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PHASE TRANSITIONS FOR QUANTUM XY-MODEL ON THE CAYLEY TREE OF ORDER THREE IN QUANTUM MARKOV CHAIN SCHEME

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Abstract

In the present paper we study forward Quantum Markov Chains (QMC) defined on a Cayley tree. Using the tree structure of graphs, we give a construction of quantum Markov chains on a Cayley tree. By means of such constructions we prove the existence of a phase transition for the XY-model on a Cayley tree of order three in QMC scheme. By the phase transition we mean the existence of two distinct QMC for the given family of interaction operators $\{K_{<x,y>}\}$.

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1 Introduction

It is known that Markov fields play an important role in classical probability, in physics, in biological and neurological models and in an increasing number of technological problems such as image recognition. Therefore, it is quite natural to forecast that the quantum analogue of these models will also play a relevant role. The quantum analogues of Markov processes were first constructed in [1], where the notion of quantum Markov chain on infinite tensor product algebras was introduced. Nowadays, quantum Markov chains have become a standard computational tool in solid state physics, and several natural applications have emerged in quantum statistical mechanics and quantum information theory. The reader is referred to [26, 27, 28, 36] and the references cited therein, for recent developments of the theory and the applications.

First attempts to construct a quantum analogue of classical Markov fields have been done in [31], [4], [6],[9]. In these papers the notion of *quantum Markov state*, introduced in [8], extended to fields as a sub–class of the *quantum Markov chains* introduced in [1]. In [7] it has been proposed a definition of quantum Markov states and chains, which extend a proposed one in [35], and includes all the presently known examples. Note that in the mentioned papers quantum Markov fields were considered over multidimensional integer lattice $\mathbb{Z}^d$. This lattice has so-called amenability property. On the other hand, there do not exist analytical solutions (for example, critical temperature) on such lattice. But investigations of phase transitions of spin models on hierarchical lattices showed that there are exact calculations of various physical quantities (see for example, [13, 37]). Such studies on the hierarchical lattices begun with the development of the Migdal-Kadanoff renormalization group method where the lattices emerged as approximants of the ordinary crystal ones. On the other hand, the study of exactly solved models deserves some general interest in statistical mechanics [13]. Therefore, it is natural to investigate quantum Markov fields over hierarchical lattices. For example, a Cayley tree is the simplest hierarchical lattice with non-amenable graph structure. This means that the ratio of the number of boundary sites to the number of interior sites of the Cayley tree tends to a nonzero constant in the thermodynamic limit of a large system, i.e. the ratio W_n/V_n (see section 2 for the definitions) tends to $(k-1)/(k+1)$ as $n \to \infty$, where k is the order of the tree. Nevertheless, the Cayley tree is not a realistic lattice, however, its amazing topology makes the exact calculation of various quantities possible. First attempts to investigate quantum Markov chains over such trees was done in [12], such studies were related to investigate thermodynamic limit of valence-bond-solid models on a Cayley tree [20]. The mentioned considerations naturally suggest the study of the following problem: the extension to fields of the notion of generalized Markov chain. In [11] we have introduced a hierarchy of notions of Markovianity for states on discrete infinite tensor products of C^*–algebras and for each of these notions we constructed some explicit examples. We showed that the construction of [8] can be generalized to trees. It is worth to note that, in a different context and for quite different purposes, the special role of trees was already emphasized in [31]. Note that in [20] finitely correlated states are constructed as ground states of VBS-model on a Cayley tree. Such shift invariant d-Markov chains can be considered as an extension of C^*-finitely correlated states defined in [21] to the Cayley trees. Note that a noncommutative extension of classical Markov fields, associated with Ising and Potts models on a Cayley tree, were investigated in [33, 34]. In the classical case, Markov fields on trees are also considered in [38]-[43].

If a tree is not one-dimensional lattice, then it is expected (from a physical point of view) that for quantum Markov chains constructed over such a tree there occurs a phase transition. Therefore, our goal in this paper is to establish the existence of phase transition for quantum Markov chains associated with XY-model on a Cayley tree. Note that phase transitions in a quantum setting play an important role to understand quantum spin systems (see for example [14],[23]). In this paper, using a tree structure of graphs, we give a construction of quantum Markov chains on a Cayley tree, which generalizes the construction of [2] to trees. Here, we involve some methods used in the theory of Gibbs measures on trees (see [25]). By means of such constructions we prove the existence of a phase transition for the XY-model on a Cayley tree of order three in QMC scheme. By the phase transition we means the existence of two distinct QMC for the given family of interaction operators $\{K_{<x,y>}\}$. Note that in [10] we have established the uniqueness of for the same model on the Cayley tree of order two. Hence, results of the present paper totaly differ from [10] and show by increasing the dimension of the tree we are getting the phase transition. We have to stress here that the constructed QMC associated with XY-model, is different from thermal states of that model, since such states correspond to $\exp(-\beta\sum_{<x,y>} H_{<x,y>})$, which is different from a product of $\exp(-\beta H_{<x,y>})$. Roughly speaking, if we consider the usual Hamiltonian system $H(\sigma) = -\beta\sum_{<x,y>} h_{<x,y>}(\sigma)$, then its Gibbs measure is defined by the fraction

$$\mu(\sigma) = \frac{e^{-H(\sigma)}}{\sum_\sigma e^{-H(\sigma)}}. \tag{1.1}$$

Such a measure can be viewed in another way as well. Namely,

$$\mu(\sigma) = \frac{\prod_{<x,y>} e^{\beta h_{<x,y>}(\sigma)}}{\sum\limits_\sigma \prod_{<x,y>} e^{\beta h_{<x,y>}(\sigma)}}. \tag{1.2}$$

A usual quantum mechanical definition of the quantum Gibbs states based on equation (1.1). In this paper, we use an alternative way to define the quantum Gibbs states based on (1.2). Note that whether or not the resulting states have a physical interest is a question that cannot be solved on a purely mathematical ground.

2 Preliminaries

Let $\Gamma^k_+ = (L, E)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (i.e. each vertex of Γ^k_+ has exactly $k + 1$ edges, except for the root x^0, which has k edges). Here L is the set

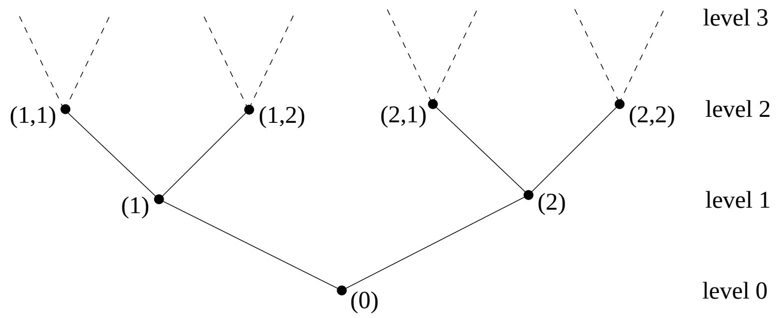

Figure 1: The first levels of Γ^2_+

of vertices and E is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l =< x, y >$ if there exists an edge connecting them. A collection of the pairs $< x, x_1 >, \dots, < x_{d-1}, y >$ is called a *path* from the point x to the point y. The distance $d(x, y), x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y.

Recall a coordinate structure in Γ^k_+: every vertex x (except for x^0) of Γ^k_+ has coordinates $(i_1, \dots, i_n)$, here $i_m \in \{1, \dots, k\}$, $1 \leq m \leq n$ and for the vertex x^0 we put (0). Namely, the symbol (0) constitutes level 0, and the sites $(i_1, \dots, i_n)$ form level n (i.e. $d(x^0, x) = n$) of the lattice (see Fig. 1).

Let us set

$$W_n = \{x \in L \,:\, d(x, x_0) = n\}, \qquad \Lambda_n = \bigcup_{k=0}^{n} W_k, \qquad \Lambda_{[n,m]} = \bigcup_{k=n}^{m} W_k, \ (n < m)$$

$$E_n = \big\{ < x, y > \in E \,:\, x, y \in \Lambda_n \big\}, \qquad \Lambda^c_n = \bigcup_{k=n}^{\infty} W_k$$

For $x \in \Gamma^k_+$, $x = (i_1, \dots, i_n)$ denote

$$S(x) = \{(x, i) : \ 1 \leq i \leq k\},$$

here (x, i) means that $(i_1, \dots, i_n, i)$. This set is called a set of *direct successors* of x.

The algebra of observables $\mathcal{B}_x$ for any single site $x \in L$ will be taken as the algebra M_d of the complex $d \times d$ matrices. The algebra of observables localized in the finite volume $\Lambda \subset L$ is then given by $\mathcal{B}_\Lambda = \bigotimes_{x \in \Lambda} \mathcal{B}_x$. As usual if $\Lambda^1 \subset \Lambda^2 \subset L$, then $\mathcal{B}_{\Lambda^1}$ is identified as a subalgebra of $\mathcal{B}_{\Lambda^2}$ by tensoring with unit matrices on the sites $x \in \Lambda^2 \setminus \Lambda^1$. Note that, in the sequel, by $\mathcal{B}_{\Lambda,+}$ we denote the positive part of $\mathcal{B}_\Lambda$. The full algebra $\mathcal{B}_L$ of the tree is obtained in the usual manner by an inductive limit

$$\mathcal{B}_L = \overline{\bigcup_{\Lambda_n} \mathcal{B}_{\Lambda_n}}.$$

In what follows, by $\mathcal{S}(\mathcal{B}_\Lambda)$ we will denote the set of all states defined on the algebra $\mathcal{B}_\Lambda$.

Consider a triplet $\mathcal{C} \subset \mathcal{B} \subset \mathcal{A}$ of unital C^*-algebras. Recall that a *quasi-conditional expectation* with respect to the given triplet is a completely positive (CP) identity preserving linear map $\mathcal{E} : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{E}(ca) = c\mathcal{E}(a)$, for all $a \in \mathcal{A}$, $c \in \mathcal{C}$.

A state φ on $\mathcal{B}_L$ is called a *forward quantum d-Markov chain (QMC)*, associated to $\{\Lambda_n\}$, on $\mathcal{B}_L$ if for each Λ_n, there exist a quasi-conditional expectation $\mathcal{E}_{\Lambda_n^c}$ with respect to the triplet $\mathcal{B}_{\Lambda_{n+1}^c} \subseteq \mathcal{B}_{\Lambda_n^c} \subseteq \mathcal{B}_{\Lambda_{n-1}^c}$ and a state $\hat{\varphi}_{\Lambda_n^c} \in \mathcal{S}(\mathcal{B}_{\Lambda_n^c})$ such that for any $n \in \mathbb{N}$ one has

$$\hat{\varphi}_{\Lambda_n^c} | \mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n} = \hat{\varphi}_{\Lambda_{n+1}^c} \circ \mathcal{E}_{\Lambda_{n+1}^c} | \mathcal{B}_{\Lambda_{n+1} \setminus \Lambda_n} \tag{2.1}$$

and

$$\varphi = \lim_{n \to \infty} \hat{\varphi}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_n^c} \circ \mathcal{E}_{\Lambda_{n-1}^c} \circ \cdots \circ \mathcal{E}_{\Lambda_1^c} \tag{2.2}$$

in the weak-* topology.

Note that (2.1) is an analogue of the DRL equation from classical statistical mechanics [19, 25], and QMC state is thus the counterpart of the infinite-volume Gibbs measure.

Remark 2.1. *We point out that in [11] a forward QMC was called a generalized quantum Markov state, and the existence of the limit* (2.2) *under the condition* (2.1) *was proved there as well.*

3 Constructions of quantum d-Markov chains on the Cayley tree

In this section, we recall a construction of forward quantum d-Markov chain (see [10]).

Let us rewrite the elements of W_n in the following order, i.e.

$$\overrightarrow{W_n} := \left(x^{(1)}_{W_n}, x^{(2)}_{W_n}, \cdots, x^{(|W_n|)}_{W_n}\right), \quad \overleftarrow{W_n} := \left(x^{(|W_n|)}_{W_n}, x^{(|W_n|-1)}_{W_n}, \cdots, x^{(1)}_{W_n}\right).$$

Note that $|W_n| = k^n$. Vertices $x^{(1)}_{W_n}, x^{(2)}_{W_n}, \cdots, x^{(|W_n|)}_{W_n}$ of W_n can be represented in terms of the coordinate system as follows

$$\begin{array}{lll} x^{(1)}_{W_n} = (1,1,\cdots,1,1), & x^{(2)}_{W_n} = (1,1,\cdots,1,2), \quad \cdots & x^{(k)}_{W_n} = (1,1,\cdots,1,k,), \\ x^{(k+1)}_{W_n} = (1,1,\cdots,2,1), & x^{(2)}_{W_n} = (1,1,\cdots,2,2), \quad \cdots & x^{(2k)}_{W_n} = (1,1,\cdots,2,k), \end{array}$$

$$\vdots$$

$$x^{(|W_n|-k+1)}_{W_n} = (k,k,,\cdots,k,1), \ x^{(|W_n|-k+2)}_{W_n} = (k,k,\cdots,k,2), \quad \cdots x^{|W_n|}_{W_n} = (k,k,\cdots,k,k).$$

Analogously, for a given vertex x, we shall use the following notation for the set of direct successors of x:

$$\overrightarrow{S(x)} := ((x,1),(x,2),\cdots(x,k)), \quad \overleftarrow{S(x)} := ((x,k),(x,k-1),\cdots(x,1)).$$

In what follows, for the sake of simplicity, we will use notation $i \in \overrightarrow{S(x)}$ (resp. $i \in \overleftarrow{S(x)}$ instead of $(x,i) \in \overrightarrow{S(x)}$ (resp. $(x,i) \in \overleftarrow{S(x)}$).

