Available at: http://publications.ictp.it IC/2010/043

United Nations Educational, Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

OUTER FUNCTIONS IN ANALYTIC WEIGHTED LIPSCHITZ ALGEBRAS

Brahim Bouya[1]
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

The analytic weighted Lipschitz algebra Λ_ω is the Banach algebra of all analytic functions on the unit disk $\mathbb{D}$, that are continuous on $\overline{\mathbb{D}}$ and such that

$$\sup_{\substack{z,w\in\overline{\mathbb{D}}\\ z\neq w}} \frac{|f(z)-f(w)|}{\omega(|z-w|)} < +\infty,$$

where ω is a modulus of continuity. We give a new characterization of outer functions in Λ_ω, by their modulus in $\mathbb{T}$. As application, we obtain a refinement of Shirokov's construction of outer functions in Λ_ω vanishing on a given $\omega-$Carleson set. We obtain also an extension of Havin-Shamoyan-Carleson-Jacobs Theorem to an arbitrary modulus of continuity.

MIRAMARE – TRIESTE
July 2010

[1]brahimbouya@gmail.com

1. Introduction and statement of main result

Let ω be a *modulus of continuity*, i.e., a nondecreasing continuous real-valued function on $[0, 2]$ with $\omega(0) = 0$, $\omega(1) = 1$ and such that $t \mapsto \omega(t)/t$ is a non increasing function. It is called *fast* if

$$\int_0^s \frac{\omega(t)}{t} dt \leq C_\omega \, \omega(s), \qquad s \in]0, 2], \tag{1.1}$$

and *slow* if

$$\int_s^2 \frac{\omega(t)}{t^2} dt \leq c_\omega \frac{\omega(s)}{s}, \qquad s \in]0, 2], \tag{1.2}$$

for some constants C_ω and $c_\omega > 0$ independent of s (see [4]). Associated to such modulus of continuity ω we define the following

$$\omega^*(t) := \inf\{u \in [0, 1] \ : \ \omega(u) = t\}, \qquad t \in [0, 1].$$

It is clear that ω^* is an increasing function such that $\omega(\omega^*(t)) = t$ and $\omega^*(\omega(t)) \leq t$, $t \in [0, 1]$. Since $\omega(t)/t$ is non increasing, then $\omega^*(t)/t$ is non decreasing.

Now, let $\mathbb{D}$ be the open unit disk of the complex plane and $\mathbb{T}$ its boundary, and denote by $\mathcal{A}(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$ that are continuous on $\overline{\mathbb{D}}$. For given continuous function f defined on a compact subset $\mathbb{G}$ of $\overline{\mathbb{D}}$ and given modulus of continuity ω, we set

$$\omega_{\mathbb{G}}(f) := \sup_{\substack{z,w \in \mathbb{G} \\ z \neq w}} \frac{|f(z) - f(w)|}{\omega(|z - w|)}.$$

The analytic weighted Lipschitz algebra Λ_ω of $\mathbb{D}$ consists of all functions in $\mathcal{A}(\mathbb{D})$ satisfying $\omega_{\overline{\mathbb{D}}}(f) < +\infty$, i.e,

$$\Lambda_\omega := \Big\{ f \in \mathcal{A}(\mathbb{D}) : \omega_{\overline{\mathbb{D}}}(f) < +\infty \Big\}.$$

For an arbitrary modulus of continuity ω, Tamrazov [10] proved that

$$\omega_{\overline{\mathbb{D}}}(f) < +\infty \quad \Longleftrightarrow \quad \omega_{\mathbb{T}}(f) < +\infty, \qquad f \in \mathcal{A}(\mathbb{D}). \tag{1.3}$$

A complex analytic proof of (1.3) can be found in [1, Appendix A].

Let $h : \mathbb{T} \to \mathbb{R}^+$ be a nonnegative continuous function satisfying

$$\frac{1}{2\pi} \int_{\mathbb{T}} \log h(\xi) |d\xi| > -\infty, \tag{1.4}$$

We associate to h the outer function O_h defined by

$$O_h(z) := \exp\{u_h(z) + i v_h(z)\}, \qquad z \in \mathbb{D},$$

where

$$u_h(z) := \frac{1}{2\pi} \int_{\mathbb{T}} \mathrm{Re}\Big(\frac{\xi + z}{\xi - z}\Big) \log h(\xi) |d\xi|, \quad z \in \mathbb{D},$$

and v_h is the harmonic conjugate of the harmonic function u_h given by

$$v_h(z) := \frac{1}{2\pi} \int_{\mathbb{T}} \mathrm{Im}\Big(\frac{\xi + z}{\xi - z}\Big) \log h(\xi) |d\xi|, \qquad z \in \mathbb{D}.$$

Since h is continuous and satisfies (1.4), then $|O_h| = \exp\{u_h\}$ is also continuous on $\overline{\mathbb{D}}$ and $O_h \in \mathcal{H}^\infty(\mathbb{D})$ is a bounded analytic function. The non tangential limits of v_h exist and coincide a.e on $\mathbb{T}$ with the following function (see [5, page 99])

$$v_h(\theta) := -\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\varepsilon\le|t|\le\pi}\frac{\log h(\theta+t)}{\tan(\frac{1}{2}t)}dt, \qquad e^{i\theta}\in\mathbb{T}\setminus E_h, \tag{1.5}$$

where $h(s) := h(e^{is}),\ s\in[0,2\pi[$, and $E_h \neq \varnothing$ is the zeros set of h.
We suppose that h satisfies

$$\psi_{\mathbb{T}}(h) < +\infty, \tag{1.6}$$

for all $(\theta,s)\in[0,2\pi[\times[0,2\pi[$ such that $e^{i\theta}\in\mathbb{T}\setminus E_h$, $0<s\le\psi^*\big(\frac{h(\theta)}{2\psi_{\mathbb{T}}(h)}\big)$,

$$\lim_{\varepsilon\to 0}\int_\varepsilon^s\frac{|h(\theta+t)-h(\theta-t)|}{t}dt \le C_h\ \psi(s), \tag{1.7}$$

and

$$\frac{1}{h(\theta)}\int_s^{\psi^*\left(\frac{h(\theta)}{2\psi_{\mathbb{T}}(h)}\right)}\frac{|h(\theta+t)\times h(\theta-t)-h^2(\theta)|}{t^2}dt \le c_h\ \frac{\psi(s)}{s}, \tag{1.8}$$

where ψ is a modulus of continuity and $C_h>0$, $c_h>0$ are constants independent of both θ and s. In section 2.2 below, we will give simple examples of functions satisfying the conditions (1.4), (1.6), (1.7) and (1.8) with respect to an arbitrary modulus of continuity. It is easy to check that if ψ is fast then (1.7) follows, while (1.8) holds when ψ is slow. We set

$$O_h(\xi) := \begin{cases} h(\xi)\exp\{iv_h(\xi)\}, & \xi\in\mathbb{T}\setminus E_h,\\ 0, & \xi\in E_h.\end{cases} \tag{1.9}$$

In the proof of Lemmas 3.5 and 4.1 below, we will see that the conditions (1.4), (1.6), (1.7) and (1.8) ensure the continuity of v_h on $\overline{\mathbb{D}}\setminus E_h$ and by consequence the continuity of O_h on $\overline{\mathbb{D}}$.

