Available at: http://publications.ictp.it IC/2010/035

United Nations Educational, Scientific and Cultural Organization
and
International Atomic Energy Agency
THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON P-ADIC QUASI GIBBS MEASURES FOR $Q+1$-STATE POTTS MODEL ON THE CAYLEY TREE

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Abstract

In the present paper we introduce a new class of p-adic measures, associated with $q+1$-state Potts model, called *p-adic quasi Gibbs measure*, which is totally different from the p-adic Gibbs measure. We establish the existence p-adic quasi Gibbs measures for the model on a Cayley tree. If q is divisible by p, then we prove the occurrence of a strong phase transition. If q and p are relatively prime, then there is a quasi phase transition. These results are totally different from the results of [F.M.Mukhamedov, U.A. Rozikov, *Indag. Math. N.S.* **15**(2005) 85–100], since q is divisible by p, which means that $q+1$ is not divided by p, so according to a main result of the mentioned paper, there is a unique and bounded p-adic Gibbs measure (different from p-adic quasi Gibbs measure).

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June 2010

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1. INTRODUCTION

Due to the assumption that p-adic numbers provide a more exact and more adequate description of microworld phenomena, starting in the 1980s, various models described in the language of p-adic analysis have been actively studied [5],[11],[27],[36]. The well-known studies in this area are primarily devoted to investigating quantum mechanics models using equations of mathematical physics [4, 37, 35]. Furthermore, numerous applications of the p-adic analysis to mathematical physics have been proposed in [6],[18],[19]. One of the first applications of p-adic numbers in quantum physics appeared in the framework of quantum logic in [7]. This model is especially interesting for us because it could not be described by using conventional real valued probability. Besides, it is also known [19, 24, 27, 31, 34, 35] that a number of p-adic models in physics cannot be described using ordinary Kolmogorov's probability theory. After that in [23] an abstract p-adic probability theory was developed by means of the theory of non-Archimedean measures [31]. Using that measure theory in [20],[26] the theory of stochastic processes with values in p-adic and more general non-Archimedean fields having probability distributions with non-Archimedean values, has been developed. In particular, a non-Archimedean analogue of the Kolmogorov theorem was proven (see also [12]). Such a result allows us to construct wide classes of stochastic processes using finite dimensional probability distributions[1]. Therefore, this result gives us a possibility to develop the theory of statistical mechanics in the context of the p-adic theory, since it lies on the basis of the theory of probability and stochastic processes. Note that one of the central problems of such a theory is the study of infinite-volume Gibbs measures corresponding to a given Hamiltonian, and a description of the set of such measures. In most cases such an analysis depends on specific properties of Hamiltonian, and a complete description is often a difficult problem. This problem, in particular, relates to a phase transition of the model (see [13]).

The aim of this paper is devoted to the development of p-adic probability theory approaches to study $q+1$-state nearest-neighbor p-adic Potts model on a Cayley tree (see [39]). We are especially interested in the construction of p-adic quasi Gibbs measures for the mentioned model, since such measures present more natural concrete examples of p-adic Markov processes (see [20], for definitions). In [29, 30] we have studied p-adic Gibbs measures and the existence of phase transitions for the q-state Potts models on the Cayley tree[2]. It was established that a phase transition occurs [3] if q is divisible by p. This shows that the transition depends on the number of spins q.

[1]We point out that stochastic processes in the field $\mathbb{Q}_p$ of p-adic numbers with values of real numbers have been studied by many authors, for example, [1, 2, 3, 8, 25, 38]. In those investigations wide classes of Markov processes over $\mathbb{Q}_p$ were constructed and studied. In our case the situation is different, since probability measures take their values in $\mathbb{Q}_p$. This leads our investigation to some difficulties. For example, there is no information about the compactness of p-adic values probability measures.

[2]The classical (real value) counterparts of such models were considered in [39]

[3]Here the phase transition means the existence of two distinct p-adic Gibbs measures for the given model.

In the present paper we introduce a new class of p-adic measures, associated with $q+1$-state Potts model, called *p-adic quasi Gibbs measure*, which is totaly different from the p-adic Gibbs measures considered in [29]. In Section 3, we establish the existence of p-adic quasi Gibbs measures for the said model on a Cayley tree of order two, and give concepts of *strong phase transition*, *phase transition* and *quasi phase transition* for the given model in terms of p-adic quasi Gibbs measures. Later on, in Section 4, we shall prove the occurrence of the strong phase transition, whenever q is divisible by p. When q and p are relatively prime, then we establish the existence of the quasi phase transition. These results totally different from the results of [29, 30], since q is divisible by p means that $q+1$ is not divided by p, which according to [29] means that uniqueness and boundedness of p-adic Gibbs measure.

2. PRELIMINARIES

In what follows p will be a fixed prime number, and $\mathbb{Q}_p$ denotes the field of p-adic field, formed by completing $\mathbb{Q}$ with respect to the unique absolute value satisfying $|p|_p = 1/p$. The absolute value $|\cdot|_p$, is non-Archimedean, meaning that it satisfies the ultrametric triangle inequality $|x+y|_p \leq \max\{|x|_p, |y|_p\}$.

