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ON PIECEWISE SMOOTHNESS OF CONJUGACY OF CLASS P CIRCLE HOMEOMORPHISMS TO DIFFEOMORPHISMS AND ROTATIONS

Abdelhamid Adouani[1]
Bizerte Preparatory Engineering Institute, 7021 Zarzouna, Tunisia

and

Habib Marzougui[2]
Department of Mathematics, Faculty of Science of Bizerte, 7021 Zarzouna, Tunisia
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

We give a characterization of piecewise C^1 class P homeomorphism f of the circle with irrational rotation number and finitely many break points which is piecewise C^1 conjugate to a C^1-diffeomorphism. The following properties are equivalent:

(i) f is conjugate to a C^1-diffeomorphism of the circle by a piecewise C^1 homeomorphism.

(ii) the product of jumps of f in the break points contained in a same orbit is trivial.

(iii) f is conjugate to a C^1-diffeomorphism of the circle by a PL homeomorphism or a piecewise quadratic homeomorphism.

For a PL-homeomorphism f having the property (ii): f is conjugate to a rotation by either a PL homeomorphism or a piecwise analytic homeomorphism.

MIRAMARE – TRIESTE
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[1]arbi.abdelhamid@gmail.com
[2]Regular Associate of ICTP. hmarzoug@ictp.it

1. Introduction

Denote by $S^1 = \mathbb{R}/\mathbb{Z}$ the circle and $p : \mathbb{R} \longrightarrow S^1$ the canonical projection. Let f be an orientation preserving homeomorphism of S^1. The homeomorphism f admits a lift $\widetilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ that is an increasing homeomorphism of $\mathbb{R}$ such that $p \circ \widetilde{f} = f \circ p$. Conversely, the projection of such a homeomorphism of $\mathbb{R}$ is an orientation preserving homeomorphism of S^1.

Let $x \in S^1$. We call:

- *orbit* of x by f the subset $O_f(x) = \{f^n(x) : \ n \in \mathbb{Z}\}$
- *positive orbit* from x by f the subset $O^+_f(x) = \{f^n(x) : \ n \in \mathbb{N}\}$
- *negative orbit* from x by f the subset $O^-_f(x) = \{f^n(x) : \ n \in -\mathbb{N}\}$

A segment of the orbit $O_f(d)$ containing d is a subset of the form $\{f^s(d) : \ -k \leq s \leq n-k\}$, noted $[f^{-k}(d), \dots, f^{n-k}(d)]$, $k, \ n \in \mathbb{N}$.

Historically, the dynamic study of circle homeomorphisms was initiated by H. Poincaré ([12], 1886), he introduced the rotation number of a homeomorphism f of S^1 as

$$\rho(f) = \lim_{n \longrightarrow +\infty} \frac{\tilde{f}^n(x) - x}{n} \ (mod\ 1)$$

Poincaré shows that this limit exists and does not depend on x and the lift $\tilde{f}$ of f. Assuming f is a C^r-diffeomorphism ($r \geq 2$) and $\rho(f)$ is irrational, A. Denjoy ([4]) proved:

Theorem 1.1 (Denjoy, [4]). *Every C^r-diffeomorphism f $(r \geq 2)$ of S^1 with irrational rotation number $\rho(f)$ is topologically conjugate to rotation $R_{\rho(f)}$.*

This means that there exists an orientation preserving homeomorphism h of S^1 such that $f = h^{-1} \circ R_{\rho(f)} \circ h$. Denjoy noted that this result can be extended (with the same proof) to a large class of circle homeomorphisms: *the class P* (see [6], Chapter VI).

Definition 1.2. *An orientation preserving homeomorphism f of S^1 is called a* class P *homeomorphism if it is derivable except at finitely or countably many points called break points of f at which f admits left and right derivatives (denoted, respectively, by Df_- and Df_+) and such that the derivative $Df : \ S^1 \longrightarrow \mathbb{R}^\star_+$ has the following properties:*

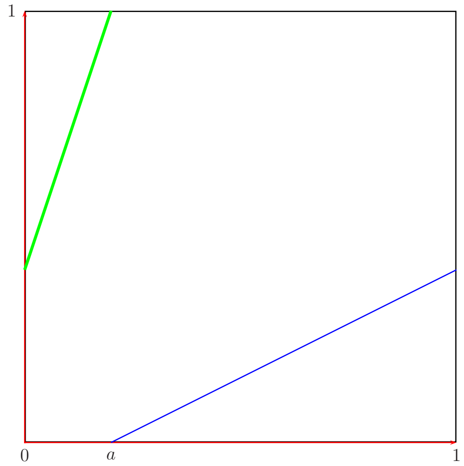
- *There exist two constants $0 < a < b < +\infty$ such that:*

$a < Df(x) < b$, for every x where Df exists, $a < Df_+(c) < b$, and $a < Df_-(c) < b$ at the break point c.

- *$\log Df$ has bounded variation on S^1*

Denote by $\sigma_f(c) := \frac{Df_-(c)}{Df_+(c)}$ called the *f-jump* in c.

We notice that if f is a class P homeomorphism of S^1 which is C^1 on S^1 then f is a C^1-diffeomorphism of S^1. Among the simplest examples of class P homeomorphisms, we mention:



Affine standard Boshernitzan

$$f(x) = \begin{cases} 3x+4 & x \in [0, 0.2] \\ 0.5x-1 & x \in [0.2, 1] \end{cases}$$

FIGURE 1. A standard affine Boshernitzan

- The C^2-diffeomorphisms.

- The piecewise linear (PL) homeomorphisms; these are not C^2-diffeomorphisms. An orientation preserving homeomorphism f of S^1 is called *PL homeomorphism* if f is derivable except in many finitely break points $(c_i)_{1\leq i\leq p}$ of S^1 such that the derivative Df is constant on each $]c_i,\ c_{i+1}[$.

Notice that analytic homeomorphisms are not in general of class P. By Yoccoz [13], they satisfy the conclusion of the Denjoy theorem but not Denjoy theory.

Definition 1.3. *A homeomorphism $B \in \mathcal{P}(S^1)$ with two break points x_0 and $B(x_0)$ is called a* general Boshernitzan *of S^1 of* last break point $B(x_0)$*. If $B(x_0) = 0$, B is called a* standard general Boshernitzan.

If B_0 is a standard general Boshernitzan with $B_0(x_0) = 0$ then $R_{x_0} \circ B_0 \circ R_{x_0}^{-1}$ is a general Boshernitzan of last break point x_0 and conversely. The lift $\widetilde{B_0}$ (mod 1) of B_0 on $[0, 1[$ is a general exchange of 2 intervals on $[0, 1[$. The standard affine Boshernitzan corresponds to affine exchange map of 2 intervals on $[0, 1[$ (see Figure 1).

Notice that Boshernitzan was the first who studied these examples in order to build examples of "rational" PL-homeomorphisms with irrational rotation numbers (cf. [3]).

Definition 1.4. *A class P homeomorphism of S^1 is called a piecewise C^1 class P homeomorphism if f is C^1 except at finitely or countably many break points of f. Furthermore, at a break point, f admits left and right derivatives that respectively coincide with the limits on the left and on the right of the derivative at this point.*

Throughout the paper, we assume that all class P homeomorphisms have *finitely* many break points. Then we deal with

Denote by

- $\mathrm{Homeo}_+(S^1)$ the group of orientation-preserving homeomorphisms of S^1.
- $\mathcal{P}(S^1)$ the set of class P homeomorphisms of S^1, it is a subgroup of $\mathrm{Homeo}_+(S^1)$.
- $\mathcal{P}^1(S^1)$ the set of piecewise C^1 class P homeomorphisms of S^1, it is a subgroup of $\mathcal{P}(S^1)$.
- $PL_+(S^1)$ the set of PL-homeomorphisms of S^1, it is a subgroup of $\mathcal{P}(S^1)$ which contains rotations.