Assume that for each edge $< x,y > \in E$ of the tree an operator $K_{<x,y>} \in \mathcal{B}_{\{x,y\}}$ is assigned. We would like to define a state on $\mathcal{B}_{\Lambda_n}$ with boundary conditions $w_0 \in \mathcal{B}_{(0),+}$ and $\mathbf{h} = \{h_x \in \mathcal{B}_{x,+}\}_{x \in L}$.

Let us denote

$$
\begin{aligned}
K_{[m-1,m]} &:= \prod_{x\in\overrightarrow{W}_{m-1}} \prod_{y\in\overrightarrow{S(x)}} K_{<x,y>}, && (3.1)\\
\mathbf{h}_n^{1/2} &:= \prod_{x\in\overrightarrow{W}_n} h_x^{1/2}, \qquad \mathbf{h}_n := \mathbf{h}_n^{1/2}(\mathbf{h}_n^{1/2})^*, && (3.2)\\
K_n &:= w_0^{1/2} K_{[0,1]} K_{[1,2]} \cdots K_{[n-1,n]} \mathbf{h}_n^{1/2}, && (3.3)\\
\mathcal{W}_{n]} &:= K_n K_n^*, && (3.4)
\end{aligned}
$$

It is clear that $\mathcal{W}_{n]}$ is positive.

If $K_{<x,y>}$ is a self-adjoint operator for every $x, y \in L$, then we have

$$
\mathcal{W}_{n]} = w_0^{1/2} K_{[0,1]} K_{[1,2]} \cdots K_{[n-1,n]} \mathbf{h}_n K_{[n-1,n]} \cdots K_{[1,2]} K_{[0,1]} w_0^{1/2}.
$$

In what follows, by $\mathrm{Tr}_\Lambda : \mathcal{B}_L \to \mathcal{B}_\Lambda$ we mean normalized partial trace, for any $\Lambda \subseteq_{\text{fin}} L$. For the sake of shortness we put $\mathrm{Tr}_{n]} := \mathrm{Tr}_{\Lambda_n}$.

Let us define a positive functional $\varphi^{(n,f)}_{w_0,\mathbf{h}}$ on $\mathcal{B}_{\Lambda_n}$ by

$$
\varphi^{(n,f)}_{w_0,\mathbf{h}}(a) = \mathrm{Tr}(\mathcal{W}_{n+1]}(a \otimes \mathbb{1}_{W_{n+1}})), \tag{3.5}
$$

for every $a \in \mathcal{B}_{\Lambda_n}$, where $\mathbb{1}_{W_{n+1}} = \bigotimes_{y\in W_{n+1}} \mathbb{1}$. Note that here, Tr is a normalized trace on $\mathcal{B}_L$.

To get an infinite-volume state $\varphi^{(f)}$ on $\mathcal{B}_L$ such that $\varphi^{(f)}\lceil_{\mathcal{B}_{\Lambda_n}} = \varphi^{(n,f)}_{w_0,\mathbf{h}}$, we need to impose some constrains to the boundary conditions $\{w_0, \mathbf{h}\}$ so that the functionals $\{\varphi^{(n,f)}_{w_0,\mathbf{h}}\}$ satisfy the compatibility condition, i.e.

$$
\varphi^{(n+1,f)}_{w_0,\mathbf{h}}\lceil_{\mathcal{B}_{\Lambda_n}} = \varphi^{(n,f)}_{w_0,\mathbf{h}}. \tag{3.6}
$$

Theorem 3.1 ([10])**.** *Assume that $K_{<x,y>}$ is self-adjoint for every $< x, y >\in E$. Let the boundary conditions $w_0 \in \mathcal{B}_{(0),+}$ and $\mathbf{h} = \{h_x \in \mathcal{B}_{x,+}\}_{x\in L}$ satisfy the following conditions:*

$$
\mathrm{Tr}(w_0 h_0) = 1 \tag{3.7}
$$

$$
\mathrm{Tr}_{x]}\left[\prod_{y\in\overrightarrow{S(x)}} K_{<x,y>} \prod_{y\in\overrightarrow{S(x)}} h^{(y)} \prod_{y\in\overleftarrow{S(x)}} K_{<x,y>}\right] = h^{(x)} \quad \text{for every } \ x \in L. \tag{3.8}
$$

Then the functionals $\{\varphi^{(n,f)}_{w_0,\mathbf{h}}\}$ satisfy the compatibility condition (3.6)*. Moreover, there is a unique forward quantum d-Markov chain $\varphi^{(b)}_{w_0,\mathbf{h}}$ on $\mathcal{B}_L$ such that $\varphi^{(f)}_{w_0,\mathbf{h}} = w - \lim_{n\to\infty} \varphi^{(n,f)}_{w_0,\mathbf{h}}$.*

From direct calculation we can derive the following

Proposition 3.2. *If* (3.7) *and* (3.8) *are satisfied then one has $\varphi^{(n,f)}_{w_0,\mathbf{h}}(a) = \mathrm{Tr}(\mathcal{W}_{n]}(a))$ for any $a \in \mathcal{B}_{\Lambda_n}$.*

Our goal in this paper is to establish the existence of phase transition for the given family $\{K_{<x,y>}\}$ of operators. Heuristically, the phase transition means the existence of two distinct QMC for the given $\{K_{<x,y>}\}$. Let us provide a more exact definition.

Definition 3.3. *We say that there exists a phase transition for a family of operators $\{K_{<x,y>}\}$ if (3.7), (3.8) have at least two $(u_0, \{h_x\}_{x\in L})$ and $(v_0, \{s_x\}_{x\in L})$ solutions such that the corresponding quantum d-Markov chains $\varphi_{u_0,\mathbf{h}}$ and $\varphi_{v_0,\mathbf{s}}$ are not quasi equivalent. Otherwise, we say there is no phase transition.*

Remark 3.4. *In the classical case, i.e. the interaction operators commute with each other and belong to commutative part of $\mathcal{B}_L$, the provided definition coincides with the known definition of the phase transition for models with nearest-neighbor interactions on the tree (see for example [13, 25, 38]).*

4 Quantum d-Markov chains associated with XY-model

In this section, we define the model and shall formulate the main results of the paper. In what follows we consider a semi-infinite Cayley tree $\Gamma^3_+ = (L, E)$ of order 3. Our starting C^*-algebra is the same $\mathcal{B}_L$ but with $\mathcal{B}_x = M_2(\mathbb{C})$ for $x \in L$. By $\sigma_x^{(u)}, \sigma_y^{(u)}, \sigma_z^{(u)}$ we denote the Pauli spin operators at site $u \in L$. Here

$$\sigma_x^{(u)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y^{(u)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z^{(u)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.1}$$

For every edge $< u, v >\in E$ put

$$K_{<u,v>} = \exp\{\beta H_{<u,v>}\}, \quad \beta > 0, \tag{4.2}$$

where

$$H_{<u,v>} = \frac{1}{2}\big(\sigma_x^{(u)}\sigma_x^{(v)} + \sigma_y^{(u)}\sigma_y^{(v)}\big). \tag{4.3}$$

Such kind of Hamiltonian is called *quantum XY-model* per edge $< x, y >$.

Now taking into account the following equalities

$$H^{2m}_{<u,v>} = H^2_{<u,v>} = \frac{1}{2}\big(\mathbb{1} - \sigma_z^{(u)}\sigma_z^{(v)}\big), \quad H^{2m-1}_{<u,v>} = H_{<u,v>}, \quad m \in \mathbb{N},$$

one finds

$$K_{<u,v>} = \mathbb{1} + \sinh\beta H_{<u,v>} + (\cosh\beta - 1)H^2_{<u,v>}. \tag{4.4}$$

The main results of the paper concern the existence of the phase transition for the model (4.2). Namely, we have

Theorem 4.1. *Let $\{K_{<x,y>}\}$ be given by (4.2) on the Cayley tree of order three. Then there are two positive numbers β_* and β^* such that* *(i) if* $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$*, then there is a unique forward quantum d-Markov chain associated with* (4.2)*;*

(ii) if $\beta \in (\beta_*, \beta^*)$*, then there is a phase transition for a given model, i.e. there are two distinct forward quantum d-Markov chains.*

To prove the theorem, we need first the existence of forward QMC for the model. To this end, we next sections investigate dynamical system associated with the equations (3.7),(3.8).

5 A dynamical system related to (3.7),(3.8)

In this section we shall reduce equations (3.7),(3.8) to some dynamical system. Our goal is to describe all solutions $\mathbf{h} = \{h_x\}$ and w_0 of those equations.

Furthermore, we shall assume that $h_x = h_y$ for every $x, y \in W_n$, $n \in \mathbb{N}$. Hence, we denote $h_x^{(n)} := h_x$, if $x \in W_n$. Now from (4.2),(4.3) one can see that $K_{<u,u>} = K^*_{<u,v>}$, therefore, equation (3.8) can be rewritten as follows

$$Tr_x(K_{<x,y>}K_{<x,z>}K_{<x,v>}h_y^{(n)}h_z^{(n)}h_v^{(n)}K_{<x,v>}K_{<x,z>}K_{<x,y>}) \quad = \quad h_x^{(n-1)}, \tag{5.1}$$

for every $x \in L$.

After small calculations equation (5.1) reduces to the following system

$$\begin{cases} \left(\dfrac{a_{11}^{(n)} + a_{22}^{(n)}}{2}\right)^3 \cosh^6\beta + a_{12}^{(n)}a_{21}^{(n)}\left(\dfrac{a_{11}^{(n)} + a_{22}^{(n)}}{2}\right)\sinh^2\beta\cosh^2\beta(1+2\cosh\beta) = a_{11}^{(n-1)} \\ a_{12}^{(n)}\left(\left(\dfrac{a_{11}^{(n)} + a_{22}^{(n)}}{2}\right)^2 \sinh\beta\cosh^2\beta(1+\cosh\beta+\cosh^2\beta) + a_{12}^{(n)}a_{21}^{(n)}\sinh^3\beta\cosh\beta\right) = a_{12}^{(n-1)} \\ a_{21}^{(n)}\left(\left(\dfrac{a_{11}^{(n)} + a_{22}^{(n)}}{2}\right)^2 \sinh\beta\cosh^2\beta(1+\cosh\beta+\cosh^2\beta) + a_{12}^{(n)}a_{21}^{(n)}\sinh^3\beta\cosh\beta\right) = a_{21}^{(n-1)} \\ \left(\dfrac{a_{11}^{(n)} + a_{22}^{(n)}}{2}\right)^3 \cosh^6\beta + a_{12}^{(n)}a_{21}^{(n)}\left(\dfrac{a_{11}^{(n)} + a_{22}^{(n)}}{2}\right)\sinh^2\beta\cosh^2\beta(1+2\cosh\beta) = a_{22}^{(n-1)} \end{cases} \tag{5.2}$$

here

$$h_x^{(n-1)} = \begin{pmatrix} a_{11}^{(n-1)} & a_{12}^{(n-1)} \\ a_{21}^{(n-1)} & a_{22}^{(n-1)} \end{pmatrix}, \qquad h_y^{(n)} = h_z^{(n)} = h_v^{(n)} = \begin{pmatrix} a_{11}^{(n)} & a_{12}^{(n)} \\ a_{21}^{(n)} & a_{22}^{(n)} \end{pmatrix}.$$

From (5.2) we immediately get that $a_{11}^{(n)} = a_{22}^{(n)}$ for all $n \in \mathbb{N}$.

Self-adjointness of $h_x^{(n)}$ (i.e. $\overline{a_{12}^{(n)}} = a_{21}^{(n)}$, for any $n \in \mathbb{N}$) allows us to reduce the system (5.2) to

$$\begin{cases} (a_{11}^{(n)})^3\cosh^6\beta + |a_{12}^{(n)}|^2 a_{11}^{(n)}\sinh^2\beta\cosh^2\beta(1+2\cosh\beta) = a_{11}^{(n-1)} \\ a_{12}^{(n)}\Big((a_{11}^{(n)})^2\sinh\beta\cosh^2\beta(1+\cosh\beta+\cosh^2\beta) + |a_{12}^{(n)}|^2\sinh^3\beta\cosh\beta\Big) = a_{12}^{(n-1)} \end{cases} \tag{5.3}$$

From the representation $a_{12}^{(n)}=|a_{12}^{(n)}|\exp(i\varphi_n)$ the last system (5.3) reduces to

$$\begin{cases} (a_{11}^{(n)})^3\cosh^6\beta + a_{11}^{(n)}|a_{12}^{(n)}|^2\sinh^2\beta\cosh^2\beta(1+2\cosh\beta) = a_{11}^{(n-1)} \\ |a_{12}^{(n)}|\Big((a_{11}^{(n)})^2\sinh\beta\cosh^2\beta(1+\cosh\beta+\cosh^2\beta)+|a_{12}^{(n)}|^2\sinh^3\beta\cosh\beta\Big) = |a_{12}^{(n-1)}| \\ \varphi_n = \varphi_{n-1} \end{cases} \tag{5.4}$$

From (5.4) it follows that $\varphi_n=\varphi_0$, whenever $n\in\mathbb{N}$. Therefore, we shall study the following system

$$\begin{cases} (a_{11}^{(n)})^3\cosh^6\beta + a_{11}^{(n)}|a_{12}^{(n)}|^2\sinh^2\beta\cosh^2\beta(1+2\cosh\beta) = a_{11}^{(n-1)} \\ |a_{12}^{(n)}|\Big((a_{11}^{(n)})^2\sinh\beta\cosh^2\beta(1+\cosh\beta+\cosh^2\beta)+|a_{12}^{(n)}|^2\sinh^3\beta\cosh\beta\Big) = |a_{12}^{(n-1)}| \end{cases} \tag{5.5}$$

Remark 5.1. *Note that according to the positivity of $h_x^{(n)}$ and $a_{11}^{(n)}=a_{22}^{(n)}$ we conclude that $a_{11}^{(n)}>|a_{12}^{(n)}|$ for all $n\in\mathbb{N}$.*

Now we are going to investigate the derived system (5.5). To do this, let us define a mapping $f:(x,y)\in\mathbb{R}_+^2\to(x',y')\in\mathbb{R}_+^2$ by

$$\begin{cases} (x')^3\cosh^6\beta + x'(y')^2\sinh^2\beta\cosh^2\beta(1+2\cosh\beta) = x \\ (x')^2y'\sinh\beta\cosh^2\beta(1+\cosh\beta+\cosh^2\beta)+(y')^3\sinh^3\beta\cosh\beta = y \end{cases} \tag{5.6}$$

Furthermore, due to Remark 5.1, we restrict the dynamical system (5.6) to the following domain

$$\Delta=\{(x,y)\in\mathbb{R}_+^2 : x>y\}.$$

Denote

$$A_1=\sinh^3\beta\cosh\beta,\quad B_1=\sinh\beta\cosh^2\beta(1+\cosh\beta+\cosh^2\beta), \tag{5.7}$$

$$A_2=\sinh^2\beta\cosh^2\beta(1+2\cosh\beta),\quad B_2=\cosh^6\beta, \tag{5.8}$$

$$P_9(t)=t^9-t^8-t^7-t^6+2t^4+2t^3-t-1, \tag{5.9}$$

$$E:=\frac{1}{\sinh^2\beta\cosh^2\beta(1+2\cosh\beta)+D\cosh^6\beta}, \tag{5.10}$$

$$D:=\frac{A_2-A_1}{B_1-B_2}. \tag{5.11}$$

Further, we will need the following auxiliary facts.

