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# NONULTRALOCAL QUANTUM ALGEBRA AND 1D ANYONIC QUANTUM INTEGRABLE MODELS

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**Abstract**

Applying braided Yang-Baxter equation quantum integrable and Bethe ansatz, solvable 1D anyon lattice and field models are constructed. Along with known models we discover novel lattice anyon and q-anyon models as well as nonlinear Schödinger equation (NLS) and the derivative NLS anyon quantum field models, N-particle sectors of which yield the well-known anyon gases, interacting through $\delta$ and derivative $\delta$ function potentials. As a byproduct we discover a new anyon quantum group Hopf algebra with unusual braided multiplication.

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# 1 Introduction

Anyons [1] are receiving renewed attention after their experimental confirmation [2] and the promise of potential applications in quantum computation [3]. Although the anyons live in two space-dimensions, they remarkably retain their basic properties when projected to one-dimension (1D) [4]. Therefore, since exactly solvable models can be constructed in 1D, interacting 1D anyon models introduced in [5, 6, 7] are becoming increasingly popular in recent years [4, 8]. However, though these models capture the exchange algebra of anyon operators at the space-separated points $x \neq y$, they cannot reproduce the required anyonic commutation relation (CR) at $x = y$ and hence they behave like bosons [5, 6, 4, 8] or fermions [7] at the coinciding points and consequently cannot interpolate between bosons and fermions in the entire domain. This remains as a major drawback of the existing 1D anyon models.

Another unsettled problem is that, unlike the nonlinear Schödinger equation (NLS) and the derivative NLS, which give the known solvable Bose gases [9, 10] at their N-particle sectors, no anyon quantum field models are discovered yet, which could yield the known anyon gases at the N-particle case, [5, 6].

Our aim here is to resolve both these unsolved problems by finding quantum integrable novel anyonic lattice and field models, based on a braided extension of the Yang-Baxter equation (BYBE) [11, 12]. Note that the anyons, not commuting at space-separated points, belong to nonultralocal models and go beyond the standard formulation of the quantum inverse scattering method [13].

Remarkably, our approach through the BYBE leads also to the discovery of a new anyon quantum group exhibiting Hopf algebra properties with nonultralocal braiding relations.

The arrangement of the paper is as follows. Sect. 2 describes the brief history of 1D anyon models. Sect. 3 describes the BYBE and the reletad quantum integrability. Subsections 3.4-5 construct the quantum integrable lattice anyon and the NLS anyon field model. Subsections 3.6-8 account for the q-anyon and the derivative NLS anyon field model. Sect. 4 presents the anyon quantum group with Hopf algebra structure. Sect. 5 is the concluding section followed by the bibliography.

# 2 Exactly solvable 1D boson and anyon gases

Surprisingly, anyons continue to exhibit nontrivial exchange and cross-over properties even when projected to 1D, with the 2-particle anyon wave function showing

$$\Phi(x_1, x_2) = e^{-i\theta}\Phi(x_2, x_1), \tag{1}$$

under exchange and an intriguing sensitivity on the boundary condition on a chain of length $L$ [4]

$$\Phi(x_1 + L, x_2) = e^{-2i\theta}\ \Phi(x_1, x_2 + L), \tag{2}$$

confirming that the *passing* of particle 1 through 2 in 1D is not the same as 2 *passing* through 1. This reflects the known property of the standard 2D anyon, where the effect of particle 1 going around 2 is different from that of 2 going round 1. Note that from (1 -2) one recovers the usual bosonic behavior at $\theta = 0$,, while $\theta = \pi$ corresponds to the fermions. The focus on 1D anyons has been intensified during the recent years, since together with the preserving of the basic properties of the standard anyons, they could be exactly solvable in 1D, offering detailed analytic result, which should be valuable for analysing standard anyons.

Apart from the well-known Calogero model with anyon-type exchange statistics, there is an interesting history of 1D anyon models belonging to the exactly solvable class.

