\leq \alpha\phi(p,x_n)+(1-\alpha)\phi(p,y_n)-\alpha(1-\alpha)M_0^2\frac{\delta_E\Big(\frac{||Jx_n-JT_{n+1}y_n||}{2M_0}\Big)}{2L^*}\\
&\leq \alpha\phi(p,x_n)+(1-\alpha)\phi(p,x_n)-\alpha(1-\alpha)M_0^2\frac{\delta_E\Big(\frac{||Jx_n-JT_{n+1}y_n||}{2M_0}\Big)}{2L^*}\\
&= \phi(p,x_n)-\alpha(1-\alpha)M_0^2\frac{\delta_E\Big(\frac{||Jx_n-JT_{n+1}y_n||}{2M_0}\Big)}{2L^*}, \qquad (19)
\end{aligned}
$$

for some constant $M_0>0$ such that $\sup\limits_{n\geq 0}\{\|x_n\|, \max\limits_{1\leq j\leq \omega}\|T_jx_n\|\}\leq M_0$. Thus,

$$
\delta_E\Big(\frac{||Jx_n-JT_{n+1}y_n||}{2M_0}\Big)\leq \frac{2L^*}{M_0^2\alpha(1-\alpha)}\Big(\phi(p,x_n)-\phi(p,u_n)\Big), \forall\ n\geq 0. \qquad (20)
$$

Using (17) and (18) and definition of ϕ, it is easy to see that

$$
\lim_{n\to\infty}\big(\phi(p,x_n)-\phi(p,u_n)\big)=0. \qquad (21)
$$

Thus, we obtain from (20) using (21) that

$$
\lim_{n\to\infty}\delta_E\Big(\frac{||Jx_n-JT_{n+1}y_n||}{2M_0}\Big)=0. \qquad (22)
$$

By the properties of δ_E we get that $\lim\limits_{n\to\infty}||Jx_n-JT_{n+1}y_n||=0$, and since J^{-1} is norm-to-norm uniformly continuous on bounded subsets of E^*, we have that

$$
\lim_{n\to\infty}||x_n-T_{n+1}y_n||=0. \qquad (23)
$$

Moreover, using inequalities (8) and (9), we obtain that

$$
\begin{aligned}
\phi(p,u_n) &\leq \alpha\phi(p,x_n)+(1-\alpha)\phi(p,y_n)\\
&\leq \alpha\phi(p,x_n)+(1-\alpha)\Big[\phi(p,x_n)+2\beta\Big(\frac{2}{c^2}\beta-\gamma\Big)||A_{n+1}x_n||^2\Big]\\
&= \phi(p,x_n)+2(1-\alpha)\beta\Big(\frac{2}{c^2}\beta-\gamma\Big)||A_{n+1}x_n||^2. \qquad (24)
\end{aligned}
$$

Inequality (24) implies that

$$
2(1-\alpha)\beta(\gamma-\frac{2}{c^2}\beta)||A_{n+1}x_n||^2\leq \phi(p,x_n)-\phi(p,u_n). \qquad (25)
$$

Thus, we obtain from (25) using (21) that

$$\lim_{n\to\infty} ||A_{n+1}x_n||^2 = 0. \tag{26}$$

Furthermore, since $x_n \in C$ for all $n \geq 0$, then using Lemmas 4, 5 and 12, we get

$$\begin{aligned}
\phi(x_n, y_n) &= \phi(x_n, \Pi_C J^{-1}(Jx_n - \beta A_{n+1}x_n)) \\
&\leq \phi(x_n, J^{-1}(Jx_n - \beta A_{n+1}x_n)) \\
&= V(x_n, Jx_n - \beta A_{n+1}x_n) \\
&\leq V(x_n, (Jx_n - \beta A_{n+1}x_n + \beta A_{n+1}x_n)) \\
&\quad -2\big\langle J^{-1}(Jx_n - \beta A_{n+1}x_n) - x_n, \beta A_{n+1}x_n\big\rangle \\
&= \phi(x_n, x_n) + 2\big\langle J^{-1}(Jx_n - \beta A_{n+1}x_n) - J^{-1}Jx_n, -\beta A_{n+1}x_n\big\rangle \\
&\leq 2||J^{-1}(Jx_n - \beta A_{n+1}x_n) - J^{-1}Jx_n|| \times \beta||A_{n+1}x_n|| \\
&\leq \frac{4}{c^2}\beta^2||A_{n+1}x_n||^2. \qquad (27)
\end{aligned}$$

