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APPLICATION OF THE (G'/G)-EXPANSION METHOD TO NONLINEAR BLOOD FLOW IN LARGE VESSELS

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Abstract

The (G'/G)-expansion method is applied to the one-dimensional Navier-Stokes equations that usually model blood flow in large vessels. A nonlinear stress-strain condition is applied and different types of travelling waves solutions are found. The shape of the pressure of blood waves and their biological implications are discussed.

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The importance of nonlinear evolution equations is now established, this because these equations arise in various areas of science and engineering, especially in fluid mechanics, solid state physics, biophysics and so on. As a key problem, finding their analytical solutions is of a great interest and this is actually performed through various powerful and exact methods such as the homogeneous balance method [1], the ansatz method [2, 3], the Jacobi elliptic function expansion method [4], the extended tanh method [5], the tanh-coth method and the sech methods[6, 7], just to name a few. Recently, a new method, known as the (G'/G)-expansion method, has been proposed by Wang et al. [8]. It is actually widely applied to real problems. In so doing, as in the case of many problems of life sciences, the phenomenon of blood flow in vessels is quite complex and is not yet fully understood. Furthermore, because of its pulsatile character and due to physiological constraints, blood flow is expected to display different behaviors in terms of nonlinear waves and solitons as substantially developed since the pioneering works of Hashimuze [9] and Yomosa [10]. These authors showed in fact that the dynamics of blood waves are governed by the KdV or the modified KdV equations. Regardless the fact that these equations have been successfully used to explain some natural behaviors of blood flows [9, 10, 11, 12], it remains obvious that they are only approximated from Navier-Stokes and continuity equations. In this letter, we intend to avoid all approximations or expansions of the corresponding equations in order to investigate blood waves in large vessels through the (G'/G)-expansion method. As a consequence some features of the pressure of blood waves related to the founded solutions are discussed.

We consider the fluid in the vessel to be Newtonian, viscous, homogeneous and incompressible. The whole dynamics of blood flow in vessels is governed by the conservation of mass, momenta and forces balance equation for radial motion of the wall [13, 14]

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial z}(AW) = 0, \tag{1}$$

$$\frac{\partial W}{\partial t} + W\frac{\partial W}{\partial z} + \frac{\partial P}{\partial z} = 0, \tag{2}$$

where z is the axial coordinate, t is time, W is the axial component of the fluid velocity, A is the cross-sectional area of the vessel and P is the pressure inside the vessel.

For the modeling of the wall dynamics, the second law of Newton is used on a portion of the vessel wall and the equation of the dynamics of wall takes the form [13, 14]

$$P = \frac{2}{1+A}\frac{\partial^2 A}{\partial t^2} + 2(A-1)\frac{(2+a(A-1))}{(A+1)^2}, \tag{3}$$

where a is the coefficient representing the nonlinear coefficient of the vessel wall. The variables

$$z' = L_0 z, \quad t' = T_0 t, \quad w' = w_0 W, \quad P' - p_e = p_0 P, \ \text{ and } \ S = S_0 A \tag{4}$$

are dimensionless, with the characteristic parameters are

$$L_0 = (R_0 h_0 \rho_0 / 2\rho)^{1/2}, \quad T_0 = (\rho_0 R_0^2/E_0)^{1/2}, \quad w_0 = L_0/T_0, \quad S_0 = \pi R_0^2, \ \text{ and } \ p_0 = \rho v_0^2. \tag{5}$$

R_0 and h_0 are the equilibrium values of radius R and thickness h of the wall. ρ and ρ_0 are respectively, the density of the fluid and the wall. E_0 is the Young modulus, S and w' are the dimensional cross-section area of the vessel and the axial longitudinal flow velocity respectively. Combining the independent variables z and t into one variable $\eta = z - Vt$, Eqs.(1)-(3) become

$$-VA_\eta + (AW)_\eta = 0, \tag{6}$$

$$-VW_\eta + WW_\eta + 2\left(\frac{V^2}{(1+A)}A_{\eta\eta} + (A-1)\frac{(2+a(A-1))}{(A+1)^2}\right)_\eta = 0. \tag{7}$$

