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SOME RESULTS ON THE ASYMPTOTIC BEHAVIOUR OF HYPERBOLIC SINGULAR PERTURBATIONS PROBLEMS

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Abstract

Anisotropic singular perturbations of linear hyperbolic problems are considered. A description of the asymptotic behaviour of the solution as $\varepsilon \to 0$ is given. In the case of cylindrical domains, we improve the rate of convergence in regions far from the lateral boundary.

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1. Introduction

The purpose of this study is to analyze the asymptotic behaviour of the solutions of hyperbolic problems when some coefficients become very small. This is what we call an anisotropic singular perturbations problem. As a typical example, we can take on $(0,T) \times (0,1)^2$, a wave problem where the propagation speed of the wave is very small in the x_1 direction, defined as

$$\begin{cases} u''_\varepsilon - \varepsilon^2 \partial^2_{x_1} u_\varepsilon - \partial^2_{x_2} u_\varepsilon = f & \text{in } (0,T) \times (0,1)^2, \\ u_\varepsilon = 0 & \text{on } (0,T) \times \partial (0,1)^2, \\ u_\varepsilon(0) = u^0, u'_\varepsilon(0) = u^1 & \text{in } (0,1)^2, \end{cases}$$

where $T > 0$, $\varepsilon > 0$ is a small parameter, u^0, u^1 are initial conditions and f represents the source term (for instance f represents the force driving a wave on a membrane, the charge and the current density in the Lorenz gauge of electromagnetism). We are interested in the limit behaviour of u_ε when $\varepsilon \to 0$ and the natural candidate is $\tilde{u}$ solution to

$$\begin{cases} \tilde{u}'' - \partial^2_{x_2} \tilde{u} = f & \text{in } (0,T) \times (0,1), \\ \tilde{u} = 0 & \text{on } (0,T) \times \{0,1\}, \\ \tilde{u}(0) = u^0, \tilde{u}'(0) = u^1 & \text{in } (0,1). \end{cases}$$

The case of elliptic as well as of parabolic boundary value problems, approximately invariant under arbitrary translations in some directions (cylindrical symmetry) and defined on cylindrical domains, has been considered in [3, 7, 9]. In the same framework, hyperbolic boundary value problems are discussed in [1, 7] where a polynomial rate of convergence is shown for elementary models. Recently, more general results for elliptic problems defined on arbitrary domains have been shown in [4]. In the latter paper as well as here the limit solution depends on all directions since we do not have, as in the previous papers, a cylindrical symmetry. This negatively influenced the rate of convergence. However, some difficulties arise, especially here for the hyperbolic problems.

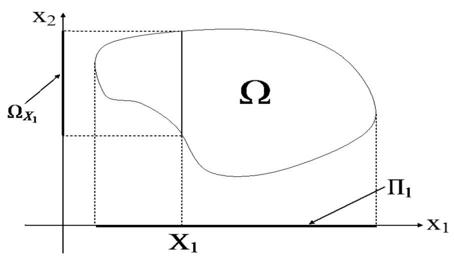
The paper is organized as follows: In the following section, we consider the general linear hyperbolic problems on arbitrary domain Ω and we show the convergence of the solutions of such problems towards solutions of other problems of the same type defined on the sections of Ω. A detailed study is given in the third section to analyze the rate of convergence in the case of cylindrical domains.

Let Ω be a bounded open subset of $\mathbb{R}^n$. We denote by $x = (x_1, \ldots, x_n) = (X_1, X_2)$ the points in $\mathbb{R}^n$ where $X_1 = (x_1, \ldots, x_p)$ and $X_2 = (x_{p+1}, \ldots, x_n)$. With this notation we set

$$\nabla u = (\partial_{x_1} u, \ldots, \partial_{x_n} u)^T = \begin{pmatrix} (\partial_{x_1} u, \ldots, \partial_{x_p} u)^T \\ (\partial_{x_{p+1}} u, \ldots, \partial_{x_n} u)^T \end{pmatrix} = \begin{pmatrix} \nabla_{X_1} u \\ \nabla_{X_2} u \end{pmatrix}.$$

(n and p are integer constants). We denote by Π_{X_1} the orthogonal projection from $\mathbb{R}^n$ onto the space $X_2 = 0$. For any $X_1 \in \Pi_{X_1}(\Omega) = \Pi_1$, we denote by Ω_{X_1} the section of Ω above X_1, i.e.

$$\Omega_{X_1} = \{ X_2 \mid (X_1, X_2) \in \Omega \}.$$



For a positive constant T, we set

$$Q = (0,T) \times \Omega, \ \ Q_{X_1} = (0,T) \times \Omega_{X_1}.$$

Let us denote by $A = (a_{ij}(t,x))$ a $n \times n$ matrix such that

$$a_{ij} \in C^1(\bar{Q}), \qquad a_{ij} = a_{ji}, \qquad\qquad \forall\, i,j = 1, \cdots, n \tag{1.1}$$

and for some $\lambda > 0$ we have the hyperbolicity hypothesis

$$A\xi \cdot \xi \geq \lambda |\xi|^2 \quad \forall\, \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega, \quad \forall t \in (0,T)\,. \tag{1.2}$$

We decompose A into four blocks by writing

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A' = \frac{\partial}{\partial t} A = \begin{pmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{pmatrix}, \tag{1.3}$$

where A_{11} and A_{22} are respectively $p \times p$ and $(n-p) \times (n-p)$ matrices. We then set for every $0 < \varepsilon < 1$

$$A_\varepsilon = A_\varepsilon(t,x) = \begin{pmatrix} \varepsilon^2 A_{11} & \varepsilon A_{12} \\ \varepsilon A_{21} & A_{22} \end{pmatrix}. \tag{1.4}$$

Therefore, there exists $\lambda > 0$ such that for a.e. $(t,x) \in Q$

$$A_\varepsilon \xi \cdot \xi \geq \lambda |\xi_\varepsilon|^2 = \lambda \left(\varepsilon^2 |\bar{\xi}_1|^2 + |\bar{\xi}_2|^2 \right), \qquad\qquad \forall \xi \in \mathbb{R}^n, \tag{1.5}$$

$$A_{22} \bar{\xi}_2 \cdot \bar{\xi}_2 \geq \lambda |\bar{\xi}_2|^2, \qquad\qquad \forall \bar{\xi}_2 \in \mathbb{R}^{n-p}, \tag{1.6}$$

where we have set $\xi = \left(\bar{\xi}_1, \bar{\xi}_2\right)^T$ with $\bar{\xi}_1 = (\xi_1, \ldots, \xi_p)^T$, $\bar{\xi}_2 = (\xi_{p+1}, \ldots, \xi_n)^T$ and $\xi_\varepsilon = (\varepsilon \bar{\xi}_1, \bar{\xi}_2)^T$. We make the following assumptions on the other data

$$u^0, u^0_\varepsilon \in H^1_0(\Omega), \quad u^1, u^1_\varepsilon \in L^2(\Omega), \quad f \in L^2(0,T; L^2(\Omega)) = L^2(Q), \tag{1.7}$$

and

$$u^0_\varepsilon \to u^0 \text{ in } H^1_0(\Omega), \quad u^1_\varepsilon \to u^1 \text{ in } L^2(\Omega). \tag{1.8}$$

(The subscript should not be confused with the power notation.)

2. Problems in general domains

In this section we would like to consider the following problem

$$\begin{cases} u'' - \nabla \cdot (A_\varepsilon \nabla u) = f & \text{in} \quad Q, \\ u = 0 & \text{on} \quad (0,T) \times \partial\Omega, \\ u\,(0) = u^0_\varepsilon, \quad u'\,(0) = u^1_\varepsilon & \text{in} \quad \Omega. \end{cases} \tag{2.1}$$

Under the assumptions (1.1), (1.2) and (1.7), there exists a weak solution u_ε satisfying

$$\begin{cases} u_\varepsilon \in C([0,T];H_0^1(\Omega)), \quad u'_\varepsilon \in C([0,T];L^2(\Omega)), \quad u''_\varepsilon \in L^2(0,T;H^{-1}(\Omega)), \\ \left\langle u''_\varepsilon(t,x), v\right\rangle + \displaystyle\int_\Omega A_\varepsilon(t,x)\nabla u_\varepsilon(t,x)\cdot\nabla v dx = \int_\Omega f(t,x)v dx, \quad \forall t\in(0,T), \quad \forall v\in H_0^1(\Omega), \\ u_\varepsilon(0) = u_\varepsilon^0, \quad u'_\varepsilon(0) = u_\varepsilon^1 \text{ in } \Omega. \end{cases} \tag{2.2}$$

($|\cdot|_E$ denotes the norm of a space E and $\langle\cdot,\cdot\rangle$ denotes its duality product.) The natural candidate limit of u_ε is $\tilde u$ defined for a.e. $X_1\in\Pi_1$ as a solution to

$$\begin{cases} \tilde u''(X_1,\cdot) - \nabla_{X_2}\cdot(A_{22}(X_1,\cdot)\nabla_{X_2}\tilde u(X_1,\cdot)) = f(X_1,\cdot) & \text{in} \quad Q_{X_1}, \\ \tilde u(X_1,\cdot) = 0 & \text{on} \quad (0,T)\times\partial\Omega_{X_1}, \\ \tilde u(0,X_1,\cdot) = u^0(X_1,\cdot) \quad \text{and} \quad \tilde u'(0,X_1,\cdot) = u^1(X_1,\cdot) & \text{in} \quad \Omega_{X_1}. \end{cases} \tag{2.3}$$