Dyakonov [3] and Pavlović [7] have proved the following equivalence

$$\omega_{\mathbb{D}}(f) < +\infty \Longleftrightarrow \omega_{\mathbb{D}}(|f|) < +\infty, \qquad f\in\mathcal{A}(\mathbb{D}), \tag{1.10}$$

where ω is supposed here to be both fast and slow modulus of continuity. In the case when h satisfies (1.4) and (1.6) and also ω is both fast and slow, Shirokov [8] showed that $O_h\in\Lambda_\omega$ if and only if (see [3, Theorem A])

$$\sup_{\substack{z\in\mathbb{D}\\ M_z\ge\omega(1-|z|)}}\big|\log\frac{O_h(z)}{M_z}\big| < +\infty, \tag{1.11}$$

where

$$M_z := \max\{h(\xi)\ :\ \xi\in\mathbb{T},\ |\xi-z|\le 2(1-|z|)\}, \qquad z\in\mathbb{D}.$$

In Section 4.1 we have to prove our main result (Theorem 1.1 below). For exact statement, we set

$$a_h(\xi) := \int_{\mathbb{T}\setminus\Psi_\xi}\frac{\big|\log\frac{h(\zeta)}{h(\xi)}\big|}{|\zeta-\xi|^2}|d\zeta|, \qquad \xi\in\mathbb{T},$$

with

$$\Psi_\xi := \{\zeta\in\mathbb{T}\ :\ |\zeta-\xi|\le\psi^*\big(\frac{h(\xi)}{2\psi_{\mathbb{T}}(h)}\big)\}, \qquad \xi\in\mathbb{T}.$$

Theorem 1.1. *Let $h : \mathbb{T} \mapsto \mathbb{R}^+$ be a continuous nonnegative function satisfying* (1.4), (1.6), (1.7) *and* (1.8). *Let $\rho \geq 1$ be a real number and ω be an arbitrary modulus of continuity such that*

$$\omega(t) \geq \psi(t), \qquad t \in [0,2].$$

The following four assertions are equivalents

1. $O_h^\rho \in \Lambda_\omega$.
2. $\omega_{\mathbb{D}}(|O_h^\rho|) < +\infty$.
3. *There exists a real number $\delta > 0$ such that*

$$\sup_{\xi \in \mathbb{T}} \Big\{ \sup_{z \in \mathbb{D}_\xi} \big| \log \frac{|O_h(z)|}{h(\xi)} \big| \Big\} < +\infty,$$

where

$$\mathbb{D}_\xi := \{ z \in \overline{\mathbb{D}} \ : \ |z - \xi| \leq \omega^* \big(\delta h^\rho(\xi)\big) \}, \qquad \xi \in \mathbb{T}.$$

4.

$$\sup_{\xi \in \mathbb{T}} \frac{h^\rho(\xi)}{\omega(\min\{1, a_h^{-1}(\xi)\})} < +\infty.$$

In particular, when $\rho = 1$ and $\omega = \psi$ is both a fast and slow modulus of continuity, the equivalence between 1 and 3 of Theorem 1.1 gives a refinement of the above Shirokov's result (1.11). Also, in this particular case, the equivalence between 2 and 3 gives a connection between the above Dyakonov's result (1.10) and Shirokov's one.

2. Some applications

2.1. **Havin-Shamoyan-Carleson-Jacobs Theorem.** We say that a modulus of continuity ω is *ρ-slow* if the following condition holds

$$\inf_{t \in [0,2]} \frac{\omega(t^2)}{\omega^\rho(t)} \geq \eta_\rho > 0, \tag{2.1}$$

where $1 \leq \rho \leq 2$ is a real number. As an example, we can take $\chi(t) := 1/(1 + |\log(t)|)$ and $\omega_\alpha(t) := t^\alpha$, $0 < \alpha < 1$. It is easy to check that χ is ρ–slow and not fast, while ω_α is fast but not ρ–slow for any $1 \leq \rho < 2$.

As a simple consequence of Theorem 1.1, we have the following

Corollary 2.1. *Let $h : \mathbb{T} \mapsto \mathbb{R}^+$ be a nonnegative continuous function satisfying* (1.4), (1.6), (1.7) *and* (1.8). *Suppose that there exists a real number $1 \leq \rho \leq 2$ such that ψ is ρ-slow. Then $O_h^\rho \in \Lambda_\psi$ and $O_h \in \Lambda_{\psi_\rho}$, where*

$$\psi_\rho(t) := \psi^{\frac{1}{\rho}}(t), \qquad t \in [0,2].$$

The particular case of $\rho = 2$ and ψ being fast and slow (take for example $\psi = \omega_\alpha$) in Corollary 2.1 gives a simple proof of the Havin-Shamoyan-Carleson-Jacobs Theorem (see for instance [3, 6]).

Proof of Corollary 2.1: Using Theorem 4.1, we have

$$\begin{aligned} O_h^\rho \in \Lambda_\psi \quad & \iff \quad \sup_{\xi\in\mathbb{T}} \frac{h^\rho(\xi)}{\psi(\min\{1, a_h^{-1}(\xi)\})} < +\infty \\ & \iff \quad \sup_{\xi\in\mathbb{T}} \frac{h(\xi)}{\psi^{\frac{1}{\rho}}(\min\{1, a_h^{-1}(\xi)\})} < +\infty \\ & \iff \quad O_h \in \Lambda_{\psi_\rho}. \end{aligned}$$

We fix $\xi \in \mathbb{T} \setminus E_h$. It is easy to show that

$$a_h(\xi) \le \frac{c_h}{\left(\psi^*(\frac{h(\xi)}{2\psi_{\mathbb{T}}(h)})\right)^2}, \tag{2.2}$$

where c_h is a constant independent of ξ. Using (2.1) and (2.2), we derive

$$\begin{aligned} h^\rho(\xi) &= (2\psi_{\mathbb{T}}(h))^\rho \ \psi^\rho(\psi^*(\frac{h(\xi)}{2\psi_{\mathbb{T}}(h)})) \\ &\le \frac{(2\psi_{\mathbb{T}}(h))^\rho}{\eta_\rho} \ \psi\Big(\big(\psi^*(\frac{h(\xi)}{2\psi_{\mathbb{T}}(h)})\big)^2\Big) \\ &\le c_{\rho,h} \ \psi(\min\{1, a_h^{-1}(\xi)\}), \end{aligned}$$

where $c_{\rho,h}$ is a constant independent of ξ. This finishes the proof of Corollary 2.1.

2.2. **Boundary zeros set of functions in** Λ_ω **.** Let $\mathbb{E} \subseteq \mathbb{T}$ be a closed set of zero Lebesgue measure. It is clear that we can write $\mathbb{T}\setminus\mathbb{E} = \bigcup_{n=0}^{\infty} \gamma_n$, where $\gamma_n = (a_n, b_n) \subseteq \mathbb{T}\setminus\mathbb{E}$ is an open arc joining the points $a_n, b_n \in \mathbb{E}$. The closed set $\mathbb{E}$ is called ω-*Carleson* if it satisfies the following condition

$$\sum_{n\in\mathbb{N}} |b_n - a_n| \log \omega(|b_n - a_n|) > -\infty. \tag{2.3}$$

In the case when $\omega = \omega_\alpha$, Carleson [2] proved that the condition (2.3) is necessary and sufficient for $\mathbb{E}$ being zero set of a function $f \in \Lambda_\omega \setminus \{0\}$. In [8, 9], Shirokov obtained a generalization of Carleson's result to an arbitrary modulus of continuity. However, Shirokov's construction is different and much more complicated than Carleson's one. Namely, he proved that if there exists a set of real numbers $\mathcal{R} := \{r_n > 0 \ : \ n \in \mathbb{N}\}$ satisfying some properties then $O_{h_\mathcal{R}} \in \Lambda_\omega$, where

$$h_\mathcal{R}(\xi) := \begin{cases} r_n \dfrac{|\xi - b_n|^2|\xi - a_n|^2}{|b_n - a_n|^4}, & \xi \in \gamma_n, \\ 0, & \xi \in \mathbb{E}. \end{cases}$$