Lemma 5.2. *Let A_1,B_1,A_2,B_2,D be numbers defined by* (5.7), (5.8), (5.11) *and $P_9(t)$ be polynomial given by* (5.9), *where $\beta>0$. Then the following statements hold true:*

(i) The polynomial $P_9(t)$ has only three positive roots 1, t_*, and t^* such that $1.05<t_*<1.1$ and $1.5<t_*<1.6$. Moreover, if $t\in(1,t_*)\cup(t^*,\infty)$ then $P_9(t)>0$ and $t\in(t_*,t^*)$ then $P_9(t)<0$. Denote by $\beta_*=\cosh^{-1}t_*$ and $\beta^*=\cosh^{-1}t^*$;

(ii) For any $\beta\in(0,\infty)$ we have $A_1<A_2$;

(iii) *If* $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$ *then* $B_1 \leq B_2$ *and If* $\beta \in (\beta_*, \beta^*)$ *then* $B_1 > B_2$;

(iv) *For any* $\beta \in (0, \infty)$ *we have* $A_1 + B_1 < A_2 + B_2$;

(v) *If* $\beta \in (\beta_*, \beta^*)$ *then* $D > 1$ *and* $E > 0$;

(vi) *For any* $\beta \in (0, \infty)$ *we have* $A_1A_2 < B_1B_2$ *and* $A_1B_2 < A_2B_1$;

(vii) *If* $\beta \in (\beta_*, \beta^*)$ *then* $A_2B_1 < A_1A_2 + 3A_1B_2 + B_1B_2$ *and* $2A_1A_2 + 3A_1B_2 < A_2B_1$;

(viii) *For any* $\beta \in (0, \infty)$ *we have* $0 < \sinh\beta(1 + \cosh\beta) < \cosh^3\beta$.

The proof is provided in the Appendix.

6 Fixed points and asymptotical behavior of f. Existence of forward QMC

In this section we shall find fixed points of (5.6) and the absence of periodic points. Moreover, we investigate asymptotical behavior one. Note that every solution of (5.6) defines (see Theorem 3.1) forward QMC, therefore, the existence of fixed points of (5.6) implies the existence of forward quantum d-Markov chain.

First let us find all of the fixed points of the dynamical system.

Theorem 6.1. *Let* f *be a dynamical system given by* (5.6)*. Then the following assertions hold true:*

(i) *If* $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$ *then there is a unique fixed point* $\left(\frac{1}{\cosh^3\beta}, 0\right)$ *in the domain* Δ*;*

(ii) *If* $\beta \in (\beta_*, \beta^*)$ *then there are two fixed points in the domain* Δ*, which are* $\left(\frac{1}{\cosh^3\beta}, 0\right)$ *and* $(\sqrt{DE}, \sqrt{E})$.

Proof. Assume that (x, y) is a fixed point, i.e.

$$\begin{cases} x^3\cosh^6\beta + xy^2\sinh^2\beta\cosh^2\beta(1 + 2\cosh\beta) = x \\ x^2y\sinh\beta\cosh^2\beta(1 + \cosh\beta + \cosh^2\beta) + y^3\sinh^3\beta\cosh\beta = y \end{cases} \tag{6.1}$$

Consider two different cases with respect to y.

Case (a). Let $y = 0$. Then one finds that either $x = 0$ or $x = \frac{1}{\cosh^3\beta}$. But, only the point $(\frac{1}{\cosh^3\beta}, 0)$ belongs to the domain Δ.

Case (b). Now suppose $y > 0$. Since $x > y > 0$ one finds

$$\begin{cases} x^2\cosh^6\beta + y^2\sinh^2\beta\cosh^2\beta(1 + 2\cosh\beta) = 1 \\ x^2\sinh\beta\cosh^2\beta(1 + \cosh\beta + \cosh^2\beta) + y^2\sinh^3\beta\cosh\beta = 1, \end{cases}$$

hence, due to (5.7) and (5.8) we obtain

$$(B_1 - B_2)x^2 = (A_2 - A_1)y^2.$$

According to Lemma 5.2 $(ii), (iii), (v)$ we infer that if $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$ then $B_1 \leq B_2$, $A_1 < A_2$, and if $\beta \in (\beta_*, \beta^*)$ then $B_1 > B_2$, $A_1 < A_2$, and

$$\frac{x^2}{y^2} = \frac{A_2 - A_1}{B_1 - B_2} = D > 1.$$

Therefore, if $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$ then the dynamical system (5.6) has a unique fixed point $(\frac{1}{\cosh^3 \beta}, 0)$. If $\beta \in (\beta_*, \beta^*)$ then the dynamical system (5.6) has two fixed points $(\frac{1}{\cosh^3 \beta}, 0)$ and $(\sqrt{DE}, \sqrt{E})$. $\square$

To investigate an asymptotical behavior of the dynamical system on Δ we need some auxiliary facts.

Let $g_\beta : [0, 1] \to \mathbb{R}_+$ be a function given by

$$g_\beta(t) = \frac{A_1 t^3 + B_1 t}{A_2 t^2 + B_2}, \tag{6.2}$$

where $\beta \in (0, \infty)$.

Proposition 6.2. *Let $g_\beta : [0, 1] \to \mathbb{R}_+$ be the function given by* (6.2) *and $\beta \in (\beta_*, \beta^*)$. Then the following assertions hold true:*

(i) g_β is an increasing function on $[0, 1]$;

(ii) If $t \in [0, \frac{1}{\sqrt{D}}]$, then $g_\beta(t) \geq t$. If $t \in [\frac{1}{\sqrt{D}}, 1]$ then $g_\beta(t) \leq t$;

(iii) If $0 \leq g_\beta(t) \leq \frac{1}{\sqrt{D}}$ then $0 \leq t \leq \frac{1}{\sqrt{D}}$ and if $\frac{1}{\sqrt{D}} \leq g_\beta(t) \leq 1$ then $\frac{1}{\sqrt{D}} \leq t \leq 1$.

Proof. Let us prove (i). We know that

$$g'_\beta(t) = \frac{A_1 A_2 t^4 + (3A_1 B_2 - A_2 B_1)t^2 + B_1 B_2}{(A_2 t^2 + B_2)^2}.$$

Let us denote

$$\hat{g}(t) = A_1 A_2 t^4 + (3A_1 B_2 - A_2 B_1)t^2 + B_1 B_2.$$

It is enough to show that $\hat{g}(t) > 0$, for any $t \in [0, 1]$. To do so, we will show that $\min\limits_{t \in [0,1]} \hat{g}(t) > 0$. It follows from Lemma 5.2 (vii) that $\hat{g}(0) = B_1 B_2 > 0$ and $\hat{g}(1) > 0$. It is clear that

$$\hat{g}'(t) = 2t(2A_1 A_2 t^2 - (A_2 B_1 - 3A_1 B_2))$$

Since $A_2 B_1 - 3A_1 B_2 - 2A_1 A_2 > 0$ (see Lemma 5.2 (vii)) one has

$$t^2 = \frac{A_2 B_1 - 3A_1 B_2}{2A_1 A_2} > 1.$$

So, $\min\limits_{t \in [0,1]} \hat{g}(t) > 0$, and hence $\hat{g}(t) > 0$ for any $t \in [0, 1]$. Therefore, $g'_\beta(t) > 0$ for any $t \in [0, 1]$, and this proves the assertion.

(ii). One can see that

$$g_\beta(t) - t = -\frac{(A_2 - A_1)t(t^2 - \frac{1}{D})}{A_2 t^2 + B_2} \tag{6.3}$$

Therefore, we find that if $t \in [0, \frac{1}{\sqrt{D}}]$ then $g_\beta(t) \geq t$, and if $t \in [\frac{1}{\sqrt{D}}, 1]$ then $g_\beta(t) \leq t$.

(iii). It follows from (6.3) that the function $g_\beta(t)$ has two fixed points $t = 0$ and $t = \frac{1}{\sqrt{D}}$. Let $0 \leq g_\beta(t) \leq \frac{1}{\sqrt{D}}$, and suppose that $t > \frac{1}{\sqrt{D}}$. Due to (i) and $t = \frac{1}{\sqrt{D}}$ is fixed point, we obtain $g_\beta(t) > \frac{1}{\sqrt{D}}$, which is impossible. Similarly, one can show that $\frac{1}{\sqrt{D}} \leq g_\beta(t) \leq 1$ implies $\frac{1}{\sqrt{D}} \leq t \leq 1$. □

Let us start to study the asymptotical behavior of the dynamical system $f : \Delta \to \mathbb{R}_+$ given by (5.6)

Theorem 6.3. *The dynamical system $f : \Delta \to \mathbb{R}^2_+$, given by (5.6) with $\beta \in (0, \infty)$, does not have any k $(k \geq 2)$ periodic points in Δ.*

Proof. Assume that the dynamical system f has a periodic point $(x^{(0)}, y^{(0)})$ with a period of k in Δ, where $k \geq 2$. This means that there are points

$$(x^{(0)}, y^{(0)}), (x^{(1)}, y^{(1)}), \dots, (x^{(k-1)}, y^{(k-1)}) \in \Delta,$$

such that they satisfy the following equalities

$$\begin{cases} (x^{(i)})^3 \cosh^6\beta + x^{(i)}(y^{(i)})^2 \sinh^2\beta \cosh^2\beta(1 + 2\cosh\beta) = x^{(i-1)} \\ (x^{(i)})^2 y^{(i)} \sinh\beta \cosh^2\beta(1 + \cosh\beta + \cosh^2\beta) + (y^{(i)})^3 \sinh^3\beta \cosh\beta = y^{(i-1)} \end{cases} \tag{6.4}$$

where $i = \overline{1,k}$, i.e. $f\left(x^{(i-1)}, y^{(i-1)}\right) = \left(x^{(i)}, y^{(i)}\right)$, with $x^{(k)} = x^{(0)}$, $y^{(k)} = y^{(0)}$.

Now again consider two different cases with respect to $y^{(0)}$.

Case (a). Let $y^{(0)} > 0$. Then $x^{(i)}, y^{(i)}$ should be positive for all $i = \overline{1,k}$. Let us look for different cases with respect to β.

Assume that $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$. We then have

$$\frac{x^{(i-1)}}{y^{(i-1)}} = \frac{B_2}{B_1} \cdot \frac{x^{(i)}}{y^{(i)}} + \frac{(A_2 B_1 - A_1 B_2) x^{(i)} y^{(i)}}{B_1\left(B_1 (x^{(i)})^2 + A_1 (y^{(i)})^2\right)}$$

where $i = \overline{1,k}$.

Due to Lemma 5.2 (vi) and $x^{(i)}, y^{(i)} > 0$ for all $i = \overline{1,k}$, one finds

$$\frac{x^{(i-1)}}{y^{(i-1)}} > \frac{B_2}{B_1} \cdot \frac{x^{(i)}}{y^{(i)}}, \tag{6.5}$$

for all $i = \overline{1,k}$.

Iterating (6.5) we get

$$\frac{x^{(0)}}{y^{(0)}} > \left(\frac{B_2}{B_1}\right)^k \cdot \frac{x^{(0)}}{y^{(0)}}.$$

But, the last inequality is impossible, since Lemma 5.2 (iii) implies

$$\frac{B_2}{B_1} \geq 1.$$

Hence, in this case, the dynamical system (5.6) does not have any periodic points with $k \geq 2$.

Let $\beta \in (\beta_*, \beta^*)$, then one finds

$$\frac{y^{(i-1)}}{x^{(i-1)}} = g_\beta\left(\frac{y^{(i)}}{x^{(i)}}\right), \quad i = \overline{1,k}.$$

This means that $\frac{y^{(0)}}{x^{(0)}}$ is a k periodic point for the function $g_\beta(t)$. But this is a contradiction, since according to Proposition 6.2 (i) the function $g_\beta(t)$ is increasing, and it does not have any periodic point on the segment $[0,1]$.

