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**ENERGY-MOMENTUM TRANSPORT THROUGH SOLITON  
IN A SITE-DEPENDENT FERROMAGNET**

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## Abstract

The study of energy-momentum transport phenomena between two interacting spins in a ferromagnetic system provides a good realization of physical mechanism behind it. We investigate the nonlinear spin dynamics of a site-dependent ferromagnet with inhomogeneous exchange interaction in the presence of relativistic Gilbert-damping. We demonstrate the influence of inhomogeneity and damping on the evolution of energy and current densities of the magnetization. It is found that the inhomogeneity and damping support the loss-less energy-momentum transport along the site-dependent spin chain.

# 1 Introduction

Both theoretical and experimental interests in energy transport in one-dimensional (1D) systems have initiated related work for many years but more recently the field experienced a renaissance because of the availability of new theoretical methods and of experimental observations of anomalous features of energy transport in a few classes of new materials which provide good physical realizations of idealized 1D model systems. Recently, transport properties of various low-dimensional spin systems were investigated by considering the integrability of corresponding models [1-6]. Many important 1D model spin systems, such as the anisotropic Heisenberg model with nearest-neighbor interaction, were shown to be integrable, and the prediction of ballistic energy transport in those idealized systems seem to be undisputed [2,5]. The theoretical estimates have shown that for 1D spin systems the energy propagation occurs ballistically. The character of this energy transport differs for different anisotropies of the spin-spin interactions. It was also shown that in an external magnetic field, the energy transport remains non-diffusive in the XY  $S = \frac{1}{2}$  chain [7] but the energy diffusion is expected for isotropic  $S = \frac{1}{2}$  chains [8]. If next-nearest-neighbor interactions are not negligible, a diffusive spin-mediated energy transport is established [8]. The analysis of dynamical correlation functions of the spin and energy densities for the 1D  $S = \frac{1}{2}$  Ising model also indicates the absence of energy diffusion [9]. This result is closely related to the fact that the energy is a constant of motion for the above model. The thermal conductivity of spin chains has also been of theoretical interest because of the possibility to study nonlinear excitations (solitons), typical for 1D magnetic spin systems [10-11]. The motion of solitons is generally unaffected by interactions with other quasi-particles including other solitons and therefore, a robust transport of energy via solitons is expected. Wysin and Kumar [12] considered the heat energy transport in a two-component ideal gas of magnons and solitons and developed a theory for the thermal conductivity of a one-dimensional easy-plane classical ferromagnet in an external magnetic field.

For the past few decades, several attempts have been made to study the dynamics of different magnetic interactions such as bilinear isotropic exchange, single ion anisotropy due to crystal field effect, inhomogeneity in the exchange interaction and interaction with external magnetic field, etc, have been identified as integrable models with localized spin excitations such as soliton, domain wall under the different circumstances [13-19] in the classical continuum limit. The nonlinear dynamics of inhomogeneous systems have been widely investigated and expected to have many applications in the construction of magnetic memory devices, logic gates and so on. Furthermore, it has been proved that the inhomogeneous one dimensional as well as radially symmetric spin system, inhomogeneous compressible biquadratic Heisenberg ferromagnetic spin chain with harmonic lattice vibration, inhomogeneous vortex filaments and inhomogeneous exchange interactions are found to be integrable under certain conditions and exhibiting nonlinear spin excitations in terms of solitons [20-24]. In Ref. [25], it is pointed out that the

site-dependent biquadratic ferromagnetic spin chain with different type of inhomogeneities exhibits soliton excitations and further exploited for the magnetization reversal process. Also, it has been demonstrated that in Ref. [26] the site-dependent ferromagnetic spin chain with linear inhomogeneity admits shape changing property during its evolution. This shape changing property can be exploited to reverse the magnetization without loss of energy which may have potential applications in magnetic memory and recording devices. In some other context, it has been manifested that different types of nonlinear inhomogeneities have been shown to support soliton creation and annihilation in a site-dependent biquadratic ferromagnetic medium [27].

In this paper, we demonstrate the energy transport phenomenon and evolution of magnetization vector of an 1D classical Heisenberg site-dependent ferromagnetic spin chain with the Gilbert damping under the influence of both inhomogeneity and damping. In section 2, we formulate the associated dynamical equation for the site-dependent ferromagnetic spin chain. In section 3, we derive the expressions for the energy and current densities and the evolution of magnetization vector components is also constructed. Finally the results are concluded in section 4.

## 2 Formulation of the dynamical model

The equation of motion governing the dynamics of an inhomogeneous one-dimensional ferromagnetic spin chain with phenomenological damping known as Gilbert damping in the classical continuum limit can be written as

$$\vec{S}_t = \vec{S} \wedge \vec{F}_{eff} + \lambda[\vec{S} \wedge \vec{S} \wedge \vec{F}_{eff}] = \vec{S} \wedge \vec{F}_{eff} + \lambda[(\vec{S} \cdot \vec{F}_{eff})\vec{S} - \vec{F}_{eff}], \quad \vec{S}^2 = 1, \quad (1)$$

where,  $\vec{S}=(S^x, S^y, S^z)$  represents the classical three component spin (magnetization) vector and  $\lambda$  is a small parameter. Conventionally  $\lambda$  is identified as  $\alpha\gamma$ , where  $\alpha$  is the dimensionless Gilbert damping parameter and  $\gamma$  is the gyromagnetic ratio. Eq. (1) is commonly known as Landau-Lifshitz-Gilbert (LLG) equation in which the first term describes the precession of magnetization vector about the effective field  $\vec{F}_{eff}$  at the angular frequency  $\vec{\omega} = -\gamma\vec{F}_{eff}$  and the second term (the term proportional to  $\lambda$ ) describes the change of magnetization vector due to the damping parameter, causing  $\vec{S}$  to turn toward the direction of  $\vec{F}_{eff}$  for a positive value of damping parameter  $\alpha$ . In Eq. (1) the effective field  $\vec{F}_{eff}$  contribution may come from the exchange interaction, crystalline anisotropy, magnetostatic self energy, external magnetic fields, thermal fluctuations, and so on. Most of the studies on magnetic spin chains have been based on the homogeneous Heisenberg Hamiltonian, where the exchange interaction coupling between nearest-neighbour pair of spins is a single constant  $J$  or at most two constants as in the case of uniaxial anisotropy. But the presence of the magnetic defects introduced inhomogeneity in the exchange interaction. Generally inhomogeneity in magnetic materials arise because of the following two factors (i) if the distance between neighbouring magnetic atoms varies along the chain depending on the distance between the spins (see Fig. (1)) and the degree of overlapping of electronic wave

function varies from site to site. Thus the interaction between the spins depends upon the site in the crystal lattice, which is known as site-dependent interaction. To illustrate this type of inhomogeneity, charge transfer complexes TCNQ,  $Ni(CN)_4$ , organo-metallic insulators, TTF-bisdithiolenes and  $Ni(Co)_4$  in which the characterizing inhomogeneous parameter alternates between two values as we move along the spin chain. (ii) If the atomic wave function itself varies from site to site although the atoms themselves may equally be spaced. This type of inhomogeneity occurs when magnetic insulators placed in a weak, static and inhomogeneous electric field. It can also be simulated by the deliberate introduction of imperfections (impurities or organic complexes) in the vicinity of a bond so as to alter the electronic wave functions without causing appreciable lattice distortion.

Conceived by the above in mind, we would like to investigate the evolution of energy and current densities in a site-dependent ferromagnetic spin chain especially for the case of linear inhomogeneity under the influence of Gilbert-damping parameter. In particular, we consider the bilinear exchange varying interaction represented by the Hamiltonian  $H = -J \sum_{i,j} h_i \vec{S}_i \cdot \vec{S}_j$ , where the function  $h_i$  characterize the variation of exchange interaction along the spin chain. After suitable rescaling and redesignation of the parameter, we write down the corresponding evolution equation in the continuum limit using Eq. (1)

$$\vec{S}_t = \vec{S} \wedge (h_x \vec{S}_x + h \vec{S}_{xx}) + \lambda [h(\vec{S} \cdot \vec{S}_{xx}) \vec{S} - h_x \vec{S}_x - h \vec{S}_{xx}]. \quad (2)$$

In Eq. (2), the internal effective field due to the site-dependent interaction is  $\vec{F}_{eff} = h_x \vec{S}_x + h \vec{S}_{xx}$ . The terms proportional to the Gilbert damping torque will make the magnetization to an orientation so that the magnetic energy is minimized. When the exchange interaction is invariant ( $h(x)=\text{constant}$ ) and  $\lambda = 0$ , Eq. (2) describes the evolution of the isotropic Heisenberg spin system and associated N-solitons and is completely integrable [28-29] and whose soliton solutions can be found by the IST [30]. For the past three decades, many attempts have been made to investigate the nature of nonlinear wave propagation in inhomogeneous media [31-37]. For the homogeneous exchange interaction, Eq. (2) has already been well studied by Lakshmanan et al [32] by treating the terms proportional to  $\lambda$  as perturbation to the cubic nonlinear Schrödinger equation. In the absence of Gilbert damping, Lakshmanan and Bullough [33] have considered the more general spin evolution equation (2) involving gradient  $\vec{S}_x$  terms and linearly x-dependent coefficients, which has been shown to be equivalent to a certain generalized NLS again with linearly x-dependent coefficients. The latter equation for linear  $h(x)$  in turn has been shown to support soliton solution by Calogero and Degasperis [34], and the same has been solved exactly for certain classes of the function  $h(x)$ , by applying an extension of the AKNS formalism and IST by Radha Balakrishnan [35].

