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S-MATRIX FOR MAGNONS IN THE D1-D5 SYSTEM

Justin R. David[1]
Centre for High Energy Physics, Indian Institute of Science,
C.V. Raman Avenue, Bangalore 560012, India

and

Bindusar Sahoo[2]
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

We show that integrability and symmetries of the near horizon geometry of the D1-D5 system determine the S-matrix for the scattering of magnons in this system completely up to a phase. Using semi-classical methods we evaluate the phase to the leading and to the one-loop approximation in the strong coupling expansion. We then show that the phase obeys the unitarity constraint implied by the crossing relations to the one-loop order. We also verify that the dispersion relation obeyed by these magnons is one-loop exact at strong coupling which is consistent with their BPS nature.

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[1]justin@cts.iisc.ernet.in
[2]bsahoo@ictp.it

1 Introduction

Integrability has played an important role in understanding the spectrum of $\mathcal{N}=4$ super Yang-Mills in the planar limit. For this system integrable structures were discovered both in the field theory and for strings propagating on $AdS_5 \times S^5$, its holographic dual. In fact the integrable structures found on both sides of the holographic duals played complementary roles in discovering the exact spectrum of planar $\mathcal{N}=4$ super Yang-Mills. For a recent review and a complete list of references see [1]. Another well studied dual pair is the duality between the IIB string theory on $AdS_3 \times S^3 \times \mathcal{M}$ where $\mathcal{M}$ could be T^4 or K^3 and the $\mathcal{N}=(4,4)$ superconformal field theory on a resolution of the symmetric product [2]

$$\mathcal{M}^N/S(N). \tag{1.1}$$

In this paper we will restrict our attention to the case when $\mathcal{M}$ is T^4. $AdS_3 \times S^3 \times T^4$ arises as a near horizon limit of the system of Q_1 D1-branes and Q_5 D5-branes wrapped on T^4, then N in (1.1) is given by Q_1Q_5. String theory on $AdS_3 \times S^3$ is known to be classically integrable [3]. There has been recent progress in organizing conformal perturbation theory on the symmetric product [4, 5, 6] however similar integrable structures have not yet been discovered for the $\mathcal{N}=(4,4)$ on the symmetric product.

One direction to explore the role of integrability in this system is to precisely determine the implication of integrability on its spectrum. Recently [7] has put forward a proposal for the quantum Bethe equations from examining classical Bethe equations of the string theory on $AdS_3 \times S^3$. In this paper we will study the implication of integrability on a certain class of excitations which are present both in the symmetric product as well as on the string theory on $AdS_3 \times S^3 \times T^4$. These excitations are similar to the magnon excitations in $\mathcal{N}=4$ Yang-Mills which played a crucial role in understanding integrability for this case. For the D1-D5 system they were studied in [8] and were argued to be BPS in a centrally extended $SU(1|1) \times SU(1|1)$ algebra. The BPS condition then determined their dispersion relation. Briefly they are states of the form

$$J^{-1}_{p_1} J^{-1}_{p_2} \cdots J^{-1}_{p_j} |0\rangle \otimes |0\rangle, \tag{1.2}$$

where the vacuum denotes $\mathbb{Z}_J$ twisted chiral primary of the $\mathcal{N}=(4,4)$ conformal field theory with R-charge $(\frac{J-1}{2}, \frac{J-1}{2})$. J^{-1}_p are operators which lower the left J^3 quantum number and carry momentum p in the $\mathbb{Z}_J$ twisted sector. Under the action of an element of $\mathbb{Z}_J$, $J^{-1}_p|0\rangle \otimes |0\rangle$ picks up a phase proportional to integer multiples of p. These states were argued to be BPS and as a result the dispersion relation for a single magnon is given by

$$\Delta - J = \sqrt{1 + 16 g^2 \sin^2 \frac{p}{2}}, \tag{1.3}$$

where Δ is the sum of the left and right moving conformal dimension of the state and J is the sum of the left and right moving J^3 charge. g is a function of the parameters of the D1-D5 system.

A similar dispersion relation can be written for a bound state of Q magnons, the dyonic magnon which is given by

$$\Delta - J = \sqrt{Q + 16g^2 \sin^2 \frac{p}{2}}. \tag{1.4}$$

Both at strong coupling that is at the zeroth order in the string sigma model and at first order in conformal perturbation theory on the symmetric product it was found that

$$16g^2 = \frac{g_6^2 Q_1 Q_5}{\pi^2}. \tag{1.5}$$

Here g_6 is the 6-dimensional string coupling.