Case (b). Now suppose that $y^{(0)} = 0$. Since $k \geq 2$ we have $x^{(0)} \neq \frac{1}{\cosh^3\beta}$. So, from (6.4) one finds that $y^{(i)} = 0$ for all $i = \overline{1,k}$. Then again (6.4) implies that

$$(x^{(i)})^3 \cosh^6\beta = x^{(i-1)}, \quad \forall i = \overline{1,k},$$

which means

$$x^{(i)} = \frac{1}{\cosh^2\beta}\sqrt[3]{x^{(i-1)}}, \quad \forall i = \overline{1,k}.$$

Hence, we have

$$x^{(0)} = \frac{1}{\cosh^3\beta}\sqrt[3^{k+1}]{x^{(0)}\cosh^3\beta}.$$

This yields either $x^{(0)} = 0$ or $x^{(0)} = \frac{1}{\cosh^3\beta}$, which is a contradiction. □

Theorem 6.4. *Let $f : \Delta \to \mathbb{R}^2_+$ be the dynamical system given by* (5.6) *and* $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$. *Then the following assertions hold true:*

(i) if $y^{(0)} > 0$ then the trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^\infty$ of f starting from the point $(x^{(0)}, y^{(0)})$ is finite.

(ii) if $y^{(0)} = 0$ then the trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^\infty$ starting from the point $(x^{(0)}, y^{(0)})$ has the following form

$$\begin{cases} x^{(n)} = \dfrac{\sqrt[3^n]{x^{(0)}\cosh^3\beta}}{\cosh^3\beta} \\ y^{(n)} = 0, \end{cases}$$

and it converges to the fixed point $(\frac{1}{\cosh^3\beta}, 0)$.

Proof. (i) Let $y^{(0)} > 0$ and suppose that the trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^\infty$ of the dynamical system starting from the point $(x^{(0)}, y^{(0)})$ is infinite. This means that the points $(x^{(n)}, y^{(n)})$ are well defined and belong to the domain Δ for all $n \in \mathbb{N}$. Since $y^{(0)} > 0$ we have $y^{(n)} > 0$ for all $n \in \mathbb{N}$. Then, it follows from (5.6) that

$$\frac{x^{(n-1)}}{y^{(n-1)}} = \frac{B_2}{B_1} \cdot \frac{x^{(n)}}{y^{(n)}} + \frac{(A_2B_1 - A_1B_2)\dfrac{x^{(n)}}{y^{(n)}}}{B_1^2\left(\dfrac{x^{(n)}}{y^{(n)}}\right)^2 + A_1B_1} \quad \text{for all} \quad n \in \mathbb{N}. \tag{6.6}$$

It yields that

$$\frac{x^{(n-1)}}{y^{(n-1)}} > \frac{B_2}{B_1} \cdot \frac{x^{(n)}}{y^{(n)}},$$

and

$$\frac{x^{(0)}}{y^{(0)}} > \left(\frac{B_2}{B_1}\right)^n \cdot \frac{x^{(n)}}{y^{(n)}}, \quad \text{for all} \quad n \in \mathbb{N}. \tag{6.7}$$

It follows from (6.7) and Lemma 5.2 (*iii*) that

$$\frac{x^{(n)}}{y^{(n)}} < \left(\frac{B_1}{B_2}\right)^n \cdot \frac{x^{(0)}}{y^{(0)}} \leq \frac{x^{(0)}}{y^{(0)}}, \tag{6.8}$$

for all $n \in \mathbb{N}$. Using (6.6) and (6.8) one gets

$$\frac{x^{(n-1)}}{y^{(n-1)}} > \left(\frac{B_2}{B_1} + \frac{(A_2B_1 - A_1B_2)(y^{(0)})^2}{B_1^2(x^{(0)})^2 + A_1B_1(y^{(0)})^2}\right) \cdot \frac{x^{(n)}}{y^{(n)}},$$

and

$$\frac{x^{(n)}}{y^{(n)}} < \left(\frac{B_2}{B_1} + \frac{(A_2B_1 - A_1B_2)(y^{(0)})^2}{B_1^2(x^{(0)})^2 + A_1B_1(y^{(0)})^2}\right)^{-n} \cdot \frac{x^{(0)}}{y^{(0)}}.$$

We know that if $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$ then due to Lemma 5.2 (*iii*) one finds

$$\frac{B_2}{B_1} + \frac{(A_2B_1 - A_1B_2)(y^{(0)})^2}{B_1^2(x^{(0)})^2 + A_1B_1(y^{(0)})^2} > \frac{B_2}{B_1} \geq 1.$$

Therefore, we conclude that, for all $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$

$$\frac{x^{(n)}}{y^{(n)}} \to 0$$

as $n \to \infty$.

On the other hand, due to $(x^{(n)}, y^{(n)}) \in \Delta$, we have

$$\frac{x^{(n)}}{y^{(n)}} \geq 1,$$

for all $n \in \mathbb{N}$. This contradiction shows that the trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^{\infty}$ must be finite.

(ii) Now let $y^{(0)} = 0$, then (5.6) implies $y^{(n)} = 0$ for all $n \in \mathbb{N}$. Hence, from (5.6) one finds

$$x^{(n)} \cosh^3 \beta = \sqrt[3]{x^{(n-1)} \cosh^3 \beta}.$$

So iterating the last equality we obtain

$$x^{(n)} \cosh^3 \beta = \sqrt[3^n]{x^{(0)} \cosh^3 \beta},$$

which yields the desired equality and the trajectory $\{(x^{(n)}, 0)\}_{n=0}^{\infty}$ converges to the fixed point $(\frac{1}{\cosh^3 \beta}, 0)$. $\square$

Theorem 6.5. *Let $f : \Delta \to \mathbb{R}_+^2$ be the dynamical system given by* (5.6) *and $\beta \in (\beta_*, \beta^*)$. Then the following assertions hold true:*

(i) There are two invariant lines $l_1 = \{(x, y) \in \Delta : y = 0\}$ and $l_2 = \{(x, y) \in \Delta : y = \frac{x}{\sqrt{D}}\}$ w.r.t. f;

(ii) if an initial point $(x^{(0)}, y^{(0)})$ belongs to the invariant lines l_k, then its trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^{\infty}$ converges to the fixed point belonging to the line l_k, where $k = \overline{1, 2}$;

(iii) if an initial point $(x^{(0)}, y^{(0)})$ satisfies the following condition

$$\frac{y^{(0)}}{x^{(0)}} \in \left(0, \frac{1}{\sqrt{D}}\right),$$

then its trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^{\infty}$ converges to the fixed point $(\frac{1}{\cosh^3 \beta}, 0)$ which belongs to l_1;

(iv) if an initial point $(x^{(0)}, y^{(0)})$ satisfies the following condition

$$\frac{y^{(0)}}{x^{(0)}} \in \left(\frac{1}{\sqrt{D}}, 1\right),$$

then its trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^{\infty}$ is finite.

Proof. (i). It follows from (5.6) that if $y = 0$ then $y' = 0$, which means l_1 is an invariant line. Let $\frac{y}{x} = \frac{1}{\sqrt{D}}$. Again from (5.6) it follows that $\frac{1}{\sqrt{D}} = \frac{y}{x} = g_\beta(\frac{y'}{x'})$. Since $g_\beta(t)$ is the increasing function on segment $[0, 1]$ and $t = \frac{1}{\sqrt{D}}$ is its fixed point, we then get $\frac{y'}{x'} = \frac{1}{\sqrt{D}}$, which yields that l_2 is an invariant line for f.

(ii). Let us consider a case when an initial point $(x^{(0)}, y^{(0)})$ belongs to l_k. Let (x_k, y_k) be the fixed point of f belonging to l_k ($k = \overline{1, 2}$). It follows from (5.6) that

$$\frac{y_k}{x_k} = \frac{y^{(0)}}{x^{(0)}} = g_\beta^{(n)}\left(\frac{y^{(n)}}{x^{(n)}}\right). \tag{6.9}$$

for all $n \in \mathbb{N}$. Since $g_\beta(t)$ is increasing and $t = \frac{y_k}{x_k}$ is its fixed point, we have

$$\frac{y_k}{x_k} = \frac{y^{(n)}}{x^{(n)}}, \tag{6.10}$$

for all $n \in \mathbb{N}$. We know that $\frac{y_1}{x_1} = 0$ and $\frac{y_2}{x_2} = \frac{1}{\sqrt{D}}$.

In the case when $\frac{y_1}{x_1} = 0$, one gets

$$\left(x^{(n)}, y^{(n)}\right) = \left(\frac{\sqrt[3^n]{x^{(0)} \cosh^3 \beta}}{\cosh^3 \beta}, 0\right),$$

hence the trajectory converges to the fixed point $(x_1, y_1) = (\frac{1}{\cosh^3 \beta}, 0)$. Clearly, it belongs to l_1.

In the case when $\frac{y_2}{x_2} = \frac{1}{\sqrt{D}}$, we have

$$\left(x^{(n)}, y^{(n)}\right) = \left(\sqrt{DE}\sqrt[3^n]{\frac{x^{(0)}}{\sqrt{DE}}}, \sqrt{E}\sqrt[3^n]{\frac{y^{(0)}}{\sqrt{E}}}\right),$$

and the trajectory converges to the fixed point $(x_2, y_2) = (\sqrt{DE}, \sqrt{E})$ which belongs to the line l_2.

(iii). Assume that an initial point $(x^{(0)}, y^{(0)})$ satisfies

$$\frac{y^{(0)}}{x^{(0)}} \in \left(0, \frac{1}{\sqrt{D}}\right). \tag{6.11}$$

It then follows from (5.6) that

$$\frac{y^{(n-1)}}{x^{(n-1)}} = g_\beta\left(\frac{y^{(n)}}{x^{(n)}}\right),$$

for all $n \in \mathbb{N}$. Since (6.11) and due to Proposition (6.2) (ii), we conclude that

$$\frac{y^{(n)}}{x^{(n)}} \in \left(0, \frac{1}{\sqrt{D}}\right),$$

for all $n \in \mathbb{N}$. According to Proposition 6.2(iii) we get

$$\frac{y^{(0)}}{x^{(0)}} > \frac{y^{(1)}}{x^{(1)}} > \cdots > \frac{y^{(n)}}{x^{(n)}} > \cdots,$$

and the sequence

$$c_n := \frac{y^{(n)}}{x^{(n)}}$$

converges to 0.

Let us denote

$$b_n := \frac{1}{B_2 + c_n A_2}.$$

From (5.6), one can easily get

$$x^{(n)} = \sqrt[3]{b_n \sqrt[3]{b_{n-1} \sqrt[3]{\cdots \sqrt[3]{b_1 x^{(0)}}}}}$$

and

$$\lim_{n\to\infty} x^{(n)} = \lim_{n\to\infty} b_n = \frac{1}{\sqrt{B_2}} = \frac{1}{\cosh^3 \beta}.$$

Therefore, the trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^{\infty}$ converges to the fixed point $(\frac{1}{\cosh^3\beta}, 0)$ which belongs to l_1.

(iv) Now assume that

$$\frac{y^{(0)}}{x^{(0)}} \in \left(\frac{1}{\sqrt{D}}, 1\right). \tag{6.12}$$

We suppose that the trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^{\infty}$ is infinite. This means that the points $(x^{(n)}, y^{(n)})$ are well defined and belong to the domain Δ for all $n \in \mathbb{N}$. Then, it follows from (5.6) that

$$\frac{y^{(n-1)}}{x^{(n-1)}} = g_\beta\left(\frac{y^{(n)}}{x^{(n)}}\right),$$

for all $n \in \mathbb{N}$. Since (6.12) and due to Proposition (6.2) (ii), we conclude that

$$\frac{y^{(n)}}{x^{(n)}} \in \left(\frac{1}{\sqrt{D}}, 1\right),$$

for all $n \in \mathbb{N}$. According to Proposition 6.2(iii) one finds

$$\frac{y^{(0)}}{x^{(0)}} < \frac{y^{(1)}}{x^{(1)}} < \cdots < \frac{y^{(n)}}{x^{(n)}} < \cdots.$$

Since $(x^{(n)}, y^{(n)}) \in \Delta$ and the sequence $\dfrac{y^{(n)}}{x^{(n)}}$ is bounded, so it converges to some point $\tilde{t} \in (\frac{1}{\sqrt{D}}, 1]$. We know that the point $\tilde{t}$ should be a fixed point of $g_\beta(t)$ on $(\frac{1}{\sqrt{D}}, 1]$. However, the function $g_\beta(t)$ does not have any fixed points on $(\frac{1}{\sqrt{D}}, 1]$. Hence, this contradiction shows that the trajectory $\{(x^{(n)}, y^{(n)})\}_{n=0}^{\infty}$ must be finite. □

7 Uniqueness of QMC

In this section we prove the first part of the main theorem (see Theorem 4.1), i.e. we show the uniqueness of the forward quantum d-Markov chain in the regime $\beta \in (0, \beta_*) \cup [\beta^*, \infty)$.

So, assume that $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$. From Theorem 6.4, we infer that equations (3.7),(3.8) have a lot of parametrical solutions $(w_0(\alpha), \{h_x(\alpha)\})$ given by

$$w_0(\alpha) = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}, \quad h_x^{(n)}(\alpha) = \begin{pmatrix} \frac{\sqrt[3^n]{\alpha\cosh^3\beta}}{\cosh^3\beta} & 0 \\ 0 & \frac{\sqrt[3^n]{\alpha\cosh^3\beta}}{\cosh^3\beta} \end{pmatrix}, \tag{7.1}$$

for every $x \in V$, here α is any positive real number.

The boundary conditions corresponding to the fixed point of (5.6) are the following ones:

$$w_0 = \begin{pmatrix} \cosh^3\beta & 0 \\ 0 & \cosh^3\beta \end{pmatrix}, \quad h_x^{(n)} = \begin{pmatrix} \frac{1}{\cosh^3\beta} & 0 \\ 0 & \frac{1}{\cosh^3\beta} \end{pmatrix}, \quad \forall x \in V, \tag{7.2}$$

which correspond to the value of $\alpha_0 = \dfrac{1}{\cosh^3\beta}$ in (7.1). Therefore, in the sequel we denote such operators by $w_0(\alpha_0)$ and $h_x^{(n)}(\alpha_0)$, respectively.