### 2.1 Exactly solvable Bose gases

Bose gas interacting through $\delta$-function potential is a celebrated exactly solvable model introduced by Lieb & Liniger [9]

$$H_N^{(b1)} = -\sum_k^N \partial_{x_k}^2 + \sum_{<k,l>} c\delta(x_k - x_l). \tag{3}$$

After about thirty years another solvable Bose gas model, interacting through derivative $\delta$-function potential was also proposed [10]

$$H_N^{(b2)} = -\sum_k^N \partial_{x_k}^2 + \sum_{<k,l>} i\kappa\delta(x_k - x_l)\left((\partial_{x_k} + \partial_{x_l})\right) \tag{4}$$

### 2.2 Exactly solvable anyon gases and $N$-particle anyon models

After the success of Bose gases with singular potentials, there were naturally attempts to build Bose gases with even higher singular potentials, like *double* $\delta$-function potentials

$$\gamma_1 \sum_{<j,k,l>} \delta(x_j - x_k)\delta(x_l - x_k) + \gamma_2 \sum_{<k,l>} (\delta(x_k - x_l))^2,$$

which, however, remained unsuccessful until the introduction of a $\delta$-function anyon gas model [5]. This exactly solvable 1D anyon model was shown, in fact, to be equivalent to a double $\delta$-Bose gas with its coupling constants related by $\gamma_1 = \gamma_2 = \kappa^2$.

Recently, another exactly solvable 1D anyon gas model interacting through derivative-$\delta$-function potential is also proposed [6]. These anyon gas models, with different research groups focusing on their various aspects, are gaining growing popularity [8]. In a recent work [7] other nearest neighbor lattice anyon models, solvable by the algebraic Bethe ansatz, were proposed. However, as mentioned, all 1D anyon models have an inherent drawback: they behave like bosons or fermions at the coinciding points.

It is well known that there are quantum integrable field models, $N$-particle sectors of which correspond to the exactly solvable Bose gases. In fact the nonlinear Schödinger equation (NLS)

$$H^{(b1f)} = \int dx(\psi_x^\dagger \psi_x + c(\psi^\dagger \psi)^2) \tag{5}$$

in bosonic field $[\psi(x), \psi^\dagger(y)] = \delta(x-y)$ corresponds to the $\delta$-Bose gas, while the $N$-particle model of the derivative NLS to the derivative $\delta$-function Bose gas.

However, an important question surrounding the 1D anyon gases, that remained unanswered up to this date is that, what are the anyonic QFT models, $N$-particle sectors of which would generate the anyon gases interacting through $\delta$ and derivative $\delta$ -function potentials. Our aim here is to take up this challenging problem and discover the needed quantum integrable anyonic QFT models. We also intend to construct integrable anyonic lattice and field models in a systematic way through Yang-Baxter equation, so as to guarantee anyonic commutation relations at all points, including the coinciding points, thus resolving the existing problem of the anyon models, not obeying anyon exchange at the coinciding points.

# 3 Construction of integrable anyon models through braided Yang-Baxter equation

Anyon models due to the noncommutation of anyon fields at space seperated points belong to the problem of nonultralocal models and go beyond the standard formulation of quantum integrable systems based on the YBE. We have to use therefore an extension of the YBE with nontrivial braiding (BYBE) developed by us [11], for systematic generation of the anyon commutation relations as well as for the construction of quantum integrable anyon models.

## 3.1 Braided Yang-Baxter equation

The BYBE represents two different commutation relations for the Lax operator $L_{aj}(u)$, given at the coinciding and at noncoinciding points, expressed through the standard quantum $R(u-v)$-matrix in addition to a braiding matrix $Z$:

$$R_{12}(u-v)Z_{21}^{-1}(L_{1j}(u)Z_{21}L_{2j}(v) = Z_{12}^{-1}L_{2j}(v)Z_{12}L_{1j}(u)R_{12}(u-v), \tag{6}$$

at the lattice sites $j = 1, 2, \ldots .N$, together with the braiding relation (BR):

$$L_{2k}(v)Z_{21}^{-1}L_{1j}(u) = Z_{21}^{-1}L_{1j}(u)Z_{21}L_{2k}(v)Z_{21}^{-1} \tag{7}$$

for $k > j$, representing nonultralocality, i.e. noncommutativity at space separated points. Recall that the quantum $R(u-v)$ matrix is a $4 \times 4$ matrix

$$R(\lambda) = \begin{pmatrix} a(\lambda) & & & \\ & b(\lambda) & c & \\ & c & b(\lambda) & \\ & & & a(\lambda) \end{pmatrix} \tag{8}$$

with rational:

$$a(\lambda) = \lambda + \alpha, \; b(\lambda) = \lambda, \; c = \alpha \tag{9}$$

or trigonometric:

$$a(\lambda) = \sin(\lambda + \alpha), \; b(\lambda) = \sin\lambda, \; c = \sin\alpha \tag{10}$$