Note that X_1 plays the role of a parameter in this problem. Taking into account (1.7), it follows that -see [4]-

$$u^0(X_1,\cdot)\in H_0^1(\Omega_{X_1}), u^1(X_1,\cdot)\in L^2(\Omega_{X_1}) \text{ and } f(X_1,\cdot)\in L^2(0,T;L^2(\Omega_{X_1}))$$

for a.e. $X_1\in\Pi_1$. Then the problem (2.3) has a unique solution in the sense

$$\begin{cases} \tilde u(X_1,\cdot)\in C([0,T];H_0^1(\Omega_{X_1})), \quad \tilde u'(X_1,\cdot)\in C([0,T];L^2(\Omega_{X_1})), \\ \tilde u''(X_1,\cdot)\in L^2(0,T;H^{-1}(\Omega_{X_1})), \\ \left\langle \tilde u''(t,X_1,X_2), v\right\rangle + \displaystyle\int_{\Omega_{X_1}} A_{22}(t,X_1,X_2)\nabla_{X_2}\tilde u(t,X_1,X_2)\cdot\nabla_{X_2}v dX_2 \\ \qquad\qquad = \displaystyle\int_{\Omega_{X_1}} f(t,X_1,X_2)v dX_2, \qquad \forall t\in(0,T), \quad \forall v\in H_0^1(\Omega_{X_1}), \\ \tilde u(0,X_1,\cdot) = u^0(X_1,\cdot), \quad \tilde u'(0,X_1,\cdot) = u^1(X_1,\cdot) \quad \text{in } \Omega_{X_1}, \end{cases} \tag{2.4}$$

for a.e. $X_1\in\Pi_1$. Next, we introduce the following useful space

$$\mathcal V(\Omega) := \left\{ v\in L^2(\Omega) \,|\, \nabla_{X_2}v\in L^2(\Omega), \ v(X_1,\cdot)\in H_0^1(\Omega_{X_1}) \text{ a.e. } X_1\in\Pi_1 \right\}$$

equipped with the norm

$$|v|^2_{\mathcal V(\Omega)} := |v|^2_{L^2(\Omega)} + |\nabla_{X_2}v|^2_{L^2(\Omega)}. \tag{2.5}$$

It is clear that $\mathcal V(\Omega)$ is a Hilbert space and the mapping $v\to|\nabla_{X_2}v|_{L^2(\Omega)}$ is a norm equivalent to (2.5). In fact we can choose a Poincaré constant in

$$|v(X_1,\cdot)|_{L^2(\Omega_{X_1})} \le C\,|\nabla_{X_2}v(X_1,\cdot)|_{L^2(\Omega_{X_1})}, \quad \forall v\in\mathcal V(\Omega), \text{ a.e. } X_1$$

independent of X_1, by extending the elements of $\mathcal V(\Omega)$ by zero outside of Ω then integrating on Π_1. We can easily check that $\mathcal D(\Omega)$ is dense in $\mathcal V(\Omega)$, hence it holds that

$$H_0^1(\Omega)\subset\mathcal V(\Omega)\subset L^2(\Omega)\subset\mathcal V'(\Omega)\subset H^{-1}(\Omega)$$

with continuous injections. Note that the embedding of $\mathcal V(\Omega)$ in $L^2(\Omega)$ is not compact.

We are now ready to state the main result in this section.

Theorem 1. *Under the assumptions above, we have for every $t\in[0,T]$*

$$\begin{aligned} u_\varepsilon(t)\to\tilde u(t) \quad &\textit{in} \quad \mathcal V(\Omega), \\ \varepsilon\nabla_{X_1}u_\varepsilon(t)\to 0, \quad u'_\varepsilon(t)\to\tilde u'(t) \quad &\textit{in} \quad L^2(\Omega), \end{aligned} \tag{2.6}$$

with $\tilde{u}$ (resp. u_ε) is the solution of (2.4) (resp. (2.2)). Moreover, $\tilde{u}$ is also the solution of the following problem

$$\begin{cases} u \in C([0,T];\mathcal{V}(\Omega)), \quad u' \in C([0,T];L^2(\Omega)), \quad u'' \in L^2(0,T;\mathcal{V}'(\Omega)), \\ \langle u'', v\rangle + \int_\Omega A_{22}\nabla_{X_2}u\cdot\nabla_{X_2}v dx = \int_\Omega fv dx, \qquad \forall v\in\mathcal{V}(\Omega), \\ u(0) = u^0, \quad u'(0) = u^1 \qquad in \ \ \Omega. \end{cases} \tag{2.7}$$

The above convergences are vectorial convergences in $L^2(\Omega)$, i.e. component by component convergences. First, we establish the following lemma.

Lemma 1. *The problem (2.7) has a unique solution satisfying the following energy equality*

$$\begin{aligned} |u'(t)|^2_{L^2(\Omega)} + \int_\Omega A_{22}(t)\nabla_{X_2}u(t)\cdot\nabla_{X_2}u(t)dx = |u^1|^2_{L^2(\Omega)} + \int_\Omega A_{22}(0)\nabla_{X_2}u^0\cdot\nabla_{X_2}u^0 dx \\ +2\int_0^t\int_\Omega fu'dxds + \int_0^t\int_\Omega A'_{22}\nabla_{X_2}u\cdot\nabla_{X_2}udxds, \qquad \forall t\in[0,T]. \end{aligned} \tag{2.8}$$

Proof. Applying Theorem 8.1 and Lemma 8.3 in [11, Chap. 3] to this special case, (for the spaces $\mathcal{V}(\Omega)$, $L^2(\Omega)$), then the lemma follows. □

Proof of Theorem 1. We split the proof in several steps.

I) *A priori estimates.* The energy equality for the problem (2.2) is given by (see [11])

$$\begin{aligned} |u'_\varepsilon(t)|^2_{L^2(\Omega)} + \int_\Omega A_\varepsilon\nabla u_\varepsilon(t)\cdot\nabla u_\varepsilon(t)dx = |u^1_\varepsilon|^2_{L^2(\Omega)} + \int_\Omega A_\varepsilon(0)\nabla u^0_\varepsilon\cdot\nabla u^0_\varepsilon dx \\ + \int_0^t\int_\Omega A'_\varepsilon\nabla u_\varepsilon\cdot\nabla u_\varepsilon dxds + 2\int_0^t\int_\Omega fu'_\varepsilon dxds, \end{aligned} \tag{2.9}$$

for every $t\in[0,T]$. Using (1.1) and (1.5), we get

$$\begin{aligned} |u'_\varepsilon(t)|^2_{L^2(\Omega)} + \lambda\left(\varepsilon^2|\nabla_{X_1}u_\varepsilon(t)|^2_{L^2(\Omega)} + |\nabla_{X_2}u_\varepsilon(t)|^2_{L^2(\Omega)}\right) \le |u^1_\varepsilon|^2_{L^2(\Omega)} + C\varepsilon^2|\nabla_{X_1}u^0_\varepsilon|^2_{L^2(\Omega)} \\ + C|\nabla_{X_2}u^0_\varepsilon|^2_{L^2(\Omega)} + 2\int_0^t\int_\Omega|fu'_\varepsilon|dxds + C\int_0^t\varepsilon^2|\nabla_{X_1}u_\varepsilon|^2_{L^2(\Omega)} + |\nabla_{X_2}u_\varepsilon|^2_{L^2(\Omega)}ds, \end{aligned}$$

where here and in the following C is a positive constant independent of ε that can take different values in this paper. For ε small enough and thanks to (1.8), we derive

$$|u^1_\varepsilon|^2_{L^2(\Omega)} \le 2|u^1|^2_{L^2(\Omega)} \text{ and } \varepsilon^2|\nabla_{X_1}u^0_\varepsilon|^2_{L^2(\Omega)} + |\nabla_{X_2}u^0_\varepsilon|^2_{L^2(\Omega)} \le 2|\nabla_{X_2}u^0|^2_{L^2(\Omega)}. \tag{2.10}$$

This implies

$$\begin{aligned} |u'_\varepsilon(t)|^2_{L^2(\Omega)} + \lambda\left(\varepsilon^2|\nabla_{X_1}u_\varepsilon(t)|^2_{L^2(\Omega)} + |\nabla_{X_2}u_\varepsilon(t)|^2_{L^2(\Omega)}\right) \le 2|u^1|^2_{L^2(\Omega)} + C|\nabla_{X_2}u^0|^2_{L^2(\Omega)} \\ + C|f|^2_{L^2(Q)} + C\int_0^t|u'_\varepsilon|^2_{L^2(\Omega)} + \lambda\left(\varepsilon^2|\nabla_{X_1}u_\varepsilon|^2_{L^2(\Omega)} + |\nabla_{X_2}u_\varepsilon|^2_{L^2(\Omega)}\right)ds. \end{aligned}$$

Using the Gronwall inequality we get for every $t\in[0,T]$

$$|u'_\varepsilon(t)|^2_{L^2(\Omega)} + \lambda\left(\varepsilon^2|\nabla_{X_1}u_\varepsilon(t)|^2_{L^2(\Omega)} + |\nabla_{X_2}u_\varepsilon(t)|^2_{L^2(\Omega)}\right) \le C. \tag{2.11}$$

Thus

$$\begin{aligned} (u_\varepsilon)_\varepsilon \text{ is bounded in } \quad & L^\infty(0,T;\mathcal{V}(\Omega)), \\ (u'_\varepsilon)_\varepsilon \text{ and } (\varepsilon\nabla_{X_1}u_\varepsilon)_\varepsilon \text{ are bounded in } \quad & L^\infty(0,T;L^2(\Omega)). \end{aligned} \tag{2.12}$$

Going back to (2.2), we get

$$\begin{aligned}\left|\left\langle u_\varepsilon'', v\right\rangle\right| &\leq \left|\int_\Omega f v dx\right| + \left|\int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla v dx\right| \\ &\leq C\left(|f|_{L^2(\Omega)} + \varepsilon\left|\nabla_{X_1} u_\varepsilon\right|_{L^2(\Omega)} + \left|\nabla_{X_2} u_\varepsilon\right|_{L^2(\Omega)}\right)|\nabla v|_{L^2(\Omega)},\end{aligned}$$

for every $v \in H_0^1(\Omega)$, hence

$$\left|u_\varepsilon''\right|_{H^{-1}(\Omega)} \leq C\left(|f|_{L^2(\Omega)} + \varepsilon\left|\nabla_{X_1} u_\varepsilon\right|_{L^2(\Omega)} + \left|\nabla_{X_2} u_\varepsilon\right|_{L^2(\Omega)}\right).$$

Then taking the square in both sides and integrating over $(0, T)$, we obtain

$$\left|u_\varepsilon''\right|^2_{L^2(0,T;H^{-1}(\Omega))} \leq C\left(|f|^2_{L^2(Q)} + \varepsilon\left|\nabla_{X_1} u_\varepsilon\right|^2_{L^2(Q)} + \left|\nabla_{X_2} u_\varepsilon\right|^2_{L^2(Q)}\right).$$

By (2.12) the second member is bounded, then

$$\left(u_\varepsilon''\right)_\varepsilon \text{ is bounded in } L^2(0,T;H^{-1}(\Omega)). \tag{2.13}$$