### 3 Evolution of energy density and current density

From the earlier studies noted above, it is clear that the spin evolution equation Eq. (2) admits soliton solution and the energy-momentum transport along the chain takes place through solitonic evolution. In view of this, we would like to investigate the combined effect of linear inhomogeneity and relativistic damping on the energy-momentum transport by identifying the spin system in terms of a moving helical space curve in  $E^3$  [13]. A local coordinate system  $e_i$  is formed on the space curve by identifying the unit spin vector  $\vec{S}(x, t)$  with the unit tangent vector  $e_1(x, t)$  and unit principle and binormal vectors  $e_2(x, t)$ ,  $e_3(x, t)$ , respectively in the usual way. The variation of Eq. (2) along the space curve in  $E^3$  is given by the usual Serret-Frenet (SF) equation [38],

$$\begin{bmatrix} \vec{e}_{1x} \\ \vec{e}_{2x} \\ \vec{e}_{3x} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}, \quad (3)$$

and the time evolution of  $e_i$ ,  $i = 1, 2, 3$  can be evaluated by using the SF equations. The energy density associated with inhomogeneous spin chain in the classical continuum limit [35] can be written as

$$\delta(x, t) = \frac{h(x)}{2} \vec{S}_x \cdot \vec{S}_x, \quad (4)$$

and the equation of continuity  $\delta_t + \zeta_x = 0$  enables us to identify the momentum density  $\zeta$  (current density) as

$$\zeta(x, t) = h(x)^2 \vec{S} \cdot (\vec{S}_x \wedge \vec{S}_{xx}). \quad (5)$$

Further the development of classical differential geometry help us to express the energy density and current density in terms of torsion  $\tau(x, t)$  and curvature  $\kappa(x, t)$  of the space curve. This enable us to express the energy density  $\delta(x, t)$  and current density  $\zeta(x, t)$  of the spin system in terms of curvature  $\kappa(x, t)$  and torsion  $\tau(x, t)$  of the space curve as

$$\delta(x, t) = \frac{h(x)}{2} \left( \frac{\partial \vec{S}}{\partial x} \right) \cdot \left( \frac{\partial \vec{S}}{\partial x} \right) = \frac{h(x)}{2} \kappa^2(x, t), \quad (6)$$

$$\zeta(x, t) = h(x)^2 \vec{S} \cdot \left( \frac{\partial \vec{S}}{\partial x} \right) \wedge \left( \frac{\partial^2 \vec{S}}{\partial x^2} \right) = h(x)^2 \kappa^2(x, t) \tau(x, t). \quad (7)$$

After some lengthy algebraical calculations,  $e_{it}$  can be written as,

$$\begin{bmatrix} \vec{e}_{1t} \\ \vec{e}_{2t} \\ \vec{e}_{3t} \end{bmatrix} = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}, \quad (8)$$

where,

$$\begin{aligned} \omega_1 &= \frac{(h\kappa)_{xx}}{\kappa} - h\tau^2 + \frac{\lambda}{\kappa} (2\tau(h\kappa)_x + h\kappa\tau_x), \\ \omega_2 &= -(h\kappa)_x - \lambda h\kappa\tau, \\ \omega_3 &= -h\kappa\tau + \lambda(h\kappa)_x. \end{aligned}$$

The compatibility condition  $(\vec{e}_{ix})_t = (\vec{e}_{it})_x$  with the system of SF equations lead to the evolution of curvature  $\kappa(x, t)$  and torsion  $\tau(x, t)$  of the space curve as follows,

$$\kappa_t = -\tau(h\kappa)_x - \lambda h\kappa\tau^2 - \left[ h\kappa\tau - \lambda(h\kappa)_x \right]_x, \quad (9)$$

$$\tau_t = \kappa \left[ (h\kappa)_x + \lambda h\kappa\tau \right] - \left[ h\tau^2 - \frac{(h\kappa)_{xx}}{\kappa} - \frac{\lambda}{\kappa} (2\tau(h\kappa)_x + h\kappa\tau_x) \right]_x. \quad (10)$$

To identify the evolution Eqs. (9-10) with more standard nonlinear partial differential equations, we make the following complex transformation for the dependent variables,

$$u(x, t) = \frac{1}{2}\kappa(x, t)e^{i \int \tau(x, t) dx}, \quad (11)$$

and Eqs. (9-10) can be reduced to an inhomogeneous damped nonlinear Schrödinger (IDNLS) equation.

$$\begin{aligned} iu_t + (hu)_{xx} + 2h|u|^2u + 2u \int_{-\infty}^x h_{x'}|u^2|dx' - i\lambda[(hu)_{xx} \\ - 2u \int_{-\infty}^x h(uu_x^* - u^*u_x)dx] = 0. \end{aligned} \quad (12)$$

The Painlevé singularity structure analysis of Eq. (12) reveals that, it is free from movable critical manifolds only for the specific choices of the inhomogeneity  $h(x)=Ax+B$ , and  $\lambda = 0$  thereby satisfying P-property. Thus only when the inhomogeneity varies in a linear fashion, the nonlinear dynamics of a site-dependent ferromagnet found to support soliton excitations and vice versa [36]. Physically this type of linear inhomogeneity can be achieved by gradually changing the concentration of impurities in a specified manner along the spin chain. When  $\lambda = 0$ , Eq. (12), the energy momentum transport along the inhomogeneous spin chain is therefore directly related to the solution of IDNLS equation according to [35]

$$\delta(x, t) = 2h|u|^2, \quad (13)$$

$$\zeta(x, t) = 4h^2|u|^2(\arg u)_x. \quad (14)$$

Now, we consider Eq. (12) along with its complex conjugate and we can show that,

$$\begin{aligned} i \frac{d}{dt}(uu^*) + 2h_x(u_xu^* - u_x^*u) + h(u_{xx}u^* - u_{xx}^*u) - i\lambda[h(u_{xx}u^* - u_{xx}^*u) \\ + 2h_x(u_xu^* - u_xu)] = 0. \end{aligned} \quad (15)$$

By considering the total energy of the magnetic systems,

$$E = \int_{-\infty}^{\infty} \delta(x, t) dx = \frac{1}{2} \int_{-\infty}^{\infty} h\kappa^2(x, t) dx, \quad (16)$$

the one soliton solution of the undamped INLS equation (12) is well known to be of the form [37],

$$u(x, 0) = -2\beta \operatorname{sech} 2\beta x e^{-2i\alpha x}, \quad (17)$$

where,  $\alpha$  and  $\beta$  are constants. On comparing Eq. (17) and Eq. (11), it yields,

$$\kappa(x, 0) = -4\beta \operatorname{sech} 2\beta x \quad \text{and} \quad \tau(x, 0) = -2\alpha. \quad (18)$$

Now in order to study the effect of damping on the inhomogeneous spin chain, we allow  $\beta$  and  $\alpha$  to grow as function of time

$$u(x, t) = -2\beta(t) \operatorname{sech} 2(\beta(t)x) e^{-2i\alpha(t)x}. \quad (19)$$

Assuming the linear inhomogeneity,  $h(x) = Ax + B$ , where  $A$  and  $B$  are arbitrary constants and Eq. (19) in Eq. (15) it becomes,

$$\begin{aligned} & (8i\beta \frac{d\beta}{dt} - 32iA\alpha\beta^2 - 32A\lambda\alpha\beta^2) \int_{-\infty}^{\infty} \operatorname{sech}^2(2\beta x) dx \\ & + (64iA\alpha\beta^3 + 64\lambda A\alpha\beta^3 - 16i\beta^2 \frac{d\beta}{dt}) \int_{-\infty}^{\infty} x \operatorname{sech}^2(2\beta x) \tanh(2\beta x) dx \\ & + (i + \lambda) 64B\alpha\beta^3 \int_{-\infty}^{\infty} \operatorname{sech}^2(2\beta x) \tanh(2\beta x) dx = 0, \end{aligned} \quad (20)$$

and then solving Eq. (20) one can obtain,

$$\frac{d\beta}{dt} = 4A(1 - i\lambda)\alpha\beta. \quad (21)$$

Similarly using  $h(x)$  and Eq. (19) in Eq. (12), we obtain

$$\begin{aligned} & (8iA\gamma\alpha\beta - 2i \frac{d\beta}{dt} + 8B\gamma(\alpha^2\beta - \beta^3)) \operatorname{sech}(2\beta x) + (8A\gamma(\alpha^2\beta - \beta^3) - 4\beta \frac{d\alpha}{dt}) \\ & \times \operatorname{sech}(2\beta x) + (8i\beta \frac{d\beta}{dt} - 16iA\gamma\alpha\beta^2 + 32A\lambda\alpha\beta^2) x \operatorname{sech}(2\beta x) \tanh(2\beta x) \\ & + (8A\gamma\beta^2 - 16iB\gamma\alpha\beta^2 - 8A\beta^2 + 32B\lambda\alpha\beta^2) \operatorname{sech}(2\beta x) \tanh(2\beta x) \\ & + 16(\gamma - 1)A\beta^3 x \operatorname{sech}^3(2\beta x) + 16(\gamma - 1)B\beta^3 \operatorname{sech}^3 2\beta x \\ & - 16A\lambda\alpha\beta \operatorname{sech}(2\beta x) \ln \cosh(2\beta x) = 0, \end{aligned} \quad (22)$$

then multiplying by  $\operatorname{sech} 2\beta x \tanh 2\beta x$  on both sides in Eq. (22) and integrating once we obtain

$$\frac{d\alpha}{dt} = 2A\gamma(\alpha^2 + \beta^2) - 4A\beta^2 + \frac{1}{3}(32\lambda - 16i\gamma)B\alpha\beta^2. \quad (23)$$

On solving Eq. (21) and Eq. (23), we get

$$\alpha(t) = -\frac{2c^2(At - 2c_1 + \gamma c_1)}{2 - \gamma + 4c^2\gamma(At - 2c_1 + \gamma c_1)^2}, \quad (24)$$

$$\beta(t) = -\frac{c}{2 - \gamma + 4c^2\gamma(At - 2c_1 + \gamma c_1)^2}, \quad (25)$$

where,  $\gamma = 1 - i\lambda$ ,  $\lambda$  represents damping parameter and  $c_1, c$  are constants of integration and  $B = 0$ . From the above solution, it is proved that energy transport in the inhomogeneous spin chain occurs as a function of time [36].