solutions. We consider both of these forms and show that they would generate two different classes of anyonic integrable models. The braiding matrix $Z$ containing the anyonic parameter $\theta$, may be given in the graded form

$$Z = \sum_{a,b} e^{i\theta(\hat{a}\cdot\hat{b})} e_{a,b} \otimes e_{b,a}, \quad \hat{a} = 0, 1 \text{ denotes gradings,}$$

which satisfies all the relations as required for the braided generalization [11]. For a $4 \times 4$ matrix with the choice $\hat{1} = 0, \ \hat{2} = 1$ we get the simplest form

$$Z = diag(1, 1, 1, e^{i\theta}) \tag{11}$$

which we use in constructing all our anyonic models. It is evident that for $\theta = 0 : Z = I$, BYBE (6) reduces to the standard YBE $R(u - v)L_{1j}(u)L_{2j}(v) = L_{2j}(v)L_{1j}(u)R(u - v)$, while the BR (6) recovers the ultralocality condition $[L_{2k}(v), L_{1j}(u)] = 0, \ k \neq j$, related to the bosonic commutativity.

### 3.2 Quantum integrable model construction

For building the Hamiltonian of the model we have to construct conserved quantities by switching over from the local to a global picture, by defining the transfer matrix as a global quantum operator acting on the multi-particle Hilbert space:

$$\tau(u) = trace_a(L_{a1}(u) \dots L_{aN}(u)),$$

which generates all conserved quantities $\log \tau(u) = \sum_n C_n u^n$. The BYBE guarantees that $[\tau(u), \tau(v)] = 0$, and hence $[C_n, C_m] = 0$, ensuring the quantum integrability of the model, while the Hamiltonian of the model can be chosen as any of the conserved operators: $H = C_n, \ n = 1, 2, 3, \dots$. The Lax operator $L_j(u)$ may be constructed as a solution of the BYBE (6) using the known solution of the quantum $R$-matrix and the braiding matrix $Z$.

Let us consider first the rational $R$-matrix solution (8-9) together with the $Z$-matrix (11). The anyonic commutation relations (CR) are obtained directly from the BYBE and the BR, different realizations of which construct different types of anyon algebra.

### 3.3 Lattice hard-core anyon model

We can construct through the above scheme a nearest-neighbor interacting anyon model, proposed recently [7] as

$$C_1 = H^{(1a)} = \sum_{k=1}^{N} 2n_k n_{k+1} + +a_k a_{k+1}^\dagger + a_k^\dagger a_{k+1}, \quad n_k \equiv a_k^\dagger a_k, \tag{12}$$

with anyonic CR at space-separated points $k > l$:

$$a_k a_l^\dagger = e^{i\theta} a_l^\dagger a_k, \quad a_k a_l = e^{-i\theta} a_l a_k \tag{13}$$

However, at the coinciding points one gets only fermionic relations $[a_k, a_k^\dagger]_+ = 1, \quad a^2 = 0$, confirming the drawback as mentioned above.

### 3.4 Novel anyon lattice model

As a different realization of the BYBE we construct another quantum integrable anyon model with next-nearest neighbor and higher order nonlinear interactions given by the Hamiltonian

$$C_3 = \mathrm{H}^{(2a)} = \sum_k (\psi^\dagger_{k+1}\psi_{k-1} - (n_k + n_{k+1})\psi^\dagger_{k+1}\psi_k + \frac{1}{3\Delta^2} n^3_k, ) \tag{14}$$

with $n_k = p_k + \Delta^2 \psi^\dagger_k \psi_k$, with $p_k$ related to the number operator. It is remarkable that this model gives finally the needed anyonic CR at the coinciding points $k$:

$$\psi_k \psi^\dagger_k - e^{-i\theta} \psi^\dagger_k \psi_k = p_k \frac{1}{\Delta} \tag{15}$$

together with at noncoinciding points $k > j$:

$$\psi_k \psi^\dagger_j = e^{i\theta} \psi^\dagger_j \psi_k. \tag{16}$$

Thus we solve one of the existing problems of the anyon models by achieving the anyonic CR at all points.