Any p-adic number $x \in \mathbb{Q}_p$, $x \neq 0$ can be uniquely represented in the form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...), \tag{2.1}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leq x_j \leq p-1$, $x_0 > 0$, $j = 0, 1, 2, ...$ In this case $|x|_p = p^{-\gamma(x)}$.

We recall that an integer $a \in \mathbb{Z}$ is called *a quadratic residue modulo p* if the equation $x^2 \equiv a (\text{mod } p)$ has a solution $x \in \mathbb{Z}$.

Lemma 2.1. [24] *In order that the equation*

$$x^2 = a, \quad 0 \neq a = p^{\gamma(a)}(a_0 + a_1 p + ...), \quad 0 \leq a_j \leq p-1, \ a_0 > 0$$

has a solution $x \in \mathbb{Q}_p$, it is necessary and sufficient that the following conditions are fulfilled:

(i) *$\gamma(a)$ is even;*

(ii) *a_0 is a quadratic residue modulo p if $p \neq 2$, and moreover, $a_1 = a_2 = 0$ if $p = 2$.*

Note the basics of p-adic analysis, p-adic mathematical physics are explained in [24, 32, 35].

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a *p-adic measure* if for any $A_1, \dots, A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ $(i \neq j)$ the equality holds

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

A p-adic measure is called a *probability measure* if $\mu(X) = 1$. A p-adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$. Note that, in general, a p-adic probability measure need not be bounded [17, 20, 24].

For more detailed information about p-adic measures we refer the reader to [17],[22],[31].

Recall that the Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, and each vertex has exactly $k+1$ edges. Let $\Gamma^k = (V, L)$, where V is the set of vertices of Γ^k, L is the set of edges of Γ^k. The vertices x and y are called *nearest neighbors* and they are denoted by $l =< x, y >$ if there exists an edge connecting them. A collection of the pairs $< x, x_1 >, \dots, < x_{d-1}, y >$ is called a *path* from the point x to the point y. The distance $d(x, y), x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y. Now fix $x^0 \in V$, and set

$$W_n = \{x \in V | d(x, x^0) = n\}, \quad V_n = \bigcup_{m=1}^{n} W_m, \quad L_n = \{l =< x, y >\in L | x, y \in V_n\}.$$

The set of *direct successors* of x is defined by

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n. \tag{2.2}$$

Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k+1$.

3. p-adic Potts model and its p-adic quasi Gibbs measures

In this section we consider the p-adic Potts model where spin takes values in the set $\Phi = \{0, 1, 2, \cdots, q\}$, here $q \geq 1$, (Φ is called a *state space*) and is assigned to the vertices of the tree $\Gamma^k = (V, \Lambda)$. A configuration σ on V is then defined as a function $x \in V \to \sigma(x) \in \Phi$; in a similar manner one defines configurations σ_n and ω on V_n and W_n, respectively. The set of all configurations on V (resp. V_n, W_n) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}, \ \ \Omega_{W_n} = \Phi^{W_n}$). One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Using this, for given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x), & \text{if } \ x \in V_{n-1}, \\ \omega(x), & \text{if } \ x \in W_n. \end{cases}$$

It is clear that $\sigma_{n-1} \vee \omega \in \Omega_{V_n}$.

The Hamiltonian $H_n : \Omega_{V_n} \to \mathbb{Q}_p$ of the p-adic $q+1$-*state Potts* model has the form

$$H_n(\sigma) = N \sum_{<x,y>\in L_n} \delta_{\sigma(x),\sigma(y)}, \quad \sigma \in \Omega_{V_n}, \ n \in \mathbb{N}, \tag{3.1}$$

where δ is the Kronecker symbol and the coupling constant N belongs to $\mathbb{Z}$.

Note that when $q = 1$, then the corresponding model reduces to the p-adic Ising model. Such a model was investigated in [12, 14].

Now let us construct p-adic quasi Gibbs measures corresponding to the model.

Assume that $\mathbf{h} : V \setminus \{x^{(0)}\} \to \mathbb{Q}_p^{\Phi}$ is a function, i.e. $\mathbf{h}_x = (h_{0,x}, h_{1,x}, \dots, h_{q,x})$, where $h_{i,x} \in \mathbb{Q}_p$ ($i \in \Phi$) and $x \in V \setminus \{x^{(0)}\}$. Given $n \in \mathbb{N}$, let us consider a p-adic probability measure $\mu_{\mathbf{h}}^{(n)}$ on Ω_{V_n} defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{Z_n^{(\mathbf{h})}} p^{H_n(\sigma)} \prod_{x \in W_n} h_{\sigma(x),x} \tag{3.2}$$

Here, $\sigma \in \Omega_{V_n}$, and $Z_n^{(\mathbf{h})}$ is the corresponding normalizing factor called a *partition function* given by

$$Z_n^{(\mathbf{h})} = \sum_{\sigma \in \Omega_{V_n}} p^{H_n(\sigma)} \prod_{x \in W_n} h_{\sigma(x),x}, \tag{3.3}$$

here subscript n and superscript $(\mathbf{h})$ are accorded to the Z, since it depends on n and a function $\mathbf{h}$.