### 3.1 Effect of inhomogeneity and damping on the energy and current densities

In order to express evolution of curvature and torsion of the space curve, we substitute Eq. (24-25) in Eq. (18),

$$\kappa(x, t) = \frac{4c}{2 - \gamma + 4c^2\gamma(At - 2c_1 + \gamma c_1)^2} \operatorname{sech}\left(\frac{2cx}{2 - \gamma + 4c^2\gamma(At - 2c_1 + \gamma c_1)^2}\right), \quad (26)$$

$$\tau(x, t) = \frac{4c^2(At - 2c_1 + \gamma c_1)}{2 - \gamma + 4c^2\gamma(At - 2c_1 + \gamma c_1)^2}. \quad (27)$$

Using Eqs. (26-27) in Eqs. (6-7), the energy and current densities of IDNLS equation can be written as

$$\delta(x, t) = \frac{2Ac^2x}{(2 - \gamma + 4c^2\gamma(At - 2c_1 + \gamma c_1)^2)^2} \operatorname{sech}^2\left(\frac{-2cx}{2 - \gamma + 4c^2\gamma(At - 2c_1 + \gamma c_1)^2}\right), \quad (28)$$

$$\zeta(x, t) = \frac{16A^2x^2c^4(At - 2c_1 + \gamma c_1)}{(2 - \gamma + 4c^2\gamma(At - 2c_1 + \gamma c_1)^2)^3} \operatorname{sech}^2\left(\frac{-2cx}{2 - \gamma + 4c^2\gamma(At - 2c_1 + \gamma c_1)^2}\right). \quad (29)$$

We have plotted the solutions as presented in Eqs. (28-29) and its corresponding contours plots in figures 2, depicting the influence of inhomogeneity on the energy and current densities for the undamped case, by choosing the arbitrary constants  $c = 1.2$  and  $c_1 = 0.8$ . When the inhomogeneity is not present and  $\lambda \neq 0$ , the solitonic profile of energy density damps out as represented in [32]. But in our analysis the combined effect of inhomogeneity and damping eventually support the energy-momentum transport along the spin chain without any loss. Figs. (2a & 2b) represent the evolution of energy and current density for homogeneous spin chain, ( $A = 0$ ), which is in the form of solitons and trivial in nature. In the contour plots Figs. (2a & 2b) the more clear and brighter regions correspond to the higher amplitude of the energy and current densities, whereas the darker regions correspond to the lower or zero amplitude. When the inhomogeneity is slowly increased the solitonic profile of energy density is getting more localized and sharp peaked with an increase in the amplitude as seen in Figs. (3a-3c). Whereas a single solitonic profile of the current density appears with the doubly humped localization with the increase of inhomogeneity (Figs. 3d-3f). When  $A = 0.06$ , the double humps grow closer and higher in amplitude which is also evident from contour plots and Figs. (4).

Also, we are interested to study the effect of Gilbert damping on the energy-momentum transport in the presence of inhomogeneity. In this case, with the increase of the value of the damping parameter, one could observe that the solitonic profile of energy density is sharply peaked with a very small increase in amplitude as depicted in Figs. (5a-5d). From its corresponding contours, it is depicted that the increased damping parameter increases the mobility of the solitonic profile and thus enhances the speed of the transport without any loss. On the other hand, the increased damping causes the doubly humped solitonic profile of current density to become singly humped and sharply peaked with high amplitude. One hump slowly transfers its amplitude to another and finally merges together with the value of  $\lambda = 0.9$ , and it is clearly depicted in contour plots (Figs. 5e-5h) and Figs. (6).

### 3.2 Effect of inhomogeneity and damping on the magnetization density

Having known the curvature and torsion of the space curve, one can able to construct the orthogonal trihedral uniquely from the well known classical differential geometry procedures. Moreover, using the solutions of energy and current densities of inhomogeneous damped spin chain, the associated solutions for the spin components may be obtained by noting that Eqs. (3 & 8) are equivalent to a set of two Riccati equations in terms of Darboux vector  $z_l$  [39].

$$z_l = \frac{e_{2l} + ie_{3l}}{1 - e_{1l}}, \quad e_{1l}^2 + e_{2l}^2 + e_{3l}^2 = 1, \quad (30)$$

In terms of  $z_l$ , Eqs. (8-9) are equivalently written as

$$z_{lx} = -i\tau z_l + \frac{1}{2}(z_l^2 + 1), \quad (31)$$

$$\begin{aligned} z_{lt} = & -i \left[ -\tau^2 h + \frac{(\kappa h)_{xx}}{\kappa} + \lambda \left( 2 \frac{\kappa_x \tau h}{\kappa} + \tau h_x + (\tau h)_x \right) \right] z_l \\ & + \frac{1}{2} \left[ -\kappa \tau h + \lambda (\kappa h)_x - i((\kappa h)_x - \lambda(\kappa \tau h)) \right] z_l^2 \\ & + \frac{1}{2} (-\kappa \tau h + \lambda (\kappa h)_x + i((\kappa h)_x - \lambda(\kappa \tau h))). \end{aligned} \quad (32)$$

By solving Eqs. (31-32), it is possible to construct the solution for the magnetization density (spin vector)  $\vec{S}(x, t)$  in an explicit manner. The solution to Eqs. (31-32) takes the form

$$z_l = \frac{\phi P + Q}{\phi R + T}, \quad (33)$$

where,  $\phi$  is an arbitrary function independent of  $x$  and  $P$ ,  $Q$ ,  $R$  and  $T$  are the functions to be evaluated. With the knowledge of  $P$ ,  $Q$ ,  $R$  and  $T$ , the space curve corresponding to spin system may be referred to rectangular coordinate system and we derive the coordinates as

$$r_1 = \int^x e_{1x} dx' = \int^x \frac{\phi(P^2 - R^2) - \phi^{-1}(Q^2 - T^2)}{2(PT - QR)} dx', \quad (34)$$

$$r_2 = \int^x e_{2x} dx' = i \int^x \frac{\phi(P^2 - R^2) - \phi^{-1}(Q^2 - T^2)}{2(PT - QR)} dx', \quad (35)$$

$$r_3 = \int^x e_{3x} dx' = \int^x \frac{\phi(RT - PQ)}{2(PT - QR)} dx'. \quad (36)$$

The components of the spin vector can be written in terms of the coordinates as  $S^x = r_{1x}$ ,  $S^y = r_{2x}$  and  $S^z = r_{3x}$ . Knowing  $\kappa$  and  $\tau$ , the explicit form of the functions  $P$ ,  $Q$ ,  $R$  and  $T$  and hence the spin components can be evaluated as,

$$\begin{aligned} S^x = & \frac{2\beta(t)}{\alpha(t)^2 + \beta(t)^2} \left\{ \beta(t) \tanh(2\beta(t)x) \sin(2\alpha(t)x) \operatorname{sech}(2\beta(t)x) \right. \\ & \left. - \alpha(t) \cos(2\alpha(t)x) \operatorname{sech}(2\beta(t)x) \right\}, \end{aligned} \quad (37)$$

$$S^y = -\frac{2\beta(t)}{\alpha(t)^2 + \beta(t)^2} \left\{ \beta(t) \tanh(2\beta(t)x) \cos(2\alpha(t)x) \operatorname{sech}(2\beta(t)x) + \alpha(t) \sin(2\alpha(t)x) \operatorname{sech}(2\beta(t)x) \right\}, \quad (38)$$

$$S^z = 1 - \frac{2\beta(t)^2}{\alpha(t)^2 + \beta(t)^2} \operatorname{sech}^2(2\beta(t)x). \quad (39)$$

In order to investigate the effect of inhomogeneity and damping on the magnetization components, we have plotted  $S^x$ ,  $S^y$  and  $S^z$  and its contour plots in Figs. (7 and 8) by choosing the arbitrary constants  $c = 1.1$  and  $c_1 = 0.01$ . From the Figs. (7a, 7c, 7e), it is obvious that, in the absence of inhomogeneity and damping, the  $x$  and  $y$  components of spin soliton assume the usual soliton form and  $z$  component assumes the anti-soliton shape. This implies that the spin vector precesses about the effective field arises because of the homogeneous exchange interaction with an invariant velocity. But when the inhomogeneity is introduced for the undamped case the amplitude of the spin solitons increase appreciably and at the same time the soliton components suffer with the multiple folded excitations on the robust shape leading to the enhancement in the precessing speed about the effective field (see Figs. (7b, 7d, 7f)).