Thus, from (26) and (27) we have that $\lim_{n\to\infty} \phi(x_n, y_n) = 0$; and this (by Lemma 6) implies that

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{28}$$

Moreover, $||y_n - T_{n+1}y_n|| \leq ||y_n - x_n|| + ||x_n - T_{n+1}y_n||$. Thus, using (23) and (28) we have that

$$\lim_{n\to\infty} ||y_n - T_{n+1}y_n|| = 0. \tag{29}$$

Also, since $\lim_{n\to\infty} x_n = x^*$, we easily see from (28) that $\lim_{n\to\infty} y_n = x^*$. So, for every subsequence $\{y_{n_s}\}_{s\geq 1}$ of $\{y_n\}_{n\geq 0}$, $\lim_{s\to\infty} y_{n_s} = x^*$ and

$$\lim_{s\to\infty} ||y_{n_s} - T_{n_s+1}y_{n_s}|| = 0. \tag{30}$$

Since the family $T_j : C \to E, j = 1, 2, ..., \omega$ is a finite pool of weak relatively nonexpansive mappings, let $\{n_k\}_{k\geq 1} \subset \mathbb{N}$ be an increasing sequence of natural numbers such that $T_{n_k+1} = T_1, \forall\ k \in \mathbb{N}$. Then $\lim_{k\to\infty} ||y_{n_k} - x^*|| = 0$ and

$$0 = \lim_{k\to\infty} ||y_{n_k} - T_{n_k+1}y_{n_k}|| = \lim_{k\to\infty} ||y_{n_k} - T_1y_{n_k}||. \tag{31}$$

This implies that $x^* \in \tilde{F}(T_1) = F(T_1)$.

Next, let $\{n_l\}_{l\geq 1} \subset \mathbb{N}$ be an increasing sequence of natural numbers such that $T_{n_l+1} = T_2, \forall\ l \in \mathbb{N}$. Then $\lim_{l\to\infty} ||y_{n_l} - x^*|| = 0$ and

$$0 = \lim_{l\to\infty} ||y_{n_l} - T_{n_l+1}y_{n_l}|| = \lim_{l\to\infty} ||y_{n_l} - T_2y_{n_l}||. \tag{32}$$

Thus, $x^* \in \tilde{F}(T_2) = F(T_2)$.

Similar argument shows that $x^* \in \tilde{F}(T_j) = F(T_j), \forall\ j = 1, 2, 3, ..., \omega$. Hence

$$x^* \in \bigcap_{j=1}^{\omega} F(T_j). \tag{33}$$

Step 4: Next, we show that $x^* \in \bigcap_{i=1}^{m} A_i^{-1}(0)$.

It follows from (26) that

$$\lim_{n\to\infty} ||A_{n+1}x_n|| = 0. \tag{34}$$

Now, since $\lim_{n\to\infty} x_n = x^*$ we have that for every subsequence $\{x_{n_j}\}_{j\geq 1}$ of $\{x_n\}_{n\geq 0}$, $\lim_{j\to\infty} x_{n_j} = x^*$ and

$$\lim_{j\to\infty} A_{n_j+1}x_{n_j} = 0. \tag{35}$$

Let $\{n_q\}_{q\geq 1} \subset \mathbb{N}$ be an increasing sequence of natural numbers such that $A_{n_q+1} = A_1, \forall\, q \in \mathbb{N}$. Then $\lim_{q\to\infty} ||x_{n_q} - x^*|| = 0$ and

$$0 = \lim_{q\to\infty} A_{n_q+1}x_{n_q} = \lim_{q\to\infty} A_1 x_{n_q}. \tag{36}$$

Since A_1 is γ-inverse strongly monotone, it is Lipschitz continuous and thus $A_1x^* = A_1(\lim_{q\to\infty} x_{n_q}) = \lim_{q\to\infty} A_1x_{n_q} = 0$. Hence,

$$x^* \in A_1^{-1}(0). \tag{37}$$

Continuing this process, we obtain that $x^* \in A_i^{-1}(0)\,\forall\; i = 1, 2, 3, ..., m$. Hence,

$$x^* \in \bigcap_{i=1}^{m} A_i^{-1}(0). \tag{38}$$

Step 5: Finally, we show that $x^* \in GMEP(f, \Phi, B)$.