Let us consider the states $\varphi^{(n,b)}_{w_0(\alpha),\mathbf{h}(\alpha)}$ corresponding to the solutions $(w_0(\alpha), \{h_x^{(n)}(\alpha)\})$. By definition we have

$$\begin{aligned}
\varphi^{(n,f)}_{w_0(\alpha),\mathbf{h}(\alpha)}(x) &= \mathrm{Tr}\left(w_0^{1/2}(\alpha)\prod_{i=0}^{n-1}K_{[i,i+1]}\prod_{x\in\overrightarrow{W}_n}h_x^{(n)}(\alpha)\prod_{i=1}^{n}K_{[n-i,n+1-i]}w_0^{1/2}(\alpha)x\right)\\
&= \frac{\left(\sqrt[2^{n+1}]{\alpha\cosh^4\beta}\right)^{2^{n+1}}}{\alpha(\cosh^4\beta)^{2^{n+1}}}\mathrm{Tr}\left(\prod_{i=0}^{n-1}K_{[i,i+1]}\prod_{i=1}^{n}K_{[n-i,n+1-i]}x\right)\\
&= \frac{\alpha_0^{2^{n+1}}}{\alpha_0}\mathrm{Tr}\left(\prod_{i=0}^{n-1}K_{[i,i+1]}\prod_{i=1}^{n}K_{[n-i,n+1-i]}x\right)\\
&= \mathrm{Tr}\left((w_0^{1/2}(\alpha_0)\prod_{i=0}^{n-1}K_{[i,i+1]}\prod_{x\in\overrightarrow{W}_n}h_x^{(n)}(\alpha_0)\prod_{i=1}^{n}K_{[n-i,n+1-i]}w_0^{1/2}(\alpha_0)x\right)\\
&= \varphi^{(n,f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}(x),
\end{aligned}$$

for any α. Hence, from the definition of quantum $d-$Markov chain we find that $\varphi^{(f)}_{w_0(\alpha),\mathbf{h}(\alpha)} = \varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}$, which yields that the uniqueness of forward quantum d-Markov chain associated with the model (4.2).

Hence, Theorem 4.1 (i) is proved.

8 Existence of phase transition

This section is devoted to the proof of part (ii) of Theorem 4.1. We shall prove the existence of the phase transition in the regime $\beta \in (\beta_*, \beta^*)$.

In this section, for the sake of simplicity of formulas, we use the following notations for the Pauli matrices:

$$\sigma_0 := \mathbb{1}, \quad \sigma_1 := \sigma_x, \quad \sigma_2 := \sigma_y, \quad \sigma_3 := \sigma_z$$

According to Theorem 6.1 in the considered regime there are two fixed points of the dynamical system (5.6). Then the corresponding solutions of equations (3.7),(3.8) can be written as follows: $(w_0(\alpha_0), \{h_x(\alpha_0)\})$ and $(w_0(\gamma), \{h_x(\gamma)\})$, where

$$w_0(\alpha_0) = \frac{1}{\alpha_0}\sigma_0, \quad h_x(\alpha_0) = \alpha_0\sigma_0^{(x)}$$

$$w_0(\gamma) = \frac{1}{\gamma_0}\sigma_0, \quad h_x(\gamma) = \gamma_0\sigma_0^{(x)} + \gamma_1\sigma_1^{(x)}$$

here $\alpha_0 = \frac{1}{\cosh^3\beta}$, $\gamma = (\gamma_0, \gamma_1)$ with $\gamma_0 = \sqrt{DE}$, $\gamma_1 = \sqrt{E}$.

By $\varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\gamma)}$, $\varphi^{(f)}_{w_0(\gamma),\mathbf{h}(\gamma)}$ we denote the corresponding forward quantum $d-$Markov chains. To prove the existence of the phase transition, we need to show that these two states are not quasi-equivalent. To do so, we will need some auxiliary facts and results.

Denote

$$A = \begin{pmatrix} \cosh^6\beta\gamma_0^2 + \sinh^2\beta\cosh^3\beta\gamma_1^2 & \gamma_0\gamma_1\sinh^2\beta\cosh^2\beta(1+\cosh\beta) \\ \gamma_0\gamma_1\sinh\beta\cosh^2\beta(1+\cosh\beta) & \sinh\beta\cosh^4\beta\gamma_0^2 + \sinh^3\beta\cosh\beta\gamma_1^2 \end{pmatrix}. \tag{8.1}$$

Let us study some properties of the matrix A. One can easily check out that the matrix A given by (8.1) can be written as follows

$$A = \begin{pmatrix} \dfrac{\cosh\beta(\sinh\beta+\cosh^3\beta)}{\sinh\beta(1+\cosh\beta)^2} & \dfrac{\sqrt{(A_2-A_1)(B_1-B_2)}}{\sinh\beta\cosh^2\beta(1+\cosh\beta)^2} \\ \dfrac{\sqrt{(A_2-A_1)(B_1-B_2)}}{\sinh^2\beta\cosh^2\beta(1+\cosh\beta)^2} & \dfrac{\sinh\beta+\cosh^3\beta}{\cosh\beta(1+\cosh\beta)^2} \end{pmatrix}. \tag{8.2}$$

Proposition 8.1. *If $\beta \in (\beta_*, \beta_*)$ then the following inequalities hold true*

(i) $0 < \dfrac{\cosh\beta(\sinh\beta+\cosh^3\beta)}{\sinh\beta(1+\cosh\beta)^2} < 1;$

(ii) $0 < \dfrac{\sinh\beta+\cosh^3\beta}{\cosh\beta(1+\cosh\beta)^2} < 1;$

(iii) $1 < \mathrm{Tr}(A) < 2;$

(iv) $0 < \det(A) < 1.$

Proof. (i). Since $B_2 < B_1$ (see Lemma 5.2 (*iii*)) one can see that

$$0 < \frac{\cosh\beta(\sinh\beta+\cosh^3\beta)}{\sinh\beta(1+\cosh\beta)^2} = \frac{\dfrac{B_2}{\cosh^2\beta} + \cosh\beta\sinh\beta}{\dfrac{B_1}{\cosh^2\beta} + \cosh\beta\sinh\beta} < 1.$$

(ii). The inequality $\sinh\beta < \cosh\beta$ implies that

$$0 < \frac{\sinh\beta+\cosh^3\beta}{\cosh\beta(1+\cosh\beta)^2} = \frac{\sinh\beta+\cosh^3\beta}{\cosh\beta+2\cosh^2\beta+\cosh^3\beta} < 1.$$

(iii). One can see that

$$\mathrm{Tr}(A) = \frac{(\sinh\beta+\cosh^2\beta)(\sinh\beta+\cosh^3\beta)}{\sinh\beta\cosh\beta(1+\cosh\beta)^2}. \tag{8.3}$$

Therefore, from (i), (ii) it immediately follows that $0 < \mathrm{Tr}(A) < 2$. Now we are going to show that $\mathrm{Tr}(A) > 1$. Indeed, since $\cosh^3\beta > \sinh\beta(1+\cosh\beta) > 0$ (see Lemma 5.2 (*viii*)) and $\cosh\beta > 1$ one has

$$\sinh^2\beta + \cosh^5\beta > \sinh\beta\cosh\beta(1+\cosh\beta) \tag{8.4}$$

Then, due to (8.4) we find

$$\mathrm{Tr}(A) = \frac{\sinh^2\beta+\cosh^5\beta+\sinh\beta\cosh^2\beta(1+\cosh\beta)}{\sinh\beta\cosh\beta(1+\cosh\beta)+\sinh\beta\cosh^2\beta(1+\cosh\beta)} > 1.$$

(iv). Let us evaluate the determinant $\det(A)$ of the matrix A given by (8.2). After some algebraic manipulations, one finds

$$\det(A) = \frac{\sinh^2\beta + \cosh^5\beta - \sinh\beta\cosh\beta(1+\cosh\beta)}{\sinh\beta\cosh\beta(1+\cosh\beta)^2}. \tag{8.5}$$

Due to (8.4) one can see that $\det(A) > 0$. We want to show that $\det(A) < 1$. Since $B_2 < B_1$ (see Lemma 5.2 (iii)) and $\sinh\beta < \cosh\beta$ we have

$$\cosh^5\beta < \sinh\beta\cosh\beta(1+\cosh\beta+\cosh^2\beta), \tag{8.6}$$

$$\sinh^2\beta < \sinh\beta\cosh\beta(1+2\cosh\beta). \tag{8.7}$$

From inequalities (8.6),(8.7), one gets

$$\sinh^2\beta + \cosh^5\beta < \sinh\beta\cosh\beta(2+3\cosh\beta+\cosh^2\beta). \tag{8.8}$$

Therefore, we obatain

$$\det(A) = \frac{\sinh^2\beta + \cosh^5\beta - \sinh\beta\cosh\beta(1+\cosh\beta)}{\sinh\beta\cosh\beta(2+3\cosh\beta+\cosh^2\beta) - \sinh\beta\cosh\beta(1+\cosh\beta)} < 1.$$

This completes the proof. $\square$

The next proposition deals with eigenvalues of the matrix A.

Proposition 8.2. *Let A be the matrix given by (8.2). Then the following assertions hold true:*

(i) the numbers $\lambda_1 = 1$, $\lambda_2 = \det(A)$ are eigenvalues of the matrix A;

(ii) the vectors

$$(x_1, y_1) = \left(\frac{\sqrt{(A_2-A_1)(B_1-B_2)}}{\sinh\beta\cosh^2\beta(1+\cosh\beta)^2}, \frac{B_1-B_2}{\sinh\beta\cosh^2\beta(1+\cosh\beta)^2}\right), \tag{8.9}$$

$$(x_2, y_2) = \left(\frac{B_2-B_1}{\sinh\beta\cosh^2\beta(1+\cosh\beta)^2}, \frac{\sqrt{(A_2-A_1)(B_1-B_2)}}{\sinh^2\beta\cosh^2\beta(1+\cosh\beta)^2}\right) \tag{8.10}$$

are eigenvectors of the matrix A corresponding to the eigenvalues $\lambda_1 = 1$ and $\lambda_2 = \det(A)$, respectively;

(iii) if $P = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$, where the vectors (x_1, y_1) and (x_2, y_2) are defined by (8.9), (8.10) then

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}; \tag{8.11}$$

(iv) for any $n \in \mathbb{N}$ one has

$$A^n = \begin{pmatrix} \dfrac{x_1^2 + \lambda_2^n y_1^2 \sinh\beta}{x_1^2 + y_1^2\sinh\beta} & \dfrac{x_1 y_1 \sinh\beta(1-\lambda_2^n)}{x_1^2 + y_1^2\sinh\beta} \\ \dfrac{x_1 y_1(1-\lambda_2^n)}{x_1^2 + y_1^2\sinh\beta} & \dfrac{\lambda_2^n x_1^2 + y_1^2\sinh\beta}{x_1^2 + y_1^2\sinh\beta} \end{pmatrix}. \tag{8.12}$$

Proof. (*i*) We know that the following equation

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

is a characteristic equation of the matrix A given by (8.2). From (8.3) and (8.5) one can easily see that

$$\mathrm{Tr}(A) - \det(A) = \frac{\sinh\beta\cosh^2\beta(1+\cosh\beta) + \sinh\beta\cosh\beta(1+\cosh\beta)}{\sinh\beta\cosh\beta(1+\cosh\beta)^2} = 1,$$

this means that $\lambda_1 = 1$ and $\lambda_2 = \det(A)$ are eigenvalues of the matrix A.

(*ii*) The eigenvector (x_1, y_1) of the matrix A, corresponding to $\lambda_1 = 1$ satisfies the following equation

$$\left(\frac{\cosh\beta(\sinh\beta + \cosh^3\beta)}{\sinh\beta(1+\cosh\beta)^2} - \lambda_1\right)x_1 + \frac{\sqrt{(A_2-A_1)(B_1-B_2)}}{\sinh\beta\cosh^2\beta(1+\cosh\beta)^2}y_1 = 0.$$

Then, one finds

$$\begin{cases} x_1 = \dfrac{\sqrt{(A_2-A_1)(B_1-B_2)}}{\sinh\beta\cosh^2\beta(1+\cosh\beta)^2} \\ y_1 = \lambda_1 - \dfrac{\cosh\beta(\sinh\beta+\cosh^3\beta)}{\sinh\beta(1+\cosh\beta)^2} = \dfrac{B_1 - B_2}{\sinh\beta\cosh^2\beta(1+\cosh\beta)^2}. \end{cases}$$

Analogously, one can show that the eigenvector (x_2, y_2) of the matrix A, corresponding to $\lambda_2 = \det(A)$, is equal to

$$\begin{cases} x_2 = \lambda_2 - \dfrac{\sinh\beta + \cosh^3\beta}{\cosh\beta(1+\cosh\beta)^2} = \dfrac{B_2 - B_1}{\sinh\beta\cosh^2\beta(1+\cosh\beta)^2} \\ y_2 = \dfrac{\sqrt{(A_2-A_1)(B_1-B_2)}}{\sinh^2\beta\cosh^2\beta(1+\cosh\beta)^2}. \end{cases}$$

It is worth noting that $(x_2, y_2) = \left(-y_1, \dfrac{x_1}{\sinh\beta}\right)$.

(*iii*) Let

$$P = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix},$$

where the vectors (x_1, y_1) and (x_2, y_2) are defined by (8.9), (8.10). We then get

$$P^{-1}AP = \frac{1}{\det(P)}\begin{pmatrix} y_2 & -x_2 \\ -y_1 & x_1 \end{pmatrix}\begin{pmatrix} \lambda_1 x_1 & \lambda_2 x_2 \\ \lambda_1 y_1 & \lambda_2 y_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $\det(P) = \dfrac{x_1^2}{\sinh\beta} + y_1^2 > 0$.

