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ON DIMENSION THEORY FOR A CERTAIN CLASS OF SIMPLE AH ALGEBRAS

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Abstract

A class of unital diagonal AH algebras will be studied in this paper. The density property of the set of all elements which are nilpotent up to (left and right multiple) unitaries is presented. As a consequence, these algebras have stable rank one. Section 3 also shows that an algebra in this class has the property LP (i.e., the linear span of projections is dense) provided a certain condition. Finally, restricting our attention to a special subclass which includes Villadsen algebras of the first type, we give the necessary and sufficient condition of real rank zero.

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1. Introduction

The classification program of Elliott, the goal of which is to classify amenable C^*-algebras by their K-theoretical data, has been successful for many classes of C^*-algebras, in particular for simple AH algebras with slow dimension growth ([6], [7], [9],[10], [13]). Note that all these simple AH algebras have stable rank one ([5], [1]). Therefore, the AH algebras which may be of most interest nowadays are the ones with higher dimension growth (than zero).

In [12], Goodearl introduced an interesting class of simple AH algebras without any condition on the dimension growth as follows. Let X be a nonempty separable compact Hausdorff space, and choose a subset $S = \{x_1, x_2, x_3, \ldots\}$ such that $\{x_i, x_{i+1}, \ldots\}$ is dense in X for every positive integer i. Let $k_1, k_2, \ldots$ be a sequence of positive integers such that $k_i | k_{i+1}$ for all i and denote by A_i the matrix algebra $M_{k_i}(C(X))$. Define unital block diagonal $*$-homomorphisms $\phi_i : A_i \to A_{i+1}$ by

$$\phi_i(a) = \mathrm{diag}(a, \ldots, a, a(x_i), \ldots, a(x_i)), \ \forall a \in A_i$$

Let α_i be the number copies of a appearing on the right side and we also assume that $k_{i+1}/k_i > \alpha_i > 0$ (i.e., there is at least one block a and at least one block $a(x_i)$ in $\phi_i(a)$). Goodearl showed that such an algebra has stable rank one and the property SP. Moreover, it has real rank zero if, and only if either $\lim_{i\to\infty} \omega_{1,j}$ is zero or X is totally disconnected ([12]).

Let X and Y be compact metric spaces. A $*$-homomorphism ϕ from $M_n(C(X))$ to $M_{nm}(C(Y))$ is said to be *diagonal* if there are continuous maps λ_i from Y to X such that

$$\phi(f) = \begin{pmatrix} f\circ\lambda_1 & 0 & \ldots & 0 \\ 0 & f\circ\lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & f\circ\lambda_n \end{pmatrix}, \quad \forall f \in M_n(C(X)).$$

The λ_i are called the *eigenvalue maps* (or simply *eigenvalues*) of ϕ. The family $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ is called *the eigenvalue pattern* of ϕ and is denoted by $ep(\phi)$. In addition, let P, respectively Q, be projections in $M_n(C(X))$ and $M_{nm}(C(Y))$. An $*$-homomorphism ψ from $PM_n(C(X))P$ to $QM_{nm}(C(Y))Q$ is called diagonal if there exists a diagonal $*$-homomorphism ϕ from $M_n(C(X))$ to $M_{nm}(C(Y))$ such that ψ is reduced from ϕ on $PM_n(C(X))P$ and $\phi(P) = Q$. This definition can also be extended to $*$-homomorphisms

$$\phi : \oplus_{i=1}^n P_i M_{n_i}(C(X_i))P_i \to \oplus_{j=1}^m Q_j M_{m_j}(C(Y_j))Q_j$$

by requiring that each partial map

$$\phi^{i,j} : P_i M_{n_i}(C(X_i))P_i \to Q_j M_{m_j}(C(Y_j))Q_j$$

induced by ϕ be diagonal. The C^*-algebras which can be represented by inductive limits of homogeneous C^*-algebras is called an AH algebra. An AH algebra is called *diagonal* if it can be written as an inductive limit of homogeneous C^*-algebras with diagonal connecting maps.

There are many concepts which are borrowed from dimension theory of topological spaces, such as stable rank ([15]), real rank ([3]), covering dimension ([22]). In this paper, we will study stable rank, real rank and some relative (weaker or stronger) properties (approximately nilpotent elements, the property SP) of a class of diagonal AH-algebras which can be written as inductive limits of homogeneous algebras of the forms $\oplus_{i=1}^{n} P_i M_{n_i}(C(X_i))P_i$ *without any condition on dimension*. This class includes interesting algebras: AF algebras, AI and AT algebras ([17], [8]), Goodearl algebras ([12]) and Villadsen algebras of the first type ([18], [19], [20]). Note that the algebras constructed by Toms in [18] have the same K-groups and tracial data but different Cuntz semigroups. This means that the class of C^*-algebras considered in this paper cannot be classified by K-theory and their tracial data.

The paper consists of five sections. Section 2 introduces the class of AH-algebras which will be studied in this paper. Section 3 discusses about the density of nilpotent elements relative to some unitaries, as a consequence, the algebras have stable rank one. Section 4 concerns the density of the linear span of projections. And finally, section 5 generalizes the sufficient and necessary condition of real rank zero which is a generalization of [[12], Theorem 9].

2. Preliminaries and Definitions

Notation: Denote by M_n the C^*-algebra of all $n \times n$ matrices with complex coefficients while by $M_n(A)$ the algebra of all $n \times n$ matrices with coefficients in A. By a standard basis $\{e_{ij}\}_{1 \le i,j \le n}$ of M_n we mean the set of matrices e_{ij} whose $(i,j)^{\text{th}}$-entry is equal to 1 and zero elsewhere.

We always assume the space X, Y are compact metric spaces with finitely many connected components. If X is connected and $p \in M_n(C(X))$ is a projection, then the rank function $\text{rank}(p(x))$ is constant for every $x \in X$. Hence we can say $\text{rank}(p)$ to be the rank of $p(x)$ for any $x \in X$.

For convenience, let us denote by $\tilde{A}$ the unitalization of A when A does not have the unit and A otherwise.