For $f \in \mathcal{P}(S^1)$, denote by:

- $C(f)$ the set of break points of f.
- $\pi_{s,O_f(c)}(f) := \prod_{x\in O_f(c)} \sigma_f(x) = \prod_{x\in C(f)\cap O_f(c)} \sigma_f(x)$, for every $c \in S^1$.
- $\pi_s(f) := \prod_{c\in S^1} \pi_{s,O_f(c)}(f) = \prod_{c\in C(f)} \sigma_f(c)$: the product of f-jumps in the break points of f.
- $\mathcal{P}^1_1(S^1) := \{f \in \mathcal{P}^1(S^1) : \ \pi_s(f) = 1\}$, it is a subgroup of $\mathcal{P}^1(S^1)$.

We notice that if $f \in PL_+(S^1)$ then $\pi_s(f) = 1$.

Definition 1.5. *Let $f \in \mathcal{P}(S^1)$. We say that f has the (D)-property (*[10], [11]*) if the product of f-jumps in the break points on each orbit $O_f(c)$ is trivial; that is*

$$\pi_{s,O_f(c)}(f) = \prod_{x\in C(f)\cap O_f(c)} \sigma_f(x) = 1.$$

In particular, if f has the (D)-property then $\pi_s(f) = 1$. Conversely, if all break points belong to the same orbit and $\pi_s(f) = 1$ then f has the (D)-property. In particular, an affine general Boshernitzan always has the (D)-property.

The problem of smoothness of the conjugacy of smooth diffeomorphisms to rotations is now very well understood (see for instance [2], [6], [7], [8], [13]). The situation is more complicated for circle homeomorphisms with break points.

• For PL-homeomorphisms, the problem was first studied by Herman in [6] and more generally by Liousse in [9] (see Corollary 1.7).

• For general (non PL) class P homeomorphisms with irrational rotation numbers of bounded type and with break points belonging to the same orbit, the problem has been studied by Dzhalilov in [5]:

Dzhalilov's Theorem [5]. Let $f \in \mathcal{P}(S^1)$ with irrational rotation number α. Assume that f satisfies the following conditions:

i) α is of bounded type

ii) $f \in C^{2+\varepsilon}(S^1 \backslash C(f))$, for some $\varepsilon > 0$

iii) the break points of f belong to the same orbit and the product of f-jumps in these break points is trivial

Then f is conjugate to the rotation R_α through a piecewise $C^{1+\varepsilon}$ homeomorphism.

• For general class P homeomorphisms of the circle with irrational rotation numbers not necessarily of bounded type and with break points not necessarily belonging to the same orbit, the problem for their piecewise C^r conjugation to C^r diffeomorphisms ($r \geq 1$) was initiated by the authors in [1]. Our result (stated for $r = 1$) is the following:

Adouani-Marzougui's Theorem ([1], Corollary 2.2) Let $f \in \mathcal{P}^1(S^1)$ with irrational rotation number. Then the following properties are equivalent:

(i) f is conjugated to a C^1-diffeomorphism by a homeomorphism in $\mathcal{P}^1(S^1)$

(ii) f has the (D)-property

(iii) f is conjugated to a C^1-diffeomorphism by a piecewise polynomial homeomorphism $K \in \mathcal{P}^1(S^1)$ of degree at most 3^m, $m \in \mathbb{N}^\star$.

The main purpose of this paper is to complete and improve the result above by giving the minimal smoothness of the conjugacy: we found either piecewise linear (PL) or piecewise quadratic homeomorphisms. We are thus led to the following strengthening of the theorem above:

Theorem 1.6. *Let $f \in \mathcal{P}^1(S^1)$ with irrational rotation number. The following properties are equivalent:*

(i) *f is conjugate to a C^1-diffeomorphism of S^1 by a piecewise C^1 homeomorphism.*

(ii) *the number of break points of f^n is bounded by some constant that doesn't depend on n.*

(iii) *f has the (D)-property*

(iv) *f is conjugate to a C^1-diffeomorphism of S^1 by either a PL homeomorphism or a piecewise quadratic homeomorphism.*

Corollary 1.7. *([9]) Let $f \in PL_+(S^1)$ with irrational rotation number α. The following properties are equivalent:*

(i) *f is conjugate to the rotation R_α through a piecewise C^1 homeomorphism,*

(ii) *the number of break points of f^n is bounded by some constant that doesn't depend on n,*

(iii) *f is conjugate to either a standard affine Boshernitzan or the rotation R_α through a PL homeomorphism,*

(iv) *f is conjugate to the rotation R_α through a piecewise analytic homeomorphism.*

2. Preliminaries and Notations

2.1. The invariant π_s and the (D)-property.

Proposition 2.1 (Invariance of the (D)-property)**.** *Let $f,\ g \in \mathcal{P}(S^1)$. If f and g are conjugated in $\mathcal{P}(S^1)$ by h then $\pi_{s,O_f(c)}(f) = \pi_{s,O_g(h(c))}(g)$, for every $c \in S^1$. In particular, f satisfies the (D)-property if and only if so is g.*

Proof. Let $c \in S^1$. As $g = h \circ f \circ h^{-1}$, we have for every $j \in \mathbb{Z}$:

$$\sigma_g(g^j(h(c))) = \frac{\sigma_h(f(f^j(c)))}{\sigma_h(f^j(c))}\sigma_f(f^j(c))$$

Hence

$$\begin{aligned}
\pi_{s,O_g(h(c))}(g) &= \prod_{j\in\mathbb{Z}} \sigma_g(g^j(h(c))) \\
&= \frac{\prod_{j\in\mathbb{Z}} \sigma_h(f^{j+1}(c))}{\prod_{j\in\mathbb{Z}} \sigma_h(f^j(c))} \pi_{s,O_f(c)}(f) \\
&= \pi_{s,O_f(c)}(f)
\end{aligned}$$

□

Proposition 2.2. *Let $f,\ g \in \mathcal{P}(S^1)$. Then $\pi_s(g \circ f) = \pi_s(g)\pi_s(f)$.*

Proof. Let $c \in S^1$. We have $\sigma_{g\circ f}(c) = \sigma_g(f(c))\sigma_f(c)$. So,

$$\begin{aligned}
\pi_s(g \circ f) &= \prod_{c\in C(g\circ f)} \sigma_{g\circ f}(c) \\
&= \prod_{c\in C(g\circ f)} \sigma_g(f(c))\sigma_f(c).
\end{aligned}$$

Since $C(g \circ f) \subset C(f) \cup f^{-1}(C(g))$ and $\sigma_{g\circ f}(c) = 1$ for every $c \in S^1 \backslash C(g \circ f)$, we have:

$$\begin{aligned}
\pi_s(g \circ f) &= \prod_{c\in C(f)} \sigma_g(f(c))\sigma_f(c) \prod_{c\in f^{-1}(C(g))\backslash C(f)} \sigma_g(f(c))\sigma_f(c) \\
&= \pi_s(f) \prod_{c\in C(f)} \sigma_g(f(c)) \prod_{c\in f^{-1}(C(g))\backslash C(f)} \sigma_g(f(c)) \\
&= \pi_s(f) \prod_{c\in f^{-1}(C(g))} \sigma_g(f(c)) \\
&= \pi_s(f)\pi_s(g).
\end{aligned}$$

□

Corollary 2.3 (Invariance of π_s)**.** *Let $f,\ g \in \mathcal{P}(S^1)$. If f and g are conjugated in $\mathcal{P}(S^1)$ then $\pi_s(f) = \pi_s(g)$. In particular, $f \in \mathcal{P}_1^1(S^1)$ if and only if so is g.*

2.2. **Maximal connections.**

Definition 2.4. *Let $c \in C(f)$. A* maximal f-connection *of c is a segment $[f^{-p}(c), \dots, f^{q}(c)]$ of the orbit $O_f(c)$ which contains all the break points of f contained on $O_f(c)$ and such that $f^{-p}(c)$ (resp. $f^{q}(c)$) is the first (resp. last) break point of f on $O_f(c)$.*

Hence, the negative orbit $O_f^-(f^{-p-1}(c))$ (resp. positive orbit $O_f^+(f^{\,q+1}(c))$ doesn't contain any break point of f.