In this section we give a refinement of the above Shirokov's construction. So, let $h_\mathbb{E}$ be a function defined on $\mathbb{T}$ as

$$h_\mathbb{E}(\xi) := \begin{cases} \omega(|b_n - a_n|) \dfrac{|\xi - b_n||\xi - a_n|}{|b_n - a_n|^2}, & \xi \in \gamma_n, \\ 0, & \xi \in \mathbb{E}, \end{cases}$$

where ω is a modulus of continuity. Using the equality

$$\int_a^b \log \frac{(b-t)(t-a)}{(b-a)^2} dt = 2(a-b),$$

it is easy to check that $h_\mathbb{E}$ satisfies (1.4) if and only if $\mathbb{E}$ is an ω-Carleson set. From Corollary 2.1, we obtain the following one:

Corollary 2.2. *Let ω be a ρ-slow modulus of continuity and $\mathbb{E} \subseteq \mathbb{T}$ a closed subset of zero Lebesgue measure. In order that $O^{\rho}_{h_{\mathbb{E}}}$ belongs to Λ_ω, it is necessary and sufficient that $\mathbb{E}$ is an ω–Carleson set.*

Proof. The necessary condition follows from Jensen's formula. Thus we have to prove the sufficient condition. For this, let $\mathbb{E} \subseteq \mathbb{T}$ be a closed subset of zero Lebesgue measure and assume that $\mathbb{E}$ is an ω–Carleson set. Then the continuous function $h_{\mathbb{E}}$ on $\mathbb{T}$ satisfies (1.4). A direct computation shows that $h_{\mathbb{E}} \leq 1$ and that $\omega_{\mathbb{T}}(h_{\mathbb{E}}) \leq 2$. Note that the function $k_{\mathbb{E}} : \theta \in [0, 2\pi[\mapsto h_{\mathbb{E}}(e^{i\theta})$ is a smooth function on $\mathbb{T} \setminus \mathbb{E}$. By a simple calculation, we obtain

$$\sup_{e^{i\theta} \in \gamma_n} k'_{\mathbb{E}}(\theta) < C \, \frac{\omega(|b_n - a_n|)}{|b_n - a_n|}, \qquad n \in \mathbb{N}, \tag{2.4}$$

and

$$\sup_{e^{i\theta} \in \gamma_n} k''_{\mathbb{E}}(\theta) < c \, \frac{\omega(|b_n - a_n|)}{|b_n - a_n|^2}, \qquad n \in \mathbb{N}, \tag{2.5}$$

where $C > 0$ and $c > 0$ are absolute constants. Using the fact that $\omega(t)/t$ is non increasing, we deduce that (2.4) implies (1.7) and that (2.5) implies (1.8). From Corollary 2.1, we conclude that $O^{\rho}_{h_{\mathbb{E}}} \in \Lambda_\omega$. The proof of Corollary 2.2 is completed. □

3. Technical results

Let $h : \mathbb{T} \mapsto \mathbb{R}^+$ be a continuous nonnegative function satisfying conditions (1.4) and (1.6). Fix two distinct points $\xi = e^{i\theta}$ and $\zeta = e^{i\varphi}$ such that $\theta \leq \varphi$, $(\theta, \varphi) \in [0, 2\pi[\times[0, 2\pi[$. We define $[\xi, \zeta] \subseteq \mathbb{T}$ to be the following closed arc joining the points ξ and ζ

$$[\xi, \zeta] := \{e^{is} \ : \ \theta \leq s \leq \varphi\}$$

and we set

$$A_h(\xi, \zeta) := \left| \sin \left\{ \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{\varepsilon}^{\lambda_h(\xi,\zeta)} \frac{\log \frac{h(\theta+t)}{h(\theta-t)} - \log \frac{h(\varphi+t)}{h(\varphi-t)}}{\tan(\frac{1}{2}t)} dt \right\} \right|,$$

where

$$\lambda_h(\xi, \zeta) := \inf_{\sigma \in [\xi,\zeta]} \psi^*\Big(\frac{h(\sigma)}{2\psi_{\mathbb{T}}(h)}\Big).$$

Note that if $\lambda_h(\xi, \zeta) > 0$, then $[\xi, \zeta] \subseteq \mathbb{T} \setminus \mathbb{E}_h$. In this section, we prove the following

Proposition 3.1. *Let $h : \mathbb{T} \mapsto \mathbb{R}^+$ be a continuous nonnegative function satisfying the conditions* (1.4) *and* (1.6)*, where ψ is an arbitrary modulus of continuity. We suppose that $|v_h(\xi)| < +\infty$, for all $\xi \in \mathbb{T} \setminus E_h$, (see* (1.5)*). Let $\rho \geq 1$ be a real number and let ω be an arbitrary modulus of continuity such that*

$$\omega(t) \geq \psi(t), \qquad t \in [0, 2]. \tag{3.1}$$

Then, $O^{\rho}_h \in \Lambda_\omega$ if and only if the following two conditions hold

$$\sup_{\xi \in \mathbb{T}} \frac{h^\rho(\xi)}{\omega(\min\{1, a_h^{-1}(\xi)\})} < +\infty, \tag{3.2}$$

$$\sup_{\xi,\zeta \in \mathbb{T}} \frac{h^\rho(\xi, \zeta) A_{h^\rho}(\xi, \zeta)}{\omega(|\xi - \zeta|)} < +\infty, \tag{3.3}$$

where $h(\xi,\zeta) := \inf_{\sigma\in[\xi,\zeta]} h(\sigma)$.

It is not hard to check that conditions (1.7) and (1.8) together implies (3.3), see Lemma 4.1.

3.1. **Proof of Proposition 3.1.** Let $h : \mathbb{T} \mapsto \mathbb{R}^+$ be a continuous nonnegative function satisfying conditions (1.4) and (1.6). We fix a real number $\rho \geq 1$ as well as a modulus of continuity ω satisfying (3.1). For proving Proposition 3.1 we show a series of technical lemmas.

3.1.1. ***Necessary conditions.*** We begin by the following lemma, which gives rise to the necessary condition (3.2) of Proposition 3.1.

Lemma 3.2. *Let* $h : \mathbb{T} \mapsto \mathbb{R}^+$ *be a continuous nonnegative function satisfying conditions* (1.4) *and* (1.6). *If*

$$\sup_{\xi\in\mathbb{T}}\Big\{\sup_{z\in\mathbb{D}_\xi}\Big|\log\frac{|O_h(z)|}{h(\xi)}\Big|\Big\} < +\infty, \tag{3.4}$$

where

$$\mathbb{D}_\xi = \{z \in \overline{\mathbb{D}} \ : \ |z-\xi| \leq \omega^*\big(\delta h^\rho(\xi)\big)\}, \qquad \xi \in \mathbb{T},$$

and $\delta \leq 1/\big(4\psi_{\mathbb{T}}(h)\big)^\rho$ *is a positive number. Then*

$$\sup_{\xi\in\mathbb{T}}\frac{h^\rho(\xi)}{\omega(\min\{1, a_h^{-1}(\xi)\})} < +\infty.$$

Proof. We have

$$\log|O_h(z)| = \frac{1}{2\pi}\int_{\mathbb{T}}\frac{1-|z|^2}{|\zeta-z|^2}\log h(\zeta)|d\zeta|, \qquad z \in \mathbb{D}.$$

Note that

$$h(\xi) \leq \psi_{\mathbb{T}}(h)\psi(d(\xi, E_h)) \leq 2\psi_{\mathbb{T}}(h), \qquad \xi \in \mathbb{T}, \tag{3.5}$$

where $d(\xi, E_h)$ is the Euclidean distance from the point ξ to E_h. Now, we fix a point $\xi \in \mathbb{T}\setminus E_h$ such that $a_h^{-1}(\xi) \leq 1$. It is clear

$$\begin{aligned}&\frac{1}{2\pi}\int_{\mathbb{T}\setminus\Psi_\xi}\frac{1-|z|^2}{|\zeta-z|^2}\log\frac{h(\zeta)}{h(\xi)}|d\zeta|\\ =\;& \log\frac{|O_h(z)|}{h(\xi)} - \frac{1}{2\pi}\int_{\Psi_\xi}\frac{1-|z|^2}{|\zeta-z|^2}\log\frac{h(\zeta)}{h(\xi)}|d\zeta|, \quad z \in \mathbb{D}.\end{aligned} \tag{3.6}$$