(*iv*) From (8.11) it follows that

$$A = P\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}P^{-1}.$$

Therefore, for any $n \in \mathbb{N}$ we obtain

$$\begin{aligned} A^n &= P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1} = \frac{1}{\det(P)} \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} \begin{pmatrix} y_2\lambda_1^n & -x_2\lambda_1^n \\ -y_1\lambda_2^n & x_1\lambda_2^n \end{pmatrix} \\ &= \frac{1}{\det(P)} \begin{pmatrix} x_1y_2\lambda_1^n - x_2y_1\lambda_2^n & x_1x_2(\lambda_2^n - \lambda_1^n) \\ y_1y_2(\lambda_1^n - \lambda_2^n) & x_1y_2\lambda_2^n - x_2y_1\lambda_1^n \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{x_1^2 + \lambda_2^n y_1^2 \sinh\beta}{x_1^2 + y_1^2\sinh\beta} & \dfrac{x_1y_1\sinh\beta(1-\lambda_2^n)}{x_1^2 + y_1^2\sinh\beta} \\ \dfrac{x_1y_1(1-\lambda_2^n)}{x_1^2 + y_1^2\sinh\beta} & \dfrac{\lambda_2^n x_1^2 + y_1^2\sinh\beta}{x_1^2 + y_1^2\sinh\beta} \end{pmatrix}. \end{aligned}$$

This completes the proof. $\square$

In what follows, for the sake of simplicity, let us denote

$$K_0 := \frac{1+\cosh\beta}{2}, \quad K_1 := \frac{\sinh\beta}{2}, \quad K_2 := \frac{\sinh\beta}{2}, \quad K_3 := \frac{1+\cosh\beta}{2} \tag{8.13}$$

In these notations, the operator $K_{<u,v>}$ given by (4.4) can be written as follows

$$K_{<u,v>} = \sum_{i=0}^{3} K_i \sigma_i^{(u)} \otimes \sigma_i^{(v)}. \tag{8.14}$$

Remark 8.3. *In the sequel, we will frequently use the following identities for the numbers K_i, $i = \overline{0,3}$ given by* (8.13)*:*

(i) $K_0^2 + K_1^2 + K_2^2 + K_3^2 = \cosh^2\beta$;

(ii) $2(K_0K_1 - K_2K_3) = \sinh\beta\cosh\beta$;

(iii) $2(K_0K_1 + K_2K_3) = \sinh\beta$;

(iv) $K_0^2 + K_1^2 - K_2^2 - K_3^2 = \cosh\beta$.

Proposition 8.4. *Let $K_{<u,v>}$ be given by* (8.14)*, $\overrightarrow{S(x)} = (1,2,3)$, and $\mathbf{h}^{(i)} = h_0^{(i)}\sigma_0^{(i)} + h_1^{(i)}\sigma_1^{(i)}$, where $i \in \overrightarrow{S(x)}$. Then we have*

$$\mathrm{Tr}_{x]}\left[\prod_{i\in\overrightarrow{S(x)}} K_{<x,i>} \prod_{i\in\overrightarrow{S(x)}} \mathbf{h}^{(i)} \prod_{i\in\overleftarrow{S(x)}} K_{<x,i>}\right] = h_0^{(x)}\sigma_0^{(x)} + h_1^{(x)}\sigma_1^{(x)} \tag{8.15}$$

where

$$\begin{aligned} h_0^{(x)} &= h_0^{(1)}h_0^{(2)}h_0^{(3)}\cosh^6\beta + h_0^{(1)}h_1^{(2)}h_1^{(3)}\sinh^2\beta\cosh^3\beta \\ &\quad + h_1^{(1)}h_1^{(2)}h_0^{(3)}\sinh^2\beta\cosh^3\beta + h_1^{(1)}h_0^{(2)}h_1^{(3)}\sinh^2\beta\cosh^2\beta, \end{aligned} \tag{8.16}$$

$$\begin{aligned} h_1^{(x)} &= h_0^{(1)}h_0^{(2)}h_1^{(3)}\sinh\beta\cosh^2\beta + h_0^{(1)}h_1^{(2)}h_0^{(3)}\sinh\beta\cosh^3\beta \\ &\quad + h_1^{(1)}h_0^{(2)}h_0^{(3)}\sinh\beta\cosh^4\beta + h_1^{(1)}h_1^{(2)}h_1^{(3)}\sinh^3\beta\cosh\beta \end{aligned} \tag{8.17}$$

Proof. Let us first evaluate $\mathbf{g}_3^{(x)} := \mathrm{Tr}_{x]}\left[K_{<x,3>}\mathbf{h}^{(3)}K_{<x,3>}\right]$. From (8.14) it follows that

$$
\begin{aligned}
K_{<x,3>}\mathbf{h}^{(3)}K_{<x,3>} &= \sum_{i,j=0}^{3} K_iK_j\sigma_i^{(x)}\sigma_j^{(x)} \otimes \sigma_i^{(3)}\left(h_0^{(3)}\sigma_0^{(3)} + h_1^{(3)}\sigma_1^{(3)}\right)\sigma_j^{(3)} \\
&= h_0^{(3)}\sum_{i,j=0}^{3} K_iK_j\sigma_i^{(x)}\sigma_j^{(x)} \otimes \sigma_i^{(3)}\sigma_j^{(3)} \\
&\quad + h_1^{(3)}\sum_{i,j=0}^{3} K_iK_j\sigma_i^{(x)}\sigma_j^{(x)} \otimes \sigma_i^{(3)}\sigma_1^{(3)}\sigma_j^{(3)}
\end{aligned}
$$

Therefore, one gets

$$
\mathbf{g}_3^{(x)} = g_0^{(3)}\sigma_0^{(x)} + g_1^{(3)}\sigma_1^{(x)} \tag{8.18}
$$

where

$$
g_0^{(3)} = h_0^{(3)}(K_0^2 + K_1^2 + K_2^2 + K_3^2) = h_0^{(3)}\cosh^2\beta \tag{8.19}
$$

$$
g_1^{(3)} = 2h_1^{(3)}(K_0K_1 + K_2K_3) = h_1^{(3)}\sinh\beta. \tag{8.20}
$$

Now, evaluate $\mathbf{g}_2^{(x)} := \mathrm{Tr}_{x]}\left[K_{<x,2>}\mathbf{h}^{(2)}\mathbf{g}_3^{(x)}K_{<x,2>}\right]$. Using (8.14) and (8.18) we find

$$
\begin{aligned}
K_{<x,2>}\mathbf{h}^{(2)}\mathbf{g}_3^{(x)}K_{<x,2>} &= g_0^{(3)}h_0^{(2)}\sum_{i,j=0}^{3} K_iK_j\sigma_i^{(x)}\sigma_j^{(x)} \otimes \sigma_i^{(2)}\sigma_j^{(2)} \\
&\quad + g_0^{(3)}h_1^{(2)}\sum_{i,j=0}^{3} K_iK_j\sigma_i^{(x)}\sigma_j^{(x)} \otimes \sigma_i^{(2)}\sigma_1^{(2)}\sigma_j^{(2)} \\
&\quad + g_1^{(3)}h_0^{(2)}\sum_{i,j=0}^{3} K_iK_j\sigma_i^{(x)}\sigma_1^{(x)}\sigma_j^{(x)} \otimes \sigma_i^{(2)}\sigma_j^{(2)} \\
&\quad + g_1^{(3)}h_1^{(2)}\sum_{i,j=0}^{3} K_iK_j\sigma_i^{(x)}\sigma_1^{(x)}\sigma_j^{(x)} \otimes \sigma_i^{(2)}\sigma_1^{(2)}\sigma_j^{(2)}.
\end{aligned}
$$

Hence, one has

$$
\mathbf{g}_2^{(x)} = g_0^{(2)}\sigma_0^{(x)} + g_1^{(2)}\sigma_1^{(x)} \tag{8.21}
$$

where

$$
\begin{aligned}
g_0^{(2)} &= g_0^{(3)}h_0^{(2)}(K_0^2 + K_1^2 + K_2^2 + K_3^2) + 2g_1^{(3)}h_1^{(2)}(K_0K_1 - K_2K_3) \\
&= g_0^{(3)}h_0^{(2)}\cosh^2\beta + g_1^{(3)}h_1^{(2)}\sinh\beta\cosh\beta, \qquad (8.22)
\end{aligned}
$$

$$
\begin{aligned}
g_1^{(2)} &= 2g_0^{(3)}h_1^{(2)}(K_0K_1 + K_2K_3) + g_1^{(3)}h_0^{(2)}(K_0^2 + K_1^2 - K_2^2 - K_3^2) \\
&= g_0^{(3)}h_1^{(2)}\sinh\beta + g_1^{(3)}h_0^{(2)}\cosh\beta. \qquad (8.23)
\end{aligned}
$$

Similarly, one can evaluate

$$
\mathbf{g}_1^{(x)} := \mathrm{Tr}_{x]}\left[K_{<x,1>}\mathbf{h}^{(1)}\mathbf{g}_2^{(x)}K_{<x,1>}\right] = g_0^{(1)}\sigma_0^{(x)} + g_1^{(1)}\sigma_1^{(x)} \tag{8.24}
$$

where

$$
\begin{aligned}
g_0^{(1)} &= g_0^{(2)} h_0^{(1)} \cosh^2\beta + g_1^{(2)} h_1^{(1)} \sinh\beta\cosh\beta, && (8.25)\\
g_1^{(2)} &= g_0^{(2)} h_1^{(1)} \sinh\beta + g_1^{(2)} h_0^{(1)} \cosh\beta. && (8.26)
\end{aligned}
$$

We know that

$$
\mathrm{Tr}_{x]}\left[\prod_{i\in\overrightarrow{S(x)}} K_{<x,i>} \prod_{i\in\overrightarrow{S(x)}} \mathbf{h}^{(i)} \prod_{i\in\overleftarrow{S(x)}} K_{<x,i>}\right] = \mathbf{g}_1^{(x)},
$$

and combining (8.19),(8.20),(8.22),(8.23),(8.25),(8.26), we get

$$
\begin{aligned}
g_0^{(1)} &= h_0^{(1)} h_0^{(2)} h_0^{(3)} \cosh^6\beta + h_0^{(1)} h_1^{(2)} h_1^{(3)} \sinh^2\beta\cosh^3\beta\\
&\quad + h_1^{(1)} h_1^{(2)} h_0^{(3)} \sinh^2\beta\cosh^3\beta + h_1^{(1)} h_0^{(2)} h_1^{(3)} \sinh^2\beta\cosh^2\beta,\\
g_1^{(2)} &= h_0^{(1)} h_0^{(2)} h_1^{(3)} \sinh\beta\cosh^2\beta + h_0^{(1)} h_1^{(2)} h_0^{(3)} \sinh\beta\cosh^3\beta\\
&\quad + h_1^{(1)} h_0^{(2)} h_0^{(3)} \sinh\beta\cosh^4\beta + h_1^{(1)} h_1^{(2)} h_1^{(3)} \sinh^3\beta\cosh\beta
\end{aligned}
$$

This completes the proof. $\square$

Corollary 8.5. *Let* $K_{<u,v>}$ *be given by* (8.14), $\overrightarrow{S(x)} = (1,2,3)$, *and*

$$
\mathbf{h}^{(1)} = h_1\sigma_1^{(1)}, \quad \mathbf{h}^{(2)} = \alpha_0\sigma_0^{(2)}, \quad \mathbf{h}^{(3)} = \alpha_0\sigma_0^{(3)}.
$$

Then we have

$$
\mathrm{Tr}_{x]}\left[\prod_{i\in\overrightarrow{S(x)}} K_{<x,i>} \prod_{i\in\overrightarrow{S(x)}} \mathbf{h}^{(i)} \prod_{i\in\overleftarrow{S(x)}} K_{<x,i>}\right] = \alpha_0^2 h_1 \sinh\beta\cosh^4\beta\sigma_1^{(x)} \tag{8.27}
$$

Corollary 8.6. *Let* $K_{<u,v>}$ *be given by* (8.14), $\overrightarrow{S(x)} = (1,2,3)$, *and*

$$
\begin{aligned}
\mathbf{h}^{(1)} &= h_0\sigma_0^{(1)} + h_1\sigma_1^{(1)},\\
\mathbf{h}^{(2)} &= \gamma_0\sigma_0^{(2)} + \gamma_1\sigma_1^{(2)},\\
\mathbf{h}^{(3)} &= \gamma_0\sigma_0^{(3)} + \gamma_0\sigma_0^{(3)}.
\end{aligned}
$$

Then we have

$$
\mathrm{Tr}_{x]}\left[\prod_{i\in\overrightarrow{S(x)}} K_{<x,i>} \prod_{i\in\overrightarrow{S(x)}} \mathbf{h}^{(i)} \prod_{i\in\overleftarrow{S(x)}} K_{<x,i>}\right] = \left\langle Ah, \sigma^{(x)}\right\rangle, \tag{8.28}
$$

where as before A *is a matrix given by* (8.1), *and here we assume that* $\sigma^{(x)} = \left(\sigma_0^{(x)}, \sigma_1^{(x)}\right)$, $h = (h_0, h_1)$ *are vectors and* $\left\langle \cdot, \cdot \right\rangle$ *stands for the standard inner product of vectors.*