Integrating the above Eqs.(6) and (7) with respect to η yields

$$-VA + AW + K_1 = 0, \tag{8}$$

$$\frac{2V^2}{(1+A)}A_{\eta\eta} + 2(A-1)\frac{(2+a(A-1))}{(A+1)^2} + \frac{1}{2}W^2 - VW + K_2 = 0. \tag{9}$$

For simplification, we consider the integration constant K_2 to be zero, while the constant K_1 is assumed to be different from zero. From Eq. (8), it is possible to write W as a function of A. Replacing the latter into Eq. (9) therefore leads to the equation

$$\begin{aligned} 2V^2A^2(1+A)A_{\eta\eta} + 2A^2(A-1)\left[2+a(A-1)\right] + \frac{1}{2}(VA-K_1)^2 + A^2(VA-K_1) \\ + \frac{1}{2}A^2(VA-K_1)^2 - VA(VA-K_1) - 2VA^2(VA-K_1) - VA^3(VA-K_1) = 0. \end{aligned} \tag{10}$$

As already mentioned previously, we assume that the solution of ODE (10) may be expressed by a polynomial in $\left(\frac{G'}{G}\right)$ as follows

$$A(\eta) = \sum_{m=0}^{M} \alpha_m \left(\frac{G'}{G}\right)^m, \tag{11}$$

where $G = G(\eta)$ satisfies the second order ODE in the form

$$G''(\eta) + \lambda G'(\eta) + \mu G(\eta) = 0, \tag{12}$$

α_m, λ and μ are constants to be determined later. The unwritten part in Eq. (13) is also a polynomial in $\left(\frac{G'}{G}\right)$, the degree of which is generally equal to or less than $M-1$, and the positive integer M can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms appearing in the equation to be solved. In the particular case of Eq. (10), the corresponding balance between the terms $A^3A_{\eta\eta}$, $A^2A_{\eta\eta}$, A^4, A^3, and A^2 leads to $M = 2$. This indubitably brings us to assume the solution to be of the form

$$A(\eta) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_2\left(\frac{G'}{G}\right)^2. \tag{13}$$

In what follows, we consider the coupled equations as given by Eqs. (8) and (9). It will therefore be useful to assume the same solution order for Eq. (8) that will be taken as

$$W(\eta) = \beta_0 + \beta_1\left(\frac{G'}{G}\right) + \beta_2\left(\frac{G'}{G}\right)^2, \tag{14}$$

where β_m are to be determined. By using Eq.(12), from Eq.(13) we have

$$A_{\eta\eta} = 6\alpha_2 \left(\frac{G'}{G}\right)^4 + (2\alpha_1 + 10\alpha_2\lambda)\left(\frac{G'}{G}\right)^3 + \left(8\alpha_2\mu + 3\alpha_1\lambda + 4\alpha_2\lambda^2\right)\left(\frac{G'}{G}\right)^2 + \left(6\alpha_2\mu\lambda + 2\alpha_1\mu + \alpha_1\lambda^2\right)\left(\frac{G'}{G}\right) + 2\alpha_2\mu^2 + \alpha_1\mu\lambda. \tag{15}$$

On substituting Eqs.(13)-(15) into Eqs. (8) and (9) and solving with respect to the powers of (G'/G) and setting each corresponding coefficient to zero, the algebraic equations for V, α_0, α_1, α_2, β_0, β_1, β_2, λ, μ and K_1 are obtained. Solving the later with the Maple software gives us

$$\alpha_0 = \alpha_0, \qquad \beta_0 = \beta_0, \qquad \mathrm{V} = \frac{\beta_0\alpha_0 + K_1}{\alpha_0}, \qquad \beta_1 = K_1\frac{\alpha_1}{\alpha_0}, \qquad \beta_2 = -K_1\frac{\alpha_2}{\alpha_0}. \tag{16}$$

$$\alpha_1 = \frac{-3\lambda\alpha_0^2}{\mu} \times \frac{f_1(\alpha_0,\beta_0) + K_1 f_2(\alpha_0,\beta_0)}{f_3(\alpha_0,\beta_0) + K_1 f_4(\alpha_0,\beta_0) + K_1^2 f_5(\alpha_0,\beta_0)}, \tag{17}$$