### 3.5 Quantum integrable NLS anyon field model

Taking carefully the continuum limit of the above lattice anyon model (14) with the lattice constant $\Delta \to 0$, $k \to x$, and the field $\psi_k \to A(x)$, we can derive a novel NLS anyon field model

$$\hat{H}^{(3a)} = \int dx (A^\dagger_x A_x + c(A^\dagger A)^2) \tag{17}$$

with the anyon field operator $A(x)$ obeying again the needed CR at all points, obtained from its lattice counterpart (15,16) at the coinciding points $x \to y$

$$A(x)A^\dagger(y) - e^{i\theta} A^\dagger(y)A(x) = \delta(x - y) \tag{18}$$

and at $x > y$:

$$A(x)A^\dagger(y) = e^{i\theta} A^\dagger(y)A(x), \tag{19}$$

$$A(x)A(y) = e^{-i\theta} A(y)A(x), \tag{20}$$

Clearly these anyonic CR can interpolate between bosons (at $\theta = 0$) and fermions (at $\theta = \pi$) at all points.

Now we solve another outstanding problem mentioned in the introduction by finding the $N$-particle sector

$$|N >= \int d^N x \sum_{\{x_l\}} \Phi(x_1, x_2, \ldots x_N) A^\dagger(x_1)A^\dagger(x_2) \cdots A^\dagger(x_N)|0 > \tag{21}$$

of the NLS anyon field model (17), which gives indeed the well-known $\delta$ -function anyon gas model

$$H_N = -\sum_k \partial^2_k + c \sum_{k \neq j} \delta(x_k - x_l)$$

establishing thus its missing link to a genuine 1D anyon quantum field model.

All the above anyon models are obtained using the rational $R$-matrix solution. Now we switch over to the trigonometric case.

### 3.6 Trigonometric class of anyon models

Considering the trigonometric quantum $R$-matrix (8,10) related to the $xxz$ spin-$\frac{1}{2}$ chain, but keeping the same braiding matrix $Z$ (11), we generate here a new trigonometric class of anyon lattice and field models, with a deformation parameter $q = e^{i\alpha}$.

### 3.7 q-anyon model

Following the same construction rule but using now $R$-matrix (10), we get from BYBE (6) a new *anyonic q-oscillator* model with CR:

$$\phi_k \phi_k^\dagger - e^{i\theta} \ \phi_k^\dagger \phi_k = e^{i\theta N_k} \cos 2\alpha N_k \tag{22}$$

and

$$\phi_k \phi_j^\dagger = e^{i\theta} \phi_j^\dagger \phi_k, \tag{23}$$

at the space-separated points $k > j$. We do not present here further details of this anyon q-oscillator model, however going to its continuum limit we derive another new anyonic quantum field model.

### 3.8 Derivative NLS anyon fields model

At the field limit $\Delta \to 0$, $\phi_k \to D(x)$, we obtain from the anyon q-oscillator a novel quantum integrable derivative- NLS anyon field model

$$\hat{H}^{(4a)} = \int dx (D_x^\dagger D_x + 2i\kappa (D^\dagger)^2 D D_x) \tag{24}$$

with the anyon field operator satisfying the CR

$$D(x) D^\dagger(y) - e^{i\theta} D^\dagger(y) D(x) = \kappa \delta(x-y) \tag{25}$$

and

$$D(x) D^\dagger(y) = e^{i\theta} D^\dagger(y) D(x), \tag{26}$$

at $x > y$. The $N$-particle sector $|N>$ of this anyon DNLS field model gives interestingly the recently proposed derivative- $\delta$ function anyon gas model

$$H_N^d = -\sum_k \partial_k^2 + i\kappa \sum_{k \neq j} \delta(x_k - x_l)(\partial_{x_k} + \partial_{x_l}),$$

establishing the final missing link of this anyon gas to a new quantum integrable anyon field model.

Thus we have constructed from the BYBE above, a series of anyonic and q-anyonic models, namely i) nearest-neighbor hard-core anyon, ii) next-nearest-neighbor nonlinear lattice anyon, iii) quantum NLS anyon field model, iv) anyon q-oscillator and v) anyonic DNLS quantum field model. We emphasize that all these models built systematically from the nonultralocal BYBE and the braiding relations are quantum integrable models, which are exactly solvable by the *algebraic Bethe ansatz*.

## 4 Novel anyonic quantum group with nonultralocal braiding

In constructing the anyonic q-oscillator models we have used the trigonometric $R$-matrix (8,10) together with the braiding matrix (11) in BYBE (6) and BR (7). It is intriguing that, following the same procedure but taking a different realization of the anyon algebra, we can discover a novel *anyon quantum group* algebra $A_\theta su_q(2)$, with an unusual nonultralocal braiding relation. This algebra, deformed by two independent parameters $q = e^{i\alpha}$ and the anyonic parameter $s = e^{i\theta}$ exibits also a beautiful Hopf algebra structure with nontrivial coproduct and multiplication.