Let us consider the following elements:

$$\sigma_0^{\Lambda} := \bigotimes_{x\in\Lambda} \sigma_0^{(x)} \in \mathcal{B}_{\Lambda},\ \Lambda \subset \Lambda_n, \qquad \sigma_1^{\overrightarrow{S(x)},1} := \sigma_1^{(1)} \otimes \sigma_0^{(2)} \otimes \sigma_0^{(3)} \in \mathcal{B}_{S(x)}, \tag{8.29}$$

$$\sigma_1^{\overrightarrow{W}_{n+1},1} := \sigma_1^{\overrightarrow{S(x^{(1)}_{W_n})},1} \otimes \sigma_0^{\overrightarrow{W}_{n+1}\setminus\overrightarrow{S(x^{(1)}_{W_n})}} \in \mathcal{B}_{W_{n+1}}, \tag{8.30}$$

$$a_{\sigma_1}^{\Lambda_{n+1}} := \bigotimes_{i=0}^{n} \sigma_0^{\overrightarrow{W}_i} \otimes \sigma_1^{\overrightarrow{W}_{n+1},1} \in \mathcal{B}_{\Lambda_{n+1}}. \tag{8.31}$$

Proposition 8.7. *Let $\varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}$ be a forward quantum $d-$Markov chain corresponding to the model (8.14) with boundary conditions $\mathbf{h}^{(x)} = \alpha_0\sigma_0^{(x)}$ for all $x \in L$, where $\alpha_0 = \frac{1}{\cosh^3\beta}$. Let $a_{\sigma_1}^{\Lambda_{N+1}}$ be an element given by (8.31) and $\beta \in (\beta_*, \beta^*)$. Then one has $\varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right) = 0$, for any $N \in \mathbb{N}$.*

Proof. Due to (3.8) (see Theorem 3.1) the compatibility condition holds $\varphi^{(n+1,f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}\lceil_{\mathcal{B}_{\Lambda_n}} = \varphi^{(n,f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}$. Therefore,

$$\varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right) = w - \lim_{n\to\infty} \varphi^{(n,f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right) = \varphi^{(N+1,f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right). \tag{8.32}$$

Taking into account $w_0(\alpha_0) = \frac{1}{\alpha_0}\sigma_0^{(0)}$ and due to Proposition 3.2, it is enough to evaluate the following

$$\begin{aligned}
\varphi^{(N+1,f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right) &= \mathrm{Tr}\left(\mathcal{W}_{N+1]}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right)\right) \\
&= \frac{1}{\alpha_0}\mathrm{Tr}\left[K_{[0,1]}\cdots K_{[N,N+1]}\mathbf{h}_{N+1}K^*_{[N,N+1]}\cdots K^*_{[0,1]}a_{\sigma_1}^{\Lambda_{N+1}}\right] \\
&= \frac{1}{\alpha_0}\mathrm{Tr}\Big[K_{[0,1]}\cdots K_{[N-1,N]} \\
&\qquad \mathrm{Tr}_{N]}\left[K_{[N,N+1]}\mathbf{h}_{N+1}K^*_{[N,N+1]}\sigma_1^{\overrightarrow{W}_{N+1},1}\right]K^*_{[N-1,N]}\cdots K^*_{[0,1]}\Big].
\end{aligned}$$

Now let us calculate $\widetilde{\mathbf{h}}_N := \mathrm{Tr}_{N]}\left[K_{[N,N+1]}\mathbf{h}_{N+1}K^*_{[N,N+1]}\sigma_1^{\overrightarrow{W}_{n+1},1}\right]$. Since $K_{<u,v>}$ is a self-adjoint, we then get

$$\begin{aligned}
\widetilde{\mathbf{h}}_N &= \mathrm{Tr}_{x^{(1)}_{W_N}]}\left[\prod_{y\in\overrightarrow{S(x^{(1)}_{W_N})}} K_{\left\langle x^{(1)}_{W_N},y\right\rangle} \prod_{y\in\overrightarrow{S(x^{(1)}_{W_N})}} \mathbf{h}^{(y)} \prod_{y\in\overleftarrow{S(x^{(1)}_{W_N})}} K_{\left\langle x^{(1)}_{W_N},y\right\rangle}\sigma_1^{\overrightarrow{S(x^{(1)}_{W_N})},1}\right] \otimes \\
&\quad \bigotimes_{x\in\overrightarrow{W}_N\setminus x^{(1)}_{W_N}} \mathrm{Tr}_{x]}\left[\prod_{y\in\overrightarrow{S(x)}} K_{<x,y>} \prod_{y\in\overrightarrow{S(x)}} \mathbf{h}^{(y)} \prod_{y\in\overleftarrow{S(x)}} K_{<x,y>}\right].
\end{aligned}$$

We know that

$$\mathrm{Tr}_{x]}\left[\prod_{y\in\overrightarrow{S(x)}} K_{<x,y>} \prod_{y\in\overrightarrow{S(x)}} \mathbf{h}^{(y)} \prod_{y\in\overleftarrow{S(x)}} K_{<x,y>}\right] = \mathbf{h}^{(x)}, \tag{8.33}$$

for every $x \in \overrightarrow{W}_N \setminus x^{(1)}_{W_N}$. Therefore, one can easily check that

$$\mathrm{Tr}_{x^{(1)}_{W_N}]}\left[\prod_{\overrightarrow{y\in S(x^{(1)}_{W_N})}} K_{\left\langle x^{(1)}_{W_N},y\right\rangle}\prod_{\overrightarrow{y\in S(x^{(1)}_{W_N})}}\mathbf{h}^{(y)}\prod_{\overleftarrow{y\in S(x^{(1)}_{W_N})}} K_{\left\langle x^{(1)}_{W_N},y\right\rangle}\sigma_1^{\overrightarrow{S(x^{(1)}_{W_N}),1}}\right]=\widetilde{\mathbf{h}}^{(x^{(1)}_{W_N})}, \quad (8.34)$$

where

$$\widetilde{\mathbf{h}}^{(x^{(1)}_{W_N})}=\alpha_1\sigma_1^{(x^{(1)}_{W_N})}, \qquad \alpha_1=\sinh\beta\cosh^5\beta.$$

Hence, we obtain

$$\widetilde{\mathbf{h}}_N=\widetilde{\mathbf{h}}^{(x^{(1)}_{W_N})}\bigotimes_{x\in\overrightarrow{W}_N\setminus x^{(1)}_{W_N}}\mathbf{h}^{(x)}.$$

Therefore, one finds

$$\begin{aligned}\varphi^{(N+1,f)}_{w_0,\mathbf{h}(\alpha_0)}\left(a^{\Lambda_{N+1}}_{\sigma_1}\right) &= \frac{1}{\alpha_0}\mathrm{Tr}\Big[K_{[0,1]}\cdots K_{[N-2,N-1]}\\ &\qquad \mathrm{Tr}_{N-1]}\left[K_{[N-1,N]}\widetilde{\mathbf{h}}_N K^*_{[N-1,N]}\right]K^*_{[N-2,N-1]}\cdots K^*_{[0,1]}\Big].\end{aligned}$$

So, after N times applying Corollary (8.5), we get

$$\varphi^{(N+1,f)}_{w_0,\mathbf{h}(\alpha_0)}\left(a^{\Lambda_{N+1}}_{\sigma_1}\right)=\alpha_0^{2N-1}\alpha_1^N(\sinh\beta\cosh^4\beta)^N\mathrm{Tr}(\sigma_1^{(0)})=0.$$

This completes the proof. □

Proposition 8.8. *Let $\varphi^{(f)}_{w_0(\gamma),\mathbf{h}(\gamma)}$ be a forward quantum $d-$Markov chain corresponding to the model (8.14) with boundary conditions $\mathbf{h}^{(x)}=\gamma_0\sigma_0^{(x)}+\gamma_1\sigma_1^{(x)}$ for all $x\in L$, where $\gamma_0=\sqrt{DE}$ and $\gamma_1=\sqrt{E}$. Let $a^{\Lambda_{N+1}}_{\sigma_1}$ be an element given by (8.31) and $\beta\in(\beta_*,\beta^*)$. Then one has*

$$\varphi^{(f)}_{w_0,\mathbf{h}(\gamma)}\left(a^{\Lambda_{N+1}}_{\sigma_1}\right)=\frac{1}{\gamma_0}\left\langle A^N h_{\gamma_0,\gamma_1},e\right\rangle \qquad \forall N\in\mathbb{N}, \quad (8.35)$$

where A is a matrix given by (8.1), $\left\langle\cdot,\cdot\right\rangle$ is the standard inner product of vectors and $e=(1,0)$, $h_{\gamma_0,\gamma_1}=(h_0,h_1)$ are vectors with

$$\begin{aligned} h_0 &= \gamma_0^2\gamma_1(\sinh^2\beta\cosh\beta(1+\cosh\beta)+\cosh^5\beta)+\gamma_1^3\sinh^2\beta\cosh^2\beta, & (8.36)\\ h_1 &= \gamma_0^3\sinh\beta\cosh^5\beta+\gamma_0\gamma_1(\sinh\beta\cosh^3\beta(1+\cosh\beta)+\sinh^3\beta\cosh^2\beta). & (8.37)\end{aligned}$$

Proof. Again the compatibility condition yields that

$$\varphi^{(f)}_{w_0,\mathbf{h}(\gamma)}\left(a^{\Lambda_{N+1}}_{\sigma_1}\right)=w-\lim_{n\to\infty}\varphi^{(n,f)}_{w_0,\mathbf{h}(\gamma)}\left(a^{\Lambda_{N+1}}_{\sigma_1}\right)=\varphi^{(N+1,f)}_{w_0,\mathbf{h}(\gamma)}\left(a^{\Lambda_{N+1}}_{\sigma_1}\right). \quad (8.38)$$

Noting that if $\mathbf{h}^{(0)} = \gamma_0\sigma_0^{(0)} + \gamma_1\sigma_1^{(0)}$ then one of the solutions of the equation $\mathrm{Tr}(wa_0\mathbf{h}^{(0)}) = 1$ w.r.t. w_0 is $w_0(\gamma) = \frac{1}{\gamma_0}\sigma_0^{(0)}$, and due to Proposition 3.2, it is enough to evaluate the following

$$\begin{aligned}
\varphi^{(N+1,f)}_{w_0,\mathbf{h}(\gamma)}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right) &= \mathrm{Tr}\left(\mathcal{W}_{N+1]}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right)\right)\\
&= \frac{1}{\gamma_0}\mathrm{Tr}\left[K_{[0,1]}\cdots K_{[N,N+1]}\mathbf{h}_{N+1}K^*_{[N,N+1]}\cdots K^*_{[0,1]}a_{\sigma_1}^{\Lambda_{N+1}}\right]\\
&= \frac{1}{\gamma_0}\mathrm{Tr}\Big[K_{[0,1]}\cdots K_{[N-1,N]}\\
&\qquad \mathrm{Tr}_{N]}\left[K_{[N,N+1]}\mathbf{h}_{N+1}K^*_{[N,N+1]}\sigma_1^{\overrightarrow{W}_{N+1},1}\right]K^*_{[N-1,N]}\cdots K^*_{[0,1]}\Big].
\end{aligned}$$

Let us calculate $\widetilde{\mathbf{h}}_N := \mathrm{Tr}_{N]}\left[K_{[N,N+1]}\mathbf{h}_{N+1}K^*_{[N,N+1]}\sigma_1^{\overrightarrow{W}_{n+1},1}\right]$. Self-adjointness of $K_{<u,v>}$ implies that

$$\begin{aligned}
\widetilde{\mathbf{h}}_N &= \mathrm{Tr}_{x^{(1)}_{W_N}]}\left[\prod_{\overrightarrow{y\in S(x^{(1)}_{W_N})}}K_{\left\langle x^{(1)}_{W_N},y\right\rangle}\prod_{\overrightarrow{y\in S(x^{(1)}_{W_N})}}\mathbf{h}^{(y)}\prod_{\overleftarrow{y\in S(x^{(1)}_{W_N})}}K_{\left\langle x^{(1)}_{W_N},y\right\rangle}\sigma_1^{\overrightarrow{S(x^{(1)}_{W_N})},1}\right]\otimes\\
&\bigotimes_{x\in\overrightarrow{W}_N\setminus x^{(1)}_{W_N}}\mathrm{Tr}_{x]}\left[\prod_{y\in\overrightarrow{S(x)}}K_{<x,y>}\prod_{y\in\overrightarrow{S(x)}}\mathbf{h}^{(y)}\prod_{y\in\overleftarrow{S(x)}}K_{<x,y>}\right].
\end{aligned}$$

It follows from (8.33) that

$$\mathrm{Tr}_{x^{(1)}_{W_N}]}\left[\prod_{\overrightarrow{y\in S(x^{(1)}_{W_N})}}K_{\left\langle x^{(1)}_{W_N},y\right\rangle}\prod_{\overrightarrow{y\in S(x^{(1)}_{W_N})}}\mathbf{h}^{(y)}\prod_{\overleftarrow{y\in S(x^{(1)}_{W_N})}}K_{\left\langle x^{(1)}_{W_N},y\right\rangle}\sigma_1^{\overrightarrow{S(x^{(1)}_{W_N})},1}\right] = \widetilde{\mathbf{h}}^{(x^{(1)}_{W_N})},$$

where

$$\begin{aligned}
\widetilde{\mathbf{h}}^{(x^{(1)}_{W_N})} &= h_0\sigma_0^{(x^{(1)}_{W_N})} + h_1\sigma_1^{(x^{(1)}_{W_N})},\\
h_0 &= \gamma_0^2\gamma_1(\sinh^2\beta\cosh\beta(1+\cosh\beta)+\cosh^5\beta)+\gamma_1^3\sinh^2\beta\cosh^2\beta,\\
h_1 &= \gamma_0^3\sinh\beta\cosh^5\beta+\gamma_0\gamma_1(\sinh\beta\cosh^3\beta(1+\cosh\beta)+\sinh^3\beta\cosh^2\beta).
\end{aligned}$$

Thus we obtain

$$\widetilde{\mathbf{h}}_N = \widetilde{\mathbf{h}}^{(x^{(1)}_{W_N})}\bigotimes_{x\in\overrightarrow{W}_N\setminus x^{(1)}_{W_N}}\mathbf{h}^{(x)}.$$

Therefore, one gets

$$\begin{aligned}
\varphi^{(N+1,f)}_{w_0,\mathbf{h}(\gamma)}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right) &= \frac{1}{\gamma_0}\mathrm{Tr}\Big[K_{[0,1]}\cdots K_{[N-2,N-1]}\\
&\qquad \mathrm{Tr}_{N-1]}\left[K_{[N-1,N]}\widetilde{\mathbf{h}}_NK^*_{[N-1,N]}\right]K^*_{[N-2,N-1]}\cdots K^*_{[0,1]}\Big].
\end{aligned}$$

Again applying N times Corollary (8.6), one finds

$$\varphi^{(N+1,f)}_{w_0,\mathbf{h}(\gamma)}\left(a_{\sigma_1}^{\Lambda_{N+1}}\right) = \frac{1}{\gamma_0}\mathrm{Tr}\left[\left\langle A^Nh_{\gamma_0,\gamma_1},\sigma^{(0)}\right\rangle\right] = \frac{1}{\gamma_0}\left\langle A^Nh_{\gamma_0,\gamma_1},e\right\rangle.$$

here as before $e = (1,0)$, $h_{\gamma_0,\gamma_1} = (h_0, h_1)$ are vectors, and A is a matrix given by (8.1). This completes the proof. $\square$

To prove our main result we are going to use the following theorem (see [16], Corollary 2.6.11).