where

$$f_1(\alpha_0,\beta_0) = \beta_0^2\alpha_0 + 3\beta_0^2\alpha_0^2 - 4a\alpha_0 + 4a\alpha_0^2 + 8\alpha_0 + 3\beta_0^2\alpha_0^3 + 4a\alpha_0^3 - 8\alpha_0^3 + \beta_0^2\alpha_0^4 - 4a\alpha_0^4,$$

$$f_2(\alpha_0,\beta_0) = 6\beta_0\alpha_0^2 + 6\beta_0\alpha_0 + 2\beta_0 + 2\beta_0\alpha_0^3,$$

$$f_3(\alpha_0,\beta_0) = -\beta_0^2\alpha_0^3 - 12a\alpha_0^3 + 8\mu\beta_0^2\alpha_0^3 + 16\alpha_0^3 - 8\lambda^2\alpha_0^3\beta_0^2 + 16\mu\beta_0^2\alpha_0^4 - 16\lambda^2\alpha_0^4\beta_0^2 + 4a\alpha_0^5 - \beta_0^2\alpha_0^5 + 8a\alpha_0^4 - 2\beta_0^2\alpha_0^4 + 8\mu\beta_0^2\alpha_0^5 - 8\lambda^2\alpha_0^5\beta_0^2,$$

$$f_4(\alpha_0,\beta_0) = 16\mu\alpha_0^2\beta_0 - 32\lambda^2\alpha_0^3\beta_0 + 32\mu\alpha_0^3\beta_0 - 16\lambda^2\alpha_0^4\beta_0 + 16\mu\alpha_0^4\beta_0 - 2\beta_0\alpha_0^4 - 16\lambda^2\alpha_0^2\beta_0 - 2\beta_0\alpha_0^2 - 4\beta_0\alpha_0^3,$$

$$f_5(\alpha_0,\beta_0) = -8\lambda^2\alpha_0^3 + 8\mu\alpha_0^3 - 2\alpha_0^3 - 6\alpha_0^2 - 6\alpha_0 - 2 - 8\lambda^2\alpha_0 + 8\mu\alpha_0 - 16\lambda^2\alpha_0^2 + 16\mu\alpha_0^2,$$

and

$$\alpha_2 = \frac{-0.25\alpha_1}{\mu\alpha_0} \times \frac{g_1(\alpha_0,\alpha_1,\beta_0) + K_1 g_2(\alpha_0,\alpha_1,\beta_0) + K_1^2 g_3(\alpha_0,\alpha_1,\beta_0)}{g_4(\alpha_0,\alpha_1,\beta_0) + K_1 g_5(\alpha_0,\alpha_1,\beta_0) + K_1^2 g_6(\alpha_0,\alpha_1,\beta_0)}, \tag{18}$$

where

$$g_1(\alpha_0,\alpha_1,\beta_0) = -\beta_0^2\alpha_0^3 - 4a\alpha_0^3 + 4\mu\beta_0^2\alpha_0^3 + 4\alpha_0^3 + 2\lambda^2\alpha_0^3\beta_0^2 + 4\mu\beta_0^2\alpha_0^4 + 2\lambda^2\alpha_0^4\beta_0^2 + 4a\alpha_0^4 - \beta_0^2\alpha_0^4 + 2\mu\alpha_1\lambda\beta_0^2\alpha_0^3,$$

$$g_2(\alpha_0,\alpha_1,\beta_0) = -2\beta_0\alpha_0^2 - 2\beta_0\alpha_0^3 + 4\lambda^2\alpha_0^2\beta_0 + 8\mu\alpha_0^2\beta_0 + 4\mu\alpha_1\lambda\beta_0\alpha_0^2 + 4\lambda^2\alpha_0^3\beta_0 + 8\mu\alpha_0^3\beta_0,$$

$$g_3(\alpha_0,\alpha_1,\beta_0) = -1 + 4\mu\alpha_0 + 2\lambda^2\alpha_0 + 2\lambda^2\alpha_0^2 + 4\mu\alpha_0^2 - \alpha_0^2 - 2\alpha_0 + 2\alpha_0\mu\alpha_1\lambda,$$