In this paper we use the $SU(1|1)$ symmetry and integrability to constrain the S-matrix for the scattering of these magnons up to a phase. Since we rely purely on symmetry this argument can be applied both to the string side and the conformal field theory on the symmetric product. The phase cannot be determined by symmetry considerations alone. To determine the phase we restrict ourselves to the $SU(2)$ sector and examine the dyonic magnons as classical solutions of the string sigma model. We then use the semi-classical string theory to determine this phase to leading and to one loop order in the string sigma model. To determine the one loop correction we use the method developed by [9] for the case of magnons in $AdS_5 \times S^5$. This method is based on determining the phase shifts suffered by plane waves scattering off the magnon. We then verify that this phase obeys the unitarity constraint implied by the crossing symmetry equations. We also determine corrections to the dispersion (1.3), (1.4) relations at one loop in strong coupling and show that there is no corrections to the relation (1.5) at one loop in the string sigma model.

The paper is organized as follows: The next section introduces the magnon states of the D1-D5 system both in the symmetric product and in the sigma model and discusses their dispersion relation. In section 3 we use $SU(1|1)$ symmetry and integrability to constrain the S-matrix for the scattering of these magnons up to a phase. We also determine the phase at the zeroth order in the string sigma model. In section 4 we determine the one loop correction to both the phase and the dispersion relation. In section 5 we verify that the phase determined to the one loop order satisfies the unitarity constraint implied by the crossing symmetry equations. We end with some brief conclusions. Appendix A provides the details regarding the derivation of the dispersion relation of magnons to make the paper self contained. Appendix B discusses the dressing method to obtain phase shifts. Appendix C derives the crossing relations for the S-matrix with $SU(1|1)$ symmetry using the antipode operation.

2 Magnons in the D1-D5 system

In this section we introduce the magnon excitations of the D1-D5 system both in the boundary theory as well as its semi-classical description as classical solutions of the sigma model on $AdS_3 \times S^3 \times T^4$. This section will also serve to set our notations and conventions.

2.1 Magnons in the symmetric product

The boundary theory corresponding to the system of Q_1 number of D1-branes and Q_5 number of D5-branes in type IIB on T^4 is given by the $\mathcal{N} = (4, 4)$ superconformal field theory on a resolution of the symmetric product orbifold [1].

$$\mathcal{M} = (T^4)^{Q_1 Q_5}/S(Q_1 Q_5). \tag{2.1}$$

The global part of the $\mathcal{N} = (4, 4)$ algebra is given by the supergroup $SU(1, 1|2) \times SU(1, 1|2)$. The two copies arise from the left movers and the right movers of the conformal field theory on $\mathcal{M}$. The bosonic part of the supergroup $SU(1, 1|2)$ consists of the global part of the conformal algebra $SL(2, R)$ whose generators are $L_0, L_\pm$ and the global part of the R-symmetry group $SU(2)_R$ whose generators are $J^3, J^\pm$. The 8 supercharges of $SU(1, 1|2)$ are labeled by:

$$G^{ab}_{1/2}, \qquad \text{and} \qquad G^{ab}_{-1/2}, \tag{2.2}$$

where $a \in \{+, -\}$ denotes the quantum numbers of the charges under $SU(2)_R$ and $b \in \{+, -\}$ denotes the quantum numbers of charges under $SU(2)_I$ which is an outer automorphism of the $\mathcal{N} = (4, 4)$ algebra. The subscripts $+1/2$ denotes a weight of $L_0 = -1/2$, while a $-1/2$ denotes a weight of $L_0 = 1/2$. From the $\mathcal{N} = (4, 4)$ algebra it is easy to see that the set of generators

$$\{G^{++}_{-1/2}, G^{--}_{1/2}, L_0 - J^3\} \quad \text{or} \quad \{G^{-+}_{1/2}, G^{+-}_{-1/2}, L_0 - J^3\}, \tag{2.3}$$

each form a $SU(1|1)$ sub-algebra with the common central charge $L_0 - J^3$. From the right moving copy of $SU(1, 1|2)$, we see that the set of generators

$$\{\tilde{G}^{++}_{-1/2}, \tilde{G}^{--}_{1/2}, \tilde{L}_0 - \tilde{J}^3\} \quad \text{or} \quad \{\tilde{G}^{-+}_{1/2}, \tilde{G}^{+-}_{-1/2}, \tilde{L}_0 - \tilde{J}^3\}, \tag{2.4}$$

each form a right moving $SU(1|1)$ subalgebra with the common central charge $\tilde{L}_0 - \tilde{J}^3$. We use the ˜ to denote the generator corresponding to the right movers. The generators in (2.3) and (2.4) annihilate the chiral primaries of the symmetric product CFT. Let us denote the chiral primary with $L_0 = J^3, \tilde{L}_0 = \tilde{J}^3$ with left and right J^3 charge $(\frac{J-1}{2}, \frac{J-1}{2})$ as

$$|0\rangle_J \otimes |0\rangle_J. \tag{2.5}$$

These are the ground states of the $\mathbb{Z}_J$ twisted sector. The magnons which are of interest in this paper are the following excitations above this chiral primary.