Denote by $\text{diag}(a_1, a_2, \dots, a_n)$ the block diagonal matrix

$$\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_n \end{pmatrix}$$

Let us consider a C^* algebra of the form $PM_n(C(X))P$, where P is a projection in $M_n(C(X))$. If X is not connected and if X has finitely connected components X_i ($X = \bigsqcup_{i=1}^{k} X_i$), then

$$PM_n(C(X))P = \oplus_{i=1}^{k} P_i M_n(C(X_i))P_i,$$

where P_i is the restriction of P on X_i. Therefore, for any homogeneous algebra of the form (1) below, it can be assumed that each X_i is connected.

We will study AH algebras which can be written as inductive limits of homogeneous algebras of the form:

$$\oplus_{i=1}^{k} P_i M_{n_i}(C(X_i))P_i, \tag{1}$$

where the X_i are connected compact metric spaces and the P_i are projections in the corresponding $M_{n_i}(C(X_i))$ with diagonal connecting maps defined below. The projections P_i are called the cutdown projections.

Definition 2.1. Let ϕ be a unital $*$-homomorphism

$$\phi : \oplus_{i=1}^{k} P_i M_{n_i}(C(X_i))P_i \longrightarrow \oplus_{j=1}^{t} Q_j M_{m_j}(C(Y_j))Q_j,$$

where the P_i and the Q_j are projections in the $M_{n_i}(C(X_i))$ and $M_{m_j}(C(X_j))$, respectively. Then, ϕ is said to be *diagonal* if there exists a diagonal $*$-homomorphism $\tilde{\phi}$ from $\oplus_{i=1}^{k} M_{n_i}(C(X_i))$ to $\oplus_{j=1}^{t} M_{m_j}(C(Y_j))$ such that ϕ is the restriction of $\tilde{\phi}$ on the subalgebra $\oplus_{i=1}^{k} P_i M_{n_i}(C(X_i))P_i$.

Notation: Let us denote by $\mathcal{D}$ the class of C^*-algebras which can be written as an inductive limit of homogeneous C^*-algebras of the form (1) with diagonal connecting maps. We also denote by $\mathcal{D}_1$ a subset of $\mathcal{D}$ consisting of C^*-algebras which can be written as inductive limits $\varinjlim(A_i, \phi_i)$, where the ϕ_i are diagonal and the A_i are of the form

$$A_i = \oplus_{t=1}^{k_i} M_{n_{it}}(C(X_{it})),\ X_i = \sqcup_{t=1}^{k_{it}} X_{it}, \tag{2}$$

the X_{it} are compact connected metric spaces and the n_{it}, k_i are positive integers. It means that an algebra $\varinjlim(A_i, \phi_i) \in \mathcal{D}$ belongs to $\mathcal{D}_1$ if all the cutdown projections of the block algebras A_i of the form (1) are the identity. The algebras in $\mathcal{D}_1$ form a rich class. They include AF algebras, AI and AT algebras (hence the rotation algebras) (see [8]).

Definition 2.2. A C^*-algebra A is said to have *slow dimension growth* if A can be written as an inductive limit of a sequence (A_i, ϕ_i), where

$$A_i = \oplus_{t=1}^{k_i} P_{it} C(X_{it}) \otimes M_{n_{it}} P_{it},\ \text{for } i = 1, 2, 3, \ldots,$$

such that

$$\lim_{i\to\infty} \max_{t} \frac{\dim X_{it}}{\mathrm{rank} P_{it}} = 0.$$

Proposition 2.1. *Let ϕ be a diagonal $*$-homomorphism as stated in Definition 2.1. Then, the following hold.*

(1) *ϕ is unital.*

(2) *The diagonal $*$-homomorphism $\tilde{\phi}$ needs not be unique.*

(3) *The composition of diagonal homomorphisms is again diagonal.*

Proof. (1) ϕ is unital since the map $\tilde{\phi}$ is unital and the subalgebras which are the domain and the range of ϕ have the units.

(2) For example, let's consider a diagonal $*$-homomorphism ϕ defined by

$$\phi : C[0,1] \longrightarrow e_{11} M_2(C[0,1])e_{11},\ \phi(f) = \mathrm{diag}(f, 0),$$

where $e_{11} = \mathrm{diag}(1, 0) \in M_2$. There exist two diagonal extensions of ϕ, let us denote by ψ and λ, from $C([0,1])$ to $M_2(C[0,1])$ given by

$$\psi(f) = \mathrm{diag}(f, f);\ \text{and } \lambda(f) = \mathrm{diag}(f, 1)$$

(3) Let ϕ and ψ be diagonal homomorphisms such that $\phi \circ \psi$ exists. By Definition 2.1, these maps are the restriction of some diagonal homomorphisms $\tilde{\phi}$ and $\tilde{\psi}$ whose domains and the target spaces are the corresponding full matrix algebras. Thus $\phi \circ \psi$ is diagonal, since the composition $\tilde{\phi} \circ \tilde{\psi}$ is diagonal. □

2.1. **Approximately Constant Eigenvalue.** In this subsection we wish to quote the characterization of simplicity for diagonal AH algebras from [11].

Definition 2.3. Let $A = \varinjlim(A_i, \phi_i)$ where the A_i are of the form (2). A is said to have the approximately constant eigenvalue property if for any $i \in \mathbb{N}$, element f in A_i, $\epsilon > 0$ and $x \in X_i$ there exist $j \geq i$ and unitaries $u_t \in A_{jt} = M_{n_{jt}}(C(X_{jt}))$, $t \in \{1, \ldots, k_j\}$ such that

$$||u_t \phi_{ij}^{lt}(f) u_t^* - \mathrm{diag}(f(x), b_t)|| < \epsilon$$

for some appropriately sized b_t such that $\mathrm{diag}(f(x), b_t) \in A_{jt}$.

The Villadsen algebras of the first type have approximately constant eigenvalue property ([18], [19], [20]). These algebras are simple due to the approximately constant eigenvalue property. The following will ensure the class of algebras under our consideration has the approximately constant eigenvalue property.