Similarly, the increase in the value of the damping parameter  $\lambda$ , in the presence of inhomogeneity also leads to the fluctuations which is symmetric about the effective field in nature (Figs. (9 & 10)). This implies that the damping does not destroy the localization. In summary, the combined effect of inhomogeneity and damping enhances the energy-momentum transport without any loss or dissipation along the spin chain.

## 4 Conclusions

In this paper, the energy-momentum transport phenomena in a classical continuum one dimensional isotropic site-dependent Heisenberg ferromagnetic spin chain with relativistic Gilbert damping is studied. It is found that the loss-less energy-momentum transport takes place in the form of soliton along the inhomogeneous spin chain under the influence of inhomogeneity and damping. Finally we have constructed the evolution of magnetization (spin) vector and from the nature of evolution it is found that solitons in the inhomogeneous spin chain travel without loss of its energy even in the presence of damping. Therefore this loss-less energy-momentum transport along the site-dependent ferromagnet is expected to have potential applications in magnetic memory and recording.

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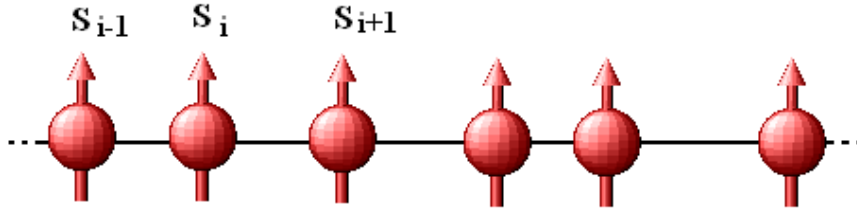


Figure 1: Inhomogeneous spin chain

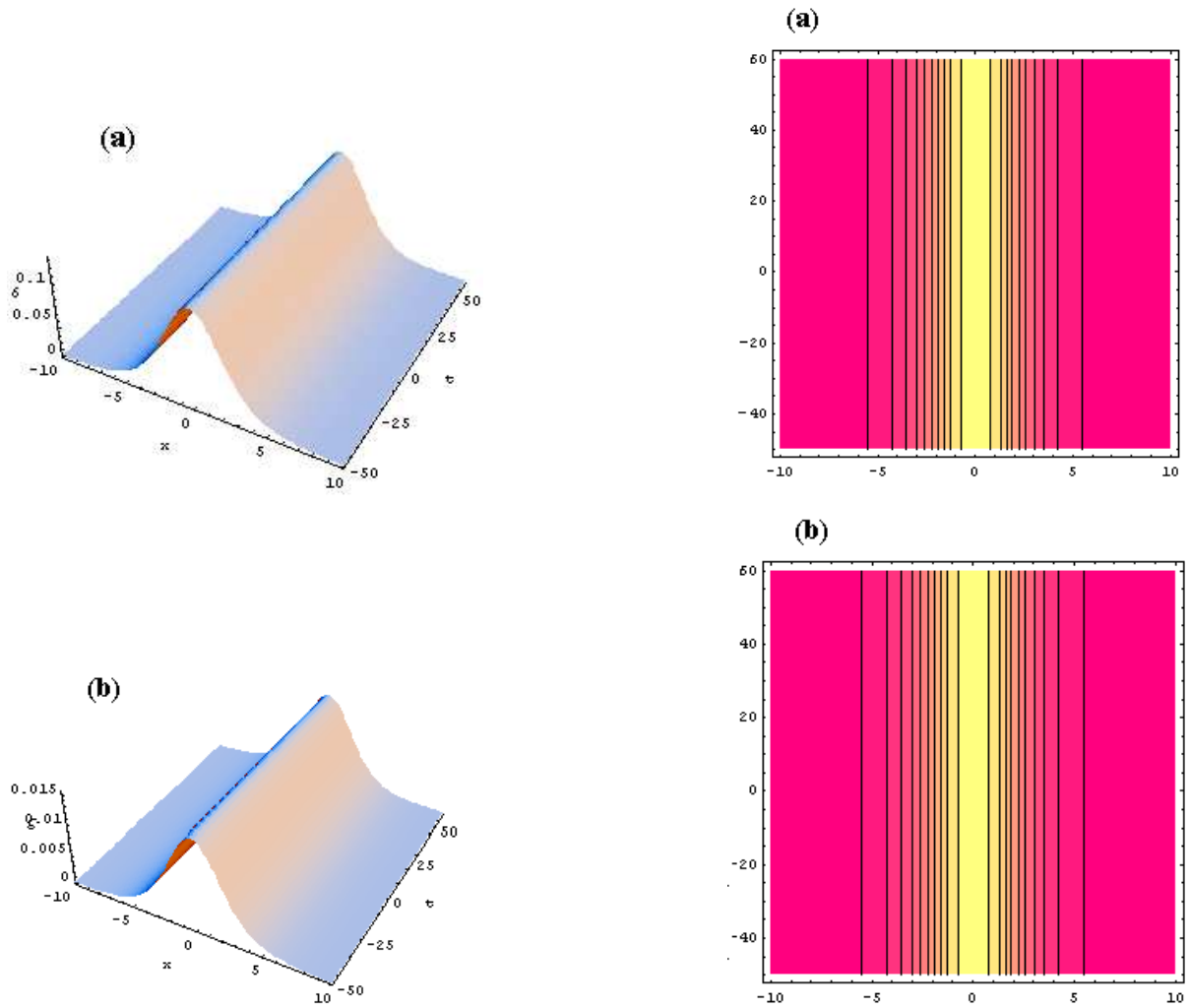
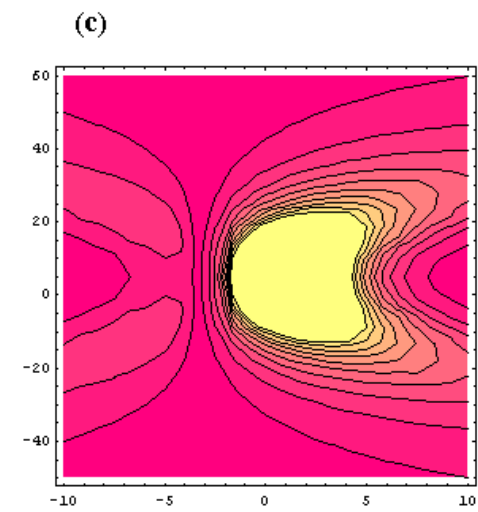
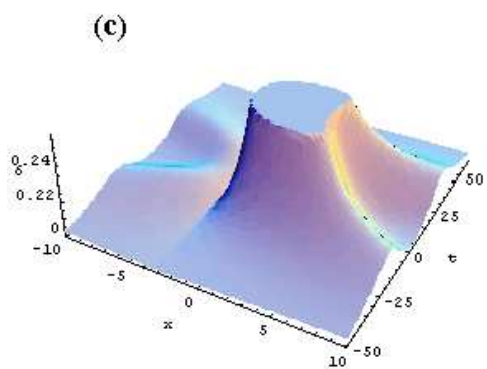
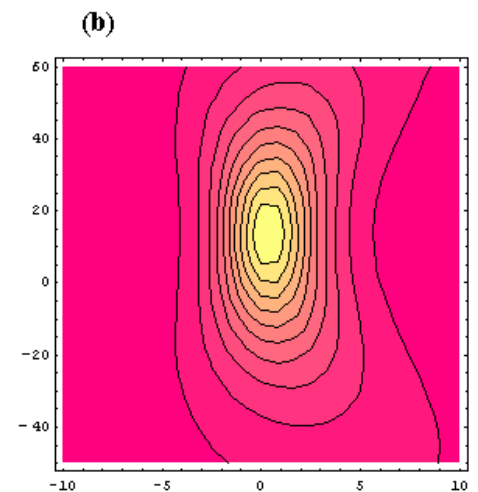
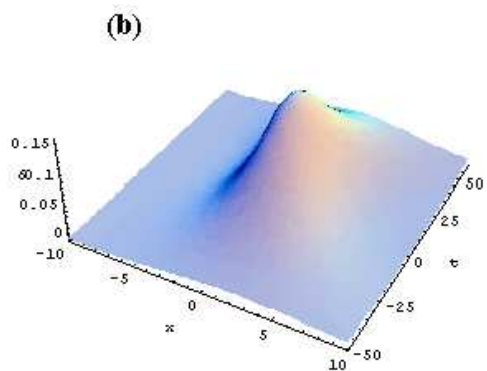
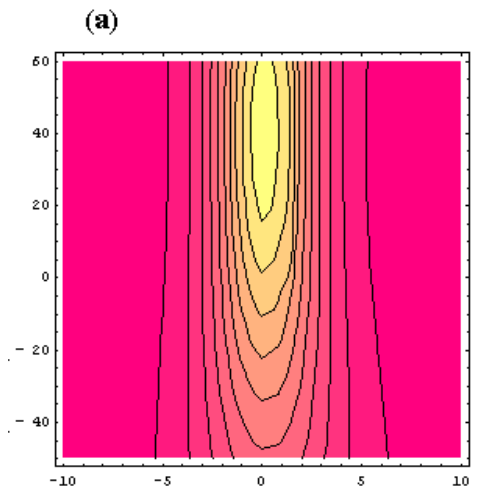
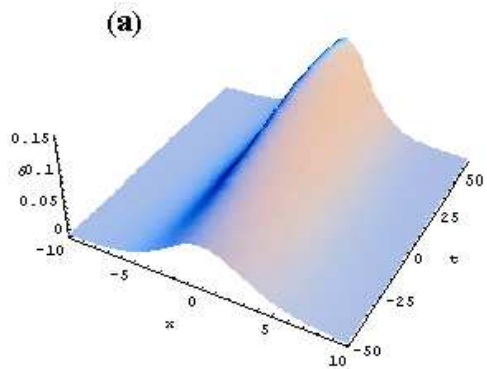


Figure 2: Evolution of energy density (a), current density (b) and its corresponding contour plots under the influence of inhomogeneity when  $A=0$  with  $\lambda = 0$ ,  $c = 1.2$  and  $c_1 = 0.8$  [32].