We have

$$\frac{1}{2\pi}\int_{\Psi_\xi}\frac{1-|z|^2}{|\zeta-z|^2}\log\frac{h(\zeta)}{h(\xi)}|d\zeta| \leq \frac{\log 2}{2\pi}\int_{\Psi_\xi}\frac{1-|z|^2}{|\zeta-z|^2}|d\zeta| \leq \log 2, \quad z \in \mathbb{D}. \tag{3.7}$$

Combining (3.4), (3.6) and (3.7) yields

$$\Big|\frac{1}{2\pi}\int_{\mathbb{T}\setminus\Psi_\xi}\frac{1-|z|^2}{|\zeta-z|^2}\log\frac{h(\zeta)}{h(\xi)}|d\zeta|\Big| \leq c_1, \qquad z \in \mathbb{D}_\xi, \tag{3.8}$$

where $c_1 > 0$ is a constant independent of ξ. By a simple calculation

$$|z-w|^2 = ||z|-|w||^2 + |zw||\frac{z}{|z|} - \frac{w}{|w|}|^2, \qquad z, w \in \overline{\mathbb{D}}. \tag{3.9}$$

Note that from (3.1)

$$\omega^*(t) \leq \psi^*(t), \qquad t \in [0,1]. \tag{3.10}$$

We set $z_\xi := \big(1-\omega^*(\delta h^\rho(\xi))\big)\xi$. It is clear that $z_\xi \in \mathbb{D}_\xi$. From (3.5), (3.10) and the fact that ψ^* is non decreasing, we deduce that $1-|z_\xi| \leq \frac{1}{2}\psi^*(\frac{h(\xi)}{2\psi_{\mathbb{T}}(h)}) := \lambda_h(\xi)$. Hence, by using (3.9),

$$\frac{1}{2}|\zeta-\xi|^2 \leq |\zeta - z_\xi|^2 \leq 2|\zeta-\xi|^2, \qquad \zeta \in \mathbb{T}\setminus\Psi_\xi. \tag{3.11}$$

Now, using the fact that $\psi(t)/t$ is non increasing, we see that under the assumption that $h(\zeta) \geq h(\xi)$ and $\zeta \in \mathbb{T}\setminus\Psi_\xi$, we have

$$\begin{aligned}\log\frac{h(\zeta)}{h(\xi)} &= \log\big(\frac{h(\zeta)-h(\xi)}{h(\xi)}+1\big)\\ &\leq \log\big(\psi_{\mathbb{T}}(h)\frac{\psi(|\xi-\zeta|)}{h(\xi)}+1\big)\\ &\leq \log\big(\frac{|\xi-\zeta|}{2\lambda_h(\xi)}+1\big).\end{aligned}$$

Consequently

$$\begin{aligned}\int_{\substack{\mathbb{T}\setminus\Psi_\xi\\ h(\zeta)\geq h(\xi)}}\frac{\lambda_h(\xi)}{|\zeta-\xi|^2}\log\frac{h(\zeta)}{h(\xi)}|d\zeta| &\leq \int_{|\zeta-\xi|\geq\lambda_h(\xi)}\frac{\lambda_h(\xi)}{|\zeta-\xi|^2}\log\big(\frac{|\xi-\zeta|}{2\lambda_h(\xi)}+1\big)|d\zeta|\\ &= c_2\int_{t\geq 1}\frac{\log t}{t^2}dt,\end{aligned} \tag{3.12}$$

where $c_2>0$ is an absolute constant. From (3.8), (3.11) and (3.12), we deduce

$$\begin{aligned}\omega^*(\delta h^\rho(\xi))a_h(\xi) &= \int_{\mathbb{T}\setminus\Psi_\xi}\frac{1-|z_\xi|}{|\zeta-\xi|^2}\Big|\log\frac{h(\zeta)}{h(\xi)}\Big||d\zeta|\\ &\leq 2\int_{\mathbb{T}\setminus\Psi_\xi}\frac{1-|z_\xi|^2}{|\zeta-z_\xi|^2}\Big|\log\frac{h(\zeta)}{h(\xi)}\Big||d\zeta|\\ &\leq 2\Big|\int_{\mathbb{T}\setminus\Psi_\xi}\frac{1-|z_\xi|^2}{|\zeta-z_\xi|^2}\log\frac{h(\zeta)}{h(\xi)}|d\zeta|\Big| + 4\int_{\substack{\mathbb{T}\setminus\Psi_\xi\\ h(\zeta)\geq h(\xi)}}\frac{1-|z_\xi|^2}{|\zeta-z_\xi|^2}\log\frac{h(\zeta)}{h(\xi)}|d\zeta|\\ &\leq 4\pi c_1 + c_3\\ &\leq c_4,\end{aligned}$$

where $c_3>0$ and $c_4>0$ are constants independent of ξ. This finishes the proof of Lemma 3.2. □

Let $f=|f|e^{iv}\in\mathcal{A}(\mathbb{D})$ be an outer function with zero set E_f. Let $\xi=e^{i\theta}$ and $\zeta=e^{i\varphi}$ be two distinct points. We suppose that $|v(\xi)|<+\infty$ and $|v(\zeta)|<+\infty$, where $v(\xi)$ and $v(\zeta)$ are the non tangential limits of v, respectively in ξ and ζ. We have

$$\begin{aligned}v(\xi)-v(\zeta) &= \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_\varepsilon^\pi\frac{\log\big|\frac{f(\theta-t)}{f(\theta+t)}\big| - \log\big|\frac{f(\varphi-t)}{f(\varphi+t)}\big|}{\tan(\frac{1}{2}t)}dt\\ &= \lim_{\varepsilon\to 0}\int_\varepsilon^\lambda + \int_\lambda^\pi\\ &=: I_f(\xi,\zeta,\lambda)+J_f(\xi,\zeta,\lambda),\end{aligned}$$

where $0 < \lambda \le \pi$ is a real number. A partial integration infers

$$\begin{aligned} J_f(\xi,\zeta,\lambda) &= \frac{1}{2\pi}\int_\lambda^\pi \frac{\log|\frac{f(\theta-t)}{f(\theta+t)}| - \log|\frac{f(\varphi-t)}{f(\varphi+t)}|}{\tan(\frac{1}{2}t)}dt \\ &= \frac{1}{2\pi}\int_\lambda^\pi \frac{1}{\tan(\frac{1}{2}t)}\frac{\partial}{\partial t}\Big(\int_\lambda^t \log|\frac{f(\theta-s)}{f(\theta+s)}| - \log|\frac{f(\varphi-s)}{f(\varphi+s)}|ds\Big)dt \\ &= \frac{1}{4\pi}\int_\lambda^\pi \frac{1}{\sin^2(\frac{1}{2}t)}\Big(\int_\lambda^t \log|\frac{f(\theta-s)}{f(\theta+s)}| - \log|\frac{f(\varphi-s)}{f(\varphi+s)}|ds\Big)dt. \end{aligned} \tag{3.13}$$

By a change of variables

$$\begin{aligned} \int_\lambda^t \log|\frac{f(\theta-s)}{f(\theta+s)}|ds &= -\int_{\theta-\lambda}^{\theta-t}\log|f(s)|ds + \int_{\theta-\lambda}^{\theta-t}\log|f(s+t+\lambda)|ds \\ &= \int_{\theta-\lambda}^{\theta-t}\log|\frac{f(s+t+\lambda)}{f(s)}|ds. \end{aligned} \tag{3.14}$$