$$g_4(\alpha_0,\alpha_1,\beta_0) = 3\beta_0^2\alpha_0^2\lambda + \alpha_1\mu\beta_0^2\alpha_0^2 + 3\beta_0^2\alpha_0^3\lambda,$$

$$g_5(\alpha_0,\alpha_1,\beta_0) = 6\lambda\alpha_0\beta_0 + 2\alpha_1\mu\beta_0\alpha_0 + 6\lambda\alpha_0^2\beta_0),$$

$$g_6(\alpha_0,\alpha_1,\beta_0) = 3\lambda + 3\lambda\alpha_0 + \alpha_1\mu.$$

The general solutions of Eqs.(1) and (2) are written as follows

$$A(\eta) = \alpha_0 + \alpha_1\left(\frac{G'}{G}\right) + \alpha_2\left(\frac{G'}{G}\right)^2, \qquad \text{and} \qquad W(\eta) = \beta_0 + K_1\frac{\alpha_1}{\alpha_0}\left(\frac{G'}{G}\right) - K_1\frac{\alpha_2}{\alpha_0}\left(\frac{G'}{G}\right)^2, \tag{19}$$

where $\eta = z - \left(\frac{\beta_0\alpha_0 + K_1}{\alpha_0}\right)t$.

The ODE (12) may then be solved exactly and admits the following solutions

$$
\begin{aligned}
G_1(\eta) &= C_1 e^{r_1\eta} + C_2 e^{r_2\eta}, \quad \text{when } \lambda^2 - 4\mu > 0\\
G_2(\eta) &= [C_1\cos(b\eta) + C_2\sin(b\eta)]e^{\sigma\eta}, \quad \text{when } \lambda^2 - 4\mu < 0\\
G_3(\eta) &= (C_1 + C_2\eta)e^{r\eta}, \quad \text{when } \lambda^2 - 4\mu = 0,
\end{aligned}
\tag{20}
$$

with

$$
r_1 = \frac{-\lambda - \sqrt{\lambda^2 - 4\mu}}{2}, \quad r_2 = \frac{-\lambda + \sqrt{\lambda^2 - 4\mu}}{2}, \quad r = \sigma = \frac{-\lambda}{2} \text{ and } b = \frac{\sqrt{\lambda^2 - 4\mu}}{2},
$$

where C_1 and C_2 are arbitrary constants. We therefore get three categories of travelling wave solution that propagate in the vessel [15]:

Case 1: Hyperbolic function travelling wave solutions

(i) If $\lambda^2 - 4\mu > 0$ and $C_1C_2 > 0$ then we have

$$
A_1(\eta) = \alpha_0 - \frac{\lambda}{4}(2\alpha_1 - \lambda\alpha_2) + (\alpha_1 - \lambda\alpha_2)\frac{\sqrt{\lambda^2 - 4\mu}}{2}\tanh(\theta_1) + \frac{\alpha_2}{4}\left(\lambda^2 - 4\mu\right)\tanh^2(\theta_1), \tag{21}
$$

$$
W_1(\eta) = \beta_0 - \frac{\lambda K_1}{4\alpha_0}(2\alpha_1 + \lambda\alpha_2) + (\alpha_1 + \lambda\alpha_2)K_1\frac{\sqrt{\lambda^2 - 4\mu}}{2\alpha_0}\tanh(\theta_1) - \frac{\alpha_2 K_1}{4\alpha_0}\left(\lambda^2 - 4\mu\right)\tanh^2(\theta_1). \tag{22}
$$

(ii) If $\lambda^2 - 4\mu > 0$ and $C_1C_2 < 0$ then we have

$$
A_2(\eta) = \alpha_0 - \frac{\lambda}{4}(2\alpha_1 - \lambda\alpha_2) + (\alpha_1 - \lambda\alpha_2)\frac{\sqrt{\lambda^2 - 4\mu}}{2}\coth(\theta_2) + \frac{\alpha_2}{4}\left(\lambda^2 - 4\mu\right)\coth^2(\theta_2), \tag{23}
$$

$$
W_2(\eta) = \beta_0 - \frac{\lambda K_1}{4\alpha_0}(2\alpha_1 + \lambda\alpha_2) + (\alpha_1 + \lambda\alpha_2)K_1\frac{\sqrt{\lambda^2 - 4\mu}}{2\alpha_0}\coth(\theta_2) - \frac{\alpha_2 K_1}{4\alpha_0}\left(\lambda^2 - 4\mu\right)\coth^2(\theta_2). \tag{24}
$$