One of the central results of the theory of probability concerns a construction of an infinite volume distribution with given finite-dimensional distributions, which is called *Kolmogorov's Theorem* [33]. Therefore, in this paper we are interested in the same question but in a p-adic context. More exactly, we want to define a p-adic probability measure μ on Ω which is compatible with defined ones $\mu_{\mathbf{h}}^{(n)}$, i.e.

$$\mu(\sigma \in \Omega : \sigma|_{V_n} = \sigma_n) = \mu_{\mathbf{h}}^{(n)}(\sigma_n), \quad \text{for all } \ \sigma_n \in \Omega_{V_n}, \ n \in \mathbb{N}. \tag{3.4}$$

In general, à priori the existence of such a kind of measure μ is not known, since there is not much information on topological properties, such as compactness, of the set of all p-adic measures defined even in compact spaces[4]. Note that certain properties of the set of p-adic measures have been studied in [16], but those properties are not enough to prove the existence of the limiting measure. Therefore, at present, we can only use the p-adic Kolmogorov extension Theorem (see [12],[20]) which is based on the so-called *compatibility condition* for the measures $\mu_{\mathbf{h}}^{(n)}$, $n \geq 1$, i.e.

$$\sum_{\omega \in \Omega_{W_n}} \mu_{\mathbf{h}}^{(n)}(\sigma_{n-1} \vee \omega) = \mu_{\mathbf{h}}^{(n-1)}(\sigma_{n-1}), \tag{3.5}$$

for any $\sigma_{n-1} \in \Omega_{V_{n-1}}$. This condition, according to the theorem, implies the existence of a unique p-adic measure μ defined on Ω with a required condition (3.4). Note that a more general theory of p-adic measures has been developed in [15].

So, if for some function $\mathbf{h}$ the measures $\mu_{\mathbf{h}}^{(n)}$ satisfy the compatibility condition, then there is a unique p-adic probability measure, which we denote by $\mu_{\mathbf{h}}$, since it depends on $\mathbf{h}$. Such a measure $\mu_{\mathbf{h}}$ is said to be *a p-adic quasi Gibbs measure* corresponding to the p-adic Potts model. By $\mathcal{QG}(H)$ we denote the set of all p-adic quasi Gibbs measures associated with functions $\mathbf{h} = \{\mathbf{h}_x, \ x \in V\}$. If there are at least two distinct p-adic quasi Gibbs measures $\mu, \nu \in \mathcal{QG}(H)$ such that μ is bounded and ν is unbounded, then we say that *a phase transition* occurs. In other words, one can find two different functions $\mathbf{s}$ and $\mathbf{h}$ defined on $\mathbb{N}$ such that there exist the corresponding measures $\mu_{\mathbf{s}}$ and $\mu_{\mathbf{h}}$, for which one is bounded, and the other is unbounded. Moreover, if there is a sequence of sets $\{A_n\}$ such that $A_n \in \Omega_{V_n}$ with $|\mu(A_n)|_p \to 0$ and $|\nu(A_n)|_p \to \infty$ as $n \to \infty$, then we say that there occurs a *strong phase transition.* If there are

[4]In the real case, when the state space is compact, then the existence follows from the compactness of the set of all probability measures (i.e. Prohorov's Theorem). When the state space is non-compact, then there is a Dobrushin's Theorem [9, 10] which gives a sufficient condition for the existence of the Gibbs measure for a large class of Hamiltonians.

two different functions $\mathbf{s}$ and $\mathbf{h}$ defined on $\mathbb{N}$ such that there exist the corresponding measures $\mu_{\mathbf{s}}$, $\mu_{\mathbf{h}}$, and they are bounded, then we say there is a *quasi phase transition*.

Remark 3.1. Note that in [29] we considered the following sequence of p-adic measures defined by

$$\mu_{\mathbf{h}}^{(n)}(\sigma) = \frac{1}{\tilde{Z}_n^{(\mathbf{h})}} \exp_p\{H_n(\sigma)\} \prod_{x \in W_n} h_{\sigma(x),x}, \tag{3.6}$$

here as usual $\tilde{Z}_n^{(\mathbf{h})}$ is the corresponding normalizing factor. A limiting p-adic measure generated by (3.6) was called *p-adic Gibbs measure*. Such kinds of measures and phase transitions, for Ising and Potts models on Cayley tree, have been studied in [12, 14, 29, 30]. When a state space Φ is countable, the corresponding p-adic Gibbs measures have been investigated in [21, 28].

Now one can ask for what kind of functions $\mathbf{h}$ the measures $\mu_{\mathbf{h}}^{(n)}$ defined by (3.2) would satisfy the compatibility condition (3.5). The following theorem gives an answer to this question.