We have the following properties:

- Two break points of f are on the same maximal f-connection, if and only if, they are on the same orbit.
- Two distinct maximal f-connections are disjoint.

Notations. Let $f \in \mathcal{P}(S^1)$. Denote by

- $C(f) = \{c_1, \dots, c_p\}$ be the set of break points, ($p \in \mathbb{N}^\star$)
- $M_i(f) = [c_i, \dots, f^{N_i}(c_i)]$, ($N_i \in \mathbb{N}$), the maximal f-connections of $c_i \in C(f)$, ($1 \leq i \leq p$).
- $M(f) = \coprod_{i=1}^{p} M_i(f)$.

So, we have the decomposition:

$$C(f) = \coprod_{i=1}^{p} C_i(f) \quad \text{where} \quad C_i(f) = C(f) \cap M_i(f),\ 1 \leq i \leq p.$$

We also have $\prod_{d \in C_i(f)} \sigma_f(d) = \prod_{d \in M_i(f)} \sigma_f(d)$.

Hence f satisfies the (D)-property means that: for every $1 \leq i \leq p$, the product of f-jumps in the break points of $C_i(f)$ is trivial; that is

$$\prod_{d \in C_i(f)} \sigma_f(d) = 1.$$

Let $n \in \mathbb{N}^\star$ and $x \in S^1$. We have:

- $C(f^n) \subset \{f^{-k}(c) : \ k = 0, 1, \dots, n-1;\ c \in C(f)\}$,
- $Df^n(x) = Df(x)Df(f(x)) \times \cdots \times Df(f^{n-1}(x))$,
- The jump of f^n in x is then:

$$\sigma_{f^n}(x) = \sigma_f(x)\sigma_f(f(x)) \times \cdots \times \sigma_f(f^{n-1}(x)).$$

Proposition 2.5. *Let $f \in \mathcal{P}(S^1)$ with irrational rotation number. Then f has the (D)-property if and only if the number of break points of f^n is bounded by some constant that doesn't depend on n.*

Proof. • Suppose that the number of break points of f^n is bounded by a constant N_0. Let $d \in C(f)$ and $M = [f^{-p}(d), \dots, f^q(d)]$ be the maximal f-connection of d. Let's show that $\prod_{\delta \in M} \sigma_f(\delta) = 1$. Let $n \in \mathbb{N}$ fixed. We have:

$$C(f^{n+1}) \subset \{f^{-k}(c) : c \in C(f),\ 0 \le k \le n\}$$

and

$$\sigma_{f^{n+1}}\left(f^{-k}(d)\right) = \sigma_f\left(f^{-k}(d)\right) \times \cdots \times \sigma_f\left(f^{n-k}(d)\right).$$

Consider

$$J_n := \{0 \le k \le n : \sigma_{f^{n+1}}\left(f^{-k}(d)\right) = 1\}.$$

By hypothesis:

$$\mathrm{card}\{0 \le k \le n\ ;\ \sigma_{f^{n+1}}\left(f^{-k}(d)\right) \ne 1\} \le N_0.$$

Then $\mathrm{card}(J_n) \ge n + 1 - N_0$. Let $n > N_0 + p$, then we have $\mathrm{card}(J_n) > p + 1 \ge \mathrm{card}(J)$ where

$$J = \{0 \le k \le p\ : \sigma_{f^{n+1}}\left(f^{-k}(d)\right) = 1\}.$$

Therefore, J is strictly contained in J_n. Take $K_n := J_n \backslash J$ and let k_n be its minimum. Since $n - k_n + 1 \ge\ card(K_n)\ \ge\ n - (N_0 + p)$, we have

$$p + 1 \le k_n \le p + N_0 + 1 \quad \text{for} \quad n > N_0 + p$$

As

$$\sigma_{f^{n+1}}\left(f^{-k_n}(d)\right) = \sigma_f\left(f^{-p}(d)\right) \times \cdots \times \sigma_f\left(f^{n-k_n}(d)\right)$$

then for every $n > N_0 + p + q$, we have

$$\sigma_f\left(f^{-p}(d)\right)\sigma_f\left(f^{-p+1}(d)\right) \times \cdots \times \sigma_f\left(f^q(d)\right) = \sigma_{f^{n+1}}\left(f^{-k_n}(d)\right).$$

Therefore:

$$\sigma_f\left(f^{-p}(d)\right)\sigma_f\left(f^{-p+1}(d)\right) \times \cdots \times \sigma_f\left(f^q(d)\right) = \prod_{\delta \in M} \sigma_f(\delta) = 1.$$

• Conversely, suppose that for every $d \in C(f)$, the product of jumps of f in the break points of the maximal f-connection of d is equal 1. Lets show that the number of break points of f^n is bounded; Let $n \in \mathbb{N}$. The point of discontinuity of Df^n are among

$$f^{-k}(c_i)\ ,\ k = -N_i\ , \dots, 0, \dots, (n-1);\ i = 1, \dots, p.$$

The jump of f^n in these points are:

$$\sigma_{f^n}\left(f^{-k}(c_i)\right) = \sigma_f\left(f^{-k}(c_i)\right) \times \cdots \times \sigma_f(c_i) \times \cdots \times \sigma_f\left(f^{n-k-1}(c_i)\right).$$

If $k \ge 0$ and $n - k - 1 \ge N_i$ then the segment of orbit

$$[f^{-k}(c_i), \dots, f^{n-k-1}(c_i)]$$

contains the maximal f-connection of c_i. Hence,

$$\sigma_{f^n}\left(f^{-k}(c_i)\right) = 1.$$

So, a necessary condition for $f^{-k}(c_i)$ to be a discontinuity point of Df^n is that $-N_i \le k < 0$ or $n - N_i - 1 < k \le n-1$. Therefore, the number of discontinuity points of Df^n, which are contained in the orbit of c_i, is less then $2N_i$. We conclude that the number of discontinuity points of Df^n is bounded by $2(N_1 + \cdots + N_p)$. □

2.3. **The invariant π and the (AM)-property.**

Definition 2.6. *Let $f \in \mathcal{P}(S^1)$ and $c \in S^1$. Suppose that f has break points on $O_f(c)$. Denote by c^* the first break point of f on $O_f(c)$. We have $(c^*)^* = c^*$. We associate the real number $\pi_{O_f(c)}(f)$ defined as:*

$$\pi_{O_f(c)}(f) := \begin{cases} \prod\limits_{j\in\mathbb{Z}} [\sigma_f(f^j(c^*))]^j, & \text{if } f \text{ has break point on } O_f(c) \\ 1, & \text{otherwise.} \end{cases}$$

In particular, we have the following properties:

- If f has only one break point on $O_f(c)$ then $\pi_{O_f(c)}(f) = 1$.
- $\pi_{O_f(c^*)}(f) = \prod\limits_{j\in\mathbb{Z}} \left(\sigma_f(f^j(c^*))\right)^j$.
- $\pi_{O_f(c_i)}(f) = \prod\limits_{j\in\mathbb{Z}} \left(\sigma_f(f^j(c_i))\right)^j$.