By (3.14) and also a change of variables

$$\begin{aligned} &\int_\lambda^t \log|\frac{f(\theta-s)}{f(\theta+s)}| - \log|\frac{f(\varphi-s)}{f(\varphi+s)}|ds \\ =& \int_{\theta-\lambda}^{\theta-t}\log|\frac{f(s+t+\lambda)}{f(s)}|ds - \int_{\varphi-\lambda}^{\varphi-t}\log|\frac{f(s+t+\lambda)}{f(s)}|ds \\ =& \int_{\theta-\lambda}^{\varphi-\lambda}\log|\frac{f(s+t+\lambda)}{f(s)}|ds - \int_{\theta-t}^{\varphi-t}\log|\frac{f(s+t+\lambda)}{f(s)}|ds \\ =& \int_\theta^\varphi \log|\frac{f(s+t)}{f(s-\lambda)}|ds - \int_\theta^\varphi \log|\frac{f(s+\lambda)}{f(s-t)}|ds \\ =& \int_\theta^\varphi \big(M_f(s,t) - M_f(s,\lambda)\big)ds, \end{aligned} \tag{3.15}$$

where $M_f(s,t)$ stands for

$$M_f(s,t) := \log|\frac{f(s+t)}{f(s)}| + \log|\frac{f(s-t)}{f(s)}|, \quad s \in [0,2\pi[\text{ and } t \in [0,\pi].$$

Using equalities (3.13) and (3.15), it follows

$$J_f(\xi,\zeta,\lambda) = \frac{1}{4\pi}\int_\theta^\varphi \int_\lambda^\pi \frac{M_f(s,t) - M_f(s,\lambda)}{\sin^2(\frac{1}{2}t)}dsdt, \qquad 0 < \lambda \le \pi. \tag{3.16}$$

The next Lemma shows that condition (3.3) of Proposition 3.1 is a necessary condition.

Lemma 3.3. *Let f be an outer function in Λ_ω such that $\psi_{\mathbb{T}}(|f|) < +\infty$. Then*

$$\sup_{\xi,\zeta\in\mathbb{T}} \frac{|f|(\xi,\zeta)A_{|f|}(\xi,\zeta)}{\omega(|\xi-\zeta|)} < +\infty.$$

Proof. Fix two distinct points $\xi = e^{i\theta}$ and $\zeta = e^{i\varphi}$, $0 \le \theta < \varphi < 2\pi$. Set $\lambda := \lambda_{|f|}(\xi,\zeta)$. We have to discuss four possible cases.

1. If $|\xi - \zeta| \ge \lambda$, then

$$|f|(\xi,\zeta)A_{|f|}(\xi,\zeta) \le |f|(\xi,\zeta) = 2\psi_{\mathbb{T}}(|f|)\ \psi(\lambda) \le 2\psi_{\mathbb{T}}(|f|)\ \psi(|\xi-\zeta|).$$

2. Now, suppose that $|\xi-\zeta|\le\lambda$. Since $\xi\neq\zeta$, then $\lambda\neq 0$. It follows that $[\xi,\zeta]\subseteq\mathbb{T}\setminus E_f$. We set

$$\mu=\mu_{|f|}(\xi,\zeta):=\inf_{\sigma\in[\xi,\zeta]}a_{|f|}^{-1}(\sigma).$$

Note that μ is a positive number.

2.1. Assume that $|\xi-\zeta|\ge\mu$. Using Lemma 3.2 and the continuity of ω, we obtain

$$|f|(\xi,\zeta)A_{|f|}(\xi,\zeta)\le|f|(\xi,\zeta)\le c_f\inf_{\sigma\in[\xi,\zeta]}\omega(a_{|f|}^{-1}(\sigma))=c_f\omega(\mu)\le c_f\ \omega(|\xi-\zeta|).$$

2.2. Now, assume that $|\xi-\zeta|\le\min\{\lambda,\ \mu\}$. Using the fact that $f\in\Lambda_\omega$, we obtain

$$|f|(\xi,\zeta)|e^{iv(\xi)}-e^{iv(\zeta)}|\le 4\omega_{\mathbb{T}}(f)\ \omega(|\xi-\zeta|). \tag{3.17}$$

Set $I_f(\xi,\zeta):=I_f(\xi,\zeta,\lambda)$ and $J_f(\xi,\zeta):=J_f(\xi,\zeta,\lambda)$. We have

$$\begin{aligned} e^{iv(\xi)}-e^{iv(\zeta)} &= 2\sin\Big(\frac{v(\xi)-v(\zeta)}{2}\Big)\\ &= 2\sin\Big(I_f(\xi,\zeta)+J_f(\xi,\zeta)\Big)\\ &= 2\sin\Big(I_f(\xi,\zeta)\Big)\cos\Big(J_f(\xi,\zeta)\Big)\\ &+ 2\cos\Big(I_f(\xi,\zeta)\Big)\sin\Big(J_f(\xi,\zeta)\Big). \end{aligned} \tag{3.18}$$

Using the fact that $|x|+|\cos x|\ge 1$, for every real number x, combined with equality (3.18), we get

$$\begin{aligned} &\big|\sin\Big(I_f(\xi,\zeta)\Big)\big|\\ \le\ &\big|\sin\Big(I_f(\xi,\zeta)\Big)J_f(\xi,\zeta)\big|+\big|\sin\Big(I_f(\xi,\zeta)\Big)\cos\Big(J_f(\xi,\zeta)\Big)\big|\\ \le\ &\big|J_f(\xi,\zeta)\big|+\frac{1}{2}\big|e^{iv(\xi)}-e^{iv(\zeta)}\big|+\big|\cos\Big(I_f(\xi,\zeta)\Big)\sin\Big(J_f(\xi,\zeta)\Big)\big|\\ \le\ &\frac{1}{2}\big|e^{iv(\xi)}-e^{iv(\zeta)}\big|+2\big|J_f(\xi,\zeta)\big|. \end{aligned} \tag{3.19}$$

From (3.17) and (3.19), we have

$$|f|(\xi,\zeta)\big|\sin\Big(I_f(\xi,\zeta)\Big)\big|\le 2\omega_{\mathbb{T}}(f)\omega(|\xi-\zeta|)+2|f|(\xi,\zeta)\big|J_f(\xi,\zeta)\big|. \tag{3.20}$$

In the other hand,

$$|f(u)|\ge\frac{1}{2}|f(s)|\ge\frac{1}{2}|f|(\xi,\zeta)=\psi_{\mathbb{T}}(|f|)\psi(\lambda),$$

with $e^{is} \in [\xi, \zeta]$ and $e^{iu} \in \Psi_{e^{is}}$. We fix two points $e^{is} \in \mathbb{T}$ and $e^{iu} \in \mathbb{T}$ such that $e^{is} \in [\xi, \zeta]$, $e^{iu} \in \Psi_{e^{is}}$ and $|e^{iu} - e^{is}| \geq \lambda$. Since $\psi(t)/t$ is non increasing, we obtain

$$\begin{aligned}
\big|\log|\frac{f(u)}{f(s)}|\big| &= \log\big(\frac{\big||f(u)|-|f(s)|\big|}{\inf\{|f(u)|,|f(s)|\}}+1\big)\\
&\leq \log\big(\frac{2\psi_{\mathbb{T}}(|f|)}{|f|(\xi,\zeta)}\omega(|e^{iu}-e^{is}|)+1\big)\\
&\leq \log\big(\frac{2\psi_{\mathbb{T}}(|f|)\psi(\lambda)}{|f|(\xi,\zeta)}\frac{|e^{iu}-e^{is}|}{\lambda}+1\big)\\
&\leq \log\big(\frac{|e^{iu}-e^{is}|}{\lambda}+1\big). \qquad (3.21)
\end{aligned}$$