The two-parameter deformed anyon quantum group may be expressed through the algebraic relations

$$S^+S^- - sS^+S^+ = [2S^3]_q s^{-S^3},$$
$$q^{S^3}S^\pm = q^{\pm 1}S^\pm q^{S^3}, \quad s^{S^3}S^\pm = s^{\pm 1}S^\pm s^{S^3} \tag{27}$$

denoting $[x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{\sin \alpha x}{\sin \alpha}$. Remarkably, this algebra also exhibits the Hopf algebra structure with an unusual braided multiplication

$$(I \otimes S^\pm)(S^\pm \otimes I) = s^{-1}(S^\pm \otimes S^\pm),$$
$$(I \otimes S^\mp)(S^\pm \otimes I) = s(S^\pm \otimes S^\mp) \tag{28}$$

and an intriguing two-parameter deformed coproduct structure

$$\Delta(S^+) = q^{-S^3} \otimes S^+ + S^+ \otimes q^{S^3}s^{-S^3},$$
$$\Delta(S^-) = q^{-S^3}s^{-S^3} \otimes S^- + S^- \otimes q^{S^3}, \quad \Delta(S^3) = S^3 \otimes I + I \otimes S^3. \tag{29}$$

It is interesting to check by direct insertion that the coproducts $\Delta(S^\pm), \Delta(S^3)$ do satisfy the same algebra, where the factors with $s$-parameter in the algebraic relations conspire with those in the multiplication and in the coproduct, such that all extra factors cancel out, similar to the standard case.

At $s = 1$ we clearly recover from (27) the standard quantum algebra

$$[S^-, S^+]_- = [2S^3]_q, \quad [S^3, S^\pm] = \pm S^3,$$

at the coinciding points and the trivial commutative multiplication

$$(I \otimes S^\pm)(S^\pm \otimes I) = (S^\pm \otimes S^\pm), \quad (I \otimes S^\mp)(S^\pm \otimes I) = (S^\pm \otimes S^\mp),$$

at different points $k > j$. At $q \to 1$ with arbitrary anyon parameter $s$ we get on the other hand a pure anyonic-deformed $A_\theta su(2)$ algebra

$$S^+S^- - sS^+S^+ = 2S^3 s^{-S^3},$$
$$s^{S^3}S^\pm = s^{\pm 1}S^\pm s^{S^3} \tag{30}$$

again with the same braided multiplication rules (28) and a twisted coproduct

$$\Delta(S^+) = I \otimes S^+ + S^+ \otimes s^{-S^3},$$
$$\Delta(S^-) = s^{-S^3} \otimes S^- + S^- \otimes I, \quad \Delta(S^3) = S^3 \otimes I + I \otimes S^3 \tag{31}$$

Note that at $s = \pm 1$ this anyonic spin algebra would be reduced to the standard or an anticommuting spin algebra, as evident from (30). The corresponding Hopf algebra structures are also obtained from (31,28).

## 5 Concluding remarks

We have constructed two classes of 1D anyon models, rational and trigonometric, in a systematic way starting from the braided YBE. The known, as well as the new anyon models that we have constructed, are all quantum integrable and exactly solvable by the algebraic Bethe Ansatz. Among the new models a next-nearest-neighbor interacting lattice anyon model and a nonlinear Schrödinger anyon quantum field model belong to the rational class, with the latter model being the missing link to the known $\delta$ anyon gas. THE q-anyon and the derivative NLS anyon field model that we discover belong to the triogonometric class, where the dNLS anyon model provides the needed link to the recently proposed derivative $\delta$- function anyon gas. Significantly, the anyons in the new models exhibit proper anyonic CR at all points, rectifying thus the deficiency of the existing anyon models.

We also obtain from the BYBE in the trigonometric case an importantly new anyon quantum group Hopf algebra, which generalizes the known quantum algebra as well as its coalgebra structures with an additional deforming anyonic parameter, resulting in a novel nonultralocal multiplication and an unusual twisted coproduct. It would be highly motivating to investigate various aspects of this Hopf algebra as well as to find representations of this new two-parameter algebra, especially at the parameters $q, s$ at the roots of unity.

Another promising line of research would be to find nonabelian realizations of the integrable 1D anyon models exploiting the BYBE, which might shed light to the nonabelian anyon models, importance of which is growing in recent years [3].

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