II) *Weak convergences.* In this step we study the weak convergence of u_ε in Q. Thanks to (2.12) and (2.13), we can extract from $(u_\varepsilon)_\varepsilon$ a weak star converging subsequence -still labeled $(u_\varepsilon)_\varepsilon$- such that

$$\varepsilon\nabla_{X_1} u_\varepsilon \overset{*}{\rightharpoonup} 0, \quad \begin{array}{rcl} u_\varepsilon \overset{*}{\rightharpoonup} z & in & L^\infty(0,T;\mathcal{V}(\Omega)), \\ u_\varepsilon' \overset{*}{\rightharpoonup} z' & in & L^\infty(0,T;L^2(\Omega)), \\ u_\varepsilon'' \rightharpoonup z'' & in & L^2(0,T;H^{-1}(\Omega)). \end{array} \tag{2.14}$$

To verify that the limits are as stated, we use the continuous injections $L^2(Q) \subset \mathcal{D}'(Q)$ and $L^2\left(0,T;H^{-1}(\Omega)\right) \subset \mathcal{D}'(Q)$ with the continuity of the derivative operator in $\mathcal{D}'(Q)$. In order to show that z satisfies the equation in (2.7), we multiply (2.2) by $\phi \in C^\infty([0,T])$ and integrate over $[0,t]$, we get

$$\int_0^t \left\langle u_\varepsilon'', v\phi(s)\right\rangle ds + \int_0^t \int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla v \phi(s)\, dx ds = \int_0^t \int_\Omega f v \phi(s)\, dx ds.$$

for every $t \in [0,T]$. Expanding this identity using the different blocks of A we derive

$$\begin{aligned}\int_0^t \left\langle u_\varepsilon'', v\phi\right\rangle ds &+ \int_0^t \int_\Omega \varepsilon^2 A_{11} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v \phi dx ds + \int_0^t \int_\Omega \varepsilon A_{12} \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} v \phi dx ds \\ &+ \int_0^t \int_\Omega \varepsilon A_{21} \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} v \phi dx ds + \int_0^t \int_\Omega A_{22} \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v \phi dx ds \\ &= \int_0^t \int_\Omega f v \phi dx ds.\end{aligned}$$

Then passing to the limit using (2.14), it follows that

$$\int_0^t \left\langle z'', v\right\rangle \phi ds + \int_0^t \int_\Omega A_{22} \nabla_{X_2} z \cdot \nabla_{X_2} v \phi dx ds = \int_0^t \int_\Omega f v \phi dx ds, \quad \forall v \in \mathcal{D}(\Omega), \forall \phi \in C^\infty([0,T]).$$

Thus z satisfies the equation in (2.7). Now, in order to check the initial conditions, we first derive from (2.14) that $z \in C([0,T]; L^2(\Omega))$ and $z' \in C([0,T]; H^{-1}(\Omega))$. So to show that

$$z(0) = u^0, \quad z'(0) = u^1, \tag{2.15}$$

we use the identities

$$\int_0^t \left\langle z'', v\phi \right\rangle ds = \left\langle z'(t), v\phi(t) \right\rangle - \left\langle z'(0), v\phi(0) \right\rangle$$
$$- \int_\Omega z(t)\, v\phi'(t) dx + \int_\Omega z(0)\, v\phi'(0) dx + \int_0^t \int_\Omega z v\phi'' dx ds \tag{2.16}$$

and

$$\int_0^t \left\langle u_\varepsilon, v\phi'' \right\rangle ds = \int_\Omega u'_\varepsilon(t)\, v\phi(t) dx - \int_\Omega u^1_\varepsilon v\phi(0) dx$$
$$- \int_\Omega u_\varepsilon(t)\, v\phi'(t) dx + \int_\Omega u^0_\varepsilon v\phi'(0) dx + \int_0^t \int_\Omega u_\varepsilon v\phi'' dx ds. \tag{2.17}$$

Then, passing to the limit in (2.17), using (1.8), (2.14) and taking $\phi(t) = \phi'(t) = 0$, we get

$$\int_0^t \left\langle z'', v\phi \right\rangle ds = - \int_\Omega u^1 v\phi(0) dx + \int_\Omega u^0 v\phi'(0) dx + \int_0^t \int_\Omega z v\phi'' dx ds. \tag{2.18}$$

By comparing (2.16) and (2.18), we deduce

$$- \left\langle z'(0), v\phi(0) \right\rangle + \int_\Omega z(0)\, v\phi'(0) dx = - \int_\Omega u^1 \phi(0) v dx + \int_\Omega u^0 v\phi'(0) dx, \quad \forall v \in \mathcal{D}(\Omega).$$

Then we conclude the first identity in (2.15) by choosing $\phi(0) = 1$, $\phi'(0) = 0$ and the second one by choosing $\phi(0) = 0$, $\phi'(0) = 1$. Thus we have shown that the unique limit of u_ε is the solution of (2.7) and the convergences (2.14) hold for the whole sequence. For a fixed $t \in [0, T]$ and since we have (2.11), there exist ζ_0, ζ_1 and ζ_2 such that -up to a subsequence-

$$\varepsilon \nabla_{X_1} u_\varepsilon(t) \rightharpoonup \zeta_2(t), u'_\varepsilon(t) \rightharpoonup \zeta_1(t) \text{ in } L^2(\Omega) \text{ and } u_\varepsilon(t) \rightharpoonup \zeta_0(t) \text{ in } \mathcal{V}(\Omega).$$

Passing to the limit in (2.17) for $\phi(0) = \phi'(0) = 0$, comparing with (2.16) and arguing as above for different choices of $\phi(t)$ and $\phi'(t)$ yield $\zeta_0(t) = z(t)$, $\zeta_1(t) = z'(t)$ and by consequence $\zeta_2(t) = 0$. Then, by the uniqueness of the limit, the following convergences hold for the whole sequence

$$\varepsilon \nabla_{X_1} u_\varepsilon(t) \rightharpoonup 0, \quad u'_\varepsilon(t) \rightharpoonup z'(t) \text{ in } L^2(\Omega) \text{ and } u_\varepsilon(t) \rightharpoonup z(t) \text{ in } \mathcal{V}(\Omega), \tag{2.19}$$

for every $t \in [0, T]$.

III) *Strong convergences.* Now we show the strong convergence of u_ε. We set for every $t \in [0, T]$

$$\begin{aligned} I_\varepsilon(t) = \left| (u_\varepsilon - z)'(t) \right|^2_{L^2(\Omega)} &+ \int_\Omega A_\varepsilon(t) \begin{pmatrix} \nabla_{X_1} u_\varepsilon(t) \\ \nabla_{X_2}(u_\varepsilon - z)(t) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_\varepsilon(t) \\ \nabla_{X_2}(u_\varepsilon - z)(t) \end{pmatrix} dx \\ &- \int_0^t \int_\Omega A'_\varepsilon \begin{pmatrix} \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2}(u_\varepsilon - z) \end{pmatrix} \cdot \begin{pmatrix} \nabla_{X_1} u_\varepsilon \\ \nabla_{X_2}(u_\varepsilon - z) \end{pmatrix} dx ds. \end{aligned} \tag{2.20}$$

Developing I_ε using the different blocks of A, we get

$$
\begin{aligned}
I_\varepsilon(t) = {} & \left|u_\varepsilon'(t)\right|_{L^2(\Omega)}^2 + \left|z'(t)\right|_{L^2(\Omega)}^2 - 2\int_\Omega u_\varepsilon'(t)\, z'(t)\, dx \\
& + \int_\Omega A_\varepsilon(t) \nabla u_\varepsilon(t) \cdot \nabla u_\varepsilon(t)\, dx - \int_0^t \int_\Omega A_\varepsilon' \nabla u_\varepsilon \cdot \nabla u_\varepsilon dx ds \\
& - \int_\Omega \varepsilon A_{12}(t) \nabla_{X_2} u_\varepsilon(t) \cdot \nabla_{X_1} z(t)\, dx - \int_\Omega \varepsilon A_{21}(t) \nabla_{X_1} u_\varepsilon(t) \cdot \nabla_{X_2} z(t)\, dx \\
& - \int_\Omega A_{22}(t) \nabla_{X_2} u_\varepsilon(t) \cdot \nabla_{X_2} z(t)\, dx + \int_\Omega A_{22}(t) \nabla_{X_2} z(t) \cdot \nabla_{X_2}(z - u_\varepsilon)(t)\, dx \\
& - \int_0^t \int_\Omega \varepsilon A_{12}' \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} z dx ds - \int_0^t \int_\Omega \varepsilon A_{21}' \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} z dx ds \\
& - \int_0^t \int_\Omega A_{22}' \nabla_{X_2} z \cdot \nabla_{X_2} u_\varepsilon dx ds + \int_0^t \int_\Omega A_{22}' \nabla_{X_2} z \cdot \nabla_{X_2}(z - u_\varepsilon)\, dx ds.
\end{aligned}
$$

Using (2.9) we derive

$$
\begin{aligned}
I_\varepsilon(t) = {} & \left|z'(t)\right|_{L^2(\Omega)}^2 - 2\int_\Omega u_\varepsilon'(t)\, z'(t)\, dx + \int_\Omega A_\varepsilon(0) \nabla u_\varepsilon^0 \cdot \nabla u_\varepsilon^0 dx + \left|u_\varepsilon^1\right|_{L^2(\Omega)}^2 + 2\int_0^t \int_\Omega f u_\varepsilon' dx ds \\
& - \int_\Omega \varepsilon A_{12}(t) \nabla_{X_2} u_\varepsilon(t) \cdot \nabla_{X_1} z(t)\, dx - \int_\Omega \varepsilon A_{21}(t) \nabla_{X_1} u_\varepsilon(t) \cdot \nabla_{X_2} z(t)\, dx \\
& - \int_\Omega A_{22}(t) \nabla_{X_2} z(t) \cdot \nabla_{X_2} u_\varepsilon(t)\, dx + \int_\Omega A_{22}(t) \nabla_{X_2} z(t) \cdot \nabla_{X_2}(z - u_\varepsilon)(t)\, dx \\
& - \int_0^t \int_\Omega \varepsilon A_{12}' \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} z dx ds - \int_0^t \int_\Omega \varepsilon A_{21}' \nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} z dx ds \\
& - \int_0^t \int_\Omega A_{22}' \nabla_{X_2} z \cdot \nabla_{X_2} u_\varepsilon dx ds + \int_0^t \int_\Omega A_{22}' \nabla_{X_2} z \cdot \nabla_{X_2}(z - u_\varepsilon)\, dx ds.
\end{aligned}
$$