Theorem 2.1. *Let $A = \varinjlim(A_i, \phi_i)$ where the A_i are of the form (2). If A has the approximately constant eigenvalue property then A is simple. The converse holds if the $*$-homomorphisms ϕ_{ij} are diagonal.*

Proof. Suppose that A has the approximately constant eigenvalue property. As in [[11], Section 2.2], all the connecting $*$-homomorphisms ϕ_{ij} can be assumed to be injective. Let a be any non-zero element in A_i for some positive integer i. Then one of the components of a is non-zero, hence we can assume that A_i has only one component, i.e., $A_i = M_{n_i}(C(X_i))$. Then there is an $x \in X_i$ such that $a(x) \neq 0$. By the definition of the approximately constant eigenvalue property, there is an integer $j > i$, and a unitary u in A_j (we can also assume that A_j has only one component $A_j = M_{n_j}(C(X_j)))$ such that

$$||u \phi_{ij}(a) u^* - \mathrm{diag}(a(x), b)|| < \epsilon$$

We can choose ϵ small enough so that $\phi_{ij}(a)(y)$ is non-zero for every $y \in X_j$. By [[5], Proposition 2.1], A is simple.
The converse is due to [[11], Theorem 3.4]. □

3. Nilpotent Elements

An element x in A is called nilpotent if there exists an n such that $x^n = 0$.
Notation: Let A be a unital C^*-algebra. Denote by $\mathcal{N}(A)$ the set of all elements x in A such that there are unitaries $u, v \in A$, uxv is nilpotent. $ZD(A)$ is denoted by the set of all zero divisors of A and $U(A)$ is the set of all unitaries of A

The fact is the set of all nilpotent elements can be approximated arbitrarily closely by the invertible. This leads us to the following.

Proposition 3.1. *Let A be a unital C^*-algebra. Then the following hold.*

(1) *Every element in $\mathcal{N}(A)$ can be approximated arbitrarily closely by the invertible.*
(2) *If every non-invertible element in A can be approximated arbitrarily closely by elements in $\mathcal{N}(A)$ then A has stable rank one.*
(3) *Suppose that $A = \varinjlim(A_n, \phi_n)$, where the A_n are unital C^*-algebras and the connecting $*$-homomorphisms $\phi_{nm} : A_n \to A_m$ $(m > n)$ are unital and injective. Then A has stable rank one if, for every positive integer n, every non-invertible element x in A_n, there exists an integer $m \geq n$ such that $\phi_{nm}(x)$ belongs to the closure of $\mathcal{N}(A_m)$ in the norm topology.*

Proof. (1) Let $x \in A$. If x is nilpotent and for any positive number ϵ, $-\epsilon = x^n - \epsilon^n 1$ is in $GL(A)$. Hence, x can be approximated by an invertible, namely $x - \epsilon$, to within 2ϵ. Furthermore, for any unitaries u, v in A, uxv is also approximated by an invertible element to within 2ϵ. Therefore, uxv belongs to $GL(A)$ if and only if x is in $GL(A)$.

(2) For any $x \in A$, a non-invertible element and any positive number ϵ, there exists an element y in $\mathcal{N}(A)$ such that $||x - y|| < \epsilon$ by (2). By (1), there is an invertible element z in A such that $||y - z|| < \epsilon$. Hence, $||x - z|| < 2\epsilon$. Thus, the invertible is dense in A.

(3) Since all the connecting homomorphisms ϕ_{nm} are injective, A_n can be identified with its image in A for any n and so A is the closure of $\cup_{n=1}^{\infty} A_n$. For any non-invertible element x in A and any positive number ϵ, there is a positive integer n and an element y in A_n such that $||x - y|| < \epsilon$. We can also assume that y is not invertible. By the hypothesis (3), there is a positive integer $m \geq n$ such that $\phi_{nm}(y)$ belongs to $\mathcal{N}(A_m)$ and hence, $\phi_{nm}(y)$ is in $\mathcal{N}(A)$. By (2), A has stable rank one. $\square$

Proposition 3.1 is usually used to prove stable rank one of a C^*-algebra. To make use of this fact, Rørdam raised a question: *If A is a simple unital C^*-algebra and $x \in ZD(A)$, can we find $u \in U(A)$ such that ux is nilpotent* ([16]). This question can be relaxed as follows without losing our purpose.

Question 3.1. *Let A be a simple unital C^*-algebra and $x \in A$ be non-invertible. Does x belongs to the closure of $\mathcal{N}(A)$?*

We can also state Question 3.1 as: *is the set $inv(A) \cup \mathcal{N}(A)$ dense in A*? where $inv(A)$ is the set of all invertible elements in A. It is obvious that stable rank of A is equal to 1 if and only if $inv(A) \cup \mathcal{N}(A)$ is dense in A by Proposition 3.1.

Example: Question 3.1 has a positive answer if A is a Goodearl algebra ([[12], Lemma 2]).

Proposition 3.2. *Let $A = \varinjlim(A_i, \phi_i)$ be a simple unital AH algebra, where the A_i are of the form (2). If A has the approximately constant eigenvalue property, then for any non invertible* *element $a \in A$ there are a positive integer n and unitaries $u, v \in U_n(A)$ such that $u(a \otimes 1_n)v$ is a nilpotent element in $M_n(A)$. (Note that the $*$-homomorphism ϕ is not necessarily diagonal).*

Proof. Let $A = \varinjlim(A_i, \phi_i)$, a non-invertible element $a \in A$ and a positive number ϵ. Since $\cup_{i>0}\phi_{i\infty}(A_i)$ is dense in A, we can assume that $a \in \cup_{i>0}\phi_{i\infty}(A_i)$. Since the images of nilpotent elements of A_i under $\phi_{i\infty}$ are again nilpotent, we can assume $a \in A_i$ for some i. An element of finitely many components can be approximated by nilpotent elements if and only if each of its components can be approximated by nilpotent elements. Hence, we can assume that each A_j has one component, i.e., $A_j = M_{n_j}(C(X_j))$. Since a is not invertible, there is a point $x \in X_i$ such that $det(a(x)) = 0$. Since A has the approximately constant eigenvalue property, there is an integer $j \geq i$ and a unitary u_0 such that

$$||u_0\phi_{i,j}(a)u_0^* - \mathrm{diag}(a(x), b_1)|| < \epsilon,$$

for some appropriately sized b_1 such that $\mathrm{diag}(a(x), b) \in A_j$. Since $a(x) \in M_{n_i}$ is not invertible, there are unitaries u_1, v_1 in M_{n_i} such that

$$u_1a(x)v_1 = \mathrm{diag}(0, b_2), \ \ b_2 \in M_{n_i-1}$$

Set $u_2 = u_0\mathrm{diag}(u_1, 1_{n_j-n_i}), \ v_2 = \mathrm{diag}(v_1, 1_{n_j-n_i})u_0^*,$ and $b = \mathrm{diag}(b_1, b_2)$. Then,

$$||u_2\phi_{ij}(a)v_2 - \mathrm{diag}(0, b)|| < \epsilon, \ \text{ where } b \in M_{n_j-1}.$$