Theorem 3.1. *The measures $\mu_{\mathbf{h}}^{(n)}$, $n = 1, 2, \dots$ (see (3.2)) satisfy the compatibility condition (3.5) if and only if for any $n \in \mathbb{N}$ the following equation holds:*

$$\hat{h}_x = \prod_{y \in S(x)} \mathbf{F}(\hat{\mathbf{h}}_y; \theta), \tag{3.7}$$

here and below $\theta = p^N$, a vector $\hat{\mathbf{h}} = (\hat{h}_1, \dots, \hat{h}_q) \in \mathbb{Q}_p^q$ is defined by a vector $\mathbf{h} = (h_0, h_1, \dots, h_q) \in \mathbb{Q}_p^{q+1}$ as follows

$$\hat{h}_i = \frac{h_i}{h_0}, \quad i = 1, 2, \dots, q \tag{3.8}$$

and mapping $\mathbf{F} : \mathbb{Q}_p^q \times \mathbb{Q}_p \to \mathbb{Q}_p^q$ is defined by $\mathbf{F}(\mathbf{x}; \theta) = (F_1(\mathbf{x}; \theta), \dots, F_q(\mathbf{x}; \theta))$ with

$$F_i(\mathbf{x}; \theta) = \frac{(\theta - 1)x_i + \sum_{j=1}^{q} x_j + 1}{\sum_{j=1}^{q} x_j + \theta}, \quad \mathbf{x} = \{x_i\} \in \mathbb{Q}_p^q, \quad i = 1, 2, \dots, q. \tag{3.9}$$

The proof consists of checking condition (3.5) for the measures (3.2) (cp. [29, 21]).

Lemma 3.2. *Let $\mathbf{h}$ be a solution of (3.7), and $\mu_{\mathbf{h}}$ be an associated p-adic quasi Gibbs measure. Then for the corresponding partition function $Z_n^{(\mathbf{h})}$ (see (3.3)) the following equality holds*

$$Z_{n+1}^{(\mathbf{h})} = A_{\mathbf{h},n} Z_n^{(\mathbf{h})}, \tag{3.10}$$

where $A_{\mathbf{h},n}$ will be defined below (see (3.13)).

Proof. Since $\mathbf{h}$ is a solution of (3.7), then we conclude that there is a constant $a_{\mathbf{h}}(x) \in \mathbb{Q}_p$ such that

$$\prod_{y \in S(x)} \sum_{j=0}^{q} p^{N\delta_{ij}} h_{j,y} = a_{\mathbf{h}}(x) h_{i,x} \tag{3.11}$$

for any $i \in \{0, \dots, q\}$. From this one gets

$$\prod_{x \in W_n} \prod_{y \in S(x)} \sum_{j=0}^{q} p^{N\delta_{ij}} h_{j,y} = \prod_{x \in W_n} a_{\mathbf{h}}(x) h_{i,x} = A_{\mathbf{h},n} \prod_{x \in W_n} h_{i,x}, \tag{3.12}$$

where

$$A_{\mathbf{h},n} = \prod_{x \in W_n} a_{\mathbf{h}}(x). \tag{3.13}$$

Given $j \in \Phi$, by $\eta^{(j)} \in \Omega_{W_n}$ we denote a configuration on W_n defined as follows: $\eta^{(j)}(x) = j$ for all $x \in W_n$.

Hence, by (3.2),(3.12) we have

$$\begin{aligned} 1 &= \sum_{\sigma \in \Omega_n} \sum_{\omega \in \Omega_{W_n}} \mu_{\mathbf{h}}^{(n+1)}(\sigma \vee \omega) \\ &= \sum_{\sigma \in \Omega_n} \sum_{\omega \in \Omega_{W_n}} \frac{1}{Z_{n+1}^{(\mathbf{h})}} p^{H(\sigma \vee \omega)} \prod_{x \in W_{n+1}} h_{\omega(x),x} \\ &= \frac{1}{Z_{n+1}^{(\mathbf{h})}} \sum_{\sigma \in \Omega_n} p^{H(\sigma)} \prod_{x \in W_n} \prod_{y \in S(x)} \sum_{j=0}^{q} p^{N\delta_{\sigma(x),j}} h_{j,y} \\ &= \frac{A_{\mathbf{h},n}}{Z_{n+1}^{(\mathbf{h})}} \sum_{\sigma \in \Omega_n} p^{H(\sigma)} \prod_{x \in W_n} h_{\sigma(x),x} \\ &= \frac{A_{\mathbf{h},n}}{Z_{n+1}^{(\mathbf{h})}} Z_n^{(\mathbf{h})} \end{aligned}$$

which implies the required relation. □

4. Existence of p-adic quasi Gibbs measures

In this section we will establish existence of p-adic quasi Gibbs measures on a Cayley tree of order 2, i.e. $k = 2$. To do it, due to Theorem 3.1 it is enough to show the existence of a solution of (3.7).