Then, by the weak limits in (2.14) and (2.19), it follows that

$$
\begin{aligned}
\lim_{\varepsilon \to 0} I_\varepsilon(t) = {} & - \left|z'(t)\right|_{L^2(\Omega)}^2 + \left|u^1\right|_{L^2(\Omega)}^2 - \int_\Omega A_{22} \nabla_{X_2} z(t) \cdot \nabla_{X_2} z(t)\, dx \\
& + \int_\Omega A_{22}(0) \nabla_{X_2} u^0 \cdot \nabla_{X_2} u^0 dx + 2\int_0^t \int_\Omega f z' dx ds + \int_0^t \int_\Omega A_{22}' \nabla_{X_2} z \cdot \nabla_{X_2} z dx ds = 0,
\end{aligned}
$$

since we already have the equality (2.8). Then using (1.1) and (1.5), we get from (2.20)

$$
\begin{aligned}
& \left|(u_\varepsilon - z)'(t)\right|_{L^2(\Omega)}^2 + \lambda\left(\varepsilon^2 \left|\nabla_{X_1} u_\varepsilon(t)\right|_{L^2(\Omega)}^2 + \left|\nabla_{X_2}(u_\varepsilon - z)(t)\right|_{L^2(\Omega)}^2\right) \\
& \qquad \le |I_\varepsilon(t)| + C\int_0^t \int_\Omega \left(\varepsilon^2 \left|\nabla_{X_1} u_\varepsilon(s)\right|^2 + \left|\nabla_{X_2}(u_\varepsilon - z)(s)\right|^2\right) dx ds, \qquad (2.21)
\end{aligned}
$$

for every $t \in [0, T]$. We set

$$
\Phi_\varepsilon(t) = \left|(u_\varepsilon - z)'(t)\right|_{L^2(\Omega)}^2 + \varepsilon^2 \left|\nabla_{X_1} u_\varepsilon(t)\right|_{L^2(\Omega)}^2 + \left|\nabla_{X_2}(u_\varepsilon - z)(t)\right|_{L^2(\Omega)}^2 .
$$

Then we can rewrite (2.21) as

$$
\Phi_\varepsilon(t) \le C\,|I_\varepsilon(t)| + C\int_0^t \Phi_\varepsilon(s)\, ds \qquad \text{where} \quad |I_\varepsilon(t)| \to 0.
$$

Applying a Gronwall's type inequality we conclude that

$$
\lim_{\varepsilon \to 0} \Phi_\varepsilon(t) = 0, \qquad \forall t \in [0, T].
$$

This means that

$$\begin{array}{rl} u_\varepsilon(t) \to z(t) \quad in \quad \mathcal{V}(\Omega), & \\ \varepsilon \nabla_{X_1} u_\varepsilon(t) \to 0, \quad u'_\varepsilon(t) \to z'(t) \quad in \quad L^2(\Omega), & \forall t \in [0,T]. \end{array} \tag{2.22}$$

IV) $z(.,X_1,.)$ *is the solution of (2.4) for a.e.* $X_1 \in \Pi_1$ *(*$z = \tilde{u}$ *a.e.).* Here for simplicity, we give the proof in the case of cylindrical domains. For general domains we can use the argument introduced in [4]. Then we set

$$\Omega = \Delta \times \omega, \tag{2.23}$$

where Δ (resp. ω) is an open subset of $\mathbb{R}^p$ (resp. $\mathbb{R}^{n-p}$). Choosing $\eta \in \mathcal{D}(\Delta)$ and $\varphi \in \mathcal{D}(\omega)$ we derive, from (2.7), that for a.e. $t \in (0,T)$,

$$\begin{aligned} &\int_\Delta \eta(X_1) \int_\omega z''(t,X_1,X_2)\varphi(X_2)\, dX_2 dX_1 \\ &+ \int_\Delta \eta(X_1) \int_\omega A_{22}(t,X_1,X_2)\nabla_{X_2} z(t,X_1,X_2) \cdot \nabla_{X_2}\varphi(X_2)\, dX_2 dX_1 \\ &= \int_\Delta \eta(X_1) \int_\omega f(t,X_1,X_2)\varphi(X_2)\, dX_2 dX_1, \qquad \forall \eta \in \mathcal{D}(\Delta). \end{aligned}$$

Then we have, for a.e. $X_1 \in \Pi_1$ and $t \in (0,T)$

$$\begin{aligned} \int_\omega z''(t,X_1,X_2)\varphi(X_2)\, dX_2 + \int_\omega A_{22}(t,X_1,X_2)\nabla_{X_2} z(X_1,X_2) \cdot \nabla_{X_2}\varphi(X_2)\, dX_2 \\ = \int_\omega f(t,X_1,X_2)\varphi(X_2)\, dX_2, \qquad \forall \varphi \in \mathcal{D}(\omega). \end{aligned}$$

Taking into account that the problems (2.4) and (2.7) have the same initial conditions, we derive that $z(X_1,.)$ is the unique solution of (2.4) for a.e. $X_1 \in \Pi_1$. This completes the proof of the theorem. □

As a consequence of Theorem 1 and (2.12) we have

Corollary 1. *For every* $1 \le p < \infty$ *we have*

$$\begin{array}{rl} u_\varepsilon \to \tilde{u} \quad in & L^p(0,T;\mathcal{V}(\Omega)), \\ \varepsilon \nabla_{X_1} u_\varepsilon \to 0, \quad u'_\varepsilon \to \tilde{u}' \quad in & L^p\left(0,T;L^2(\Omega)\right), \\ u''_\varepsilon \to \tilde{u}'' \quad in & L^p(0,T;H^{-1}(\Omega)). \end{array} \tag{2.24}$$

Proof. The first two lines of (2.24) follow from Lebesgue's theorem, (2.12) and the pointwise convergence in t given by Theorem 1. For the last limit, we subtract (2.7) from (2.2) and estimate the second time derivative of $u_\varepsilon - u$ by

$$\begin{aligned} \langle u''_\varepsilon - \tilde{u}'', v\rangle = &- \int_\Omega \varepsilon^2 A_{11}\nabla_{X_1} u_\varepsilon \cdot \nabla_{X_1} v dx + \int_\Omega \varepsilon A_{12}\nabla_{X_2} u_\varepsilon \cdot \nabla_{X_1} v dx \\ &+ \int_\Omega \varepsilon A_{21}\nabla_{X_1} u_\varepsilon \cdot \nabla_{X_2} v dx + \int_\Omega A_{22}\nabla_{X_2}(u_\varepsilon - \tilde{u}) \cdot \nabla_{X_2} v dx \\ \le & C\left\{\varepsilon |\nabla_{X_1} u_\varepsilon|_{L^2(\Omega)} + \varepsilon |\nabla_{X_2} u_\varepsilon|_{L^2(\Omega)} + |\nabla_{X_2}(u_\varepsilon - \tilde{u})|_{L^2(\Omega)}\right\} |\nabla v|_{L^2(\Omega)}, \end{aligned}$$

for every $v \in H^1_0(\Omega)$. Then it comes

$$|u''_\varepsilon - \tilde{u}''|^p_{L^p(0,T;H^{-1}(\Omega))} \le C \int_0^t \varepsilon^p |\nabla_{X_1} u_\varepsilon|^p_{L^2(\Omega)} + \varepsilon^p |\nabla_{X_2} u_\varepsilon|^p_{L^2(\Omega)} + |\nabla_{X_2}(u_\varepsilon - \tilde{u})|^p_{L^2(\Omega)}\, ds.$$

Using the first convergence in this corollary the right-hand side of the inequality above tends to zero This completes the proof. □

Remark 1. *We can replace the condition (1.8) by the following weaker one*

$$u^0_\varepsilon \to u^0 \text{ in } \mathcal{V}(\Omega) \quad \text{and} \quad \varepsilon \nabla_{X_1} u^0_\varepsilon \to 0, \quad u^1_\varepsilon \to u^1 \text{ in } L^2(\Omega).$$

In this case we have only to suppose that $u^0 \in \mathcal{V}(\Omega)$.

3. Problems in cylindrical domains

In this section we assume, as in (2.23), that the domain is cylindrical, i.e.

$$\Omega = \Delta \times \omega. \tag{3.1}$$

Then it holds that

$$\mathcal{V}(\Omega) = L^2\left(\Delta, H^1_0(\omega)\right) \text{ and } \mathcal{V}'(\Omega) = L^2\left(\Delta, H^{-1}(\omega)\right).$$

In addition we assume that the matrix A is independent of time, i.e.

$$A(t,x) = A(x). \tag{3.2}$$

For a.e. $X_1 \in \Delta$, the problem (2.4) can be stated as

$$\begin{cases} \tilde{u}(X_1,\cdot) \in C([0,T]; H^1_0(\omega)), \quad \tilde{u}'(X_1,\cdot) \in C([0,T]; L^2(\omega)), \\ \tilde{u}''(X_1,\cdot) \in L^2(0,T; H^{-1}(\omega)), \\ \langle \tilde{u}''(t,X_1,X_2), v\rangle_\omega + \displaystyle\int_\omega A_{22}(X_1,X_2)\nabla_{X_2}\tilde{u}(t,X_1,X_2)\cdot\nabla_{X_2} v dX_2 \\ \qquad\qquad = \displaystyle\int_\omega f(t,X_1,X_2) v dX_2, \qquad \forall v \in H^1_0(\omega), \\ \tilde{u}(0,X_1,\cdot) = u^0, \ \tilde{u}'(0,X_1,\cdot) = u^1 \quad \text{in } \omega. \end{cases} \tag{3.3}$$

Our aim in this section is to improve the convergence $u_\varepsilon \to \tilde{u}$, in particular the convergence of $\nabla_{X_1} u_\varepsilon$. For this reason, as a necessary condition we need to assume more regularity hypothesis, i.e. $\tilde{u}(t) \in H^1(\Omega)$ for a.e. $t \in (0,T)$. Thus we start by studying the regularity of $\tilde{u}$.