Take a positive integer n such that $m = \frac{n}{n_j} > n_j - 1$. The number of copies of 0 in $(\mathrm{diag}(0, b) \otimes 1_m)$ is at least m, hence there exists a permutation matrix $e \in M_n$ such that $e(\mathrm{diag}(0, b) \otimes 1_m) \in M_n(A)$ is an upper triangular whose main diagonal entries vanish and thus it is nilpotent. Finally, just take u and v to be $e(u_2 \otimes 1_m)$ and $v_2 \otimes 1_m$, respectively, and the proof ends. □

Theorem 3.1. *Let A be a unital simple AH algebra which can be written as an inductive limit $\varinjlim(A_i, \phi_i)$, where the ϕ_i are diagonal and the A_i are of the form (1). Suppose that $A_1 = \oplus_{t=1}^{k} P_{1t}M_{n_{1t}}(C(X_{1t}))P_{1t}$ and the cutdown projections $P_{it} \in A_1$ are Murray-von Neumann equivalent to the constant projections, then for any non-invertible element $a \in A$ and any positive number ϵ, there exists an element $a_1 \in \mathcal{N}(A)$ such that $||a - a_1|| < \epsilon$.*

In order to prove Theorem 3.1, we need the following lemma.

Lemma 3.1. *Let B be a C^*-algebra, p and q be projections in B. If p and q are Murray-von Neumann equivalent, then pBp is isomorphic to qBq.*

In particular, if $B = M_n(C(X))$ with connected compact metric spectrum X and if q is the constant projection in B of rank m, then qBq is $*$-isomorphic to the full matrix algebra $M_m(C(X))$.

Proof. There is a partial isometry v such that $p = v^*v$ and $q = vv^*$. Define a map ϕ by $\phi(x) = vxv^*$ for all $x \in pBp$. It is straightforward to check that ϕ is a $*$-homomorphism from

pBp to qBq.
Similarly, we can define a $*$-homomorphism ψ from qBq to pBp by $\psi(x) = v^*xv$, for all $x \in qBq$. For any $a \in B$,

$$\phi \circ \psi(qaq) = v(v^*qaqv)v^* = qaq, \text{ and } \psi \circ \phi(pap) = v^*(vpap)v^* = pap.$$

By the same argument, we also have that $\psi \circ \phi$ is the identity map on pBp. Thus ϕ is an isomorphism and its inverse is ψ.

In the case $B = M_n(C(X))$, and q is the constant projection in B of rank m, then pBp is $*$-isomorphic to qBq. Moreover, since $q \in M_n$ is a Hermitian, there is a unitary $u \in M_n$ such that $q = u(e_{11} + e_{22} + \ldots + e_{mm})u^*$, where $\{e_{ij}\}_{i,j=1}^n$ is the standard base of M_n. Then

$$qBq = u(\sum_{i=1}^m e_{ii})u^* Bu(\sum_{i=1}^m e_{ii})u^* = (\sum_{i=1}^m e_{ii})B(\sum_{i=1}^m e_{ii}) = M_m(C(X)).$$

$\square$

Now, we are ready to prove Theorem 3.1.

Proof. Assume $A = \varinjlim(A_i, \phi_i)$, where the A_i are of the form (1) and the ϕ_i are diagonal $*$-homomorphisms. Using the same argument as the beginning of the proof of Proposition 3.2, we can assume that each A_i has only one component, that is, $A_i = P_i M_{n_i}(C(X_i))P_i$, where P_i is a projection in A_i. By the hypothesis, the cutdown projection P_1 of A_1 is Murray-von Neumann equivalent to a projection $Q \in M_{n_1}$. Q is unitary equivalent to $Q_1 = e_{11} + e_{22} + \ldots + e_{m_1}$, where m_1 is the rank of Q and $\{e_{ij}\}_{1 \le i,j \le n_1}$ is the standard base of M_{n_1}. Hence we can say P_1 is Murray-von Neumann equivalent to Q_1. For $i > 1$, define $Q_i = \phi_{i-1}(Q_{i-1})$. Then $Q_i = \phi_{1i}(Q_1)$ is constant, since Q_1 is constant and ϕ_{1i} is diagonal. Let us denote by m_i the rank of Q_i, then $m_{i+1} | m_i$ and $\frac{m_{i+1}}{m_i} = \frac{n_{i+1}}{n_i}$ for all i. By Lemma 3.1, there are $*$-isomorphisms

$$\Theta_i : P_i M_{n_i}(C(X_i))P_i \longrightarrow Q_i M_{n_i}(C(X_i))Q_i = M_{m_i}(C(X)), \quad \Theta_i(a) = v_i a v_i^*,$$

where $P_1 = v_1^* v_1 \sim v_1 v_1^* = Q_1$ and $v_i = \phi_{1i}(v_1)$. Since each ϕ_i is diagonal, by definition, there exists a diagonal $*$-homomorphism $\tilde{\phi_i} : M_{n_i}(C(X_i)) \longrightarrow M_{n_{i+1}}(C(X_{i+1}))$ such that ϕ is the restriction of $\tilde{\phi_i}$ on A_i into A_{i+1}. Let ψ_i be the restriction of $\tilde{\phi_i}$ on $Q_i M_{n_i}(C(X_i))Q_i$, for all i. Then $\psi_i(Q_i) = Q_{i+1}$, therefore, the map ψ_i can be viewed as the map from $Q_i M_{n_i}(C(X_i))Q_i$ to $Q_{i+1} M_{n_{i+1}}(C(X_{i+1}))Q_{i+1}$ and so we have the diagonal AH-algebra $\varinjlim(M_{m_i}(C(X_i)), \psi_i)$. In addition it is straightforward to check that $\Theta_{i+1} \circ \phi_i = \psi_i \circ \Theta_i$ and hence we can assume that $A = \varinjlim(M_{m_i}(C(X_i)), \psi_i)$.