Define $\pi(f) := \prod_{i=1}^p \pi_{O_f(c_i)}(f)$.

Lemma 2.7. *Let $c \in C(f)$ and $[c, \dots, f^q(c)]$ $(q \ge 1)$ be the maximal f-connection of c. Then*

$$\pi_{O_f(c)}(f) = \prod_{k=1}^{q} \sigma_{f^{q+1}}(f^k(c)).$$

Proof. Write $\sigma_j = \sigma_f(f^j(c))$. Then

$$\begin{aligned} \prod_{k=1}^{q} \sigma_{f^{q+1}}(f^k(c)) &= \prod_{k=1}^{q}\prod_{j=k}^{q+k} \sigma_f(f^j(c)) \\ &= \prod_{k=1}^{q}\prod_{j=k}^{q+k} \sigma_j \\ &= \prod_{k=1}^{q}\prod_{j=k}^{q} \sigma_j \\ &= (\sigma_1\sigma_2\dots\sigma_q)(\sigma_2\sigma_3\dots\sigma_q)\dots(\sigma_{q-1}\sigma_q)\sigma_q \\ &= \sigma_1(\sigma_2)^2\dots(\sigma_q)^q. \end{aligned}$$

As $\sigma_{-k} = 1$ for $k \in \mathbb{N}^\star$, we get

$$\prod_{k=1}^{q} \sigma_{f^{q+1}}(f^k(c)) = \prod_{j\in\mathbb{Z}} (\sigma_j)^j = \pi_{O_f(c)}(f).$$

□

Let $f \in \mathcal{P}(S^1)$, $h \in P^1(S^1)$ and $F = h \circ f \circ h^{-1}$. For every $c \in S^1$, denote by $\upsilon(O_F(h(c)))$ the unique integer such that $(h(c^*))^* = F^{\upsilon(O_F(h(c)))}(h(c^*))$, if f has break points on $O_f(c)$ and F has break points on $O_F(h(c))$, and 0 otherwise.

We notice that if $\pi_{s,O_f(c)}(f) \neq 1$ then f has break points on $O_f(c)$ and F has break points on $\mathrm{O}_F(h(c))$.

Proposition 2.8. *Under the notations above, let $f \in \mathcal{P}(S^1)$, $h \in P^1(S^1)$ and $F = h \circ f \circ h^{-1}$. Then for every $c \in S^1$, we have:*

(i) $\pi_{O_F(h(c))}(F) = \dfrac{\left(\pi_{s,O_f(c)}(f)\right)^{-\upsilon(O_F(h(c)))}}{\pi_{s,O_f(c)}(h)} \pi_{O_f(c)}(f)$ *where* $\pi_{s,O_f(c)}(h)$ *is the product of h-jumps on the f-orbit $O_f(c)$*

(ii) $\pi(F) = \dfrac{\prod_{i=1}^{p} \left(\pi_{s,O_f(c_i)}(f)\right)^{-\upsilon(O_F(h(c_i)))}}{\pi_s(h)} \pi(f)$.

Proof. Proof of (i): For every $j \in \mathbb{Z}$, we have

$$\sigma_F(F^j(h(c))) = \frac{\sigma_h\left(f(f^j(c)\right)}{\sigma_h(f^j(c))} \sigma_f(f^j(c)).$$

Write

$$a_j = \sigma_f(f^j(c)),\ b_j = \sigma_F(h(f^j(c))) \text{ and } \lambda_j = \sigma_h(f^j(c)).$$

Then $b_j = \frac{\lambda_{j+1}}{\lambda_j} a_j$.

Case a**)** f has no break points on $O_f(c)$.
In this case, $\upsilon(O_F(h(c))) = 0$, $\pi_{O_f(c)}(f) = 1$ and $\pi_{s,O_f(c)}(f) = 1$, so $b_j = \frac{\lambda_{j+1}}{\lambda_j}$.

a-1) h has no break points on $O_f(c)$: In this case, $\lambda_j = 1$ and hence F has no break points on $h(O_f(c))$. So $\pi_{O_F(h(c))}(F) = 1$, $\pi_{s,O_f(c)}(h) = 1$, the formula (i) is then satisfied.

a-2) h has break points on $O_f(c)$: We let $[f^l(c), \dots, f^s(c)]$ be the maximal h-connection of c $(l, s \in \mathbb{Z},\ l \leq s)$. Hence,

$$b_{l-1} = \frac{\lambda_l}{\lambda_{l-1}} = \lambda_l \neq 1 \text{ and } b_s = \frac{\lambda_{s+1}}{\lambda_s} = \frac{1}{\lambda_s} \neq 1.$$

So F has break points on $h(O_f(c))$ with maximal F-connection of $h(c)$: $[h(f^{l-1}(c)),\dots,h(f^s(c))]$. By definition,

$$\begin{aligned}\pi_{O_F(h(c))}(F) &= b_l^1 b_{l+1}^2 ... b_s^{s-l+1}\\ &= \frac{\lambda_{l+1}^1\lambda_{l+2}^2...\lambda_{s+1}^{s-l+1}}{\lambda_l^1\lambda_{l+1}^2...\lambda_s^{s-l+1}}\\ &= \frac{1}{\lambda_l\lambda_{l+1}...\lambda_s}\\ &= \frac{1}{\pi_{s,O_f(c)}(h)}.\end{aligned}$$

The formula (i) in then satisfied.

Case b**)** F has no break points on $O_F(h(c))$.

We apply Case $a)$ to $(F,h^{-1},f=h^{-1}\circ F\circ h)$ instead of $(f,h,F=h\circ f\circ h^{-1})$: since $\upsilon(O_F(h(c)))=0$ and $\pi_{O_F(h(c))}(F)=1$, we have $\pi_{O_f(c)}(f)=\dfrac{1}{\pi_{s,O_F(h(c))}(h^{-1})}$. As $\pi_{s,O_F(h(c))}(h^{-1})=\dfrac{1}{\pi_{s,O_f(c)}(h)}$, the formula (i) is then satisfied.

Case c**)** f has break points on $O_f(c)$ and F has break points on $O_F(h(c))$.