3.1. Regularity results.

Proposition 1. *Under the assumptions of Theorem 1 and in addition we assume that (3.1), (3.2) hold and*

$$\partial_{x_i} u^1 \in L^2\left(\Delta, H^{-1}(\omega)\right), \ \partial_{x_i} f \in L^2\left((0,T)\times\Delta; H^{-1}(\omega)\right), \quad i = 1,\cdots,p, \tag{3.4}$$

then we have

$$\tilde{u} \in L^\infty\left(0,T; H^1(\Omega)\right), \quad \partial_{x_i}\tilde{u}' \in L^\infty\left(0,T; L^2\left(\Delta, H^{-1}(\omega)\right)\right), \quad i = 1,\cdots,p. \tag{3.5}$$

Proof. Let Δ' be an open subset such that $\Delta' \subset\subset \Delta$ ($\subset\subset$ denotes the strict inclusion). We set $h_0 = dist(\Delta', \partial\Delta)$ and for $0 < h < h_0$ we denote

$$\tau^i_h v(X_1,X_2) = v(X_1 + he_i, X_2), \quad \text{for } X_1 \in \Delta', \ i = 1,\cdots,p,$$

where e_i is the unit vector in the $i-direction$. Subtracting the identity in (3.3) from it self written for X_1+he_i, we derive

$$\begin{aligned}&\left\langle \tau_h^i\tilde{u}''-\tilde{u}'',v\right\rangle_\omega+\int_\omega A_{22}\nabla_{X_2}\left(\tau_h^i\tilde{u}-\tilde{u}\right)\cdot\nabla_{X_2}vdX_2\\&=\int_\omega\left(\tau_h^if-f\right)vdX_2-\int_\omega\left(\tau_h^iA_{22}-A_{22}\right)\nabla_{X_2}\tilde{u}\cdot\nabla_{X_2}vdX_2,\qquad\forall v\in H_0^1\left(\omega\right).\end{aligned}\tag{3.6}$$

For $s\in\left]0,T\right]$, we set

$$\begin{aligned}\psi\left(t\right)&=\begin{cases}-\int_t^s\tilde{u}\left(\sigma\right)d\sigma & t\le s,\\ 0 & t>s,\end{cases}\\ \tilde{U}\left(t\right)&=\int_0^t\tilde{u}\left(\sigma\right)d\sigma,\end{aligned}$$

then it holds that

$$\psi\left(\sigma\right)=\tilde{U}\left(\sigma\right)-\tilde{U}\left(s\right)\quad\text{if}\quad\sigma\le s.\tag{3.7}$$

Taking $v=\left(\tau_h^i\psi-\psi\right)\left(\sigma\right)\in H_0^1\left(\omega\right)$ in (3.6) and integrating over $\left(0,s\right)$, we get

$$\begin{aligned}\int_0^s\left\langle \tau_h^i\tilde{u}''-\tilde{u}'',\tau_h^i\psi-\psi\right\rangle_\omega d\sigma&+\int_0^s\int_\omega A_{22}\nabla_{X_2}\left(\tau_h^i\tilde{u}-\tilde{u}\right)\cdot\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)dX_2d\sigma\\&=\int_0^s\int_\omega\left(\tau_h^if-f\right)\left(\tau_h^i\psi-\psi\right)dX_2d\sigma\\&-\int_0^s\int_\omega\left(\tau_h^iA_{22}-A_{22}\right)\nabla_{X_2}\tilde{u}\cdot\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)dX_2d\sigma.\end{aligned}\tag{3.8}$$

Next, integrating by parts in the first integral, we obtain

$$\begin{aligned}&\int_0^s\left\langle \tau_h^i\tilde{u}''-\tilde{u}'',\tau_h^i\psi-\psi\right\rangle_\omega d\sigma=\int_\omega\left(\tau_h^i\tilde{u}'\left(s\right)-\tilde{u}'\left(s\right)\right)\left(\tau_h^i\psi\left(s\right)-\psi\left(s\right)\right)dX_2\\&-\int_\omega\left(\tau_h^iu^1-u^1\right)\left(\tau_h^i\psi\left(0\right)-\psi\left(0\right)\right)dX_2-\int_0^s\int_\omega\left(\tau_h^iu'-u'\right)\left(\tau_h^i\psi'-\psi'\right)dX_2d\sigma,\end{aligned}$$

whence

$$\begin{aligned}\int_0^s\left\langle \tau_h^i\tilde{u}''-\tilde{u}'',\tau_h^i\psi-\psi\right\rangle_\omega d\sigma&=-\frac{1}{2}\left|\left(\tau_h^i\tilde{u}-\tilde{u}\right)\left(s\right)\right|_{L^2(\omega)}^2+\frac{1}{2}\left|\tau_h^iu^0-u^0\right|_{L^2(\omega)}^2\\&+\int_\omega\left(\tau_h^iu^1-u^1\right)\left(\tau_h^i\tilde{U}-\tilde{U}\right)\left(s\right)dX_2,\end{aligned}\tag{3.9}$$

since $\psi'=\tilde{u}$, $\psi\left(s\right)=0$ and $\psi\left(0\right)=-\tilde{U}\left(s\right)$. On the other hand we have

$$\begin{aligned}&\int_0^s\int_\omega A_{22}\nabla_{X_2}\left(\tau_h^i\tilde{u}-\tilde{u}\right)\cdot\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)dX_2d\sigma\\&\qquad=-\frac{1}{2}\int_\omega A_{22}\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)\left(s\right)\cdot\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)\left(s\right)dX_2.\end{aligned}\tag{3.10}$$

We then use (3.9) and (3.10) in (3.8), it comes

$$\begin{aligned}&-\frac{1}{2}\left|\left(\tau_h^i\tilde{u}-\tilde{u}\right)\left(s\right)\right|_{L^2(\omega)}^2-\frac{1}{2}\int_\omega A_{22}\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)\left(s\right)\cdot\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)\left(s\right)dX_2\\&=-\frac{1}{2}\left|\tau_h^iu^0-u^0\right|_{L^2(\omega)}^2-\frac{1}{2}\int_\omega\left(\tau_h^iu^1-u^1\right)\left(\tau_h^i\tilde{U}-\tilde{U}\right)\left(s\right)dX_2\\&+\int_0^s\int_\omega\left(\tau_h^if-f\right)\left(\tau_h^i\psi-\psi\right)dX_2d\sigma-\int_0^s\int_\omega\left(\tau_h^iA_{22}-A_{22}\right)\nabla_{X_2}\tilde{u}\cdot\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)dX_2d\sigma.\end{aligned}$$

Applying (1.6), the Cauchy-Schwarz and Poincaré inequalities, we obtain

$$\begin{aligned}
&\frac{1}{2}\left|\left(\tau_h^i\tilde{u}-\tilde{u}\right)(s)\right|^2_{L^2(\omega)}+\frac{\lambda}{2}\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|^2_{L^2(\omega)}\\
&\qquad\leq\frac{1}{2}\left|\tau_h^i u^0-u^0\right|^2_{L^2(\omega)}+C\left|\tau_h^i u^1-u^1\right|_{H^{-1}(\omega)}\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|_{L^2(\omega)}\\
&\qquad+C\left|\tau_h^i f-f\right|_{L^2(0,T;H^{-1}(\omega))}\left|\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)\right|_{L^2((0,s)\times\omega)}\\
&\qquad+\left|\left(\tau_h^i A_{22}-A_{22}\right)\nabla_{X_2}\tilde{u}\right|_{L^2((0,T)\times\omega)}\left|\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)\right|_{L^2((0,s)\times\omega)}.
\end{aligned}$$

(In the second term of the right-hand side, we considered the duality product.) Then by Young's inequality $2ab\leq\alpha a^2+\dfrac{b^2}{\alpha}$ with convenient values of α (for instance, $\alpha=\frac{C}{\lambda}$ in the second term of the right-hand side), we derive

$$\begin{aligned}
&\left|\left(\tau_h^i\tilde{u}-\tilde{u}\right)(s)\right|^2_{L^2(\omega)}+\frac{\lambda}{2}\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|^2_{L^2(\omega)}\\
&\leq\left|\tau_h^i u^0-u^0\right|^2_{L^2(\omega)}+C\left|\tau_h^i u^1-u^1\right|^2_{H^{-1}(\omega)}+C\left|\tau_h^i f-f\right|^2_{L^2(0,T;H^{-1}(\omega))}\\
&+C\left|\left|\tau_h^i A_{22}-A_{22}\right|_*\right|^2_{L^\infty((0,T)\times\omega)}\left|\nabla_{X_2}\tilde{u}\right|^2_{L^2((0,T)\times\omega)}+\frac{\lambda}{4T}\int_0^s\left|\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)\right|^2_{L^2(\omega)}d\sigma \qquad (3.11)
\end{aligned}$$

where $|\cdot|_*$ denotes a matrix norm. We estimate the last term above, using (3.7), as follows

$$\begin{aligned}
&\int_0^s\left|\nabla_{X_2}\left(\tau_h^i\psi-\psi\right)(\sigma)\right|^2_{L^2(\omega)}d\sigma\\
&=\int_0^s\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(\sigma)-\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|^2_{L^2(\omega)}d\sigma\\
&\leq 2\int_0^s\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(\sigma)\right|^2_{L^2(\omega)}d\sigma+2T\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|^2_{L^2(\omega)}.
\end{aligned}$$

Using this in (3.11), we get

$$\begin{aligned}
&\left|\left(\tau_h^i\tilde{u}-\tilde{u}\right)(s)\right|^2_{L^2(\omega)}+\frac{\lambda}{4}\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)\right|^2_{L^2(\omega)}\\
&\qquad\leq\left|\tau_h^i u^0-u^0\right|^2_{L^2(\omega)}+C\left|\tau_h^i u^1-u^1\right|^2_{H^{-1}(\omega)}+C\left|\tau_h^i f-f\right|^2_{L^2(0,T;H^{-1}(\omega))}\\
&\qquad+C\left|\left|\tau_h^i A_{22}-A_{22}\right|_*\right|^2_{L^\infty(\omega)}\left|\nabla_{X_2}\tilde{u}\right|^2_{L^2((0,T)\times\omega)}\\
&\qquad+C\int_0^s\left|\left(\tau_h^i\tilde{u}-\tilde{u}\right)(\sigma)\right|^2_{L^2(\omega)}+\left|\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(\sigma)\right|^2_{L^2(\omega)}d\sigma.
\end{aligned}$$

For $s\in[0,T]$, we apply the Gronwall inequality and integrate on Δ', it follows that

$$\begin{aligned}
&\left|\frac{\left(\tau_h^i\tilde{u}-\tilde{u}\right)(s)}{h}\right|^2_{L^2(\Delta'\times\omega)}+\left|\frac{\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)}{h}\right|^2_{L^2(\Delta'\times\omega)}\leq\left|\frac{\tau_h^i u^0-u^0}{h}\right|^2_{L^2(\Delta'\times\omega)}\\
&+C\left|\frac{\tau_h^i u^1-u^1}{h}\right|^2_{L^2(\Delta';H^{-1}(\omega))}+C\left|\frac{\tau_h^i f-f}{h}\right|^2_{L^2((0,T)\times\Delta';H^{-1}(\omega))}+C\left|\left|\frac{\tau_h^i A_{22}-A_{22}}{h}\right|_*\right|^2_{L^\infty(\Delta'\times\omega)}.
\end{aligned}$$