For this purpose we define a function $\Gamma : C \times C \to \mathbb{R}$ by

$$\Gamma(x, y) = f(x, y) + \Phi(y) - \Phi(x) + \big\langle y - x, Bx\big\rangle\ \forall\; x, y \in C.$$

We prove that Γ satisfies conditions (A1)-(A4). We proceed as follows:

a) **Condition (A1).** Observe that $\forall\; x \in C$

$$\Gamma(x, x) = f(x, x) + \Phi(x) - \Phi(x) + \big\langle 0, Bx\big\rangle = f(x, x) = 0.$$

b) **Condition (A2).** Since f satisfies condition (A2) and B is a monotone mapping, we have that

$$\begin{aligned}
\Gamma(x, y) + \Gamma(y, x) &= f(x, y) + \Phi(y) - \Phi(x)\\
&\quad + \big\langle y - x, Bx\big\rangle + f(y, x) + \Phi(x) - \Phi(y) + \big\langle x - y, By\big\rangle\\
&= f(x, y) + f(y, x) + \big\langle y - x, Bx - By\big\rangle\\
&\leq \big\langle y - x, Bx - By\big\rangle = -\big\langle y - x, By - Bx\big\rangle \leq 0.
\end{aligned}$$

c) **Condition (A3).** Let $t \in (0,1)$, then for $x, y, z \in C$ we have that $\lim_{t\to 0}(tz + (1-t)x) = x$. Therefore, using the fact that B is continuous and Φ is lower semi-continuous we obtain

$$\begin{aligned}
\limsup_{t\downarrow 0} \Gamma(tz + (1-t)x, y) &\leq \limsup_{t\downarrow 0} f(tz + (1-t)x, y) \\
&\quad + \limsup_{t\downarrow 0} \big\langle y - (tz + (1-t)x), B(tz + (1-t)x)\big\rangle \\
&+ \Phi(y) - \liminf_{t\downarrow 0} \Phi(tz + (1-t)x) \\
&= \limsup_{t\downarrow 0} f(tz + (1-t)x, y) \\
&\quad + \lim_{t\downarrow 0} \big\langle y - (tz + (1-t)x), B(tz + (1-t)x)\big\rangle \\
&+ \Phi(y) - \liminf_{t\downarrow 0} \Phi(tz + (1-t)x) \\
&\leq f(x, y) + \Phi(y) - \Phi(x) + \big\langle y - x, Bx\big\rangle = \Gamma(x, y)
\end{aligned}$$

Thus, $\limsup_{t\downarrow 0} \Gamma(tz + (1-t)x, y) \leq \Gamma(x, y)$.

d) **Condition (A4).** For $x, y, z \in C$ and $t \in [0,1]$, we obtain using the properties of f, Φ and B that

$$\begin{aligned}
\Gamma(x, ty + (1-t)z) &= f(x, ty + (1-t)z) + \big\langle ty + (1-t)z, Bx\big\rangle \\
&\quad + \Phi(ty + (1-t)z) - \Phi(x) \\
&\leq t\Big(f(x, y) + \big\langle y - x, Bx\big\rangle + \Phi(y) - \Phi(x)\Big) \\
&+ (1-t)\Big(f(x, z) + \big\langle z - x, Bx\big\rangle + \Phi(z) - \Phi(x)\Big) \\
&= t\Gamma(x, y) + (1-t)\Gamma(x, z).
\end{aligned}$$

Thus, the function $y \to \Gamma(x, y)$ is convex. It is easy to show that $y \to \Gamma(x, y)$ is lower semi-continuous. So, Γ satisfies conditions $(A1) - (A4)$.

Now, from $u_n = T_{r_n} z_n$, $p \in F$ and Lemma 11 we have

$$\begin{aligned}
\phi(u_n, z_n) &= \phi(T_{r_n} z_n, z_n) \\
&\leq \phi(p, z_n) - \phi(p, T_{r_n} z_n) \\
&\leq \phi(p, x_n) - \phi(p, T_{r_n} z_n) \\
&= \phi(p, x_n) - \phi(p, u_n), \qquad (39)
\end{aligned}$$

which implies from (21) that $\lim_{n\to\infty} \phi(u_n, z_n) = 0$.
Since E is 2-uniformly convex and uniformly smooth Banach space and $\{u_n\}$ is bounded, we have from Lemma 6 that

$$\lim_{n\to\infty} ||u_n - z_n|| = 0. \qquad (40)$$

From $x_n \to x^*$ and $u_n \to x^*$ as $n \to \infty$, we obtain from (40) that $z_n \to x^*$ as $n \to \infty$. Since J is norm-to-norm uniformly continuous on bounded subsets of E, we get from (40)