Let $\psi_{i\infty}$ be the $*$-homomorphism from $A_i = M_{m_i}(C(X_i))$ into A. The union $\cup_{i=1}^\infty \psi_{i\infty}(A_i)$ is dense in A. An element belonging to $\mathcal{N}(A_i)$ is again in $\mathcal{N}(\phi_{i\infty}(A_i))$ and so in $\mathcal{N}(A)$. Therefore, it is sufficient to show that every non-invertible element in A_i can be approximated by elements in $\mathcal{N}(A_k)$ for some large enough integer $k \ge i$.

Suppose that $a \in A_i$ is not invertible and $\epsilon > 0$. By Theorem 2.1, A has the approximately constant eigenvalue property. Using the same argument as the proof of Proposition 3.2, there are an integer $j \geq i$ and unitaries $u_2, v_2 \in A_j$ such that $||u_2\phi_{ij}(a)v_2 - \text{diag}(0,b)|| < \epsilon$, where $b \in M_{m_j-1}$. Choose a positive integer k such that $m = \frac{m_k}{m_j} > m_j - 1$. Let $\{\lambda_t\}_{t=1}^m$ be the eigenvalue pattern of ψ_{jk}. Then $\psi_{jk}(\text{diag}(0,b)) = \text{diag}(0, b\circ\lambda_1, 0, b\circ\lambda_2, \ldots, 0, b\circ\lambda_m)$. Since the number of 0's appearing in $\psi_{jk}(\text{diag}(0,b))$ is at least m and m is strictly larger than the size of $b\circ\lambda_t$ for every t, we can choose a permutation matrix $e \in M_{n_k}$ such that $e\psi_{jk}(\text{diag}(0,b)) \in A_k$ is an upper triangular whose main diagonal entries are zero. Hence, $e\psi_{jk}(\text{diag}(0,b))$ is nilpotent. Let $a_1 = \psi_{jk}(u_2^*)\psi_{jk}(\text{diag}(0,b))\phi_{jk}(v_2^*)$. Then,

$$e\psi_{jk}(u_2)a_1\psi_{jk}(v_2^*) = e\psi_{jk}(\text{diag}(0,b))$$

is nilpotent and so $a_1 \in \mathcal{N}(A_k)$. And,

$$||\psi_{ik}(a) - a_1|| = ||\psi_{jk}(u_2)\psi_{ik}(a)\psi_{jk}(v_2) - \psi_{jk}(\text{diag}(o,b))|| \leq ||u_2\psi_{ij}(a)v_2 - \text{diag}(0,b)|| < \epsilon.$$

□

3.1. **Stable Rank.** We will now apply Theorem 3.1 to show that an algebra has stable rank one. Firstly, we recall the notion of stable rank which was introduced by Marc A. Rieffel ([15]). A unital C^*-algebra A is said to have stable rank one (denoted by $\text{sr}(A) = 1$) if the invertible is dense in A. If A is non-unital, stable rank of A (denoted by $\text{sr}(A)$) is the stable rank of its unitalization $\tilde{A}$. It is shown in [15] that for a unital C^*-algebra A, the invertible is dense in A if, and only if the set of all left-sided invertible elements is dense if, and only if the set of right-side invertible elements is dense.

It was shown in [5] and [1] that a simple AH algebra has stable rank one if the dimensions of the spectra of the building blocks in the inductive limit decomposition are bounded, respectively slow dimension growth. The following corollary does not have any assumption of dimension growth.

Corollary 3.1. *Let $A = \varinjlim(A_i, \phi_i)$ be a simple, diagonal AH algebra, where*

$$A_i = \oplus_{t=1}^{k_i} P_{i,t} M_{n_{i,t}}(C(X_{i,t}))P_{i,t}.$$

If the cutdown projections $P_{1,t}$ in A_1 are Murray-von Neumann equivalent to constant ones, then A has stable rank one.

Proof. It follows immediately from Proposition 3.1 and Theorem 3.1. □

Note There are examples of stable rank n for each positive integer n ([21]) if the hypothesis on 'dimension growth' and 'diagonal connecting homomorphisms' are ignored.

4. Linear Span of Projections

Let us recall some notations from [14]. Given a C^*-algebra A, denote by $L(A)$ the closed linear span of projections in A. In this section, we will study the density of the linear span of projections in A. More information about this problem can be found in [14].

Note: Let $A = \varinjlim(A_i, \phi_i)$ be an AH algebra. We can view the continuous functions on the spectrum of each A_i as elements in A_i by the natural embedding in the following two cases.

Case 1. If $A_i = \oplus_{t=1}^{k_i} M_{n_{it}}(C(X_{it}))$, then there is an embedding from $C(X_{it})$ into A_i defined by the composition of the following injective $*$-homomorphisms:

$$C(X_{it}) \longrightarrow M_{n_{it}}(C(X_{it})),\ f \longmapsto \operatorname{diag}(f, 0),\ \forall f \in C(X),$$

and the embedding into the k^{th}-coordinate

$$M_{n_{it}}(C(X_{it})) \longrightarrow \oplus_{t=1}^{k_i} M_{n_{it}}(C(X_{it})), a \longmapsto (0, \dots, 0, a, 0, \dots 0),\ \forall a \in M_{n_{it}}(C(X_{it}))$$

Case 2. If $A_i = \oplus_{t=1}^{k_i} P_{it} M_{n_{it}}(C(X_{it})) P_{it}$ and suppose that the projections P_{it} are Murray-von Neumann equivalent to the constant, by Lemma 3.1, A_i is isomorphic to the homogeneous algebra of form (2). Hence, we can also have the embedding from $C(X_{it})$ into A_i as Case 1.

Theorem 4.1. *Let $A = \varinjlim(A_i, \phi_i)$ be an AH algebra in $\mathcal{D}$, i.e., the ϕ_i are diagonal and the A_i are of the form*

$$A_i = \oplus_{t=1}^{k_i} P_{it} M_{n_{it}}(C(X_{it})) P_{it}.$$

Then $L(A)$ is equal to A, i.e., the linear span of projections is dense in A provided that every self-adjoint element in $\phi_{i\infty}(C(X_{it}))$ can be approximated arbitrarily closely by self-adjoint elements with finite spectra.