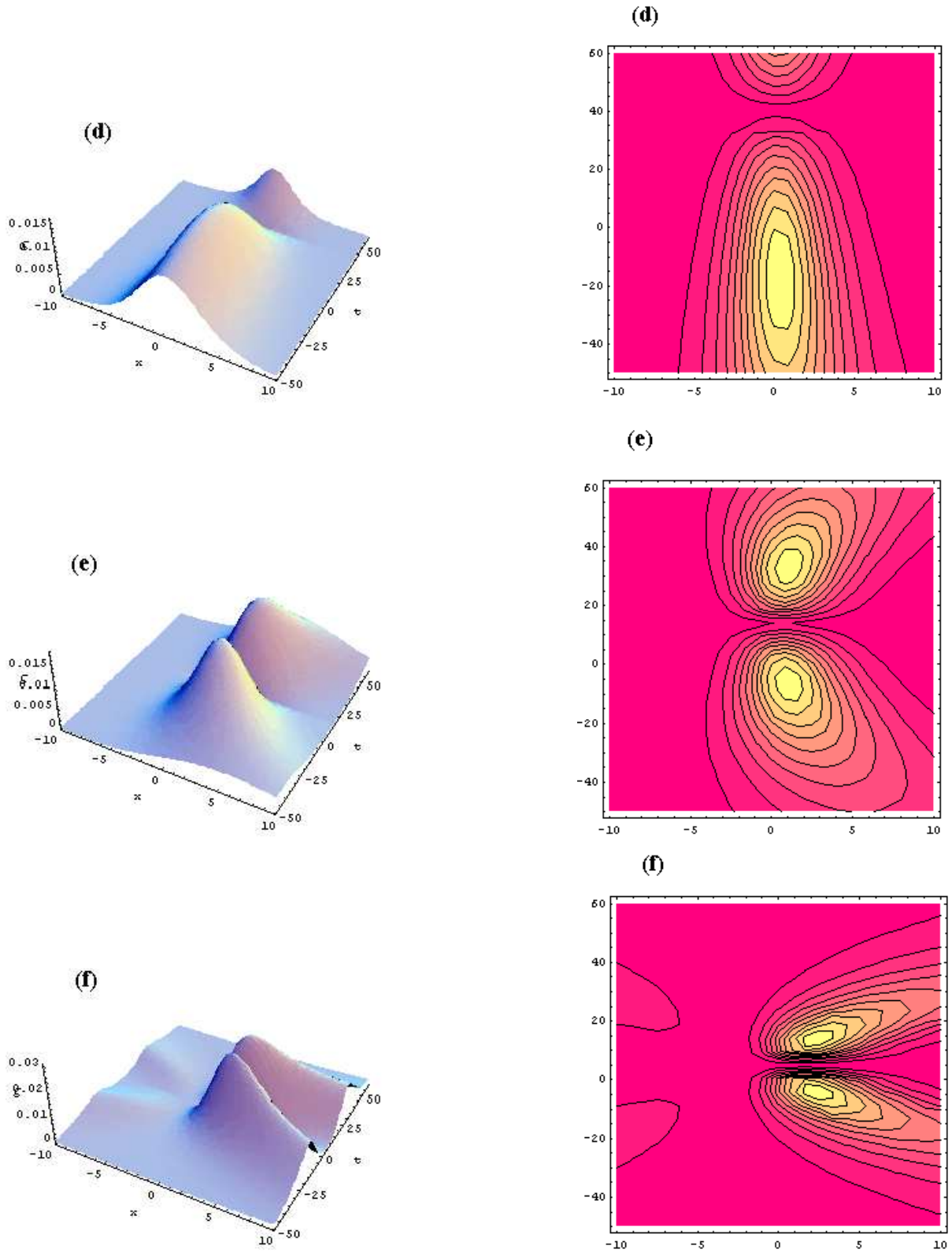


Figure 3: Evolution of energy density (a-c), current density (d-f) and its corresponding contour plots under the influence of inhomogeneity when (a,d)  $A=0.02$ , (b,e)  $A=0.06$  and (c,f)  $A=0.12$  with  $\lambda = 0$ ,  $c = 1.2$  and  $c_1 = 0.8$ .