(We divided both sides on h^2.) According to (3.4) and using the standard method in [6, Lemma 7.23], all the terms in the right-hand side are bounded independently of h, i.e.

$$\left|\frac{\left(\tau_h^i\tilde{u}-\tilde{u}\right)(s)}{h}\right|^2_{L^2(\Delta'\times\omega)}+\left|\frac{\nabla_{X_2}\left(\tau_h^i\tilde{U}-\tilde{U}\right)(s)}{h}\right|^2_{L^2(\Delta'\times\omega)}\leq C,\quad\forall s\in[0,T],$$

for every $h>0$ and $\Delta' \subset\subset \Delta$. We deduce that

$$\partial_{x_i}\tilde{u}(s),\ \partial_{x_i}\left(\nabla_{X_2}\tilde{U}\right)(s) \in L^2(\Omega), \quad i=1,\cdots,p, \quad \forall s\in[0,T]. \tag{3.12}$$

Since we already have $\partial_{x_i}\tilde{u}(s) \in L^2(\Omega)$ for $i=p+1,\cdots,n$, it follows that

$$\tilde{u}(s) \in H^1(\Omega), \qquad \forall s\in[0,T]. \tag{3.13}$$

Recall that

$$\left\langle \tilde{u}'', v\right\rangle_\omega + \int_\omega A_{22}\nabla_{X_2}\tilde{u}\cdot\nabla_{X_2}v dX_2 = \int_\omega fv dX_2, \qquad \forall v\in\mathcal{D}(\omega).$$

Integrating on $[0,s]\subset[0,T]$ and taking into account the fact that $\tilde{U}'=\tilde{u}$ and $\tilde{U}(0)=0$ yield

$$\int_\omega \tilde{u}'(s)vdX_2 = \int_\omega u^1 v dX_2 + \int_0^s\int_\omega fvdX_2 d\sigma - \int_\omega A_{22}\nabla_{X_2}\tilde{U}(s)\cdot\nabla_{X_2}vdX_2, \qquad \forall v\in\mathcal{D}(\omega).$$

Applying the derivative operator ∂_{x_i}, $(i=1,\cdots,p)$, on both sides of the identity above, we obtain

$$\begin{aligned}&\left\langle \partial_{x_i}\tilde{u}'(s), v\right\rangle_\omega = \left\langle \partial_{x_i}u^1, v\right\rangle_\omega + \int_0^s\int_\omega \partial_{x_i}fvdX_2d\sigma \\ &- \int_\omega \partial_{x_i}A_{22}\nabla_{X_2}\tilde{U}(s)\cdot\nabla_{X_2}vdX_2 - \int_\omega A_{22}\partial_{x_i}\nabla_{X_2}\tilde{U}(s)\cdot\nabla_{X_2}vdX_2, \ \ \forall v\in\mathcal{D}(\omega).\end{aligned}$$

Next, we apply the Cauchy-Schwarz inequality in each term to get

$$\begin{aligned}\left\langle \partial_{x_i}\tilde{u}'(s), v\right\rangle_\omega \ \leq\ & C\left\{\left|\partial_{x_i}u^1\right|_{H^{-1}(\omega)} + \left|\nabla_{X_2}\tilde{U}(s)\right|_{L^2(\omega)}\right. \\ &\left. + \left|\partial_{x_i}\nabla_{X_2}\tilde{U}(s)\right|_{L^2(\omega)} + \left|\partial_{x_i}f\right|_{L^2(0,T;H^{-1}(\omega))}\right\}\left|\nabla_{X_2}v\right|_{L^2(\omega)}.\end{aligned}$$

Then integrating on Δ, we derive

$$\begin{aligned}\int_\Delta \left|\partial_{x_i}\tilde{u}(s)\right|^2_{H^{-1}(\omega)}dX_1 \ \leq\ & C\left\{\left|\partial_{x_i}u^1\right|^2_{L^2(\Delta,H^{-1}(\omega))} + \left|\nabla_{X_2}\tilde{U}(s)\right|^2_{L^2(\Omega)}\right. \\ &\left. + \left|\partial_{x_i}\nabla_{X_2}\tilde{U}(s)\right|^2_{L^2(\Omega)} + \left|\partial_{x_i}f\right|^2_{L^2((0,T)\times\Delta,H^{-1}(\omega))}\right\}.\end{aligned}$$

Due to (3.4) and (3.12), we deduce that

$$\partial_{x_i}\tilde{u}'(s)\in L^2\left(\Delta;H^{-1}(\omega)\right), \quad i=1,\cdots,p., \quad \forall s\in[0,T]. \tag{3.14}$$

This ends the proof of the proposition. □

Remark 2. ***i)*** *The proof also shows that*

$$\partial_{x_i}\tilde{U}\in L^\infty\left(0,T;L^2\left(\Delta,H_0^1(\omega)\right)\right), \qquad i=1,\cdots,p.$$

ii) *Note that even we wonder if* $\tilde{u}\in C\left(0,T;H^1(\Omega)\right)$ *we have*

$$\tilde{u}(t),\tilde{U}(t)\in H_0^1(\Omega,\Delta\times\partial\omega) := \left\{v\in H^1(\Omega)\,|u_{|\Delta\times\partial\omega}=0\right\}, \quad \forall t\in[0,T]. \tag{3.15}$$

Indeed, according to [4], *since we have* $\tilde{u}(t,X_1,\cdot)\in H_0^1(\omega)$ *for a.e.* $X_1\in\Delta$, $t\in[0,T]$ *it follows from the above proposition that*

$$\tilde{u}\in L^\infty\left(0,T;H_0^1(\Omega,\Delta\times\partial\omega)\right)\cap C\left(0,T;\mathcal{V}(\Omega)\right). \tag{3.16}$$

Then, arguing as in [12, Theorem 2.1] *we derive that* $\tilde{u}(t) \in H_0^1(\Omega, \Delta \times \partial\omega)$ $\forall t \in [0, T]$. *By consequence we also have* $\tilde{U}(t) \in H_0^1(\Omega, \Delta \times \partial\omega)$ $\forall t \in [0, T]$.

Remark 3. *If the coefficients of* A_{22} *are time dependent, i.e.* $A_{22} = A_{22}(t, x)$, *Proposition 1 still holds under the assumption*

$$\partial_{x_k} a'_{ij} \in L^\infty(Q), \qquad 1 \leq k \leq p \text{ and } p+1 \leq i, j \leq n.$$

The proof is analogous, first we establish the regularity in a subinterval $[0, \tau]$ *for some* τ *sufficiently small* ($0 < \tau \leq T$), *then we repeat the same arguments for the intervals* $[\tau, 2\tau]$, $[2\tau, 3\tau], \cdots$. *We obtain the regularity on* $[0, T]$ *after a finite number of steps.*

3.2. **Convergence results.** Let Δ_0 and Δ_1 be two open subsets of $\mathbb{R}^p$ satisfying $\Delta_0 \subset\subset \Delta_1 \subset\subset \Delta$. With this notation we set

$$\Omega_i = \Delta_i \times \omega \text{ and } Q_i = (0, T) \times \Omega_i, \qquad i = 0, 1.$$

Consider a smooth function $\varrho = \varrho(X_1)$ satisfying

$$\operatorname{supp}(\varrho) \subset \Delta_1, \quad \varrho = 1 \text{ on } \Delta_0, \quad 0 \leq \varrho \leq 1 \quad \text{and} \quad |\nabla_{X_1} \varrho| \leq C. \tag{3.17}$$

For every $s \in [0, T]$, we define the functions

$$\psi_\varepsilon(t) = \begin{cases} -\displaystyle\int_t^s w_\varepsilon(\sigma)\, d\sigma & t \leq s, \\ 0 & t > s, \end{cases} \tag{3.18}$$

$$W_\varepsilon(t) = \int_0^t w_\varepsilon(\sigma)\, d\sigma \quad \text{and} \quad U_\varepsilon(t) = \int_0^t u_\varepsilon(\sigma)\, d\sigma,$$

where $w_\varepsilon = u_\varepsilon - \tilde{u}$. Then we have

Theorem 2. *Under the assumptions of Proposition 1 and if we assume that*

$$\left| u_\varepsilon^0 - u^0 \right|_{L^2(\Omega)} = O(\varepsilon) \text{ and } \left| u_\varepsilon^1 - u^1 \right|_{L^2(\Delta; H^{-1}(\omega))} = O(\varepsilon), \tag{3.19}$$

then we have

$$\sup_{t \in [0,T]} \left| (u'_\varepsilon - \tilde{u}')(t) \right|_{H^{-1}(\Omega)}, \sup_{t \in [0,T]} \left| (u_\varepsilon - \tilde{u})(t) \right|_{L^2(\Omega_0)}, \sup_{t \in [0,T]} \left| \nabla_{X_2} \left(U_\varepsilon - \tilde{U} \right)(t) \right|_{L^2(\Omega_0)} = O(\varepsilon),$$

and

$$\nabla_{X_1} U_\varepsilon(t) \rightharpoonup \nabla_{X_1} \tilde{U}(t) \quad \text{in } L^2(\Omega_0), \qquad \forall t \in [0, T],$$

where $\tilde{u}$ *(resp.* u_ε*) is the solution of problem (3.3) (resp. (2.2)).*

Proof. By comparing (2.2) and (2.7), we deduce that

$$\begin{aligned} \langle w''_\varepsilon, v \rangle + \int_\Omega A_\varepsilon \nabla w_\varepsilon \cdot \nabla v dx &= \int_\Omega \varepsilon^2 A_{11} \nabla_{X_1} \tilde{u} \cdot \nabla_{X_1} v dx \\ &+ \int_\Omega \varepsilon A_{12} \nabla_{X_2} \tilde{u} \cdot \nabla_{X_1} v dx + \int_\Omega \varepsilon A_{21} \nabla_{X_1} \tilde{u} \cdot \nabla_{X_2} v dx, \qquad \forall v \in H_0^1(\Omega). \end{aligned} \tag{3.20}$$