Proof. Every element in A can be closely approximated by elements in $\cup_i \phi_{i\infty}(A_i)$. Hence, it is sufficient to show that for any element $a \in A_i$ and any positive number ϵ, there is an $j \geq i$ such that $\phi_{ij}(a)$ can be approximated by a linear combination of projections in A_j to within ϵ. If $\phi_{ij}(a)$ has finitely many components, it is enough to show that each component can be approximated by linear combinations of projections in A_j to within ϵ. Therefore we can assume that each A_t has only one component, that is, $A_t = M_{n_t}(C(X_t))$ for every positive integer t. We can also assume that $a = a^*$ since a can be written as a sum of two self-adjoint elements. Let $a = (a_{st}) \in M_{n_i}(C(X_i))$, where the a_{st} are continuous functions on X_i. Since $a_{st} = \bar{a}_{ts}$, the a_{ss} are real valued functions and we can write $a_{st} = \bar{a}_{ts} = f_{st} + \imath g_{st}$ with $\imath^2 = -1$ and f_{st}, g_{st} are real valued functions for every $1 < s < t < n_i$. Let $\{e_{st}\}$ be the standard base of M_{n_i}. We can express the matrix a as a sum $a = B + C + D$, where

$$B = \sum_{s=1}^{n_i} a_{ss} e_{ss} \quad C = \sum_{1 \leq s < t \leq n_i} f_{st}(e_{st} + e_{ts}) \text{ and } D = \imath \sum_{1 \leq s < t \leq n_i} g_{st}(e_{st} - e_{ts})$$

Since $B = \operatorname{diag}(a_{11}, a_{22}, \dots, a_{n_i n_i})$ and each a_{ss} can be approximated by self-adjoint elements with finite spectrum, there is an integer $j_1 \geq i$ such that $\phi_{ij_1}(B)$ can be approximated by a self-adjoint element in A_{j_1} with finite spectrum to within $\frac{\epsilon}{3}$. Hence, there is a family of pairwise orthogonal projections E_t in A_{j_1} and scalars α_t such that

$$||\phi_{ij_1}(B) - b|| < \frac{\epsilon}{3}, \text{ where } b = \sum_{t=1}^{n_{j_1}} \alpha_t E_t \in A_{j_1} \tag{3}$$

To approximate C by linear combinations of projections, first we consider

$$\begin{pmatrix} 0 & f_{12} \\ f_{12} & 0 \end{pmatrix} = u^* \begin{pmatrix} -f_{12} & 0 \\ 0 & f_{12} \end{pmatrix} u = u^*(-f_{12}e_{11} + f_{12}e_{22})u, \text{ where } u = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

By hypothesis, $\phi_{i\infty}(f_{12})$ can be approximated by self-adjoint elements with finite spectra. Thus there is an $j_2 \geq i$ and a linear combination of projections $c_1 \in A_{j_2}$ such that $||\phi_{ij}(-f_{12}e_{11} + f_{12}e_{22}) - c_1|| < \frac{\epsilon}{3n_i}$. Furthermore, for each pair $s < t$, the matrix $e_{st} + e_{ts}$ is unitary equivalent to $e_{12} + e_{21}$. Hence, $\phi_{ij}(-f_{st}e_{st} + f_{st}e_{st})$ can also be approximated by a linear combination of projections in A_{j_2}, denoted by c_s, to within $\frac{\epsilon}{3n_i}$. This implies

$$||\phi_{ij_2}(C) - c|| < \frac{\epsilon}{3} \tag{4}$$

Since

$$\begin{pmatrix} 0 & g_{12} \\ -g_{12} & 0 \end{pmatrix} = v^* \begin{pmatrix} -\imath g_{12} & 0 \\ 0 & \imath g_{12} \end{pmatrix} v = \imath(v^*(-g_{12}e_{11} + g_{12}e_{22})v), \text{ where } v = \begin{pmatrix} \frac{\imath}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{\imath}{\sqrt{2}} \end{pmatrix}$$

and the same argument when we approximate C, there is an integer $j_3 \geq i$ and a linear combination of projections $d \in A_{j_3}$ such that

$$||\phi_{ij_3}(D) - d|| < \frac{\epsilon}{3} \tag{5}$$

Take $j = \max\{j_1, j_2, j_3\}$, then

$$||\phi_{ij}(a) - \phi_{j_1j}(b) - \phi_{j_2j}(c) - \phi_{j_3j}(d)|| < \epsilon,$$

and $\phi_{j_1j}(b) + \phi_{j_2j}(c) + \phi_{j_3j}(d)$ is a linear combination of projections in A_j. $\square$

Note: A unital C^*-algebra is said to have real rank zero if and only if every self-adjoint element can be approximated arbitrarily closely by self-adjoint elements whose spectra do not contain zero (or equivalently, by invertible self-adjoint elements). The fact is that a C^*-algebra has real rank zero if and only if every self-adjoint element can be approximated arbitrarily closely by a (finitely) linear combination of pairwise orthogonal projections in A. So the property (LP) (i.e., the linear span of projections is dense) is weaker than real rank zero.

5. Real Rank

5.1. **Background and Notation.** The concept of real rank of a C^*-algebra was introduced by Lawrence G. Brown and Gert K. Pedersen in [3] and since then has been developed rapidly and studied in [2], in [1], in [4], etc. Real rank might be viewed as a non-commutative dimension and is borrowed from Rieffel's ideas of stable rank. The real rank of a unital C^*-algebra B, denoted by $RR(B)$, is the least integer n such that $Lg_n(B) \cap B_{sa}^n$ is dense in B_{sa}^n, where B_{sa} denotes the set of all self-adjoint elements of B. If there is no such integer n, we say that real rank of B is infinite. For a non-unital C^*-algebra B, $RR(B)$ is defined to be $RR(\tilde{B})$.