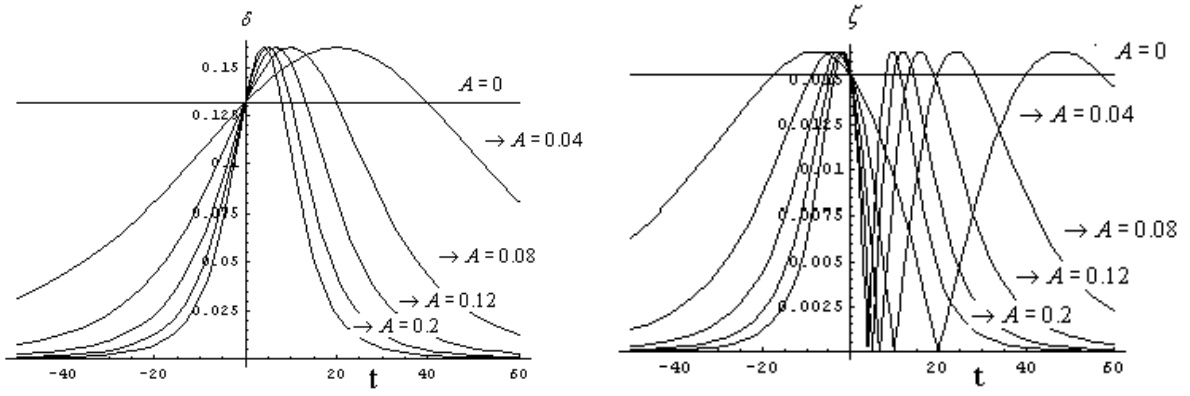
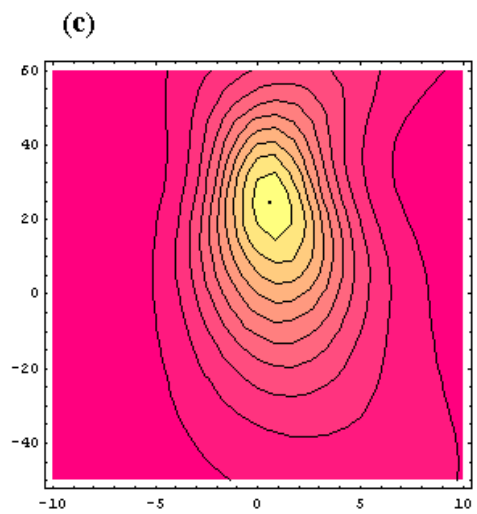
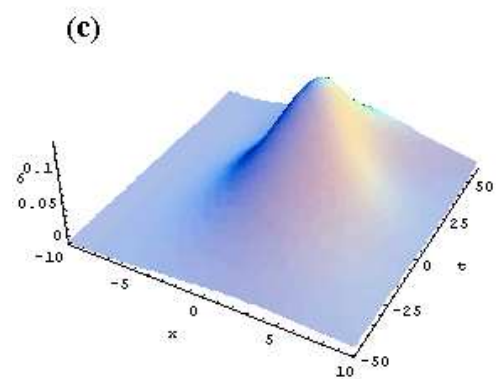
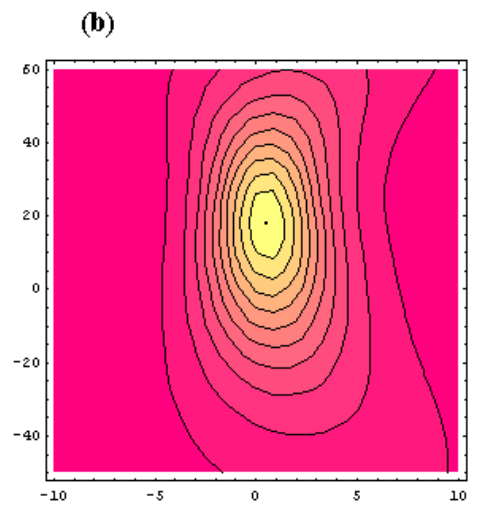
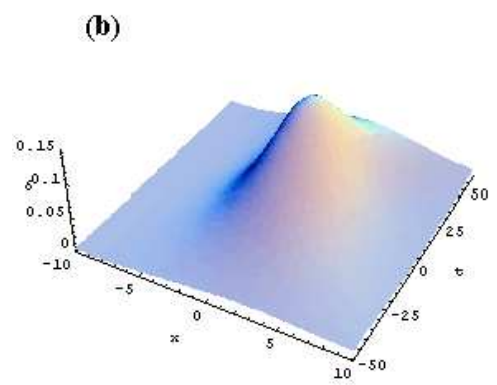
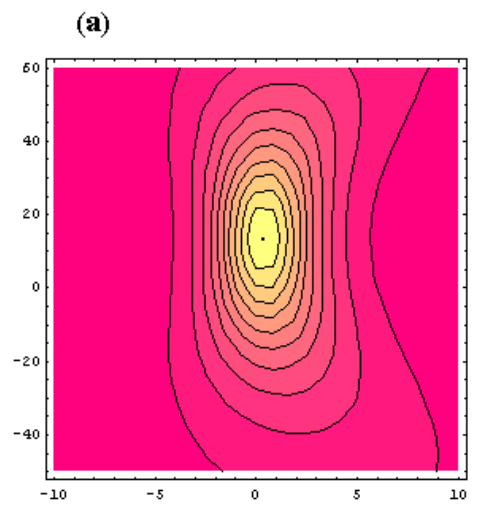
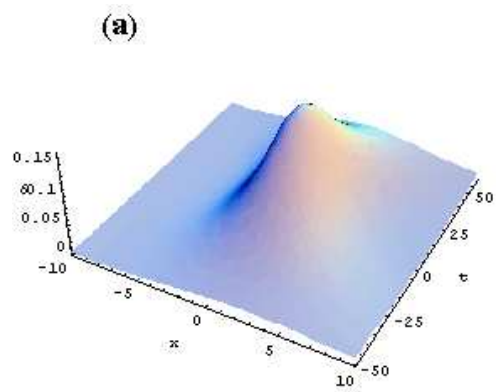
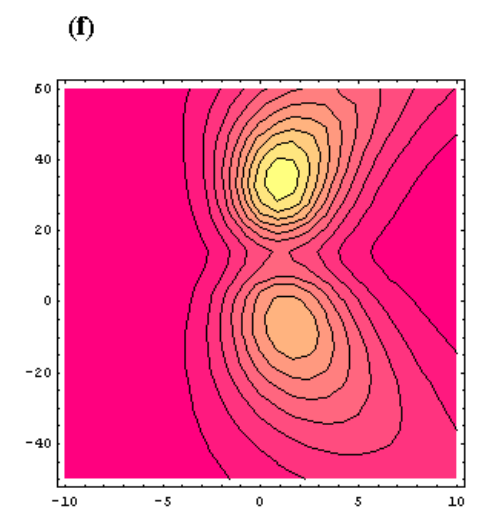
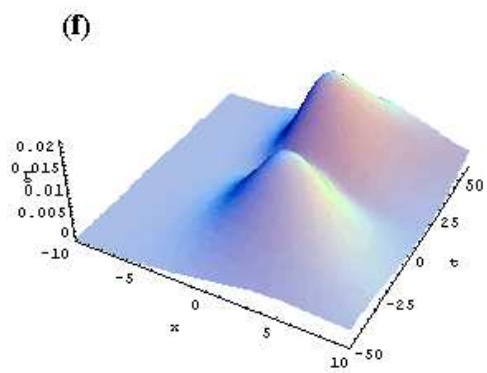
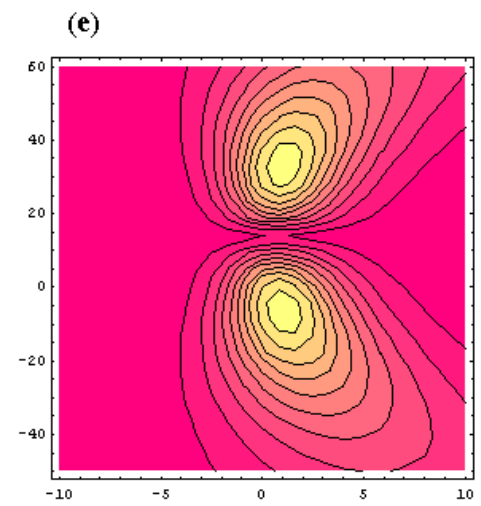
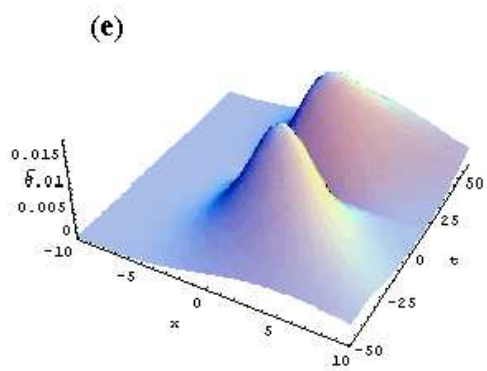
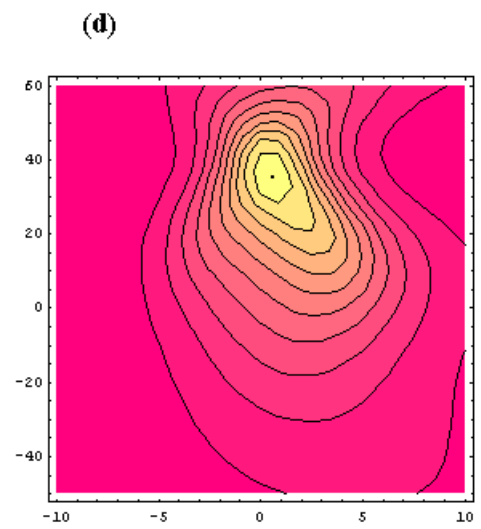
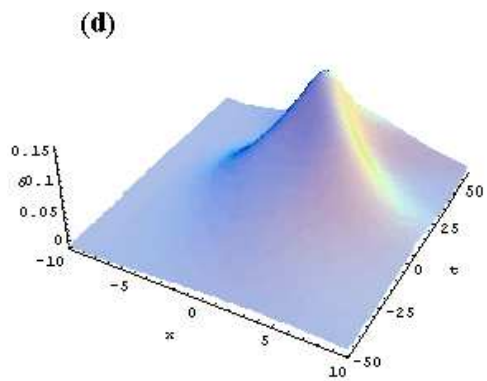


Figure 4: Evolution of energy and current densities under the influence of inhomogeneity.





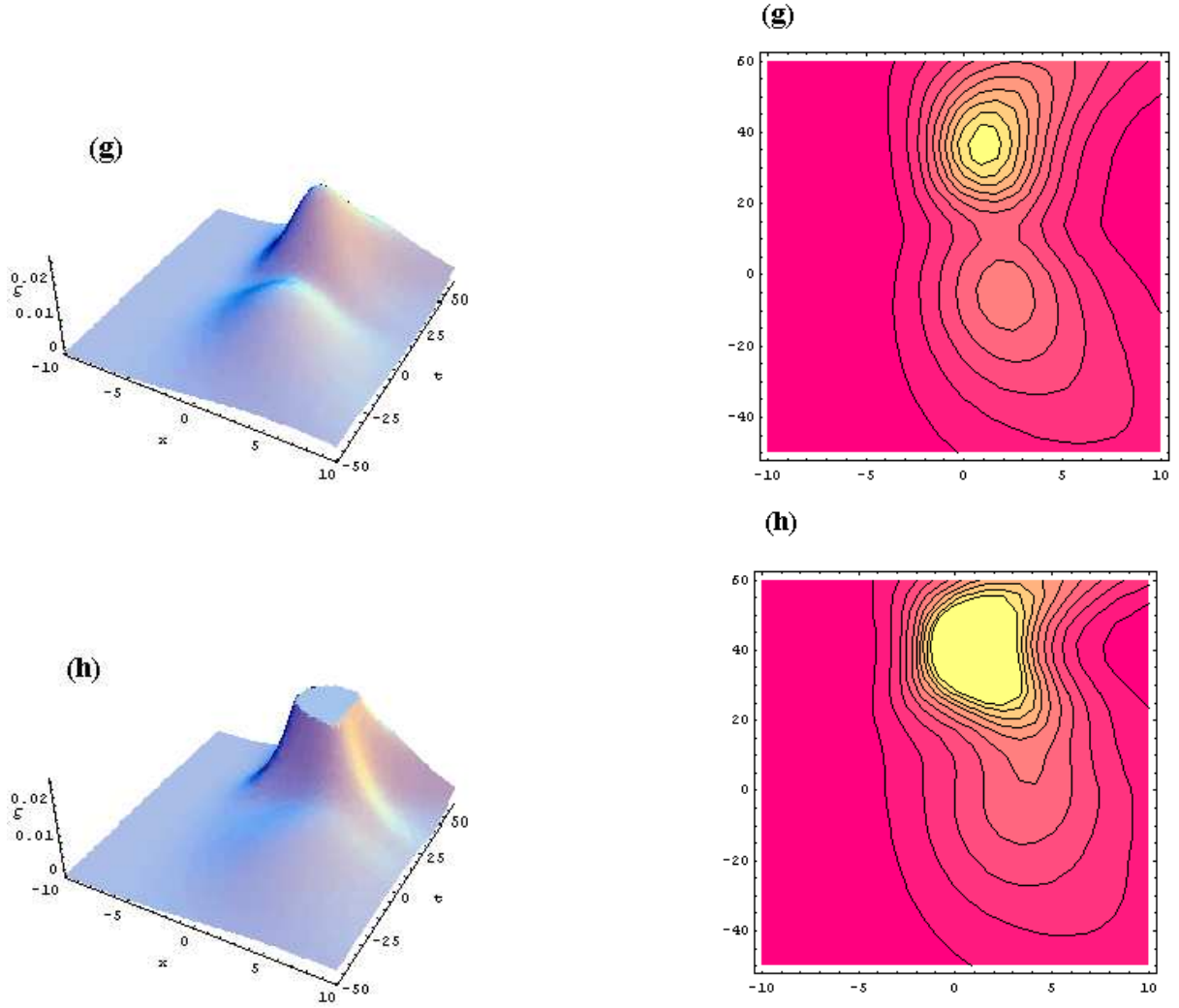


Figure 5: Evolution of energy density (a-d), current density (e-h) and its contour plots in the presence of inhomogeneity  $A = 0.06$  when (a,e)  $\lambda = 0$ , (b,f)  $\lambda = 0.4$ , (c,g)  $\lambda = 0.6$  and (d,h)  $\lambda = 0.9$  with  $c = 1.2$  and  $c_1 = 0.8$ .