According to (3.15), we have

$$\psi_\varepsilon(\sigma) \varrho^2 = 0 \text{ on } \partial\Omega, \qquad \forall \sigma \in [0, s]$$

which allows us to test (3.20) with $\psi_\varepsilon(\sigma)\varrho^2 \in H_0^1(\Omega)$. Then integrating over $[0,s]$ we get

$$
\begin{aligned}
-\int_\Omega w_\varepsilon'(s)\psi_\varepsilon(s)\varrho^2 dx - \int_\Omega w_\varepsilon^1\psi_\varepsilon(0)\varrho^2 dx - \int_0^s\int_\Omega w_\varepsilon'\psi_\varepsilon'\varrho^2 dxd\sigma + \int_0^s\int_\Omega A_\varepsilon\nabla w_\varepsilon\cdot\nabla\left(\psi_\varepsilon\varrho^2\right)dxd\sigma \\
= \int_0^s\int_\Omega \varepsilon^2 A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}\left(\psi_\varepsilon\varrho^2\right)dxd\sigma + \int_0^s\int_\Omega \varepsilon A_{12}\nabla_{X_2}\tilde{u}\cdot\nabla_{X_1}\left(\psi_\varepsilon\varrho^2\right)dxd\sigma \\
+ \int_0^s\int_\Omega \varepsilon A_{21}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_2}\left(\psi_\varepsilon\varrho^2\right)dxd\sigma, \qquad (3.21)
\end{aligned}
$$

where $w_\varepsilon^0 = u_\varepsilon^0 - u^0$ and $w_\varepsilon^1 = u_\varepsilon^1 - u^1$. The A_ε integral term can be written as

$$
\int_0^s\int_\Omega A_\varepsilon\nabla w_\varepsilon\cdot\nabla\left(\psi_\varepsilon\varrho^2\right)dxd\sigma = -\frac{1}{2}\int_\Omega A_\varepsilon\nabla W_\varepsilon(s)\cdot\nabla W_\varepsilon(s)\varrho^2 dx + \int_0^s\int_\Omega \psi_\varepsilon\varrho A_\varepsilon\nabla w_\varepsilon\cdot\nabla\varrho dxd\sigma,
$$

since $\psi_\varepsilon(s)=0$, $\psi_\varepsilon(0) = -W_\varepsilon(s)$ and $w_\varepsilon\in H^1(\Omega)$. Considering this in (3.21) and note that ϱ is independent of X_2, we obtain

$$
\begin{aligned}
&-\frac{1}{2}\left|w_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} - \frac{1}{2}\int_\Omega A_\varepsilon\nabla W_\varepsilon(s)\cdot\nabla W_\varepsilon(s)\varrho^2 dx \\
&=\frac{1}{2}\left|w_\varepsilon^0\varrho\right|^2_{L^2(\Omega)} - \int_\Omega w_\varepsilon^1 W_\varepsilon(s)\varrho^2 dx + \varepsilon^2\int_0^s\int_\Omega A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}\psi_\varepsilon\varrho^2 dxd\sigma \\
&+\varepsilon\int_0^s\int_\Omega A_{12}\nabla_{X_2}\tilde{u}\cdot\nabla_{X_1}\left(\psi_\varepsilon\varrho^2\right)dxd\sigma + \varepsilon\int_0^s\int_\Omega A_{21}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_2}\psi_\varepsilon\varrho^2 dxd\sigma \\
&+2\varepsilon^2\int_0^s\int_\Omega \varrho\psi_\varepsilon A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}\varrho dxd\sigma - 2\varepsilon^2\int_0^s\int_\Omega \psi_\varepsilon\varrho A_{11}\nabla_{X_1}w_\varepsilon\cdot\nabla_{X_1}\varrho dxd\sigma \\
&-2\varepsilon\int_0^s\int_\Omega \psi_\varepsilon\varrho A_{12}\nabla_{X_2}w_\varepsilon\cdot\nabla_{X_1}\varrho dxd\sigma. \qquad (3.22)
\end{aligned}
$$

Using the Cauchy-Schwarz, Poincaré inequalities and Young's inequality $2ab \le \alpha a^2 + \frac{b^2}{\alpha}$ with convenient choices of α, it follows that

$$
\begin{aligned}
&\left|w_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} + \lambda\varepsilon^2\left|\nabla_{X_1}W_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} + \lambda\left|\nabla_{X_2}W_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} \\
&\le \left|w_\varepsilon^0\varrho\right|^2_{L^2(\Omega)} + C\left|w_\varepsilon^1\varrho\right|^2_{L^2(\Delta;H^{-1}(\omega))} + \frac{\lambda}{4}\left|\nabla_{X_2}W_\varepsilon(s)\varrho\right|^2_{L^2(\Omega)} + C\varepsilon^2\left|\nabla_{X_1}\tilde{u}\right|^2_{L^2(Q)} \\
&+C\varepsilon^4\left|\nabla_{X_1}w_\varepsilon\right|^2_{L^2(Q)} + C\varepsilon^2\left|\nabla_{X_2}w_\varepsilon\right|^2_{L^2(Q)} + \frac{\lambda}{8T}\int_0^s \varepsilon^2\left|\nabla_{X_1}\psi_\varepsilon\varrho\right|^2_{L^2(\Omega)} + \left|\nabla_{X_2}\psi_\varepsilon\varrho\right|^2_{L^2(\Omega)}d\sigma \\
&+2\varepsilon\int_0^s\left|\int_\Omega A_{12}\nabla_{X_2}\tilde{u}\cdot\nabla_{X_1}\left(\psi_\varepsilon\varrho^2\right)dx\right|d\sigma, \qquad (3.23)
\end{aligned}
$$

Then using the density of $\mathcal{D}(\Delta_1\times\omega)$ in $H_0^1(\Delta_1\times\omega)$ we can rewrite the last integral as

$$
\begin{aligned}
\int_\Omega A_{12}\nabla_{X_2}\tilde{u}\cdot\nabla_{X_1}\left(\psi_\varepsilon(\sigma)\varrho^2\right)dx &= \sum_{i=1}^p\sum_{j=p+1}^n\int_\Omega a_{ij}\partial_{x_j}\tilde{u}\partial_{x_i}\left(\psi_\varepsilon(\sigma)\varrho^2\right)dx \\
&= \sum_{i=1}^p\sum_{j=p+1}^n\int_\Omega \partial_{x_i}\left(\varrho^2 a_{ij}\psi_\varepsilon(\sigma)\right)\partial_{x_j}\tilde{u} - \partial_{x_i}a_{ij}\partial_{x_j}\tilde{u}\varrho^2\psi_\varepsilon(\sigma)dx \\
&= \sum_{i=1}^p\sum_{j=p+1}^n\int_\Omega \varrho^2 a_{ij}\partial_{x_j}\psi_\varepsilon(\sigma)\partial_{x_i}\tilde{u} + \varrho^2\partial_{x_j}a_{ij}\psi_\varepsilon(\sigma)\partial_{x_i}\tilde{u}dx \\
&- \sum_{i=1}^p\sum_{j=p+1}^n\int_\Omega \partial_{x_i}a_{ij}\partial_{x_j}\tilde{u}\varrho^2\psi_\varepsilon(\sigma)dx.
\end{aligned}
$$

The same techniques as above yields

$$2\varepsilon\int_0^s\left|\int_\Omega A_{12}\nabla_{X_2}\tilde{u}\cdot\nabla_{X_1}\left(\psi_\varepsilon\varrho^2\right)dx\right|d\sigma \leq C\varepsilon^2\left(|\nabla_{X_1}\tilde{u}|^2_{L^2(Q_1)}+|\nabla_{X_2}\tilde{u}|^2_{L^2(Q_1)}\right) +\frac{\lambda}{8T}\int_0^s|\nabla_{X_2}\psi_\varepsilon\varrho|^2_{L^2(\Omega)}d\sigma.$$

Going back to (3.23) we derive

$$|w_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+\lambda\varepsilon^2|\nabla_{X_1}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+\frac{3\lambda}{4}|\nabla_{X_2}W(s)\varrho|^2_{L^2(\Omega)}$$
$$\leq\left|w^0_\varepsilon\right|^2_{L^2(\Omega_1)}+C\left|w^1_\varepsilon\right|^2_{L^2(\Delta;H^{-1}(\omega))}+C\varepsilon^2\left\{|\nabla\tilde{u}|^2_{L^2(Q_1)}+\varepsilon^2|\nabla_{X_1}w_\varepsilon|^2_{L^2(Q_1)}+|\nabla_{X_2}w_\varepsilon|^2_{L^2(Q_1)}\right\}$$
$$+\frac{\lambda}{4T}\int_0^s\varepsilon^2|\nabla_{X_1}\psi_\varepsilon(\sigma)\varrho|^2_{L^2(\Omega)}+|\nabla_{X_2}\psi_\varepsilon(\sigma)\varrho|^2_{L^2(\Omega)}d\sigma, \tag{3.24}$$

The last two terms of the right-hand side in (3.24) can be estimated as

$$\frac{\lambda}{4T}\int_0^s\varepsilon^2|\nabla_{X_1}\psi_\varepsilon(\sigma)\varrho|^2_{L^2(\Omega)}+|\nabla_{X_2}\psi_\varepsilon(\sigma)\varrho|^2_{L^2(\Omega)}d\sigma$$
$$\leq\frac{\lambda}{2T}\int_0^s\varepsilon^2|\nabla_{X_1}W_\varepsilon(\sigma)\varrho|^2_{L^2(\Omega)}+|\nabla_{X_2}W_\varepsilon(\sigma)\varrho|^2_{L^2(\Omega)}d\sigma$$
$$+\frac{\lambda}{2}\left(\varepsilon^2|\nabla_{X_1}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+|\nabla_{X_2}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}\right).$$

Considering this in (3.24) we get

$$|w_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+\frac{\lambda}{2}\varepsilon^2|\nabla_{X_1}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+\frac{\lambda}{4}|\nabla_{X_2}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}$$
$$\leq\left|w^0_\varepsilon\right|^2_{L^2(\Omega_1)}+C\left|w^1_\varepsilon\right|^2_{L^2(\Delta_1;H^{-1}(\omega))}+C\varepsilon^2\left\{|\nabla\tilde{u}|^2_{L^2(Q_1)}+|\nabla_{X_2}w_\varepsilon|^2_{L^2(Q_1)}+\varepsilon^2|\nabla_{X_1}w_\varepsilon|^2_{L^2(Q_1)}\right\}$$
$$+\frac{\lambda}{2T}\int_0^s\varepsilon^2|\nabla_{X_1}W_\varepsilon(\sigma)\varrho|^2_{L^2(\Omega)}+|\nabla_{X_2}W_\varepsilon(\sigma)\varrho|^2_{L^2(\Omega)}d\sigma.$$