Write $r=\upsilon(O_F(h(c)))$ and we let $d=(h(c^*))^*$ the first break-point of F on $O_F(h(c))$ where c^* is the first break-point of f on $O_f(c)$. By definition, we have $d=F^r(h(c^*))=h(f^r(c^*))$, hence

$$\begin{aligned}\pi_{O_F(h(c))}(F) &= \prod_{j\in\mathbb{Z}}\big(\sigma_F(F^j(d))\big)^j\\ &= \prod_{j\in\mathbb{Z}}\big(\sigma_F(F^{j+r}(h(c^*)))\big)^j\\ &= \prod_{j\in\mathbb{Z}}\big(\sigma_F(h(f^{j+r}(c^*)))\big)^j\\ &= \prod_{j\in\mathbb{Z}}\left(\frac{\sigma_h(f^{j+r+1}(c^*))}{\sigma_h(f^{j+r}(c^*))}\right)^j\prod_{j\in\mathbb{Z}}\big(\sigma_f(f^{j+r}(c^*))\big)^j\\ &= \frac{\prod_{j\in\mathbb{Z}}\big(\sigma_f(f^j(c^*))\big)^j}{\big(\pi_{s,O_f(c^*)}(f)\big)^r\prod_{j\in\mathbb{Z}}\sigma_h(f^j(c^*))}\\ &= \frac{\pi_{O_f(c)}(f)}{\pi_{s,O_f(c^*)}(h)}(\pi_{s,O_f(c^*)}(f))^{-r}\\ &= \frac{\pi_{O_f(c)}(f)}{\pi_{s,O_f(c)}(h)}(\pi_{s,O_f(c)}(f))^{-r}.\end{aligned}$$

Proof of (ii): We have

$$
\begin{aligned}
\pi(F) &= \prod_{c\in S^1} \pi_{O_F(h(c))}(F) \\
&= \frac{\prod_{c\in S^1} (\pi_{s,O_f(c)}(f))^{-v(O_F(h(c)))}}{\prod_{c\in S^1} \pi_{s,O_f(c)}(h)} \prod_{c\in S^1} \pi_{O_f(c)}(f) \\
&= \frac{\prod_{c\in S^1} (\pi_{s,O_f(c)}(f))^{-v(O_F(h(c)))}}{\pi_s(h)} \pi(f) \\
&= \frac{\prod_{i=1}^{p} (\pi_{s,O_f(c_i)}(f))^{-v(O_F(h(c_i)))}}{\pi_s(h)} \pi(f).
\end{aligned}
$$

□

Corollary 2.9. *Let $f \in \mathcal{P}(S^1)$ with the (D)-property. Let $h \in P^1(S^1)$ and $F = h \circ f \circ h^{-1}$. Then:*

(i) $\pi_{O_F(h(c))}(F) = \dfrac{\pi_{O_f(c)}(f)}{\pi_{s,O_f(c)}(h)}$, *for every* $c \in S^1$

(ii) $\pi(F) = \dfrac{\pi(f)}{\pi_s(h)}$

Proof. The Corollary follows from Proposition 2.8 since $\pi_{s,O_f(c)}(f) = 1$ for every $c \in S^1$. □

Definition 2.10. *Let $f \in \mathcal{P}(S^1)$. Denote by*

$$Z(f) = \sum_{i=1}^{p} \log(\pi_{s,O_f(c_i)}(f))\mathbb{Z}.$$

We say that f has the (AM)-property if $\log \pi(f) \in Z(f)$.

Here are some elementary properties of this definition.

Lemma 2.11. (i) *f satisfies the (D)-property if and only if $Z(f) = \{0\}$. In particular:*

(ii) *If f satisfies the (D)-property then f satisfies the (AM)-property if and only if $\pi(f) = 1$.*

(iii) *If $f \in \mathcal{P}(S^1)$, $h \in \mathcal{P}^1(S^1)$ and $F = h \circ f \circ h^{-1}$ then $Z(F) = Z(f)$.*

(iv) *If $\pi(f) = (\pi_s(f))^m$, for some integer m, then f has the (AM)-property. In particular:*

(v) *If $\pi(f) = 1$ (resp. $\pi(f) = \pi_s(f)$) then f has the (AM)-property.*

Proof. (i) and (ii) are obvious. Property (iii) follows from the Proposition 2.1. The property (iv) follows from the fact that

$$\log \pi(f) = \sum_{i=1}^{p} m \log(\pi_{s,O_f(c_i)}(f)) \in Z(f).$$

□

Corollary 2.12 (Invariance of the (AM)-property)**.** *Let $f,\ g \in \mathcal{P}(S^1)$. If f and g are conjugated in $\mathcal{P}_1^1(S^1)$ then f satisfies the (AM)-property if and only if so is g.*

Proof. Let $h \in \mathcal{P}_1^1(S^1)$ such that $g = h \circ f \circ h^{-1}$. Then $\pi_s(h) = 1$ and by Proposition 2.8, (ii), we have

$$\pi(g) = \prod_{i=1}^{p} \left(\pi_{s,O_f(c_i)}(f)\right)^{-\upsilon(O_g(h(c_i)))} \pi(f).$$

Hence

$$\log(\pi(g)) = \log(\pi(f)) - \sum_{i=1}^{p} \upsilon(O_g(h(c_i))) \log\left(\pi_{s,O_f(c_i)}(f)\right).$$

So, if f satisfies the (AM)-property then $\log(\pi(g)) \in Z(f)$. Since $Z(g) = Z(f)$, we have $\log(\pi(g)) \in Z(g)$ and so g satisfies the (AM)-property. Changing f by g, we obtain the converse.

□

Corollary 2.13. *Let $f,\ g \in \mathcal{P}(S^1)$ with the (D)-property. If f and g are conjugate in $\mathcal{P}_1^1(S^1)$ (and in particular in $PL_+(S^1)$) then $\pi(f) = \pi(g)$.*

Proof. If $h \in \mathcal{P}_1^1(S^1)$ conjugates f to g: $g = h \circ f \circ h^{-1}$ then $\pi_s(h) = 1$ and by Corollary 2.9, (ii) we have $\pi(g) = \dfrac{\pi(f)}{\pi_s(h)} = \pi(f)$. □

Remark 1. If $f \in \mathcal{P}^1(S^1)$ with the (D)-property and $\pi(f) \neq 1$ then f is not conjugated in $\mathcal{P}_1^1(S^1)$ to a C^1-diffeomorphism F:

Indeed, if $F = h \circ f \circ h^{-1}$ with $\pi_s(h) = 1$ then from Corollary 2.9, (ii), $\pi(F) = \dfrac{\pi(f)}{\pi_s(h)} = 1$, hence $\pi(f) = 1$, a contradiction.

Corollary 2.14. *Let $f,\ g \in \mathcal{P}^1(S^1)$ with irrational rotation numbers that are rationally independent. If $f \circ g = g \circ f$ then $f,\ g$ satisfy the (D)-property and $\pi(f) = \pi(g)$.*

Proof. By ([1], Corollary 2.9), there exists $h \in \mathcal{P}^1(S^1)$ such that $h \circ f \circ h^{-1}$ and $h \circ g \circ h^{-1}$ are C^1-diffeomorphisms. Hence $f,\ g$ satisfy the (D)-property and by Corollary 2.9, we have $\dfrac{\pi(f)}{\pi_s(h)} = 1$ and $\dfrac{\pi(g)}{\pi_s(h)} = 1$, so $\pi(f) = \pi(g)$. □

3. Reduction

The aim of this section is to prove the following proposition.

Proposition 3.1. *Let $f \in \mathcal{P}^1(S^1)$ with maximal f-connections of $c_i \in C(f)$: $M_i(f) = [c_i, \dots, f^{N_i}(c_i)]$, $N_i \geq 1,\ 1 \leq i \leq p$.*

(i) *If f satisfies the (AM)-property, then:*

there exist $k_1, \dots, k_p \in \mathbb{Z}$ and $L \in PL_+(S^1)$ that conjugates f to a class P^1 homeomorphism $F = L \circ f \circ L^{-1}$ with

$$C(F) \subset \{L(f^{k_i}(c_i)) : 1 \leq i \leq p\}$$

and $\sigma_F(L(f^{k_i}(c_i))) = \pi_{s,O_f(c_i)}(f),\ 1 \leq i \leq p$.