Case 2: Trigonometric function travelling wave solutions

(iii) If $\lambda^2 - 4\mu < 0$ and $C_1C_2 > 0$ then we have

$$
A_3(\eta) = \alpha_0 - \frac{\lambda}{4}(2\alpha_1 - \lambda\alpha_2) - (\alpha_1 - \lambda\alpha_2)\frac{\sqrt{\lambda^2 - 4\mu}}{2}\tan(\theta_3) + \frac{\alpha_2}{4}\left(\lambda^2 - 4\mu\right)\tan^2(\theta_3), \tag{25}
$$

$$
W_3(\eta) = \beta_0 - \frac{\lambda K_1}{4\alpha_0}(2\alpha_1 + \lambda\alpha_2) - (\alpha_1 + \lambda\alpha_2)K_1\frac{\sqrt{\lambda^2 - 4\mu}}{2\alpha_0}\tan(\theta_3) - \frac{\alpha_2 K_1}{4\alpha_0}\left(\lambda^2 - 4\mu\right)\tan^2(\theta_3). \tag{26}
$$

(iv) If $\lambda^2 - 4\mu < 0$ and $C_1C_2 < 0$ then we have

$$
A_4(\eta) = \alpha_0 - \frac{\lambda}{4}(2\alpha_1 - \lambda\alpha_2) - (\alpha_1 - \lambda\alpha_2)\frac{\sqrt{\lambda^2 - 4\mu}}{2}\cot(\theta_4) + \frac{\alpha_2}{4}\left(\lambda^2 - 4\mu\right)\cot^2(\theta_4), \tag{27}
$$

$$W_4(\eta) = \beta_0 - \frac{\lambda K_1}{4\alpha_0}(2\alpha_1 + \lambda\alpha_2) - (\alpha_1 + \lambda\alpha_2) K_1 \frac{\sqrt{\lambda^2 - 4\mu}}{2\alpha_0} \cot(\theta_4) - \frac{\alpha_2 K_1}{4\alpha_0}\left(\lambda^2 - 4\mu\right)\cot^2(\theta_4). \tag{28}$$

Case 3: Rational functions solutions

(v) If $\lambda^2 - 4\mu = 0$, then we have

$$A_5(\eta) = \alpha_0 - \frac{\lambda}{4}(2\alpha_1 - \lambda\alpha_2) + (\alpha_1 - \lambda\alpha_2)\frac{C_2}{C_1 + C_2\eta} + \frac{\alpha_2 C_2^2}{(C_1 + C_2\eta)^2}, \tag{29}$$

$$W_5(\eta) = \beta_0 - \frac{\lambda K_1}{4\alpha_0}(2\alpha_1 + \lambda\alpha_2) + (\alpha_1 + \lambda\alpha_2)\frac{K_1 C_2}{\alpha_0 (C_1 + C_2\eta)} - \frac{\alpha_2 K_1 C_2^2}{\alpha_0 (C_1 + C_2\eta)^2}. \tag{30}$$

where

$$\theta_1 = \frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta + \frac{1}{2}Ln\left(\frac{C_2}{C_1}\right), \qquad \theta_2 = \frac{\sqrt{\lambda^2 - 4\mu}}{2}\eta + \frac{1}{2}Ln\left(\frac{-C_2}{C_1}\right),$$

$$\theta_3 = \frac{\sqrt{4\mu - \lambda^2}}{2}\eta - \arctan\left(\frac{C_2}{C_1}\right), \qquad \theta_4 = \frac{\sqrt{4\mu - \lambda^2}}{2}\eta + \arctan\left(\frac{C_1}{C_2}\right),$$