Recall that a function $\mathbf{h} = \{\mathbf{h}_x\}_{x \in V \setminus \{x^0\}}$ is called *translation-invariant* if $\mathbf{h}_x = \mathbf{h}_y$ for all $x, y \in V \setminus \{x^0\}$. A p-adic measure $\mu_{\mathbf{h}}$, corresponding to a translation-invariant function $\mathbf{h}$, is called a *translation-invariant p-adic quasi Gibbs measure.*

In what follows, we restrict ourselves to the description of translation-invariant solutions of (3.7), namely $\mathbf{h}_x = \mathbf{h}(= (h_0, h_1, \dots, h_q))$ for all $x \in V$. Then (3.7) can be rewritten as follows

$$\hat{h}_i = \left(\frac{(\theta - 1)\hat{h}_i + \sum_{j=1}^{q} \hat{h}_j + 1}{\sum_{j=1}^{q} \hat{h}_j + \theta} \right)^2, \quad i = 1, 2, \dots, q. \tag{4.1}$$

One can see that $(\underbrace{1, \dots, 1, h}_{m}, 1, \dots, 1)$ is an invariant line for (4.1) $(m = 1, \dots, q)$. On such kind of invariant line equation (4.1) reduces to the following one

$$u = \left(\frac{\theta u + q}{u + \theta + q - 1} \right)^2. \tag{4.2}$$

A simple calculation shows that the last equality has the form

$$(u-1)(u^2+(2\theta-\theta^2+2q-1)u+q^2)=0.$$

Hence, $u_0=1$ solution defines a p-adic quasi Gibbs measure μ_0.

Now we are interested in finding other solutions of (4.2), which means we need to solve the following one

$$u^2+(2\theta-\theta^2+2q-1)u+q^2=0. \tag{4.3}$$

Observe that the solution of (4.3) can be formally written by

$$u_{1,2}=\frac{-(2\theta-\theta^2+2q-1)\pm(\theta-1)\sqrt{D(\theta,q)}}{2}, \tag{4.4}$$

where $D(\theta,q)=\theta^2-2\theta-4q+1$

So, if the defined solutions exist in $\mathbb{Q}_p$, then they define p-adic quasi Gibbs measures μ_1 and μ_2, respectively. Note that to exist such solutions the expression $\sqrt{D(\theta,q)}$ should have a sense in $\mathbb{Q}_p$, since in $\mathbb{Q}_p$ not every quadratic equation has a solution (see Lemma 2.1). Therefore, we are going to check when $\sqrt{D(\theta,q)}$ does exist.

Throughout the paper we will assume that $N>0$, this means $|\theta|_p=p^{-N}<1$. Now let us consider several distinct cases with respect to q.

Case $q=1$. Note that this case corresponds to the p-adic Ising model, and $D(\theta,1)=\theta^2-2\theta-3$.

(i) Let $p=2$. Then from $-3=1+2^2+2^3+\cdots$ one has

$$D(\theta,1)=1+2^2+2^3+2^4\epsilon-2\theta+\theta^2,$$

where $\epsilon=1+2+2^2+\cdots$. Hence, due to Lemma 2.1 one can check that for any $N\geq 1$ the $\sqrt{D(\theta,1)}$ does not exist.

(ii) Let $p=3$. Then taking into account that $\theta=p^N$ we find

$$D(\theta,1)=3(3^{2N-1}-2\cdot 3^{N-1}-1).$$

If $N=1$ then $D(\theta,1)=0$, so $\sqrt{D(\theta,1)}$ exists. If $N>1$ then due to Lemma 2.1 we conclude that $\sqrt{D(\theta,1)}$ does not exist.

(iii) Let $p\geq 5$. Then from the expression

$$-3=p-3+(p-1)p+(p-1)p^2+\cdots$$

we obtain

$$D(\theta,1)=p-3+(p-1)p\epsilon_1-2p^N+p^{2N},$$

where $\epsilon_1=1+p+p^2+\cdots$. So, according to Lemma 2.1 $\sqrt{D(\theta,1)}$ exists if and only if the equation $x^2\equiv p-3(\mathrm{mod}p)$ has a solution in $\mathbb{Z}$. It is easy to see that the last equation equivalent to $x^2+3\equiv 0(\mathrm{mod}p)$. For example, when $p=7$ the equation $x^2+3\equiv 0(\mathrm{mod}p)$ has a solution $x=2$. So, in this case $\sqrt{D(\theta,1)}$ exists.

Hence, we can formulate the following

Theorem 4.1. *Let $N \geq 1$ and $q = 1$ (Ising model). Then the following assertions hold true:*

(i) *If $p = 2$, then there is a unique translation-invariant p-adic quasi Gibbs measure μ_0;*

(ii) *Let $p = 3$. If $N = 1$, then there are three translation-invariant p-adic quasi Gibbs measures μ_0, μ_1 and μ_2, otherwise there is a unique translation-invariant p-adic quasi Gibbs measure μ_0;*

(iii) *Let $p \geq 5$, then there are three translation-invariant p-adic quasi Gibbs measures μ_0, μ_1 and μ_2 if and only if -3 is a quadratic residue of modulo p, otherwise there is a unique translation-invariant p-adic quasi Gibbs measure μ_0;*

Case $q \geq 2$. This case corresponds to $q+1$-state Potts model. Here we shall consider several cases with respect to p.