(ii) *If f does not satisfy the (AM)-property, then:*

for every integers k_1 , k_2 , … , k_p $\in \mathbb{Z}$,

$$\sigma = \left(\prod_{i=1}^{p} (\pi_{s,O_f(c_i)}(f))^{-k_i}\right) \pi(f) \neq 1$$

and there exists $L \in PL_+(S^1)$ that conjugates f to a class P homeomorphism $F_0 = L \circ f \circ L^{-1}$ with $C(F_0) \subset \{L(f^{k_i}(c_i)) : 2 \leq i \leq p\} \cup \{L(f^{k_1}(c_1)),\ L(f^{k_1+1}(c_1))\}$ and $\sigma_{F_0}(L(f^{k_i}(c_i))) = \pi_{s,O_f(c_i)}(f),\ 2 \leq i \leq p$, $\sigma_{F_0}(L(f^{k_1}(c_1))) = \frac{\pi_{s,O_f(c_1)}(f)}{\sigma}$ and $\sigma_{F_0}\left(L(f^{k_1+1}(c_1))\right) = \sigma$.

Lemma 3.2. *Let $\sigma_0, \dots, \sigma_n \in \mathbb{R}_+^\star$ and $b_0, \dots, b_n \in S^1$ ($b_{n+1} = b_0$). If $\sigma_0 \times \cdots \times \sigma_n = 1$ then there exists $L \in PL_+(S^1)$ such that $C(L) \subset \{b_0, \dots, b_n\}$ and $\sigma_L(b_i) = \sigma_i$, $i = 0, \dots, n$.*

Proof. We let $\tilde{b_0} \in \mathbb{R}$ so that $p(\tilde{b_0}) = b_0$ and $p(\tilde{b_0} + 1) = b_{n+1}$.

Define the PL-homeomorphism L of S^1 by its lift $\tilde{L} : \mathbb{R} \to \mathbb{R}$ restricted to $[\tilde{b_0}, \tilde{b_0} + 1[$ as follows:

- the slopes λ_j of $\tilde{L}$ on $[\tilde{b_j}, \tilde{b_j} + 1[$ $(0 \leq j \leq n)$, are given by

$$\lambda_j = \frac{(\sigma_0 \times \cdots \times \sigma_j)^{-1}}{\sum_{i=0}^{n} (\sigma_0 \times \cdots \times \sigma_i)^{-1} (\tilde{b_{i+1}} - \tilde{b_i})}.$$

- the images by $\tilde{L}$ of $\tilde{b_j}$ are given for any $1 \leq j \leq n+1$ by

$$\tilde{L}(\tilde{b_j}) = \tilde{L}(\tilde{b_0}) + \sum_{i=0}^{j-1} \lambda_i (\tilde{b_{i+1}} - \tilde{b_i})$$

$$= \tilde{L}(\tilde{b_0}) + \frac{\sum_{i=0}^{j-1} (\sigma_0 \times \cdots \times \sigma_i)^{-1} (\tilde{b_{i+1}} - \tilde{b_i})}{\sum_{i=0}^{n} (\sigma_0 \times \cdots \times \sigma_i)^{-1} (\tilde{b_{i+1}} - \tilde{b_i})}.$$

□

Let $k, k_1, \ldots, k_p \in \mathbb{Z}$. For $i = 1, \ldots, p$, denote by

$$a_{k,i} := \sigma_f(f^k(c_i))$$

$$\sigma_{k,i} := \begin{cases} \prod_{j \geq k} a_{j,i}, & \text{if } k > k_i \\ \dfrac{1}{\prod_{j<k} a_{j,i}}, & \text{if } k \leq k_i \end{cases}$$

$$\sigma := \prod_{i=1}^{p} \prod_{k \in \mathbb{Z}} \sigma_{k,i}$$

and

$$b_{k,i} := \frac{\sigma_{k+1,i}}{\sigma_{k,i}} a_{k,i}.$$

Then we have $a_{k,i} = 1$ if $k < 0$ (resp. $k > N_i$) and $\sigma_{0,i} = 1$ if $k_i \geq 0$.

Lemma 3.3. *Under the notation above, we have:*

(*a*) $\sigma = \pi(f) \prod_{i=1}^{p} \left(\pi_{s,O_f(c_i)}(f)\right)^{-k_i}$

(*b*) $b_{k,i} = \begin{cases} \pi_{s,O_f(c_i)}(f), & \text{if } k = k_i \\ 1, & \text{otherwise} \end{cases}$

Proof. (a): Write $m_i := \min(0, k_i)$, $n_i := \max(k_i, N_i)$. Then we have:

$\sigma_{k,i} = 1$ if $k \leq m_i$ (resp. $k \geq n_i$). Moreover, $\prod_{k \leq k_i} \sigma_{k,i} = \prod_{j \leq -1} (a_{j+k_i,i})^j$ and $\prod_{k > k_i} \sigma_{k,i} = \prod_{j \geq 1} (a_{j+k_i,i})^j$. It follows that

$$\prod_{k \in \mathbb{Z}} \sigma_{k,i} = \prod_{j \in \mathbb{Z}^*} (a_{j+k_i,i})^j = \prod_{j \in \mathbb{Z}} (a_{j+k_i,i})^j.$$

Hence

$$\prod_{k \in \mathbb{Z}} \sigma_{k,i} = \pi_{O_f(c_i)}(f) \left(\pi_{s,O_f(c_i)}(f)\right)^{-k_i}$$

and therefore

$$\sigma = \prod_{i=1}^{p} \prod_{k \in \mathbb{Z}} \sigma_{k,i} = \pi(f) \prod_{i=1}^{p} \left(\pi_{s,O_f(c_i)}(f)\right)^{-k_i}.$$

(b): We have $b_{k,i} = 1$ if $k > k_i$ (resp. $k \leq k_i - 1$). Moreover, if $k = k_i$, $b_{k,i} = \pi_{s,O_f(c_i)}(f)$, hence the formula (b) follows.

Proof of Proposition 3.1.

Proof of (i): Suppose that f satisfies the (AM)-property; that is $\log \pi(f) \in Z(f)$. Then there exist $k_1, \ldots, k_p \in \mathbb{Z}$ such that

$$\log \pi(f) = \sum_{i=1}^{p} k_i \log \left(\pi_{s,O_f(c_i)}(f)\right);$$

so

$$\pi(f) = \prod_{i=1}^{p} \left(\pi_{s,O_f(c_i)}(f)\right)^{k_i}$$

hence by Lemma 3.3, $\sigma = \prod_{i=1}^{p} \prod_{k\in\mathbb{Z}} \sigma_{k,i} = 1$. By Lemma 3.2, there exists $L \in PL_+(S^1)$ with the following properties:

(i) $L(0) = 0$,

(ii) $C(L) \subset \{f^k(c_i) : m_i \leq k \leq n_i, 1 \leq i \leq p\}$,

(iii) $\sigma_L(f^k(c_i)) = \sigma_{k,i}$

In particular, we have $\pi_s(L) = 1$.

We let $F = L \circ f \circ L^{-1}$. A priori, the break points of F are:

- The break points of L^{-1}: $L(f^k(c_i)),\ m_i \leq k \leq n_i,\ 1 \leq i \leq p$,
- The image by L of break points of f: $L(f^k(c_i)),\ 0 \leq k \leq N_i,\ 1 \leq i \leq p$,
- The image by $L \circ f^{-1}$ of break points of L: $L(f^k(c_i)),\ m_i - 1 \leq k \leq n_i - 1,\ 1 \leq i \leq p$.