If a unital C^*-algebra B is commutative, the real rank and the dimension of spectrum of B are the same. This equality comes from the fact that the covering dimension of the spectrum of B is the least positive integer n such that every continuous function from spectrum of B to

$\mathbb{R}^{n+1}$ can be approximated arbitrarily closely by nowhere zero functions from the spectrum of B to $\mathbb{R}^{n+1}$.

There is a close relation between the stable rank and the real rank of a C^*-algebra, and in particular ([3]),

$$RR(B) \leq 2sr(B) - 1. \tag{6}$$

If $sr(B)$ is equal to one, then $RR(B)$ is either equal to zero or one. Since an AI algebra is an AF algebra if and only if its real rank is zero and since there are many AI algebras which are not AF, equality can occur in (6) or not.

The most studied case is real rank zero. It can be seen in [3] that a unital C^*-algebra B has real rank zero if, and only if, any self-adjoint element can be approximated arbitrarily closely by self-adjoint elements with finite spectra if, and only if, every non-zero hereditary C^*-subalgebra has an approximate unit of projections. An immediate consequence from the results above is that every finite dimensional C^*-algebra has real rank zero, and so does any AF algebra.

Suppose that B is a simple AH algebra. Then, B has real rank zero if, and only if, its projections separate the traces provided that this algebra has slow dimension growth ([1]). This equivalence was first studied when the dimensions of the spectra of the building blocks in the inductive limit decomposition of B were no more than two, see [2] .

Let B be a C^*-algebra. Suppose that

$$B = \bigoplus_{i=1}^{k} C(X_i) \otimes M_{n_i},$$

where X_i is a connected compact metric space for every i. Set $X = \bigsqcup_{i=1}^{k} X_i$. The following notations are quoted from [2] and [1].

Let a be any self-adjoint element in B. For any x in X_i, any positive integer m, $1 \leq m \leq n_i$, let λ_m denote the m^{th} lowest eigenvalue of $a(x)$ counted with multiplicity.

The *variation of the eigenvalues* of a, denoted by $EV(a)$, is defined as the maximum of the nonnegative real numbers

$$\sup\{|\lambda_m(x) - \lambda_m(y)|\ x, y \in X_i\},$$

over all i and all possible values of m.

The *variation of the normalized trace* of a, denoted by $TV(a)$, is defined as

$$\sup_{1\leq i\leq k}\{|\mathrm{tr}(a(x)) - \mathrm{tr}(a(y))|,\ \forall x, y \in X_i\},$$

where tr denotes the normalized trace of M_n, for any positive integer n.

Proposition 5.1. *Let B be an inductive limit of homogeneous C^*-algebras B_i with morphisms ϕ_{ij} from B_i to B_j. Suppose that B_i has the form*

$$B_i = \bigoplus_{t=1}^{k_i} M_{n_{it}}(C(X_{it})),$$

where k_i and n_{it} are positive integers, and X_{it} is a connected compact metric space, for every positive integer i and $1 \leq t \leq k_i$. Consider the following conditions:

(1) *The projections of B separate the traces on B.*

(2) *For any self-adjoint element a in B_i and $\epsilon > 0$, there is a $j \geq i$ such that*

$$TV(\phi_{ij}(a)) < \epsilon.$$

(3) *For any self-adjoint element a in B_i and any positive number ϵ, there is a $j \geq i$ such that*

$$EV(\phi_{ij}(a)) < \epsilon.$$

(4) *B has real rank zero.*

(i) *The following implications hold in general.*

$$(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1).$$

(ii) *If B is simple, then the following equivalences hold.*

$$(3) \Leftrightarrow (2) \Leftrightarrow (1).$$

(iii) *If B is simple and has slow dimension growth, then all the conditions (1), (2), (3) and (4) are equivalent.*

Proof. The statements (i) and (ii) are proved in [[2], Theorem 1.3]. The statement (iii) is an immediate consequence of the statement (ii) and [[1], Theorem 2]. □

The concept of tracial rank can be found in [13]. For the convenience of readers, it will be recalled here.

Definition 5.1. Let A be a simple unital C^*-algebra. A is said to have tracial topological rank zero (written $TR(A) = 0$) if for any positive number ϵ, and finite subset $\mathcal{F}$ of A containing a non-zero positive element a, there exists a finite-dimensional C^*-subalgebra B of A with $1_B = p$ such that

(1) $||px - xp|| < \epsilon$, for all $x \in \mathcal{F}$,

(2) $||pxp - b_x|| < \epsilon$, for some element $b_x \in B$, every $x \in \mathcal{F}$ and

(3) $1 - p \preceq q$ for some projection q in the hereditary C^*-subalgebra generated by a.

5.2. **Results.** In [12], a necessary and sufficient condition for a Goodearl algebra to have real rank zero is given in terms of the limit of the *weighted identity ratio* sequence. In this subsection, we will generalize this result to a subclass of $\mathcal{D}_1$ (let us say *generalized Goodearl algebras*). This class includes the interesting algebras constructed in [18] and Villadsen algebras of the first type ([19]). Define a simple algebra in the class under consideration as $A = \varinjlim(A_i, \phi_i)$, where

$$A_i = M_{n_i}(C(X_i)) \text{ and } \phi_i(a) = \mathrm{diag}(a(x_1), \ldots, a(x_{l_i}), a \circ \lambda_1, \ldots, a \circ \lambda_{\alpha_i}),$$

and where $x_{i1}, x_{i2}, \ldots, x_{il_i}$ are elements given in X_i, and $\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{i\alpha_i}$ are nonconstant continuous maps from X_j to X_i. If all the X_i are the same as X, the sequence $x_{i1}, x_{i2}, \ldots$ is dense and all the eigenvalue maps are the identity on X, then the algebra under consideration is a Goodearl algebra.

If X_i is totally disconnected for every positive integer i, then A has real rank zero since all $RR(A_i) = 0$ and an inductive limit of algebras with real rank zero has real rank zero. For each pair of positive integers i, j, where $i < j$, set

$$\omega_{i,j} = \alpha_i \alpha_{i+1} \dots \alpha_{j-1} \frac{n_i}{n_j}.$$

Note that the number $\alpha_i \alpha_{i+1} \dots \alpha_{j-1}$ is the number of non-constant eigenvalue maps of ϕ_{ij}. So we might call $\omega_{i,j}$ the *weighted non-constant ratio* for ϕ_{ij}. It is clear that

$$\omega_{i,j+1} = \omega_{i,j} \alpha_j \frac{n_i}{n_{j+1}}$$

and $0 < \alpha_j \frac{n_i}{n_{j+1}} < 1$. Hence, the sequence $\{\omega_{i,j}\}$, for a fixed integer i, is decreasing. Thus, it is convergent as $j \to \infty$.