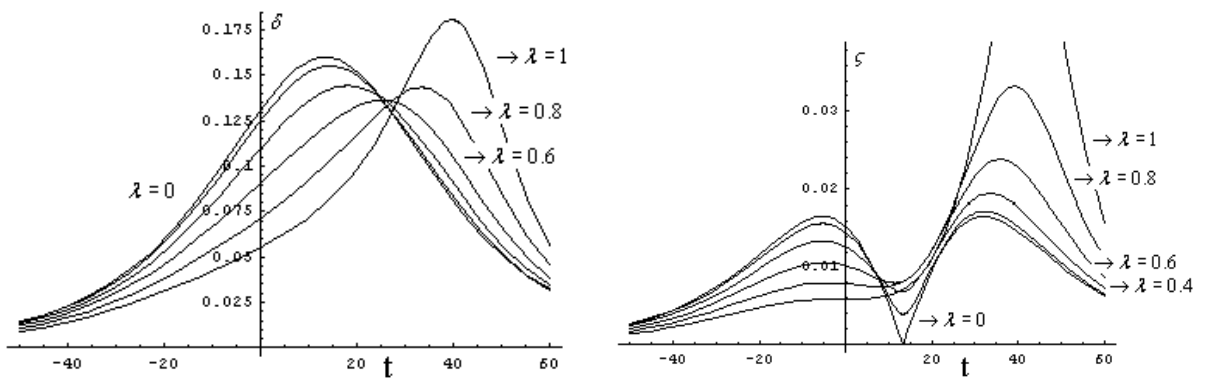


Figure 6: Evolution of energy and current densities under the influence of damping.

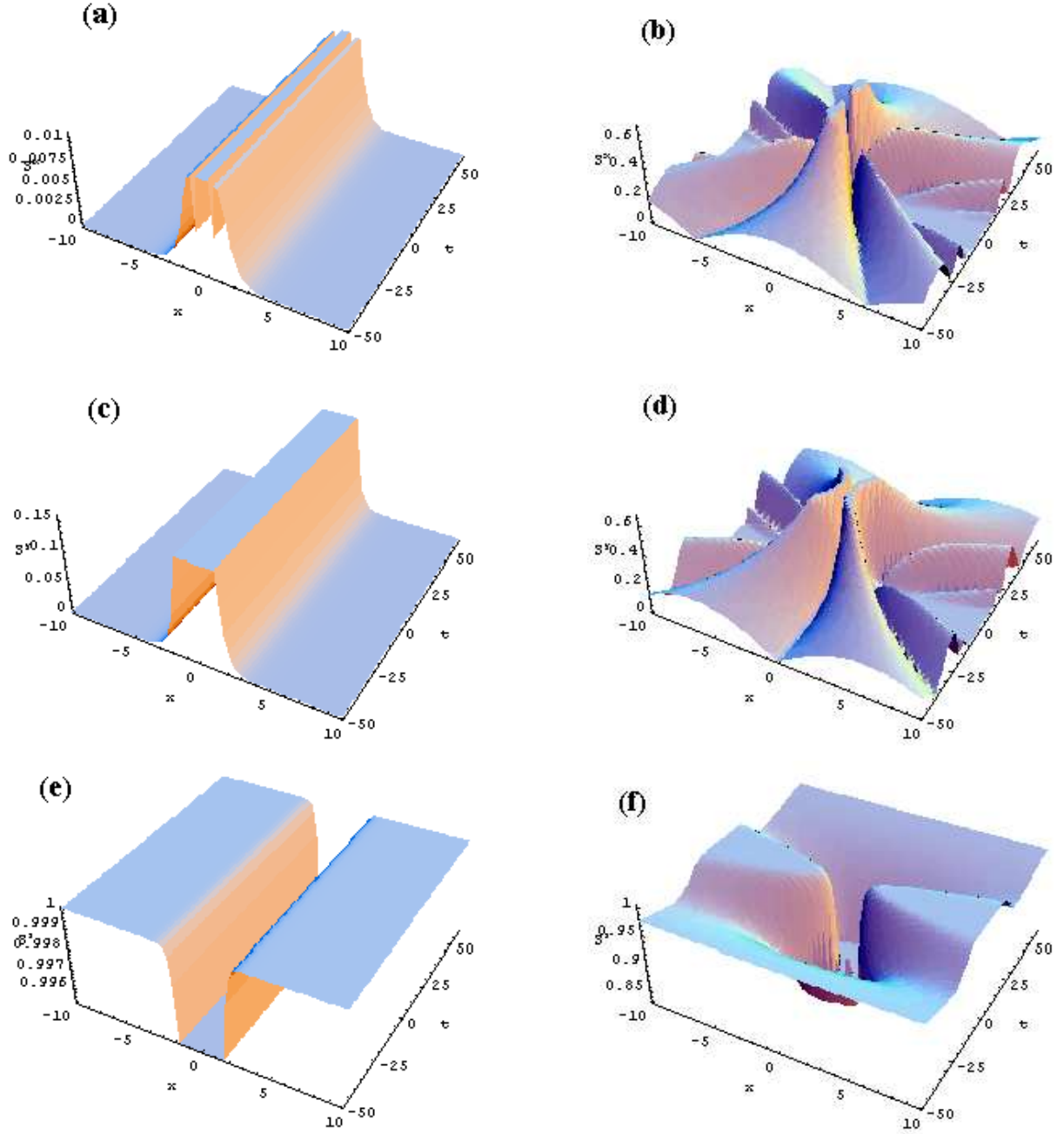


Figure 7: Evolution of spin components (1)  $S^x$  (a, b), (2)  $S^y$  (c, d) and (3)  $S^z$  (e, f) when (a, c, e)  $A = 0$  and (b, d, f)  $A = 0.2$  with  $\lambda = 0$ ,  $c = 1.1$  and  $c_1 = 0.01$ .

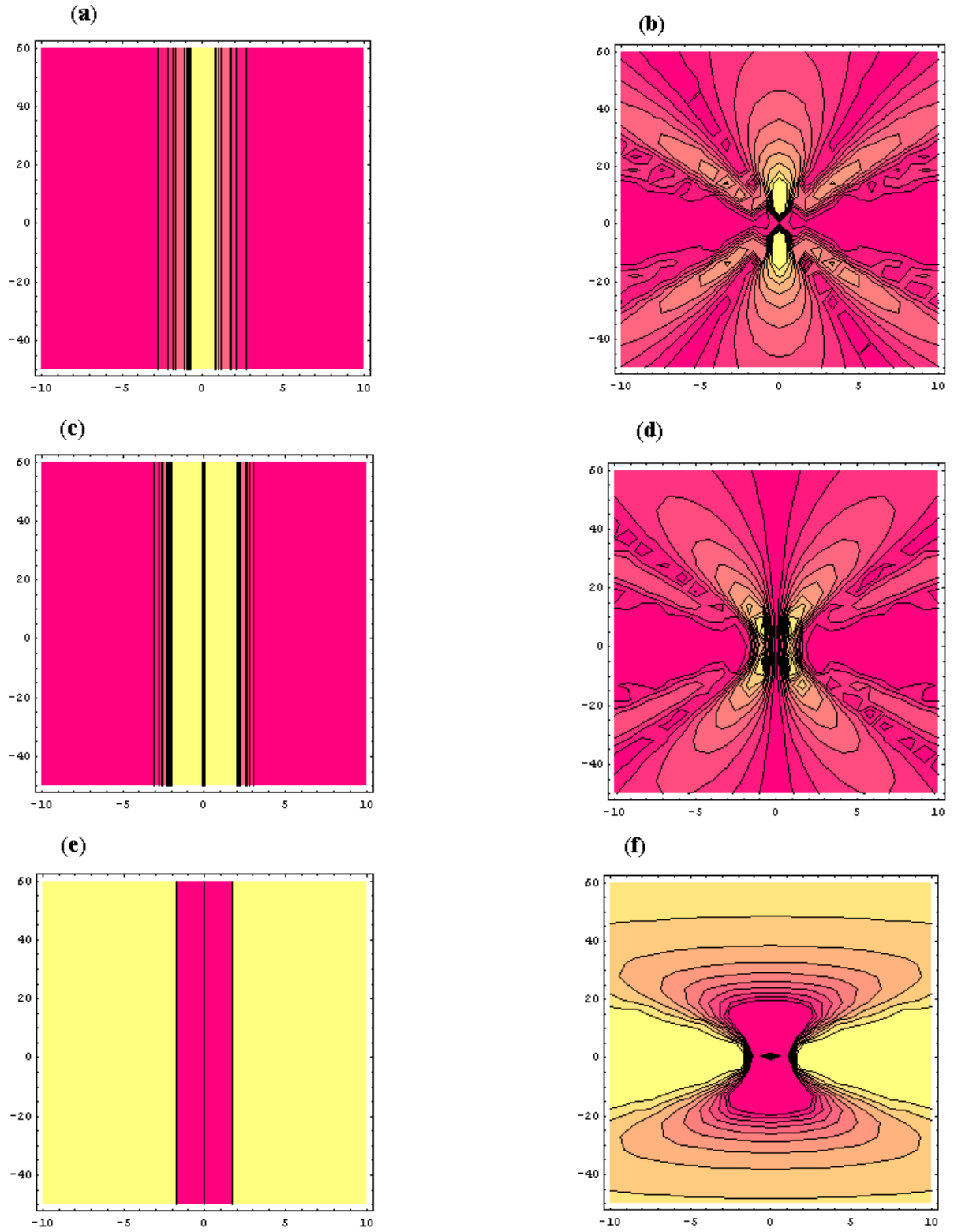


Figure 8: Contour plots for evolution of spin components (1)  $S^x$  (a, b), (2)  $S^y$  (c, d) and (3)  $S^z$  (e, f) when (a, c, e)  $A = 0$  and (b, d, f)  $A = 0.2$  with  $\lambda = 0$ ,  $c = 1.1$  and  $c_1 = 0.01$ .