The constants $c_i > 0$, $i \in \mathbb{N}$, that appears in what follows are independent of both ξ and ζ. From (3.21) and the fact that $\psi(t)/t$ is non increasing, we obtain

$$\begin{aligned}
&|f|(\xi,\zeta)\frac{1}{4\pi}\int_\theta^\varphi\int_\lambda^{\psi^*(\frac{|f(s)|}{2\psi_{\mathbb{T}}(|f|)})}\frac{|M_f(s,t)|}{\sin^2(\frac{1}{2}t)}dsdt\\
\leq\ & c_1|f|(\xi,\zeta)|\xi-\zeta|\int_{t\geq\lambda}\frac{\log\big(\frac{t}{\lambda}+1\big)}{t^2}dt\\
\leq\ & 2c_2\psi_{\mathbb{T}}(|f|)\frac{\psi(\lambda)}{\lambda}|\xi-\zeta|\\
\leq\ & c_3\psi(|\xi-\zeta|). \qquad (3.22)
\end{aligned}$$

It is clear that $|M_f(s,\lambda)| \leq 2\log 2$. Then

$$\begin{aligned}
\frac{1}{4\pi}|f|(\xi,\zeta)\int_\theta^\varphi\int_\lambda^\pi\frac{|M_f(s,\lambda)|}{\sin^2(\frac{1}{2}t)}dsdt &\leq c_4\frac{|f|(\xi,\zeta)}{\lambda}|\xi-\zeta|\\
&\leq c_5\psi(|\xi-\zeta|). \qquad (3.23)
\end{aligned}$$

From (3.16), (3.22) and (3.23), we get

$$\begin{aligned}
|f|(\xi,\zeta)|J_f(\xi,\zeta)| &\leq \frac{1}{4\pi}|f|(\xi,\zeta)\int_\theta^\varphi\int_\lambda^\pi\frac{|M_f(s,t)|+|M_f(s,\lambda)|}{\sin^2(\frac{1}{2}t)}dsdt\\
&= \frac{1}{4\pi}|f|(\xi,\zeta)\int_\theta^\varphi\int_\lambda^\pi\frac{|M_f(s,\lambda)|}{\sin^2(\frac{1}{2}t)}dsdt\\
&+ \frac{1}{4\pi}|f|(\xi,\zeta)\int_\theta^\varphi\int_\lambda^{\psi^*(\frac{|f(s)|}{2\psi_{\mathbb{T}}(|f|)})}\frac{|M_f(s,t)|}{\sin^2(\frac{1}{2}t)}dsdt\\
&+ \frac{1}{4\pi}|f|(\xi,\zeta)\int_\theta^\varphi\int_{\psi^*(\frac{|f(s)|}{2\psi_{\mathbb{T}}(|f|)})}^\pi\frac{|M_f(s,t)|}{\sin^2(\frac{1}{2}t)}dsdt\\
&\leq c_6\psi(|\xi-\zeta|)+c_7\int_\theta^\varphi|f(s)|a_{|f|}(e^{is})ds\\
&\leq c_6\psi(|\xi-\zeta|)+c_7|\xi-\zeta|\sup_{\sigma\in[\xi,\zeta]}|f(\sigma)|a_{|f|}(\sigma). \qquad (3.24)
\end{aligned}$$

Making use of (3.24), Lemma (3.2) and the fact that $\omega(t)/t$ is non increasing, it follows

$$\begin{aligned}
|f|(\xi,\zeta)|J_f(\xi,\zeta)| &\leq c_6\psi(|\xi-\zeta|)+c_8|\xi-\zeta|\frac{\omega(\min\{1,\mu\})}{\min\{1,\mu\}}\\
&\leq c_6\omega(|\xi-\zeta|)+c_8\omega(|\xi-\zeta|).\\
&\leq c_9\omega(|\xi-\zeta|). \qquad (3.25)
\end{aligned}$$

Finally, combining (3.20) and (3.25) gives rise to

$$\big|f|(\xi,\zeta)\big|\sin\Big(I_f(\xi,\zeta)\Big)\big| \leq c_{10}\omega(|\xi-\zeta|).$$

This completes the proof of Lemma 3.3.

□

3.1.2. *Sufficient conditions.* In this section we prove that (3.2) and (3.3) of Proposition 3.1 are sufficient conditions for a function h so that $O_h^\rho \in \Lambda_\omega$.We begin by the following Lemma (see [5, pages 98-100])

Lemma 3.4. *Let $h : \mathbb{T} \mapsto \mathbb{R}^+$ be a continuous nonnegative function satisfying conditions* (1.4) *and* (1.6). *Fix a point $\xi \in \mathbb{T} \setminus E_h$. Then*

$$\lim_{r\to 1} v_h(re^{i\varphi}) = v_h(e^{i\varphi}), \qquad e^{i\varphi} \in \Psi_\xi,$$

uniformly on the arc Ψ_ξ.

Proof. Define

$$Q(r,t) := \frac{2r\sin(t)}{1-2r\cos(t)+r^2}, \qquad 0 \leq r < 1 \text{ and } t \in [0, 2\pi[,$$

to be the conjugate of the following Poisson Kernel

$$P(r,t) := \frac{1-r^2}{1-2r\cos(t)+r^2}, \qquad 0 \leq r < 1 \text{ and } t \in [0, 2\pi[.$$

We fix a point $\xi \in \mathbb{T} \setminus E_h$. Let $0 < \varepsilon < \frac{1}{8}$ be a real number. Set

$$\lambda_\varepsilon(\xi) := \inf_{\zeta\in\Psi_\xi} \psi^*(\varepsilon\frac{h(\zeta)}{\psi_{\mathbb{T}}(h)}).$$

Let $z = re^{i\varphi} \in \mathbb{D}$, be a point such that $e^{i\varphi} \in \Psi_\xi$ and $1 - r \leq \lambda_\varepsilon(\xi)$. It is easy to check that

$$h(\varphi + t) \geq \frac{1}{4}h(\xi), \qquad |t| \leq \lambda_\varepsilon(\xi) \text{ and } e^{i\varphi} \in \Psi_\xi.$$

Then

$$\big|\log\frac{h(\varphi-t)}{h(\varphi+t)}\big| \leq \frac{|h(\varphi-t)-h(\varphi-t)|}{\inf\{h(\varphi+t), h(\varphi-t)\}} \leq 8\varepsilon, \qquad 0 < t \leq \lambda_\varepsilon(\xi) \text{ and } e^{i\varphi} \in \Psi_\xi.$$

In what follows, the constants $c_i > 0$, $i \in \mathbb{N}$, are independent of both φ and ε. We have

$$\begin{aligned}
& \big|v_h(z) - \frac{1}{2\pi}\int_{1-r}^{\pi} \frac{\log\frac{h(\varphi-t)}{h(\varphi+t)}}{\tan(\frac{1}{2}t)}dt\big| \\
\le\ & \frac{1}{2\pi}\int_0^{1-r} Q(r,t)\Big|\log\frac{h(\varphi-t)}{h(\varphi+t)}\Big|dt \\
+\ & \frac{1}{2\pi}\int_{1-r}^{\pi} (Q(r,t)-Q(1,t))\Big|\log\frac{h(\varphi-t)}{h(\varphi+t)}\Big|dt \\
\le\ & c_1 \sup_{0\le t\le 1-r}\Big|\log\frac{h(\varphi-t)}{h(\varphi+t)}\Big| + c_2\int_{1-r}^{\pi} P(r,t)\Big|\log\frac{h(\varphi-t)}{h(\varphi+t)}\Big|dt, \\
\le\ & 8c_1\varepsilon + c_2\Big(\int_{1-r\le t\le \lambda_\varepsilon(\xi)} + \int_{t\ge\lambda_\varepsilon(\xi)}\Big) \\
\le\ & c_3\varepsilon + c_4 \sup_{t\ge\lambda_\varepsilon(\xi)}\{P(r,t)\} \times \int_{\mathbb{T}} |\log h(\zeta)||d\zeta| \\
\le\ & c_3\varepsilon + c_5\frac{1-r}{\lambda_\varepsilon^2(\xi)}.
\end{aligned} \tag{3.26}$$