$$\lim_{n\to\infty} ||Ju_n - Jz_n|| = 0, \tag{41}$$

and since $r_n \in [d, +\infty)$ for some $d > 0$, we have that

$$\lim_{n\to\infty} \frac{||Ju_n - Jz_n||}{r_n} = 0. \tag{42}$$

Next, since $\Gamma(u_n, y) + \frac{1}{r_n}\big\langle y - u_n, Ju_n - Jz_n\big\rangle \geq 0 \; \forall \; y \in C$, this implies that

$$\frac{1}{r_n}\big\langle y - u_n, Ju_n - Jz_n\big\rangle \geq -\Gamma(u_n, y) \geq \Gamma(y, u_n), \; \forall \; y \in C. \tag{43}$$

This implies that

$$\Gamma(y, u_n) \leq \frac{1}{r_n}\big\langle y - u_n, Ju_n - Jz_n\big\rangle \leq \Big(M_1 + \|y\|\Big)\frac{\|Ju_n - Jz_n\|}{r_n}, \tag{44}$$

for some $M_1 \geq 0$. Since $y \to \Gamma(x, y)$ is a convex and lower semi-continuous, we obtain form (44) that

$$\Gamma(y, x^*) \leq \liminf_{n\to\infty} \Gamma(y, u_n) \leq 0 \; \forall \; y \in C. \tag{45}$$

Now, for $t \in (0, 1)$ and $y \in C$, let $y_t = ty + (1-t)x^*$. Since $y \in C$ and $x^* \in C$ then $y_t \in C$ and so from (45), $\Gamma(y_t, x^*) \leq 0$. But, from conditions (A1) and (A4) we have that

$$0 = \Gamma(y_t, y_t) \leq t\Gamma(y_t, y) + (1-t)\Gamma(y_t, x^*) \leq t\Gamma(y_t, y).$$

So, $\Gamma(y_t, y) \geq 0, \forall \; y \in C$ and condition (A3) implies that $\Gamma(x^*, y) \geq \limsup_{t\to 0} \Gamma(y_t, y) \geq 0 \;\; \forall \; y \in C$. Thus $x^* \in EP(\Gamma) = GMEP(f, \Phi, B)$. Hence,

$$x^* \in F := \bigcap_{j=1}^{\omega} F(T_j) \cap \bigcap_{i=1}^{m} A_i^{-1}(0)) \cap GMEP(f, \Phi, B).$$

This completes the proof. □

The following corollary easily follows from our theorem.

Corollary 14. *Let C be a closed convex nonempty subset of a 2-uniformly convex and uniformly smooth real Banach space E with dual space E^*. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4) and let $T : C \to E$ be a weak relatively nonexpansive mapping. Let $A : C \to E^*$ be a γ-inverse strongly monotone mapping. Let $\Phi : C \to \mathbb{R}$ be a lower semi-continuous convex function and let $B : C \to E^*$ be a continuous monotone function. Suppose that $F := F(T) \cap A^{-1}(0) \cap GMEP(f, \Phi, B) \neq \varnothing$, then the sequence $\{x_n\}_{n\geq 0}$ defined iteratively* *by*

$$
\begin{aligned}
x_0 &\in C_0 = C;\\
y_n &= \Pi_C(J^{-1}(Jx_n - \beta A x_n));\\
z_n &= J^{-1}(\alpha J x_n + (1-\alpha)JTy_n);\\
u_n &\in C \; s.\,t.\; G(u_n, z_n, y) \geq 0 \;\forall\; y \in C;\\
C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\};\\
x_{n+1} &= \Pi_{C_{n+1}}(x_0),\; n \geq 0 \qquad (46)
\end{aligned}
$$

(where G, r_n, $n \in \mathbb{N}$, β and α are as in Theorem 13) converges strongly to some element of F.