As a particular case, if we consider $\alpha_1 = \lambda\alpha_2$, the solution (21) becomes

$$A_6(\eta) = \alpha_0 - \mu\alpha_1 - \frac{\alpha_1}{4\lambda}\left(\lambda^2 - 4\mu\right)\operatorname{sech}^2(\theta_1), \tag{31}$$

which is the well-known solitary wave solution of the KdV equation obtained in [10, 16, 17]. In the same way, the velocity (32) reduces to

$$W_6(\eta) = \beta_0 - \frac{K_1\alpha_1}{\lambda\alpha_0}\left(\mu - \lambda^2\right) - \frac{\alpha_1}{\alpha_0}\sqrt{\lambda^2 - 4\mu}\tanh(\theta_1) + \frac{\alpha_1 K_1}{4\alpha_0\lambda}\left(\lambda^2 - 4\mu\right)\operatorname{sech}^2(\theta_1) \tag{32}$$

In what follows, we discuss some features of the nonlinear pressure using for example the solutions (21) and (22). In fact, the elucidation of the pathology of many diseases related to blood flow, such as hypertension and hypotension, requires investigations of the mechanisms responsible for the maintenance of blood pressure in the normal system, and their possible failure within the context of these diseases. Some of the control mechanisms display nonlinear features and therefore confirm the importance of the soliton-like shape of the blood pressure. As a reminder, the pressure of blood waves as defined in this work is given by Eq.(3). Writing the global expression of the pressure is a heavy task, but the second-order derivative is supposed to bring about interesting features in the system. In so doing, the expression of $\frac{\partial^2 A_1}{\partial t^2}$ is given by

$$\begin{aligned}\frac{\partial^2 A_1}{\partial t^2} = &\left(\frac{\beta_0\alpha_0 + K_1}{\alpha_0}\right)^2\left(\frac{\lambda^2}{4} - \mu\right)\Bigg[\frac{\alpha_2}{16}(\lambda^2 - 4\mu)\operatorname{sech}^4(\theta_1) \\ &- \frac{1}{2}(\alpha_1 - \lambda\alpha_2)\sqrt{\lambda^2 - 4\mu}\tanh(\theta_1)\operatorname{sech}^2(\theta_1) - \frac{\alpha_2}{4}(\lambda^2 - 4\mu)\tanh^2(\theta_1)\operatorname{sech}^2(\theta_1)\Bigg],\end{aligned} \tag{33}$$

and the corresponding features of blood pressure are depicted in Figs.1 and 2.

It has been accepted, during decades, that the blood pressure could be assimilated to a wave whose shape evolves between a solitary and a shock-like waves. The same behavior of the blood pressure is observed in Fig.1, where the obtained solution is apparently a pulse-like soliton. It is however known that under physiological conditions, anatomic activities sometimes change, and cardiovascular control in particular, consequently affecting the parameters of its dynamics. Accordingly, the pressure wave that propagates in the vessel can adopt different configuration with important biological implications. In order to illustrate such an aspect, we have plotted the blood pressure P_1 for different values of parameters as illustrated in Fig.2. The corresponding wave solution has some features of the "*kink-pulse*" soliton. Furthermore, the discussed solutions also suggest that the appropriate shape of the blood pressure wave should be combined with a characteristic velocity. This was mainly revealed by the study of Blumgart et al. [18], who showed that the velocity of blood from the harm to the heart gradually decreases due to spontaneous fluctuations in the arm. We therefore qualitatively observed from Figs.1(a) and 2(a) that the pulse-like wave is faster than the "*kink-pulse*" solution.

To summarize, the heart sends blood pressure waves in vessels that are locally expanded. Such a complex dynamics is governed by biofluid dynamics laws such as the conservation of mass, the momenta and forces balance equation for radial motion of the wall. This is explicitly modelled by the well-known Navier-Stockes equations from which few calculations and approximations lead to Kdv or modified KdV equations. Far to perform any such simplifications and approximations, we have solved analytically the generic equations of blood flow through the (G'/G)-expansion method, and particular attention has been paid to the pressure of blood waves. Several travelling wave solutions have been obtained in terms of hyperbolic, trigonometric and rational functions. Comparisons have been made between some obtained solutions and what is commonly accepted in the literature. On the other hand, the obtained solutions have not been widely discussed, since some of them are really new in this field and therefore deserve more attention in order to better bring out their proper biological importance and implication. The method thus worked out will be applied to blood flow models including cardiovascular diseases such as aneurism and many others.