(i) Let $p = 2$. Let us represent q in a 2-adic form, i.e.

$$q = k_0 + k_1 2 + \cdots + k_s 2^s, \quad s \geq 1.$$

Then we have

$$-4q = 2^2\big((2-k_0) + (1-k_1)2 + \cdots + (1-k_s)2^s\big).$$

Therefore, one has

$$D(\theta, q) = 1 + 2^2\big((2-k_0) + (1-k_1)2 + \cdots + (1-k_s)2^s\big) - 2^{N+1} + 2^{2N}.$$

Now according to Lemma 2.1 we conclude that $\sqrt{D(\theta,q)}$ exists if and only if $k_0 = 0$, which is equivalent to $|q|_2 \leq 1/2$.

(ii) Let $p = 3$. We represent q in a 3-adic form, i.e.

$$q = k_0 + k_1 3 + \cdots + k_s 3^s, \quad s \geq 0.$$

Then we have

$$\begin{aligned} D(\theta, q) &= 1 - q - q\cdot 3 - 2\cdot 3^N + 3^{2N} \\ &= 1 + (3-k_0) + (2-k_1)3 + \cdots (2-k_s)3^s - q\cdot 3 - 2\cdot 3^N + 3^{2N}. \end{aligned}$$

If $k_0 = 0$, then from Lemma 2.1 one can see that $\sqrt{D(\theta,1)}$ exists.

If $k_0 = 2$, then $\sqrt{D(\theta,q)}$ does not exist, since $x^2 \equiv 2(\mod 3)$ has no solution in $\mathbb{Z}$.

If $k_0 = 1$, then this case is more complicated. We cannot provide any certain rule to check the existence of $\sqrt{D(\theta,q)}$. But in this case, $\sqrt{D(\theta,q)}$ may exist or may not. For example, if $k_1 \neq 2$ then $\sqrt{D(\theta,q)}$ does not exist whenever $N \geq 3$. If $k_1 = 2$ and $k_2 = 2$ then $\sqrt{D(\theta,q)}$ exists whenever $N \geq 4$.

(iii) Let $p \geq 5$. Let us represent q in a p-adic expression

$$q = k_0 + k_1 p + \cdots + k_s p^s, \quad s \geq 0.$$

Then we have

$$D(\theta, 1) = 1 + 4(p-k_0) + 4(p-1-k_1)p + \cdots + 4(p-1-k_s)p^s - 2p^N + p^{2N}.$$

So, according to Lemma 2.1 $\sqrt{D(\theta,q)}$ exists if the equation $x^2 \equiv 1-4k_0(\mod p)$ has a solution in $\mathbb{Z}$ whenever $1-4k_0$ is not divided by p. It is clear that if $k_0=0$ then the equation has a solution for any value of p ($p \geq 5$). Note that if $1-4k_0$ is divided by p, then $\sqrt{D(\theta,q)}$ does not exist.

If $p=5$ and $k_0=3$, then one can check that $x^2 \equiv -11(\mathrm{mod} 5)$ has a solution $x=5n+2$. So, in this case $\sqrt{D(\theta,q)}$ exists.

So, we have the following

Theorem 4.2. *Let $N \geq 1$ and $q \geq 2$ (Potts model). Then the following assertions hold true:*

(i) *If $|q|_p < 1$, then there are three translation-invariant p-adic quasi Gibbs measures μ_0, μ_1 and μ_2;*

(ii) *Let $p=3$. If $|q-2|_p < 1$, then there is a unique translation-invariant p-adic quasi Gibbs measure μ_0; if $|q-1|_p < 1$ there is at least one translation-invariant p-adic quasi Gibbs measure μ_0;*

(iii) *Let $p \geq 5$ and $|4q-1|_p < 1$, then there is a unique translation-invariant p-adic quasi Gibbs measure μ_0;*

5. Boundedness of p-adic quasi Gibbs measures and phase transitions

In this section we study boundedness and unboundedness of the p-adic quasi Gibbs measures μ_0,μ_1 and μ_2. In what follows we consider a case $N \geq 1$, this means that $|\theta|_p \leq 1/p$.

Assume that $\hat{\mathbf{h}}_1$, $\hat{\mathbf{h}}_2$ are solutions of (3.7). In what follows, without loss of generality, we may assume that $\hat{\mathbf{h}}_1 = (\hat{h}_1, 1, \dots, 1)$ and $\hat{\mathbf{h}}_2 = (\hat{h}_2, 1, \dots, 1)$, here $\hat{h}_i \neq 1$ and $\hat{h}_i$ $(i=1,2)$ are solutions of (4.3). Therefore, we have

$$\hat{h}_1 + \hat{h}_2 = -2q + 1 + \theta^2 - 2\theta, \quad \hat{h}_1 \cdot \hat{h}_2 = q^2. \tag{5.1}$$

Furthermore, we are going to consider the p-adic quasi Gibbs measures corresponding to these solutions. Due to Lemma 3.2 the partition function $Z_{i,n}$ corresponding to the measure μ_i $(i=1,2)$ has the following form

$$Z_{i,n} = a_i^{|V_{n-1}|} \tag{5.2}$$

where $a_i = (\hat{h}_i + \theta + q - 1)^2 h_0$.