4. Application.

Let E be a real Banach space and E^* be its dual. The generalized formulation of many boundary value problems for ordinary and partial differential equations leads to operator equations of the type

$$\langle z, Ax \rangle = \langle z, b \rangle \;\; \forall\; z \in E,$$

which is equivalent to equality of functionals on E. That is, the equality of elements of E^*,

$$Ax = b, \qquad (47)$$

where A is a monotone-type operator acting from a Banach space E into E^*. Without loss of generality we may assume $b = 0$. It is well known that a solution of the equation $Ax = 0$ (i.e., $\langle z, Ax \rangle = 0 \;\forall\; z \in E$) is a solution of the variational inequality $\langle z - x, Ax \rangle \geq 0 \;\forall\; z \in E$. Therefore, the theory of monotone operators and its applications to nonlinear partial differential equations and variational inequalities are related and have evolved into a substantial topic in nonlinear functional analysis. One important application of solving (47) is finding the zeros of the so-called equation of Hammerstein-type (see e.g., [15]), where a nonlinear integral equation of Hammerstein type is one of the form:

$$u(x) + \int_\Omega k(x,y) f(y, u(y)) dy = h(x), \qquad (48)$$

where dy is a σ-finite measure on the measure space Ω; the real kernel k is defined on $\Omega \times \Omega$, f is a real-valued function defined on $\Omega \times \mathbb{R}$ and is, in general nonlinear and h is a given function on Ω. If we now define an operator K by

$$Kv(x) = \int_\omega k(x,y) v(y) dy;\;\; x \in \Omega,$$

and the so-called supperposition or Nemytskii operator by $Fu(y) := f(y, u(y))$, then the integral equation (48) can be put in an operator theoretic form as follows:

$$u + KFu = 0, \qquad (49)$$

where, without loss of generality, we have taken $h = 0$. The Nemytskii operator F is well-defined on a given space E of functions on Ω, and that for each element $u \in E$, Fu lies in a dual space

E^*. If the linear operator K maps the space E^* into the space E, the composition KF of the two operators is well-defined and maps E into itself. Given h in the function space E, the integral equation then asks for some $u \in E$ such that $(I + KF)(u) = h$. We note that, if K and F are monotone, then $A = I + KF$ need not necessarily be monotone. If $E = H$, is a Hilbert space, K and F are monotone and if K is a compact operator, Brežis and Browder [4] proved that a suitably defined Galarkin approximation converges strongly to a solution of equation (48).

Interest in equation (48) stems mainly from the fact that several problems that arise in differential equations, for instance, elliptic boundary value problems whose linear parts possess Green's functions can, as a rule be transformed into equations of the form (48) (see e.g., [22], chapter IV). Equations of the Hammerstian type play a crucial role in the theory of optimal control systems (see e.g., [13]). Several existence and uniqueness theorems have been proved for equations of the Hammerstein type (see e.g., [5, 6, 7, 9, 10, 12]).

We are now ready to give an application of Theorem 13 to an iterative solution of the operator Hammerstein equation (49).

Theorem 15. *Let E be a real Banach space with dual space E^* such that $X = E \times E^*$ (with norm $\|z\|_X^2 = \|u\|_E^2 + \|v\|_{E^*}^2,\ z = (u, v) \in X$) is a 2-uniformly convex and uniformly smooth real Banach space. Let $F : E \to E^*$ and $K : E^* \to E$ with $D(K) = F(E) = E^*$ be continuous monotone type operators such that equation (49) has a solution in E; and such that the map $A : X \to X^*$ defined by $Az := A(u, v) = (Fu - v, u + Kv)$ is γ-inverse strongly monotone. Let C be a closed convex nonempty subset of X. Let $f : C \times C :\to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4) and $T : C \to X$ be a weak relatively nonexpansive mapping. Let $\Phi : C \to \mathbb{R}$ be a lower semi-continuous convex function and $B : C \to X^*$ be a continuous monotone function. Suppose that $F := F(T) \cap A^{-1}(0) \cap GMEP(f, \Phi, B) \neq \varnothing$, then the sequence $\{x_n\}_{n\geq 0}$ defined iteratively by*

$$\begin{aligned}
x_0 &\in C_0 = C;\\
y_n &= \Pi_C(J^{-1}(Jx_n - \beta Ax_n));\\
z_n &= J^{-1}(\alpha Jx_n + (1-\alpha)JTy_n);\\
u_n &\in C\ s.\ t.\ G(u_n, z_n, y) \geq 0\ \forall\ y \in C;\\
C_{n+1} &= \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\};\\
x_{n+1} &= \Pi_{C_{n+1}}(x_0),\ n \geq 0 \qquad (50)
\end{aligned}$$

(where G, r_n, $n \in \mathbb{N}$, β and α are as in Theorem 13) converges strongly to some element $z^* \in F$.