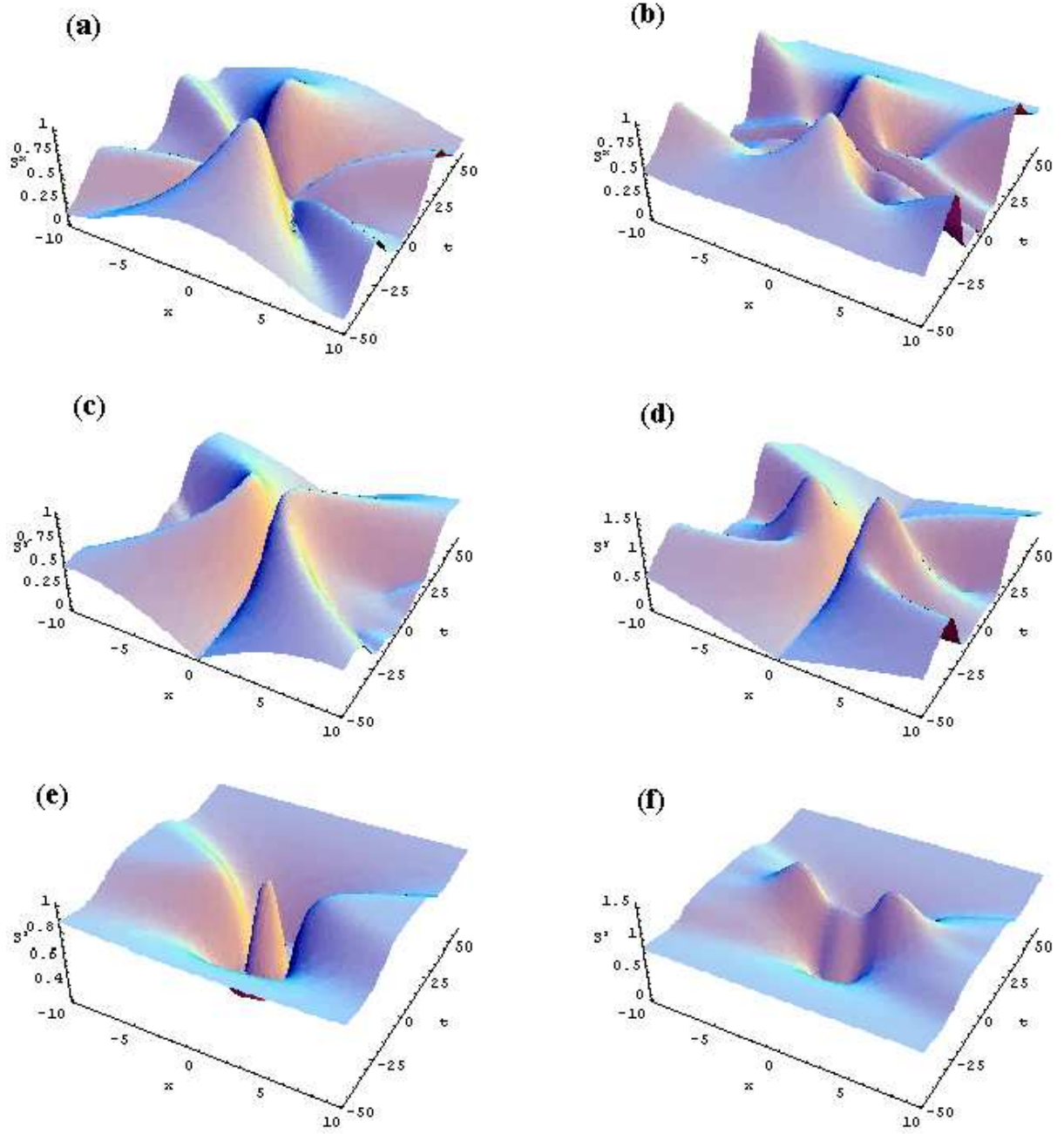


Figure 9: Evolution of spin components (1)  $S^x$  (a, b), (2)  $S^y$  (c, d) and (3)  $S^z$  (e, f) when (a, c, e)  $\lambda = 0.2$  and (b, d, f)  $\lambda = 0.9$  with  $A = 0.12$ ,  $c = 1.1$  and  $c_1 = 0.01$ .



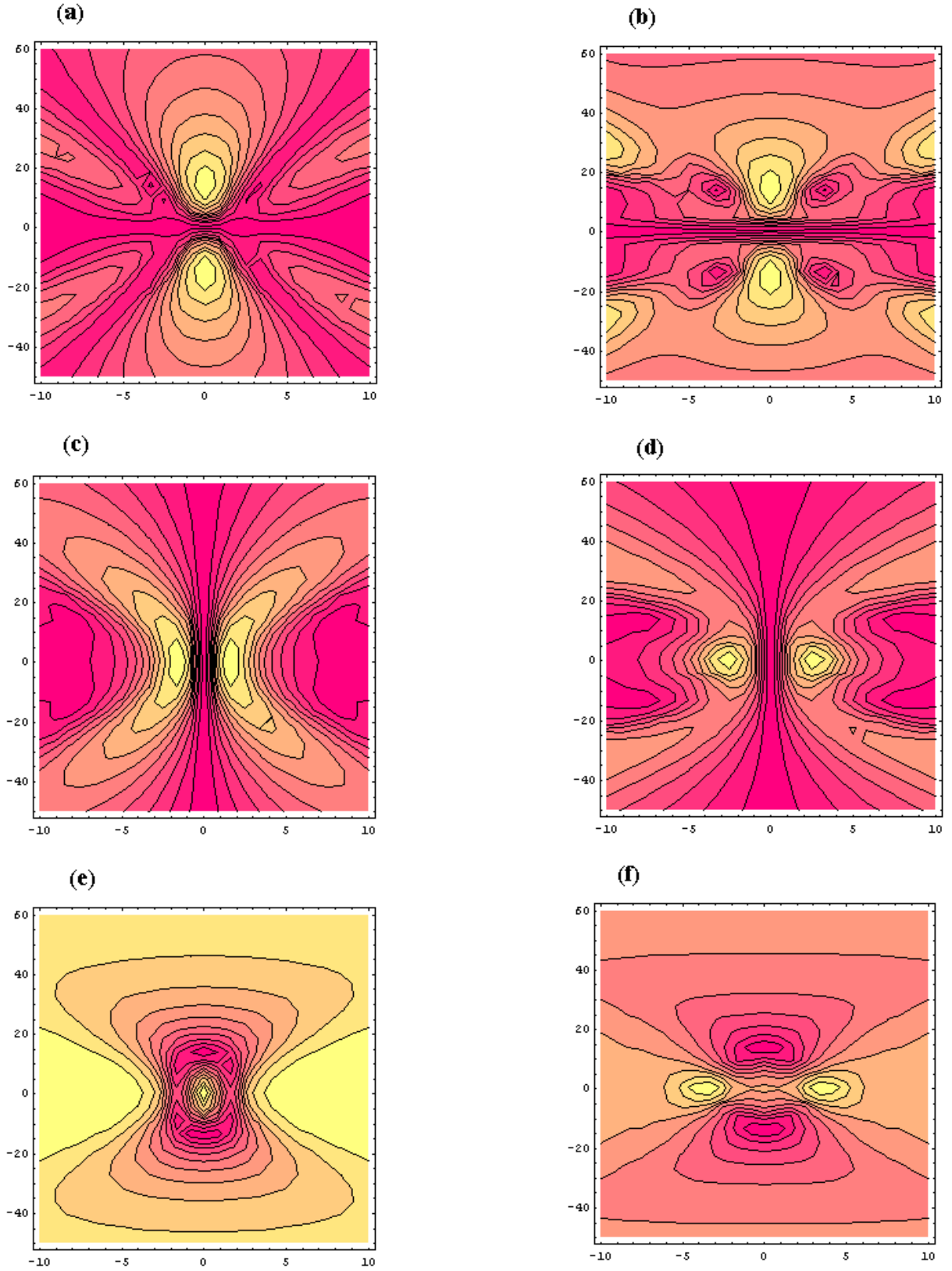


Figure 10: Contour plots for evolution of spin components (1)  $S^x$  (a, b), (2)  $S^y$  (c, d) and (3)  $S^z$  (e, f) when (a, c, e)  $\lambda = 0.2$  and (b, d, f)  $\lambda = 0.9$  with  $A = 0.12$ ,  $c = 1.1$  and  $c_1 = 0.01$ .