Therefore the possible break points of F are among:

$$L(f^k(c_i)),\ m_i - 1 \leq k \leq n_i,\ 1 \leq i \leq p,$$

Compute the jumps of F in these points:

$$\sigma_F\left(L(f^k(c_i))\right) = \frac{\sigma_L\left(f(f^k(c_i))\right)\sigma_f(f^k(c_i))}{\sigma_L(f^k(c_i))} = \frac{\sigma_{k+1,i}}{\sigma_{k,i}} a_{k,i} = b_{k,i}$$

By Lemma 3.3,

$$\sigma_F\left(L(f^k(c_i))\right) = \begin{cases} \pi_{s,O_f(c_i)}(f), & \text{if } k = k_i \\ 1, & \text{otherwise} \end{cases}$$

We conclude that

$$C(F) \subset \{L(f^{k_i}(c_i)) : 1 \leq i \leq p\} \quad \text{with} \quad \sigma_F\left(L(f^{k_i}(c_i))\right) = \pi_{s,O_f(c_i)}(f).$$

Proof of (ii). Suppose that f does not satisfy the (AM)-property, then we have $\sigma \neq 1$. By Lemma 3.2, there exists $L \in PL_+(S^1)$ with the following properties:

(i) $L(0) = 0$,

(ii) $C(L) \subset \{f^k(c_i) : m_i \leq k \leq n_i,\ 1 \leq i \leq p\}$,

(iii) $\sigma_L(f^k(c_i)) = \begin{cases} \sigma_{k,i}\ , & \text{if } (k,i) \neq (k_1+1, 1) \\ \dfrac{\sigma_{k_1+1,1}}{\sigma}\ , & \text{if } (k,i) = (k_1+1, 1) \end{cases}$

In particular, we have $\pi_s(L) = 1$. We let $F_0 = L \circ f \circ L^{-1}$. Therefore the possible break points of F_0 are among: $L(f^k(c_i)),\ m_i - 1 \leq k \leq n_i,\ 1 \leq i \leq p$.

Compute the jumps of F_0 in these points:

$$\sigma_{F_0}\left(L(f^k(c_i))\right) = \frac{\sigma_L\left(f(f^k(c_i))\right)\sigma_f(f^k(c_i))}{\sigma_L(f^k(c_i))}$$

It follows that

$$\sigma_{F_0}\Big(L(f^k(c_i))\Big) = \begin{cases} b_{k,i}, & \text{if } (k,i)\neq(k_1,1) \text{ and } (k,i)\neq(k_1+1,1)\\[2ex] \dfrac{b_{k,i}}{\sigma}, & \text{if } (k,i)=(k_1,1)\\[2ex] \sigma b_{k,i}, & \text{if } (k,i)=(k_1+1,1)\end{cases}$$

Therefore

$$\sigma_{F_0}\Big(L(f^k(c_i))\Big) = \begin{cases} \dfrac{\pi_{s,O_f(c_1)}(f)}{\sigma}, & \text{if } (k,i)=(k_1,1)\\[2ex] \sigma, & \text{if } (k,i)=(k_1+1,1)\\[2ex] \pi_{s,O_f(c_i)}(f), & \text{if } i\neq 1 \text{ and } k=k_i\\[2ex] 1, & \text{otherwise}\end{cases}$$

We conclude that

$$C(F_0)\subset\{L(f^{k_i}(c_i)) : 2\leq i\leq p\}\cup\{L(f^{k_1}(c_1)),\ L(f^{k_1+1}(c_1))\}$$

with

$$\sigma_{F_0}(L(f^{k_i}(c_i))) = \pi_{s,O_f(c_i)}(f),\ 2\leq i\leq p,$$
$$\sigma_{F_0}(L(f^{k_1}(c_1))) = \frac{\pi_{s,O_f(c_1)}(f)}{\sigma},$$

and $\sigma_{F_0}\big(L(f^{k_1+1}(c_1))\big)=\sigma.$ ☐

Corollary 3.4. *If $f\in\mathcal{P}^1(S^1)$ with $\pi(f)=1$ (resp. $\pi(f)=\pi_s(f)$) then there exists $L\in PL_+(S^1)$ with break points*

$$C(L)\subset\{f^k(c_i) : 1\leq k\leq N_i,\ 1\leq i\leq p\}$$

such that $F=L\circ f\circ L^{-1}\in\mathcal{P}^1(S^1)$ with $C(F)\subset\{L(c_i):1\leq i\leq p\}$ (resp. $C(F)\subset\{L(f(c_i)):1\leq i\leq p\}$ and $\sigma_F(L(c_i))=\pi_{s,O_f(c_i)}(f)$ (resp. $\sigma_F(L(f(c_i)))=\pi_{s,O_f(c_i)}(f)$), $1\leq i\leq p$.

Proof. Since $\pi(f)=1$ (resp. $\pi(f)=\pi_s(f)$), by Lemma 2.11, (v), f satisfies the (AM)-property with $k_i=0$ (resp. $k_i=1$) for all i. By Proposition 3.1, (i), there exists $L\in PL_+(S^1)$ satisfying the Corollary, precisely, $C(L)\subset\{f^k(c_i):1\leq k\leq N_i, 1\leq i\leq p\}$ since $\sigma_L(c_i)=\sigma_{0,i}=1$ and then $c_i\notin C(L),\ 1\leq i\leq p$. ☐

Corollary 3.5. *If $f\in\mathcal{P}^1(S^1)$ with the (D)-property then:*

(i) *If $\pi(f)=1$, f is conjugate to a C^1-diffeomorphism through a PL-homeomorphism*

(ii) *If $\pi(f) \neq 1$, f is conjugate to a standard general Boshernitzan (with the (D)-property) through a PL-homeomorphism.*

Proof. Since f satisfies the (D)-property, by Lemma 2.11, (ii), it satisfies the (AM)-property if and only if $\pi(f) = 1$. Therefore, if $\pi(f) = 1$, then by Proposition 3.1, (i), $k_i = 0$ and there exists $L \in PL_+(S^1)$ such that $F = L \circ f \circ L^{-1} \in \mathcal{P}^1(S^1)$ with $C(F) \subset \{L(c_i) : 1 \leq i \leq p\}$ and $\sigma_F(c_i) = \pi_{s,O_f(c_i)}(f) = 1$. Hence, F is a C^1-diffeomorphism. Suppose now $\pi(f) \neq 1$. By Proposition 3.1, (ii), and for $k_i = 0$ for all i, $\sigma = \pi(f)$ and there exists $L \in PL_+(S^1)$ such that $F_0 = L \circ f \circ L^{-1} \in \mathcal{P}^1(S^1)$ with $C(F_0) = \{L(c_1),\ L(f(c_1))\}$ where

$$\sigma_{F_0}(L(c_1)) = \frac{\pi_{s,O_f(c_1)}(f)}{\pi(f)} = \frac{1}{\pi(f)}$$

and $\sigma_{F_0}(L(f(c_1))) = \pi(f)$. Hence F_0 satisfies the (D)-property and has exactly two break points: $L(c_1)$ and $L \circ f(c_1)$, so F_0 is a general Boshernitzan of last break point $L(f(c_1))$. If we conjugate F_0 by a rotation $R_{L(f(c_1))}$, then $B_0 = R^{-1}_{L(f(c_1))} \circ F_0 \circ R_{L(f(c_1))}$ is a standard general Boshernitzan with the (D)-property. This completes the proof. □

As a consequence, we obtain for (PL)-homeomorphisms the following corollary.