For $s\in[0,T]$, using the Gronwall inequality we obtain

$$|w_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+\varepsilon^2|\nabla_{X_1}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+|\nabla_{X_2}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}\leq C\left(\left|w^0_\varepsilon\right|^2_{L^2(\Omega_1)}+\left|w^1_\varepsilon\right|^2_{L^2(\Delta_1;H^{-1}(\omega))}\right)$$
$$+C\varepsilon^2\left\{|\nabla_{X_1}\tilde{u}|^2_{L^2(Q_1)}+|\nabla_{X_2}\tilde{u}|^2_{L^2(Q_1)}+|\nabla_{X_2}w_\varepsilon|^2_{L^2(Q_1)}+\varepsilon^2|\nabla_{X_1}w_\varepsilon|^2_{L^2(Q_1)}\right\}, \tag{3.25}$$

where all the terms in the right-hand side are independent of s. Applying Theorem 1 and (3.19), we derive

$$\sup_{s\in[0,T]}|w_\varepsilon(s)|_{L^2(\Omega_0)},\ \sup_{s\in[0,T]}|\nabla_{X_2}W_\varepsilon(s)|_{L^2(\Omega_0)}=O(\varepsilon), \tag{3.26}$$

and by consequence we have

$$|w_\varepsilon|_{L^2(Q_0)},|\nabla_{X_2}W_\varepsilon|_{L^2(Q_0)}=O(\varepsilon). \tag{3.27}$$

We also derive from (3.25) that

$$\nabla_{X_1}W_\varepsilon(s)\text{ is bounded in }L^2(\Omega_0), \tag{3.28}$$

independently of $s\in[0,T]$. So for a fixed $s\in[0,T]$, there exist χ such that -up to a subsequence-

$$\nabla_{X_1}W_\varepsilon(s)\rightharpoonup\chi(s)\text{ in }L^2(\Omega_0),$$

and we may verify as above (see (2.14)) that $\chi(s)=0$, i.e.

$$\nabla_{X_1}W_\varepsilon(s)\rightharpoonup 0\quad\text{in }L^2(\Omega_0). \tag{3.29}$$

The convergence holds for the whole sequence, since the limit is unique, and for every $s \in [0,T]$. By Lebesgue's theorem and the pointwise convergence in s, we also obtain

$$\nabla_{X_1} W_\varepsilon \rightharpoonup 0, \quad \text{in } L^2(Q_0). \tag{3.30}$$

Integrating (3.20) over $(0,s)$, it comes

$$\int_\Omega w'_\varepsilon(s)\, v dx = \int_\Omega w^1_\varepsilon v dx - \int_\Omega A_\varepsilon \nabla W_\varepsilon(s) \cdot \nabla v dx$$
$$- \int_0^s \int_\Omega \varepsilon^2 A_{11} \nabla_{X_1} \tilde{u} \cdot \nabla_{X_1} v + \varepsilon A_{12} \nabla_{X_2} \tilde{u} \cdot \nabla_{X_1} v + \varepsilon A_{21} \nabla_{X_1} \tilde{u} \cdot \nabla_{X_2} v \; dxds, \qquad \forall v \in H^1_0(\Omega_0).$$

Applying the Cauchy-Schwarz inequality to each term, we find

$$\begin{aligned} \int_{\Omega_0} w'_\varepsilon(s)\, v dx \quad &\leq \quad C \Big\{ \left| w^1_\varepsilon \right|_{L^2(\Delta_0; H^{-1}(\omega))} + \varepsilon \left| \nabla_{X_1} W_\varepsilon(s) \right|_{L^2(\Omega_0)} \\ &\quad + \left| \nabla_{X_2} W_\varepsilon(s) \right|_{L^2(\Omega_0)} + \varepsilon \left| \nabla \tilde{u} \right|_{L^2(Q_0)} \Big\} \left| \nabla v \right|_{L^2(\Omega_0)}, \qquad \forall v \in H^1_0(\Omega_0). \end{aligned}$$

Then we conclude, after dividing on ε, that

$$\begin{aligned} \frac{1}{\varepsilon} \left| w'_\varepsilon(s) \right|_{H^{-1}(\Omega_0)} \quad &\leq \quad C \Big\{ \frac{1}{\varepsilon} \left| w^1_\varepsilon \right|_{L^2(\Delta_0; H^{-1}(\omega))} + \left| \nabla_{X_1} W_\varepsilon(s) \right|_{L^2(\Omega_0)} \\ &\quad + \frac{1}{\varepsilon} \left| \nabla_{X_2} W_\varepsilon(s) \right|_{L^2(\Omega_0)} + \left| \nabla \tilde{u} \right|_{L^2(Q_0)} \Big\}. \end{aligned} \tag{3.31}$$

Due to (3.19) and (3.26)-(3.30), the right-hand side terms of the above inequality are bounded independently of $s \in [0,T]$. Thus we get

$$\sup_{s \in [0,T]} \left| w'_\varepsilon(s) \right|_{H^{-1}(\Omega_0)} = O(\varepsilon).$$

This ends the proof of the theorem. □

When the matrix A has a diagonal structure we can improve this result.

Corollary 2 (Diagonal matrix)**.** *Under the assumptions of Theorem 2, in addition we suppose that* $A_{12} = A_{21} = 0$ *and*

$$\left| u^0_\varepsilon - u^0 \right|_{L^2(\Omega)} = o(\varepsilon), \quad \left| u^1_\varepsilon - u^1 \right|_{L^2(\Delta; H^{-1}(\omega))} = o(\varepsilon), \tag{3.32}$$

then

$$\sup_{t \in [0,T]} \left| (u'_\varepsilon - \tilde{u}')(t) \right|_{H^{-1}(\Omega_0)}, \ \sup_{t \in [0,T]} \left| (u_\varepsilon - \tilde{u})(t) \right|_{L^2(\Omega_0)}, \ \sup_{t \in [0,T]} \left| \nabla_{X_2} \left(U_\varepsilon - \tilde{U} \right)(t) \right|_{L^2(\Omega_0)} = o(\varepsilon),$$
$$\sup_{t \in [0,T]} \left| \nabla_{X_1} \left(U_\varepsilon - \tilde{U} \right)(t) \right|_{L^2(\Omega_0)} = o(1).$$

Proof. Taking into account the fact that $A_{12} = A_{21} = 0$ in the proof of Theorem 2, then (3.22) becomes

$$\begin{aligned} -\frac{1}{2} \left| w_\varepsilon(s)\, \varrho \right|^2_{L^2(\Omega)} - \frac{1}{2} \int_\Omega A_\varepsilon \nabla W_\varepsilon(s) \cdot \nabla W_\varepsilon(s)\, \varrho^2 dx \\ = -\frac{1}{2} \left| w^0_\varepsilon \varrho \right|^2_{L^2(\Omega)} - \int_\Omega w^1_\varepsilon W_\varepsilon(s)\, \varrho^2 dx - \varepsilon^2 \int_0^s \int_\Omega A_{11} \nabla_{X_1} \tilde{u} \cdot \nabla_{X_1} \psi_\varepsilon \varrho^2 dx d\sigma \\ - 2\varepsilon^2 \int_0^s \int_\Omega \varrho \psi_\varepsilon A_{11} \nabla_{X_1} \tilde{u} \cdot \nabla_{X_1} \varrho dx d\sigma - 2\varepsilon^2 \int_0^s \int_\Omega \psi_\varepsilon \varrho A_{11} \nabla_{X_1} w_\varepsilon \cdot \nabla_{X_1} \varrho dx d\sigma. \end{aligned}$$

Estimating as above, we end up with

$$\frac{1}{\varepsilon^2}|w_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+|\nabla_{X_1}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}+\frac{1}{\varepsilon^2}|\nabla_{X_2}W_\varepsilon(s)\varrho|^2_{L^2(\Omega)}$$
$$\leq\frac{C}{\varepsilon^2}\left\{|w^0_\varepsilon|^2_{L^2(\Omega)}+|w^1_\varepsilon|^2_{L^2(\Delta;H^{-1}(\omega))}\right\}+C\varepsilon^2\left\{|\nabla_{X_1}\tilde{u}|^2_{L^2(Q_1)}+|\nabla_{X_1}w_\varepsilon|^2_{L^2(Q_1)}\right\}$$
$$+C\int_0^s\left|\int_\Omega A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}\psi_\varepsilon\varrho^2dx\right|d\sigma, \tag{3.33}$$

for every $s\in[0,T]$. Thanks to Theorem 2, we use the weak convergence of $\nabla_{X_1}W_\varepsilon$ in the A_{11} integral above, it comes

$$\int_\Omega A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}\psi_\varepsilon(\sigma)\varrho^2dx=\int_\Omega A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}W_\varepsilon(\sigma)\varrho^2dx-\int_\Omega A_{11}\nabla_{X_1}\tilde{u}\cdot\nabla_{X_1}W_\varepsilon(s)\varrho^2dx\to 0,$$

for every $s\in[0,T]$. Then by Lebesgue's theorem we deduce that the A_{11} integral in (3.33) tends to zero. Thus by (3.32) we obtain

$$\sup_{s\in[0,T]}|w_\varepsilon(s)|_{L^2(\Omega_0)},\ \sup_{s\in[0,T]}|\nabla_{X_2}W_\varepsilon(s)|_{L^2(\Omega_0)}=o(\varepsilon),\ \sup_{s\in[0,T]}|\nabla_{X_1}W_\varepsilon(s)|_{L^2(\Omega_0)}\to 0. \tag{3.34}$$

Next, the equivalent of (3.31), in this case ($A_{12}=A_{21}=0$), is given by

$$\frac{1}{\varepsilon}|w'_\varepsilon(s)|_{H^{-1}(\Omega_0)}\leq C\left\{\frac{1}{\varepsilon}|w^1_\varepsilon|_{L^2(\Delta_0;H^{-1}(\omega))}+|\nabla_{X_1}W_\varepsilon(s)|_{L^2(\Omega_0)}+\frac{1}{\varepsilon}|\nabla_{X_2}W_\varepsilon(s)|_{L^2(\Omega_0)}+\varepsilon|\nabla_{X_1}\tilde{u}|_{L^2(Q_0)}\right\}.$$

It follows, by (3.32) and (3.34), that

$$\sup_{s\in[0,T]}|w'(s)|_{H^{-1}(\Omega_0)}=o(\varepsilon).$$

This completes the proof of the corollary. □

Remark 4. *As a trivial consequence of the corollary above we have*

$$\sup_{t\in[0,T]}\left|U_\varepsilon(t)-\tilde{U}(t)\right|_{H^1(\Omega_0)}\to 0.$$

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