From (3.26), we deduce that $\lim\limits_{|z|\to 1} v_h(z) = v_h(\frac{z}{|z|})$ uniformly on the arc Ψ_ξ. The proof of Lemma (3.4) is completed. □

Now we have to use Tamrazov's Theorem [10] to prove the following

Lemma 3.5. *Let $h : \mathbb{T} \mapsto \mathbb{R}^+$ be a continuous nonnegative function satisfying conditions* (1.4), (1.6), (3.2) *and* (3.3). *We suppose that $|v_h(\xi)| < +\infty$, for every $\xi \in \mathbb{T}\setminus E_h$, (see* (1.5)*). Then $O_h^\rho \in \Lambda_\omega$.*

Proof. If we prove that $O_h \in \mathcal{A}(\mathbb{D})$ and that $\omega_{\mathbb{T}}(O_h^\rho) < +\infty$, then the result follows from Tamrazov's Theorem [10]. First we prove that $\omega_{\mathbb{T}}(O_h^\rho) < +\infty$. Let $\xi = e^{i\theta} \in \mathbb{T}$ and $\zeta = e^{i\varphi} \in \mathbb{T}$ be two distinct points. We have

$$\begin{aligned}
& |O_h^\rho(\xi) - O_h^\rho(\zeta)| \\
\le\ & |h^\rho(\xi) - h^\rho(\zeta)| + h^\rho(\xi)|e^{i\rho v_h(\xi)} - e^{i\rho v_h(\zeta)}| \\
\le\ & c_\rho \psi_{\mathbb{T}}^\rho(h)\ \psi(|\xi-\zeta|) + h^\rho(\xi,\zeta)|e^{i\rho v_h(\xi)} - e^{i\rho v_h(\zeta)}| \\
\le\ & c_{\rho,h}\ \omega(|\xi-\zeta|) + \min\{2h^\rho(\xi,\zeta),\ h^\rho(\xi,\zeta)|e^{i\rho v_h(\xi)} - e^{i\rho v_h(\zeta)}|\}.
\end{aligned} \tag{3.27}$$

Fix $\lambda = \lambda_h(\xi,\zeta)$. We distingue between three cases.

1. If $|\xi-\zeta| \ge \lambda$, then

$$h(\xi,\zeta) = 2\psi_{\mathbb{T}}(h)\ \psi(\lambda) \le 2\psi_{\mathbb{T}}(h)\ \omega(|\xi-\zeta|).$$

2. Now, assume that $|\xi-\zeta| \le \lambda$ and set

$$\mu = \mu_{[\xi,\zeta]} := \inf_{\sigma\in[\xi,\zeta]} a_h^{-1}(\sigma).$$

   Since $[\xi,\zeta] \subseteq \mathbb{T}\setminus E_h$, then μ is a positive number.

2.1. If $|\xi-\zeta|\geq\mu$, then by condition (3.2) and the continuity of ω,

$$h^\rho(\xi,\zeta)\leq c_h \inf_{\sigma\in[\xi,\zeta]}\omega(a_h^{-1}(\sigma))=c_h\ \omega(\mu)\leq c_h\ \omega(|\xi-\zeta|).$$

2.2. Now, assume that $|\xi-\zeta|\leq\min\{\lambda,\ \mu\}$.

Using (3.18)

$$h^\rho(\xi,\zeta)|e^{i\rho v_h(\xi)}-e^{i\rho v_h(\zeta)}|\leq 2h^\rho(\xi,\zeta)A_{h^\rho}(\xi,\zeta)+2h^\rho(\xi,\zeta)|J_{h^\rho}(\xi,\zeta)|. \tag{3.28}$$

By (3.24)

$$h^\rho(\xi,\zeta)|J_{h^\rho}(\xi,\zeta)|\leq c_1\omega(|\xi-\zeta|)+c_2|\xi-\zeta|\sup_{\sigma\in[\xi,\zeta]}h^\rho(\sigma)a_h(\sigma). \tag{3.29}$$

Since $\omega(t)/t$ is non increasing and h satisfies (3.2), then

$$\begin{aligned}|\xi-\zeta|\sup_{\sigma\in[\xi,\zeta]}h^\rho(\sigma)a_h(\sigma) &\leq c_3|\xi-\zeta|\sup_{\sigma\in[\xi,\zeta]}\frac{\omega(a_h^{-1}(\sigma))}{a_h^{-1}(\sigma)}\\ &\leq |\xi-\zeta|\frac{\omega(\mu)}{\mu}\\ &\leq c_3\omega(|\xi-\zeta|). \end{aligned}\tag{3.30}$$

Combining (3.28), (3.29) and (3.30) and the fact that h satisfies (3.3), we get

$$h^\rho(\xi,\zeta)|e^{i\rho v_h(\xi)}-e^{i\rho v_h(\zeta)}|\leq c_4\omega(|\xi-\zeta|).$$

From (3.27) and the above three cases, we deduce that $\omega_{\mathbb{T}}(O_h^\rho)<+\infty$. It remains to show that O_h is continuous on $\overline{\mathbb{D}}$. Since $|O_h|$ is continuous on $\overline{\mathbb{D}}$ and O_h is analytic in $\mathbb{D}$, then O_h is continuous on $\mathbb{D}\cup E_h$. We have to show that O_h is continuous on every point $\xi\in\mathbb{T}\setminus E_h$. Fix $\xi\in\mathbb{T}\setminus E_h$ and $z\in\mathbb{D}$ such that $\frac{z}{|z|}\in\Psi_\xi$. It is clear that

$$|O_h^\rho(z)-O_h^\rho(\xi)|\leq ||O_h^\rho(z)|-h^\rho(\frac{z}{|z|})|+\rho h^\rho(\frac{z}{|z|})|v(z)-v(\frac{z}{|z|})|+|O_h^\rho(\frac{z}{|z|})-O_h^\rho(\xi)|.$$

Hence, from Lemma 3.4 and the fact that $\omega_{\mathbb{T}}(O_h)<+\infty$, we deduce the continuity of O_h at ξ. The proof of Lemma 3.5 is completed. □

4. Proof of Theorem 1.1

We need the following Lemma

Lemma 4.1. *Let $h:\mathbb{T}\mapsto\mathbb{R}^+$ be a continuous nonnegative function satisfying conditions* (1.4), (1.6), (1.7) *and* (1.8). *Then* (3.3) *is fulfilled and $|v_h(\xi)|<+\infty$, for all $\xi\in\mathbb{T}\setminus E_h$.*