Corollary 3.6. *If $f \in PL_+(S^1)$ with the (D)-property then:*

(i) *If $\pi(f) = 1$, f is conjugate to a rotation through a PL-homeomorphism*

(ii) *If $\pi(f) \neq 1$, f is conjugate to a standard affine Boshernitzan through a PL-homeomorphism.*

4. Proof of Theorem 1.6 and Corollary 1.7

Let first introduce the following basic class P homeomorphism.

Let $\sigma \in \mathbb{R}_+^\star \setminus \{1\}$. Let g_σ denote the orientation preserving homeomorphism of S^1 with lift $\widetilde{h_\sigma} : \mathbb{R} \longrightarrow \mathbb{R}$ restricted to $[0, 1[$ is given by:

$$\widetilde{g_\sigma}(x) = \left(\frac{1-\sigma}{\sigma+1}\right)\left(x^2 + \frac{2\sigma}{1-\sigma}x\right),\ x \in [0, 1[.$$

We identify g_σ with its lift $\widetilde{g_\sigma}$. Since $g_\sigma(0) = 0$, $g_\sigma(1) = 1$ and $\sigma > 1$, $g_\sigma \in \mathcal{P}(S^1)$ with one break point 0 and such that $\sigma_{g_\sigma}(0) = \sigma$. Moreover, g_σ is quadratic on $S^1 \backslash \{0\}$.

As a consequence of the Proposition 3.1, we obtain:

Corollary 4.1. *Let $f \in \mathcal{P}^1(S^1)$ with maximal f-connections $c_i \in C(f)$: $M_i(f) = [c_i, \ldots, f^{N_i}(c_i)]$, $N_i \geq 1$, $1 \leq i \leq p$. If f does not satisfy the (AM)-property, then for every $k_1, \ldots, k_p \in \mathbb{Z}$, $\sigma := \prod_{i=1}^{p} \left(\pi_{s,O_f(c_i)}(f)\right)^{-k_i} \pi(f) \neq 1$ and there exists a piecewise quadratic class P homeomorphism φ_σ that conjugates f to a class P homeomorphism $F = \varphi_\sigma \circ f \circ \varphi_\sigma^{-1}$ with $C(F) \subset \{\varphi_\sigma(f^{k_i}(c_i)) : 1 \leq i \leq p\}$ where $\sigma_F(\varphi_\sigma(f^{k_i}(c_i))) = \pi_{s,O_f(c_i)}(f),\ 1 \leq i \leq p$.*

Proof. Let $L \in PL_+(S^1)$ and $F_0 = L \circ f \circ L^{-1}$ be as in the proof of Proposition 3.1, (ii). We let $b = L(f^{k_1}(c_1))$ and $b' = F_0(b) = L(f^{k_1+1}(c_1))$. Then $b \neq b'$ (otherwise, $f(c_1) = c_1$ and $N_1 = 0$, a contradiction) and then $F_0(b') \neq b'$. Define $g_{\sigma,b'} = R_{b'} \circ g_\sigma \circ (R_{b'})^{-1}$. Then $g_{\sigma,b'}$ is a piecewise quadratic class P homeomorphism with one break point b' and such that: $\sigma_{g_{\sigma,b'}}(b') = \sigma$. We let

$$F = g_{\sigma,b'} \circ F_0 \circ (g_{\sigma,b'})^{-1}.$$

We have:

$$\sigma_F(g_{\sigma,b'}(b')) = \frac{\sigma_{g_{\sigma,b'}}(F_0(b'))\sigma_{F_0}(b')}{\sigma_{g_{\sigma,b'}}(b')}.$$

We have $\sigma_{g_{\sigma,b'}}(F_0(b')) = 1$ and $\sigma_{F_0}(b') = \sigma$, so $\sigma_F(g_{\sigma,b'}(b')) = 1$.

On the other hand, we have:

$$\sigma_F(g_{\sigma,b'}(b)) = \frac{\sigma_{g_{\sigma,b'}}(b')\sigma_{F_0}(b)}{\sigma_{g_{\sigma,b'}}(b)}.$$

As $\sigma_{g_{\sigma,b'}}(b') = 1$, $\sigma_{g_{\sigma,b'}}(b) = 1$ and $\sigma_{F_0}(b) = \frac{\pi_{s,O_f(c_1)}(f)}{\sigma}$ then

$$\sigma_F(g_{\sigma,b'}(b)) = \pi_{s,O_f(c_1)}(f).$$

We conclude that $C(F) \subset \{g_{\sigma,b'}(L(f^{k_i}(c_i))) : 1 \leq i \leq p\}$ with

$$\sigma_F\left(g_{\sigma,b'}(L(f^{k_i}(c_i)))\right) = \pi_{s,O_f(c_i)}(f),\ 1 \leq i \leq p.$$

By taking $\varphi_\sigma = g_{\sigma,b'} \circ L$, the corollary follows. □

Corollary 4.2. *Let $f \in \mathcal{P}^1(S^1)$ with the (D)-property. If $\pi(f) \neq 1$ then f is conjugated to a C^1-diffeomorphism through a piecewise quadratic class P homeomorphism.*

Proof of Theorem 1.6. (i) $\Longrightarrow$ (ii). Let h be a piecewise C^1 homeomorphism that conjugates f to a diffeomorphism F: $f = h^{-1} \circ F \circ h$. Since the rotation number is irrational, h^{-1} is also piecewise C^1. Then $f^n = h^{-1} \circ F^n \circ h$. As h and h^{-1} have the same number l of break points then f^n has at most $2l$ break points for every $n \in \mathbb{Z}$.

(ii) $\Longrightarrow$ (iii) is the Proposition 2.5.

(iii) $\Longrightarrow$ (iv): this follows from Corollary 3.5, (i) and Corollary 4.2.

(iv) $\Longrightarrow$ (i) is obvious. □

To prove Corollary 1.7, we consider the following basic homeomorphism:

Let $\sigma \in \mathbb{R}_+^\star \setminus \{1\}$. Let h_σ denote the homeomorphism of S^1 with lift $\widetilde{h_\sigma} : \mathbb{R} \longrightarrow \mathbb{R}$ restricted to $[0,1[$ is given by:

$$\widetilde{h_\sigma}(x) = \frac{\sigma^x - 1}{\sigma - 1},\ x \in [0,1[.$$

We identify h_σ with its lift $\widetilde{h_\sigma}$. Then $h_\sigma \in \mathcal{P}(S^1)$ with one break point 0 and such that $\sigma_{h_\sigma}(0) = \sigma$. Moreover, h_σ is analytic on $S^1 \backslash \{0\}$.

Lemma 4.3. *([3]). If B_0 is a standard affine Boshernitzan with slopes (λ, λ') then $B_0 = h_\sigma \circ R_\alpha \circ h_\sigma^{-1}$ where $\sigma = \frac{\lambda}{\lambda'}$ and R_α is a rotation of angle $\alpha = \frac{\log \lambda}{\log \lambda - \log \lambda'}$.*

Proof of Corollary 1.7. $(i) \Rightarrow (ii)$: If f is conjugate to a rotation by a piecewise C^1 homeomorphism h, then h^{-1} is piecewise C^1 since the rotation number of f is irrational, so by Proposition 2.1, f satisfies the (D)-property and so (ii) follows from Proposition 2.5.

$(ii) \Rightarrow (iii)$: this follows from Proposition 2.5 and then by Corollary 3.6.

$(iv) \Rightarrow (i)$ is clear.

$(iii) \Rightarrow (iv)$: follows from Corollary 3.6 and Lemma 4.3.

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