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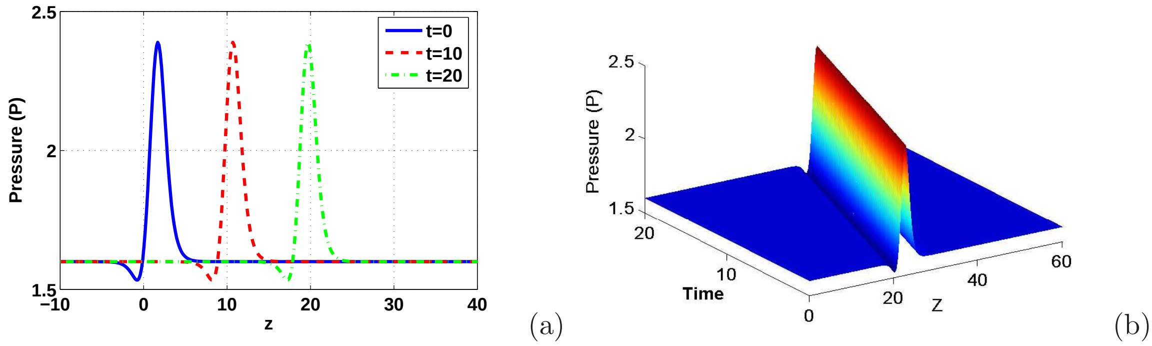

Figure 1: The panels display the features of the blood pressure according to the solutions A_1 and W_1. In panel (a) the pressure is plotted in a moving frame for t=0 (blue line), t=10 (red dashed line) and t=20 (green dashed-dotted line). Panel (b) depicts the 3D propagation of the blood pressure. values of parameters are: $\alpha_0 = 0.005$, $\beta_0 = 0.5$, $K_1 = 0.002$, $a = 0.8$, $\mu = 0.2$, $\lambda = 1.8$, and $C_1 = C_2 = 1$.

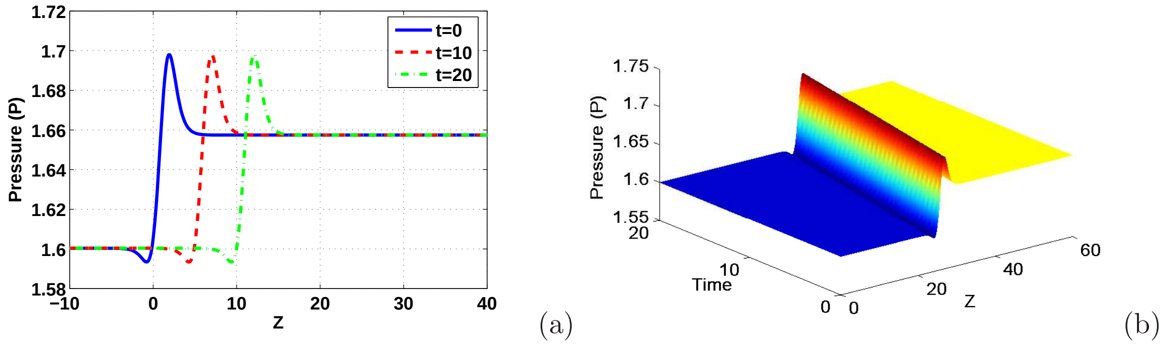

Figure 2: The panels display the features of the blood pressure according to the solutions A_1 and W_1. In panel (a) the pressure is plotted in a moving frame for t=0 (blue line), t=10 (red dashed line) and t=20 (green dashed-dotted line). Panel (b) depicts the 3D propagation of the blood pressure. values of parameters are: $\alpha_0 = 0.25$, $\beta_0 = 0.5$, $K_1 = 0.002$, $a = 0.8$, $\mu = 0.2$, $\lambda = 1.8$, and $C_1 = C_2 = 1$.