For a given configuration $\sigma \in \Omega_{V_n}$ denote

$$\#\sigma = \{x \in W_n : \ \sigma(x) = 1\}.$$

From (3.2),(3.8) and (5.2) we find

$$
\begin{aligned}
|\mu_1(\sigma)|_p &= \frac{1}{Z_{1,n}} \cdot \frac{1}{p^{H(\sigma)}} \prod_{x\in W_n} \left|\frac{h_{\sigma(x),x}}{h_0}\right|_p |h_0|_p^{|W_n|} \\
&= \frac{|h_0|_p^{|W_n|-|V_{n-1}|}}{|\hat{h}_1+\theta+q-1|_p^{2|V_{n-1}|}} \cdot \frac{|\hat{h}_1|_p^{\#\sigma}}{p^{H(\sigma)}} \\
&= \frac{|h_0|_p^2}{|\hat{h}_1+\theta+q-1|_p^{2|V_{n-1}|}} \cdot \frac{|\hat{h}_1|_p^{\#\sigma}}{p^{H(\sigma)}}, \qquad (5.3)
\end{aligned}
$$

where we have used the equality $|W_n| - |V_{n-1}| = 2$.

Similarly, one gets

$$
|\mu_2(\sigma)|_p = \frac{|h_0|_p^2}{|\hat{h}_2+\theta+q-1|_p^{2|V_{n-1}|}} \cdot \frac{|\hat{h}_2|_p^{\#\sigma}}{p^{H(\sigma)}}, \qquad (5.4)
$$

Now assume that q is divided by p, i.e. $|q|_p \le 1/p$. Note that according to Theorem 4.2 in the current case (i.e. $|q|_p < 1$) there exist the solutions $\hat{h}_1$ and $\hat{h}_2$. Hence, from (5.1) we conclude that $|\hat{h}_1+\hat{h}_2|_p = 1$ and $|\hat{h}_1 \cdot \hat{h}_2|_p = |q^2|_p \le p^{-2}$. From the last equality, without loss of generality, it yields that

$$
|\hat{h}_1|_p = |q^2|_p < 1, \quad |\hat{h}_2|_p = 1. \qquad (5.5)
$$

Hence, we obtain $|\hat{h}_1+\theta+q-1|_p = 1$, therefore, from (5.3) one gets

$$
|\mu_1(\sigma)|_p = \frac{|h_0|_p^2}{p^{H(\sigma)}} \cdot |\hat{h}_1|_p^{\#\sigma} \le |h_0|_p^2, \qquad (5.6)
$$

which implies that the measure μ_1 is bounded.

The equality (5.1) implies that $\hat{h}_2 - 1 = \theta^2 - 2\theta - 2q - \hat{h}_1$, this with the strong triangle inequality and (5.5) yields

$$
|\hat{h}_2+\theta+q-1|_p = |\theta^2-\theta-q-\hat{h}_1|_p = |q|_p, \qquad (5.7)
$$

if $|\theta|_p \le |q|_p^2$.

Hence, from (5.4) with (5.7),(5.5) we find

$$
\begin{aligned}
|\mu_2(\sigma)|_p &= \frac{|h_0|_p^2}{|q|_p^{2|V_{n-1}|}} \cdot \frac{1}{p^{H(\sigma)}} \\
&\ge |h_0|_p^2 p^{2|V_{n-1}|-H(\sigma)} \qquad (5.8)
\end{aligned}
$$

Now let us choose $\sigma_{0,n} \in \Omega_{V_n}$ as follows $\sigma_{0,n}(x) = 1$ for all $x \in V_n$. Then one can see that $H(\sigma_{0,n}) = 0$, therefore from (5.8) one gets

$$
|\mu_2(\sigma_{0,n})|_p \ge |h_0|_p^2 p^{2|V_{n-1}|} \to \infty \ \text{ as } \ n \to \infty.
$$

This yields that the measure μ_2 is not bounded.

Let us consider the measure μ_0. Similarly, we obtain

$$\begin{aligned} |\mu_0(\sigma)|_p &= \frac{|h_0|_p^2}{|\theta+q|_p^{2|V_{n-1}|}}\cdot\frac{1}{p^{H(\sigma)}} \\ &= \frac{|h_0|_p^2}{|q|_p^{2|V_{n-1}|}}\cdot\frac{1}{p^{H(\sigma)}} \\ &\geq |h_0|_p^2 p^{2|V_{n-1}|-H(\sigma)} \end{aligned} \tag{5.9}$$

so, we immediately find that $|\mu_0(\sigma_{0,n})|_p \to \infty$ as $n \to \infty$.

From (5.8), (5.9) we immediately find

$$\left|\frac{\mu_0(\sigma)}{\mu_2(\sigma)}\right|_p = 1.$$

Now let us compare μ_1 and μ_2. From (5.6),(5.8) with (5.5) one finds

$$\begin{aligned} |\mu_1(\sigma_{0,n})\mu_2(\sigma_{0,n})|_p &= \frac{|h_0|_p^4|\hat{h}_1|_p^{\#\sigma_{0,n}}}{|q|_p^{2|V_{n-1}|}} \\ &= |h_0|_p^4|q|_p^{2(|W_n|-|V_{n-1}|)} \\ &= |h_0|_p^4|q|_p^4. \end{aligned} \tag{5.10}$$

This implies that $|\mu_1(\sigma_{0,n})|_p \to 0$ as $n \to \infty$.