Proof. Using (1.6), we obtain

$$\frac{1}{2}h(\theta)\leq h(\theta+t)\leq\frac{3}{2}h(\theta), \tag{4.1}$$

for every θ and t such that $e^{i\theta}\in\mathbb{T}\setminus E_h$ and $|t|\leq\psi^*\big(\frac{h(\theta)}{2\psi_{\mathbb{T}}(h)}\big)$. Next, fix $e^{i\theta}\in\mathbb{T}\setminus E_h$ and let $0<s\leq\psi^*\big(\frac{h(\theta)}{2\psi_{\mathbb{T}}(h)}\big)$ be a real number. By (4.1) and (1.7), we obtain

$$\begin{aligned}|\lim_{\varepsilon\to 0}\int_{\varepsilon\leq|t|\leq s}\frac{\log h(\theta+t)}{\tan(\frac{1}{2}t)}dt| &\leq \frac{2}{h(\theta)}\lim_{\varepsilon\to 0}\int_{\varepsilon\leq t\leq s}\frac{|h(\theta+t)-h(\theta-t)|}{\tan(\frac{1}{2}t)}dt\\ &\leq c\frac{\psi(s)}{h(\theta)},\end{aligned}$$

where c is a constant that not depend on s. Hence $|v_h(\xi)| < +\infty$, for all $\xi \in \mathbb{T} \setminus E_h$. Now, we prove the estimate (3.3). Let $\xi = e^{i\theta} \in \mathbb{T}$ and $\zeta = e^{i\varphi} \in \mathbb{T}$ be two distinct points such that $|\xi - \zeta| \leq \lambda_h(\xi, \zeta) := \lambda$. We have

$$\begin{aligned}
& A_h(\xi,\zeta) \\
\leq\ & c_1 \lim_{\varepsilon\to 0} \frac{1}{2\pi} \int_{\varepsilon}^{|\xi-\zeta|} \frac{\left|\log \frac{h(\theta+t)}{h(\theta-t)}\right| + \left|\log \frac{h(\varphi+t)}{h(\varphi-t)}\right|}{t} dt \\
+\ & \Big|\frac{1}{2\pi} \int_{|\xi-\zeta|}^{\lambda} \frac{\log \frac{h(\theta+t)}{h(\theta-t)} - \log \frac{h(\varphi+t)}{h(\varphi-t)}}{\tan(\frac{1}{2}t)} dt\Big|. \qquad (4.2)
\end{aligned}$$

Using (1.7), we obtain

$$\begin{aligned}
& h(\xi,\zeta) \lim_{\varepsilon\to 0} \int_{\varepsilon}^{|\xi-\zeta|} \frac{\left|\log \frac{h(\theta+t)}{h(\theta-t)}\right| + \left|\log \frac{h(\varphi+t)}{h(\varphi-t)}\right|}{t} dt \\
\leq\ & 2 \lim_{\varepsilon\to 0} \int_{\varepsilon}^{|\xi-\zeta|} \frac{\left|h(\theta+t) - h(\theta-t)\right|}{t} dt + 2 \lim_{\varepsilon\to 0} \int_{\varepsilon}^{|\xi-\zeta|} \frac{\left|h(\varphi+t) - h(\varphi-t)\right|}{t} dt \\
\leq\ & c_2 \psi(|\xi-\zeta|). \qquad (4.3)
\end{aligned}$$

By (3.16)

$$\begin{aligned}
& \frac{1}{2\pi} \int_{|\xi-\zeta|}^{\lambda} \frac{\log \frac{h(\theta+t)}{h(\theta-t)} - \log \frac{h(\varphi+t)}{h(\varphi-t)}}{\tan(\frac{1}{2}t)} dt \\
=\ & J_h(\xi,\zeta,|\xi-\zeta|) - J_h(\xi,\zeta,\lambda). \\
=\ & \frac{1}{4\pi} \int_{\theta}^{\varphi} \int_{|\xi-\zeta|}^{\lambda} \frac{M_h(s,t)}{\sin^2(\frac{1}{2}t)} ds dt \\
-\ & \frac{1}{4\pi} \int_{\theta}^{\varphi} \int_{|\xi-\zeta|}^{\pi} \frac{M_h(s,|\xi-\zeta|)}{\sin^2(\frac{1}{2}t)} ds dt \\
+\ & \frac{1}{4\pi} \int_{\theta}^{\varphi} \int_{\lambda}^{\pi} \frac{M_h(s,\lambda)}{\sin^2(\frac{1}{2}t)} ds dt. \qquad (4.4)
\end{aligned}$$

Since

$$|M_h(s,\lambda)| \leq 2\log 2, \quad \text{and} \quad |M_h(s,|\xi-\zeta|)| \leq 4\psi_{\mathbb{T}}(h) \frac{\psi(|\xi-\zeta|)}{h(\xi,\zeta)}, \qquad e^{is} \in [\xi,\zeta].$$

Then

$$h(\xi,\zeta) \int_{\theta}^{\varphi} \int_{|\xi-\zeta|}^{\pi} \frac{|M_h(s,|\xi-\zeta|)|}{\sin^2(\frac{1}{2}t)} ds dt \leq c_3 \psi(|\xi-\zeta|), \qquad (4.5)$$

and

$$h(\xi,\zeta) \int_{\theta}^{\varphi} \int_{\lambda}^{\pi} \frac{|M_h(s,\lambda)|}{\sin^2(\frac{1}{2}t)} ds dt \leq c_4 \psi(|\xi-\zeta|). \qquad (4.6)$$

From (1.8), we deduce

$$\begin{aligned}
& h(\xi,\zeta) \int_{\theta}^{\varphi} \int_{|\xi-\zeta|}^{\lambda} \frac{|M_h(s,t)|}{\sin^2(\frac{1}{2}t)} ds dt \\
\leq\ & c_5 \int_{\theta}^{\varphi} \Big(\frac{1}{h(s)} \int_{|\xi-\zeta|}^{\lambda} \frac{|h(s+t) \times h(s-t) - h^2(s)|}{t^2} dt\Big) ds \\
\leq\ & c_6 \psi(|\xi-\zeta|). \qquad (4.7)
\end{aligned}$$

□

By combining (4.2), (4.3), (4.4), (4.5), (4.6) and (4.7) we deduce the desired result. This completes the proof of Lemma 4.1.

4.1. **Proof of Theorem 1.1.** Let $h : \mathbb{T} \mapsto \mathbb{R}^+$ be a continuous nonnegative function satisfying conditions (1.4), (1.6), (1.7) and (1.8). It is trivial that $1 \Longrightarrow 2 \Longrightarrow 3$, where $\delta = 1/\max\{2, 2\omega_{\mathbb{D}}(|O_h^\rho|)\}$. Using Lemma 3.2 we deduce that $3 \Longrightarrow 4$. From Lemmas 3.5 and 4.1, we deduce that $4 \Longrightarrow 1$. This completes the proof of Theorem 1.1.

References

[1] B. Bouya, *Closed ideals in weighted Lipschitz algebras.* Advances in Math. 219 (2008) 1446–1468.

[2] L. Carleson, *Sets of uniqueness for functions regular in the unit circle.* Acta Math. 87 (1952) 325–345.

[3] K. M. Dyakonov, *Equivalent norms on Lipschitz-type spaces of holomorphic functions.* Acta Math. 178 (1997), no. 2, 143-167.

[4] K. M. Dyakonov, *Holomorphic functions and quasiconformal mappings with smooth moduli.* Adv. Math. 187 (2004), no. 1, 146–172.

[5] J. Garnett, *Bounded Analytic Functions.* Springer, New York, 2007.

[6] V. P. Havin; F. A. Shamoyan, *Analytic functions with a Lipschitzian modulus of the boundary values.* Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 19 (1970) 237–239.

[7] M. Pavlović, *On K. M. Dyakonov's paper: "Equivalent norms on Lipschitz-type spaces of holomorphic functions".* Acta Math. 183 (1999), no. 1, 141–143.

[8] N. A. Shirokov, *Analytic functions smooth up to the boundary.* Lecture Notes in Mathematics, 1312. Springer-Verlag, Berlin, 1988.

[9] N. A. Shirokov, *Zeros sets for functions from* Λ_ω. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 107 (1982) 178–188, 232.

[10] P. M. Tamrazov, *Contour and solid structural properties of holomorphic functions of a complex variable.* (Russian) Uspehi Mat. Nauk 28 (1973), no. 1(169), 131–161.