Remark 16. Observe that $z^* \in F$ implies, in particular, that $z^* \in A^{-1}(0) \Leftrightarrow Az^* = 0$. But $z^* = (u^*, v^*)$ for some $u^* \in E$ and $v^* \in E^*$; Moreover, $Az^* = A(u^*, v^*) = (Fu^* - v^*, u^* + Kv^*)$. So, $Az^* = 0$ implies that $(Fu^* - v^*, u^* + Kv^*) = (0, 0)$. This implies that $Fu^* - v^* = 0$ and

$u^* + Kv^* = 0$. Thus, $v^* = Fu^*$ and this implies that $u^* + KFu^* = 0$. Hence, $u^* \in E$ solves the Hammerstein-type equation (49).

The following example gives a prototype of operators F, K and A satisfying the conditions of Theorem 15.

Example 17. Let $F, K : (\mathbb{R}, |.|) \to (\mathbb{R}, |.|)$ be define (respectively) by $F(x) = 2x + c$ for some real constant c and $K(x) = 2x$. Observe that $(I + KF)(x) = x + 4x + 2c = 5x + 2c$. Clearly F and K are continuous monotone mappings. Define $A : (\mathbb{R} \times \mathbb{R}, \|.\|_2) \to (\mathbb{R} \times \mathbb{R}, \|.\|_2)$ by

$$A(x, y) = (Fx - y, x + Ky) = (2x + c - y, x + 2y).$$

Then $A(x_1, x_2) = (2x_1 + c - x_2, x_1 + 2x_2)$, $A(y_1, y_2) = (2y_1 + c - y_2, y_1 + 2y_2)$ and $A(x_1, x_2) - A(y_1, y_2) = \Big(2(x_1 - y_1) - (x_1 - y_1), (x_1 - y_1) + 2(x_2 - y_2)\Big)$. This implies that

$$\begin{aligned} \|A(x_1, x_2) - A(y_1, y_2)\|_2^2 &= \Big[2(x_1 - y_1) - (x_1 - y_1)\Big]^2 + \Big[(x_1 - y_1) + 2(x_2 - y_2)\Big]^2 \\ &= 4(x_1 - y_1)^2 - 4(x_1 - y_1(x_2 - y_2) + (x_2 - y_2)^2 \\ &+ (x_1 - y_1)^2 + 4(x_1 - y_1)(x_2 - y_2) + 4(x_2 - y_2)^2 \\ &= 5\Big[(x_1 - y_1)^2 + (x_2 - y_2)^2\Big]. \end{aligned} \tag{51}$$

On the other hand, we have that

$$\begin{aligned} &\Big\langle (x_1, x_2) - (y_1, y_2), A(x_1, x_2) - A(y_1, y_2)\Big\rangle \\ &= \Big\langle (x_1, x_2) - (y_1, y_2), (2x_1 + c - x_2, x_1 + 2x_2) - (2y_1 + c - y_2, y_1 + 2y_2)\Big\rangle \\ &= \Big\langle (x_1 - y_1, x_2 - y_2), \Big(2(x_1 - y_1) - (x_2 - y_2), (x_1 - y_1 + 2(x_2 - y_2)\Big)\Big\rangle \\ &= 2(x_1 - y_1)^2 - (x_1 - y_1)(x_2 - y_2) + (x_2 - y_2)(x_1 - y_1) + 2(x_2 - y_2)^2 \\ &= 2\Big[(x_1 - y_1)^2 + (x_2 - y_2)^2\Big] = \frac{2}{5}\Big\|A(x_1, x_2) - A(y_1, y_2)\Big\|_2^2. \end{aligned}$$

Hence, A is a γ-inverse strongly monotone mapping with $\gamma \in \Big(0, \frac{2}{5}\Big]$.

Remark 18. In the results obtained in [8, 17, 24, 29] and many other results published in this direction, the assumptions $VI(C, A) \neq \emptyset$ and $\|Ax\| \leq \|Ax - Au\| \;\; \forall \;\; x \in C, \;\; u \in VI(C, A)$ were made. It is of interest to mention here that these two assumptions are equivalent to the single assumption $A^{-1}(0) \neq \emptyset$. This could be easily verified.

Remark 19. It is easy to see that the iteration process studied in this paper seems simpler than the schemes studied by [8, 17, 24, 29]. Moreover, Theorem 13 generalizes the corresponding results of Chang *et al.* [8], Kang, et al. [17], Takahashi and Zambayeshi [24], Zegeye [29] and a host of other authors to approximation of common element of null spaces of finite family of γ inverse strongly monotone mappings, a set of common fixed point of finite family of weak relatively nonexpansive mappings and a solution set of generalized mixed equilibrium problem. Theorem 15 is of independent interest.

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