Theorem 8.9. *Let φ_1, φ_2 be two states on a quasi-local algebra $\mathfrak{A} = \cup_\Lambda \mathfrak{A}_\Lambda$. The states φ_1, φ_2 are quasi-equivalent if and only if for any given $\varepsilon > 0$ there exists a finite volume $\Lambda \subset L$ such that $\|\varphi_1(a) - \varphi_2(a)\| < \varepsilon \|a\|$ for all $a \in B_{\Lambda'}$ with $\Lambda' \cap \Lambda = \emptyset$.*

Theorem 8.10. *Let $\beta \in (\beta_*, \beta^*)$ and $\varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}$, $\varphi^{(f)}_{w_0(\gamma),\mathbf{h}(\gamma)}$ be two forward quantum $d-$Markov chains corresponding to the model (8.14) with two boundary conditions $\mathbf{h}^{(x)} = \alpha_0 \sigma_0^{(x)}$, $\forall x \in L$ and $\mathbf{h}^{(x)} = \gamma_0 \sigma_0^{(x)} + \gamma_1 \sigma_1^{(x)}$, $\forall x \in L$, respectively, here as before $\alpha_0 = \frac{1}{\cosh^3 \beta}$, $\gamma_0 = \sqrt{DE}$, and $\gamma_1 = \sqrt{E}$. Then $\varphi^{(f)}_{w_0,\mathbf{h}(\alpha_0)}$ and $\varphi^{(f)}_{w_0,\mathbf{h}(\gamma)}$ are not quasi-equivalent.*

Proof. Let $a_{\sigma_1}^{\Lambda_{N+1}}$ be an element given by (8.31). It is clear that $\left\| a_{\sigma_1}^{\Lambda_{N+1}} \right\| = 1$, for all $N \in \mathbb{N}$.

If $\beta \in (\beta_*, \beta^*)$, then according to Propositions 8.7 and 8.8, we have

$$\varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)} \left(a_{\sigma_1}^{\Lambda_{N+1}} \right) = 0, \tag{8.39}$$

$$\varphi^{(f)}_{w_0(\gamma),\mathbf{h}(\gamma)} \left(a_{\sigma_1}^{\Lambda_{N+1}} \right) = \frac{1}{\gamma_0} \left\langle A^N h_{\gamma_0,\gamma_1}, e \right\rangle \tag{8.40}$$

for all $N \in \mathbb{N}$, here as before $e = (1,0)$, $h_{\gamma_0,\gamma_1} = (h_0, h_1)$ (see (8.36),(8.37)) and A is given by (8.1). Then from (8.40) with Proposition 8.2 one finds

$$\varphi^{(f)}_{w_0(\gamma),\mathbf{h}(\gamma)} \left(a_{\sigma_1}^{\Lambda_{N+1}} \right) = \frac{x_1^2 h_1 + x_1 y_1 \sinh \beta h_2}{\gamma_0 (x_1^2 + y_1^2 \sinh \beta)} + \frac{y_1^2 \sinh \beta h_1 - x_1 y_1 \sinh \beta h_2}{\gamma_0 (x_1^2 + y_1^2 \sinh \beta)} \lambda_2^N, \tag{8.41}$$

where λ_2 is an eigenvalue of A and (x_1, y_1) is an eigenvector of the matrix A corresponding to the eigenvalue $\lambda_1 = 1$ (see Proposition 8.2). Due to Propositions 8.1(iv) and 8.2 one has $0 < \lambda_2 < 1$, which implies the existence $N_0 \in \mathbb{N}$ such that

$$\left| \frac{x_1^2 h_1 + x_1 y_1 \sinh \beta h_2}{\gamma_0 (x_1^2 + y_1^2 \sinh \beta)} + \frac{y_1^2 \sinh \beta h_1 - x_1 y_1 \sinh \beta h_2}{\gamma_0 (x_1^2 + y_1^2 \sinh \beta)} \lambda_2^N \right| \geq \frac{x_1^2 h_1 + x_1 y_1 \sinh \beta h_2}{2\gamma_0 (x_1^2 + y_1^2 \sinh \beta)} \tag{8.42}$$

for all $N > N_0$.

Now putting $\varepsilon_0 = \dfrac{x_1^2 h_1 + x_1 y_1 \sinh \beta h_2}{2\gamma_0 (x_1^2 + y_1^2 \sinh \beta)}$ and using (8.39), (8.41), (8.42) we obtain

$$\left| \varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)} \left(a_{\sigma_1}^{\Lambda_{N+1}} \right) - \varphi^{(f)}_{w_0(\gamma),\mathbf{h}(\gamma)} \left(a_{\sigma_1}^{\Lambda_{N+1}} \right) \right| \geq \varepsilon_0 \left\| a_{\sigma_1}^{\Lambda_{N+1}} \right\|,$$

for all $N > N_0$, which means $\varphi^{(f)}_{w_0(\alpha_0),\mathbf{h}(\alpha_0)}$ and $\varphi^{(f)}_{w_0(\gamma),\mathbf{h}(\gamma)}$ are not quasi-equivalent. This completes the proof. $\square$

From the proved theorem we immediately get the occurrence of the phase transition for the model (8.14) on the Cayley tree of order 3 in the regime $\beta \in (\beta_*, \beta^*)$. This completely proves our main Theorem 4.1.

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A Proof of Lemma 5.2

(i) Let $P_9(t) = t^9 - t^8 - t^7 - t^6 + 2t^4 + 2t^3 - t - 1$. One can check that

$$P_9(t) = (t-1)(t^8 - t^6 - 2t^5 - 2t^4 + 2t^2 + 2t + 1)$$

and $t = 1$ is a root of the polynomial $P_9(t)$. It is easy to see that $P_9(1.05) > 0$, $P_9(1.1) < 0$, $P_9(1.5) < 0$, $P_9(1.6) > 0$. This means $P_9(t)$ has two roots t_* and t^* such that $1.05 < t_* < 1.1$ and $1.5 < t_* < 1.6$. On the other side, due to Descartes theorem, the number of positive roots of $P_9(t)$ is at most the number of exchanging signs of its coefficients $1, -1, -1, -1, 2, 2, -1, -1$. So, $P_9(t)$ has exactly three roots $1, t_*, t^*$. It is evident that if $t \in (1, t_*) \cup (t^*, \infty)$ then $P_9(t) > 0$ and $t \in (t_*, t^*)$ then $P_9(t) < 0$.

(ii) Since $\beta > 0$ and $\cosh\beta > \sinh\beta > 0$, we get

$$A_2 - A_1 = \sinh^2\beta\cosh^2\beta(2\cosh^2\beta + \cosh\beta - \sinh\beta) > 0.$$

(iii) Let us denote by $t = \cosh\beta$ and $\beta_* = \cosh^{-1} t_*$, $\beta^* = \cosh^{-1} t^*$. One can check that

$$B_2 - B_1 \geq 0 \qquad \Leftrightarrow \qquad P_9(t) \geq 0,$$

and

$$B_2 - B_1 < 0 \qquad \Leftrightarrow \qquad P_9(t) < 0.$$

So, from (i) it follows that if $\beta \in (0, \beta_*] \cup [\beta^*, \infty)$ then $B_1 \leq B_2$ and if $\beta \in (\beta_*, \beta^*)$ then $B_1 > B_2$.

(iv) Let us denote by $t = \cosh\beta$, and

$$Q_{10}(t) = t^{10} + 4t^9 + 5t^8 - 4t^7 - 14t^6 - 6t^5 + 11t^4 + 8t^3 - 3t^2 - 2t + 1.$$

One can see that

$$A_2 + B_2 > A_1 + B_1 \qquad \Leftrightarrow \qquad Q_{10}(t) > 0.$$

It is clear that if $\beta > 0$ then $t > 1$. One can easily get that if $t > 1$ then

$$Q_{10}(t) = t(t-1)\big((t-1)(t^7 + 6t^6 + 16t^5 + 22t^4 + 11t^3 + 3t(t^2-1)) + 2(t+1)\big) + 1 > 0.$$

(v) If $\beta \in (\beta_*, \beta^*)$ then $B_1 - B_2 > 0$. From (iv) it follows that $A_2 - A_1 > B_1 - B_2$. This means that $D > 1$.

(vi) Since $1 + \cosh\beta + \cosh^2\beta > 1 + 2\cosh\beta$ and $\cosh\beta > \sinh\beta > 0$ we get

$$B_1B_2 - A_1A_2 = \sinh\beta\cosh^3\beta\left(\cosh^5\beta(1 + \cosh\beta + \cosh^2\beta) - \sinh^4\beta(1 + 2\cosh\beta)\right) > 0.$$

It is easy to see that

$$A_2B_1 - A_1B_2 = \sinh^3\beta\cosh^4\beta(1 + 3\cosh\beta + 3\cosh^2\beta + \cosh^3\beta) > 0$$

(vii) Let

$$Q_7(t) = t^7 + 2t^6 - 3t^4 - 2t^3 + t^2 + 3t + 1.$$

Then, one can easily check that

$$A_1A_2 + 3A_1B_2 - A_2B_1 + B_1B_2 \quad = \quad \sinh\beta\cosh^3\beta Q_7(\cosh\beta).$$

If $\beta \in (\beta_*, \beta^*)$ then $t \in (t_*, t^*)$ and

$$Q_7(t) = t(t-1)(t^5 + 3t^4 + 2t(t^2 - 1) + t^3 - 1) + 2t + 1 > 0.$$

here $t_* > 1$.

Let

$$Q_4(t) = -t^4 - t^3 + t^2 + 5t + 2.$$

Then, we get

$$A_2B_1 - 3A_1B_2 - 2A_1A_2 = \sinh^3\beta\cosh^3\beta Q_4(\cosh\beta).$$

One can check that $Q_4(1.7) > 0$ and $Q_4(1.8) < 0$. Due to Descartes Theorem we conclude that $Q_4(t)$ has a unique positive root $\hat{t}$ such that $1.7 < \hat{t} < 1.8$.

If $\beta \in (\beta_*, \beta^*)$ then $t \in (t_*, t^*)$ and $t^* < 1.7 < \hat{t}$. Then, for any $t \in (t_*, t^*)$ we have $Q_4(t) > 0$.

$(viii)$ It is clear that, if $\beta > 0$, then

$$\sinh\beta\cosh\beta(1 + \cosh\beta) > 0.$$

Now we are going to show that

$$\sinh\beta(1 + \cosh\beta) < \cosh^3\beta. \tag{A.1}$$

Noting

$$\sinh\beta = \frac{e^\beta - e^{-\beta}}{2}, \quad \cosh\beta = \frac{e^\beta + e^{-\beta}}{2}.$$

and letting $t = e^\beta$, we reduce inequality (A.1) to

$$t^6 - 2t^5 - t^4 + 7t^2 + 2t + 1 \quad > \quad 0 \tag{A.2}$$

Since $\beta > 0$, then $t > 1$. Therefore, we shall show that (A.2) is satisfied whenever $t > 1$. Now consider several cases with respect to t.

Case I. Let $t \geq 1 + \sqrt{2}$. Then we have

$$t^6 - 2t^5 - t^4 + 7t^2 + 2t + 1 = t^4\big(t - (1+\sqrt{2})\big)\big(t - (1-\sqrt{2})\big) + 7t^2 + 2t + 1 > 0$$

Case II. Let $2 \leq t \leq 1 + \sqrt{2}$. Then it is clear that $t < \sqrt{7}$. Therefore,

$$t^6 - 2t^5 - t^4 + 7t^2 + 2t + 1 = t^5(t-2) + t^2(7 - t^2) + 2t + 1 > 0$$

Case III. Let $\sqrt{\frac{7}{2}} \leq t \leq 2$. Then one gets

$$\begin{aligned} 2(t^6 - 2t^5 - t^4 + 7t^2 + 2t + 1) &= 2t^4\left(t^2 - \frac{7}{2}\right) + \frac{5}{2}t^4(2-t) \\ &+ \frac{3}{2}t^2(8 - t^3) + 2t^2 + 4t + 2 > 0 \end{aligned}$$

Case IV. Let $1 < t \leq \sqrt{\frac{7}{2}}$. Then we have

$$t^6 - 2t^5 - t^4 + 7t^2 + 2t + 1 = t^4(t-1)^2 + t^2(7 - 2t^2) + 2t + 1 > 0$$

Hence, the inequality (A.1) is satisfied for all $\beta > 0$.

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