We are now ready to state the theorem of this section. This theorem is a generalization of [[12], Theorem 9].

Theorem 5.1. *Let A be a simple AH algebra as above. The following statements are true.*

(1) *If $\lim\limits_{i \to \infty} \omega_{1,i} = 0$, then A has tracial topological rank zero. In particular, it has real rank zero and weakly unperforated $K_0(A)$.*

(2) *If A has real rank zero, then $\lim\limits_{i \to \infty} \omega_{1,i} = 0$ provided that A satisfies the following conditions:*
   - (i) *For any positive integer i and any point x_0 in X_i, there is an open neighbourhood U of x_0 such that the union of $\lambda^{-1}(U)$ for all non-constant eigenvalue maps λ of ϕ_{ij} is not the same as X_j for any positive integer $j > i$.*
   - (ii) *For any positive integer i and any integer $j_0 > i$, there is a $j \geq j_0$ with a point x_j in the spectrum of A_j and a point y_0 in the spectrum of A_i such that $\lambda(x_j) = y_0$, for every non constant eigenvalue map λ of ϕ_{ij}.*

Proof. (1). Suppose that $\lim\limits_{i \to \infty} \omega_{1,i} = 0$. Hence, given any positive integer i, $\lim\limits_{j \to \infty} \omega_{i,j} = 0$. For any positive integer ϵ, there is a positive integer $j > i$ such that $\omega_{i,j} < \epsilon$. Let p denote the projection in M_{n_j} corresponding to the constant eigenvalue maps of ϕ_{ij}, that is,

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

where the matrix 1 on the upper left-hand corner is the identity of $M_{n_j - \alpha_i \alpha_{i+1} \dots \alpha_{j-1}}$. Then p is the required projection in the definition of tracial topological rank zero.

(2). Suppose that A satisfies the conditions (i) and (ii) of Theorem 5.1(2). Without loss of generality, we can assume that $n_1 = 1$, i.e., $A_1 = C(X_1)$. Let $i > 1$ be any positive integer. By the condition (ii) of (2), passing to a subsequence if necessary, there is a point x_i in X_i and a point y_0 in X_1 such that $f(x_i) = y_0$ for every non constant eigenvalue map f of ϕ_{1i}. By (i) there is a neighbourhood U of y_0 such that the union of the pre-image of U of all non-constant eigenvalue maps of ϕ_{1i} is not equal to X_i. Hence, there is a point z_i in X_i such that $\lambda(z_i)$ does not belong to U, for every non-constant eigenvalue map λ of ϕ_{1i}. Therefore, the variation of the normalized trace of $\phi_{1i}(a)$ is larger than $\omega_{1,i}$, where a is a continuous function defined on X_1 such that $a(y_0) = 1$ and a vanishes outside the neighbourhood U. If $\lim_{i\to\infty} \omega_{1,i} \neq 0$, then the sequence $\{TV(\phi_{1i}(a))\}$ does not converge to zero as $i \to \infty$. By Proposition 5.1, the real rank of A is not zero while A has stable rank one by Corollary 3.1. Therefore, A has real rank one. $\square$

Corollary 5.1. *Let A be as in Theorem 5.1 and such that A also satisfies conditions (i) and (ii) in this theorem. Then the following are equivalent.*

(i) *A has real rank zero.*
(ii) $\lim_{i\to\infty} \omega_{1,i} = 0$.
(iii) $TR(A) = 0$.
(iv) *A has slow dimension growth and projections in A separate the traces.*

If any statement above holds, then A has fundamental comparison property.

Proof. (i) $\leftrightarrow$ (ii) $\to$ (iii) follows from Theorem 5.1.

(iii) $\to$ (iv) $\to$ (i) follows from Corollary 2.9 of [13], which states that given any unital simple AH algebra A, A has tracial topological rank zero if, and only if, A has slow dimension growth and projections in A separate the traces if, and only if, A has real rank zero, stable rank one and weakly unperforated $K_0(A)$. $\square$

Remark 5.1. When the space X_i in Theorem 5.1 is the same as a connected compact metric space X for every i, and supposing that every non-constant eigenvalue map is the identity, then the algebra discussed in Theorem 5.1 is a Goodearl algebra. The identity map on X obviously satisfies the conditions (i) and (ii) of the theorem. Therefore the sufficient and necessary condition, for a Goodearl algebra to have real rank zero in [12], is just a special case of Theorem 5.1.

Example: The Villadsen algebras of the first type ([19]) satisfy the hypothesis of Theorem 5.1, since the nonconstant eigenvalue maps are just the coordinate-projections. Moreover, if the spectrum of each building block in the inductive limit decompositions of these algebras are just the products of $[0, 1]$, then we obtain the algebras constructed in [18]. This shows that the class of algebras satisfying the hypotheses of Theorem 5.1 contains many interesting algebras.

If a Goodearl algebra, or more generally, an algebra that satisfies the hypothesis of Corollary 5.1 has real rank zero, then it has slow dimension growth. Therefore, any algebra, which satisfies the hypothesis of Corollary 5.1 and does not have slow dimension growth, has real rank one. However, for the algebras under consideration with real rank one, we still do not know if there is any algebra with a unique trace, or more generally, any algebra whose projections separate the traces.

In the absence of the hypothesis of the dimension growth, statement (iii) in Proposition 5.1 fails. The counter-examples can be found in [21]. In this paper, for each positive integer $n \geq 2$, Villadsen constructed a simple AH algebra with unique trace and its real rank is either n or $n-1$. However, the question *'Suppose that projections of a simple inductive limit of homogeneous C^*-algebras with diagonal $*$-homomorphisms between the building blocks separate the traces, does it have real rank zero?'* is still open for simple AH algebras with diagonal morphisms between the building blocks in the inductive limit decompositions but non-zero dimension growth.

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