Consequently, we can formulate the following

Theorem 5.1. *Let $N \geq 1$, $|q|_p < 1$ and $|\theta|_p \leq |q|_p^2$. Assume that μ_0,μ_1 μ_2 are p-adic quasi Gibbs measures for the Potts model* (3.1)*. Then the measure μ_1 is bounded, while the measures μ_0 and μ_2 are unbounded. Moreover, there is a strong phase transition.*

Now assume that $|q|_p = 1$ and there exist solutions $\hat{h}_1$ and $\hat{h}_2$. Note that, in general, the solutions may not exist (see Theorems 4.1 and 4.2). Then from (5.1) we find that

$$|\hat{h}_1+\hat{h}_2|_p \leq 1, \tag{5.11}$$

$$|\hat{h}_1\cdot\hat{h}_2|_p = 1. \tag{5.12}$$

In this case, one has $|\hat{h}_1|_p = 1$ and $|\hat{h}_2|_p = 1$. Indeed, assume that $|\hat{h}_1|_p < 1$, then the equality (5.12) yields $|\hat{h}_2|_p > 1$. Due to the strong triangle inequality we get $|\hat{h}_1+\hat{h}_2|_p > 1$ which contradicts (5.11).

So, we have $|\theta\hat{h}_i+q|_p = 1$, since $|\theta|_p < 1$. On the other hand, we know that $\hat{h}_i$ $(i = 1, 2)$ are solutions (4.2), therefore, from (4.2) one gets

$$|\hat{h}_i+\theta+q-1|_p^2 = \frac{|\theta\hat{h}_i+q|_p^2}{|\hat{h}_i|_p} = 1. \tag{5.13}$$

Hence, (5.3), (5.4) with (5.13) imply

$$|\mu_i(\sigma)|_p = \frac{|h_0|_p^2|\hat{h}_i|_p^{\#\sigma^{(i)}}}{p^{H(\sigma)}} = \frac{|h_0|_p^2}{p^{H(\sigma)}} \leq |h_0|_p^2 \quad (i = 1, 2). \tag{5.14}$$

Let us consider the following difference

$$|\mu_0(\sigma)-\mu_i(\sigma)|_p=\frac{|h_0|_p^2}{p^{H(\sigma)}}\left|(\theta+q-1+\hat{h}_i)^{2|V_{n-1}|}-\hat{h}_i^{\#\sigma}(\theta+q)^{2|V_{n-1}|}\right|_p. \tag{5.15}$$

Denoting

$$x=\theta+q-1,\quad y=\hat{h}_i,\quad N=2|V_{n-1}|,\quad k=\#\sigma$$

and taking into account $|x|_p\leq 1$ and $|y|_p=1$, the right-hand side of (5.15) can be estimated as follows

$$\begin{aligned}
|(x+y)^N-y^k(x+1)^N|_p &= \left|\sum_{\ell=0}^{N} C_N^\ell x^\ell(y^{N-\ell}-y^k)\right|_p\\
&= \left|\sum_{\ell=0}^{N} C_N^\ell x^\ell y^{\min\{N-\ell,k\}}(1-y^{M_\ell})\right|_p\\
&= \left|(1-y)\sum_{\ell=0}^{N} C_N^\ell x^\ell y^{\min\{N-\ell,k\}}\Big(\sum_{j=0}^{M_\ell} y^j\Big)\right|_p\\
&\leq |1-y|_p\max_{0\leq\ell\leq N}\left\{\left|C_N^\ell x^\ell y^{\min\{N-\ell,k\}}\Big(\sum_{j=0}^{M_\ell} y^j\Big)\right|_p\right\}\\
&\leq |1-y|_p, \qquad (5.16)
\end{aligned}$$

here $M_\ell=\max\{N-\ell,k\}-\min\{N-\ell,k\}$.

From (5.16) with (5.15) we immediately find

$$|\mu_0(\sigma)-\mu_i(\sigma)|_p\leq\frac{|h_0|_p^2|1-\hat{h}_i|_p}{p^{H(\sigma)}}\quad (i=1,2). \tag{5.17}$$

Using the same argument one can find

$$|\mu_1(\sigma)-\mu_2(\sigma)|_p\leq\frac{|h_0|_p^2|\hat{h}_1-\hat{h}_2|_p}{p^{H(\sigma)}}. \tag{5.18}$$

Consequently, we can formulate the following

Theorem 5.2. *Let $N\geq 1$, $|q|_p=1$ and μ_0,μ_1 μ_2 be p-adic quasi Gibbs measures for the Potts model* (3.1)*. Then the measures μ_k $(k=0,1,2)$ are bounded. Moreover, the inequalities* (5.17),(5.18) *hold. In this case, there is a quasi phase transition.*

Acknowledgments

The present study has been done within the grant FRGS0409-109 of Malaysian Ministry of Higher Education. The final part of the present work was done while the author was visiting the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy as a Junior Associate. He would like to thank the Centre for hospitality and financial support.

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