$$|\phi_{p_1}\phi_{p_2}\cdots\phi_{p_j}\rangle_J \otimes |0\rangle_J = J^-_{p_1} J^-_{p_2} \cdots J^-_{p_j} |0\rangle_J \otimes |0\rangle_J \tag{2.6}$$

with J large and J^-_p is given by

$$J^-_p = \sum_{k=1}^{J} e^{ipk} J^-_{(k)}, \tag{2.7}$$

[1]See [10] for a review.

and $J^-_{(k)}$ is the lowering operator of the left moving $SU(2)$ R-current of the k-th copy of the torus involved in the $\mathbb{Z}_J$ twisted sector. To satisfy orbifold group invariance condition we need to impose the condition

$$\sum_i p_i = 0. \tag{2.8}$$

At the free orbifold point of the symmetric product, the magnon states in (2.6) are non-chiral in the left moving sector while it is still chiral in the right moving sector. On perturbing the symmetric product CFT by the marginal operator which is a singlet of both the $SU(2)_R$ and $SU(2)_I$ belonging to the $\mathbb{Z}_2$ twisted sector, the magnons in (2.6) pick up anomalous dimensions. The anomalous dimensions of these operators was evaluated to the leading order in conformal perturbation theory by [11] using methods developed in [12, 13, 14, 15] [2]. In [8] it was argued that these magnons are BPS in a centrally extended $SU(1|1) \times SU(1|1)$ algebra. The dispersion relation of these magnons was obtained as a result of the BPS condition in this extended algebra. To make this paper self contained, Appendix A. contains a brief review of this result. In this section it suffices to introduce this centrally extended algebra. The generators of the centrally extended $SU(1|1) \times SU(1|1)$ are composed of the set

$$\{G^{++}_{-1/2}, G^{--}_{1/2}, L_0 - J^3\} \quad \text{and} \quad \{\tilde{G}^{+-}_{-1/2}, \tilde{G}^{-+}_{1/2}, \tilde{L}_0 - \tilde{J}^3\}. \tag{2.9}$$

To unclutter our notation we will define the following generators.

$$\begin{aligned} G^{++}_{-1/2} &\to Q_1, & \tilde{G}^{+-}_{-1/2} &\to Q_2, \\ G^{-+}_{1/2} &\to S_1, & \tilde{G}^{-+}_{1/2} &\to S_2, \\ L_0 - J^3 &\to C_1, & \tilde{L}_0 - \tilde{J}^3 &\to C_2. \end{aligned} \tag{2.10}$$

Then the centrally extended algebra is given by

$$\begin{aligned} \{Q_1, S_1\} &= C_1, & \{Q_2, S_2\} &= C_2, \\ \{Q_1, Q_2\} &= C_3 - iC_4, & \{S_1, S_2\} &= C_3 + iC_4, \\ \{Q_1, S_2\} &= 0, & \{S_1, Q_2\} &= 0. \end{aligned} \tag{2.11}$$

where C_3, C_4 are the two additional central charges. Note that

$$Q_a^\dagger = S_a, \qquad a \in \{1, 2\} \tag{2.12}$$

This central extension of $SU(1|1) \times SU(1|1)$ can be viewed as a $\mathcal{N} = 2$ Poincaré superalgebra in 3-dimensions with one central charge [8]. The BPS condition for the extended algebra is given by

$$\frac{1}{4}(C_1 + C_2)^2 = \frac{1}{4}(C_1 - C_2)^2 + C_3^2 + C_4^2. \tag{2.13}$$

Consider a single magnon state given by

$$|\phi_p\rangle_J \otimes |0\rangle_J = J^-_p |0\rangle_J \otimes |0\rangle_J. \tag{2.14}$$

[2] See [16, 17, 18] for related work.

In [8] it was argued that this magnon is a BPS state and carries the following values of the central charges in the extended algebra (2.11)

$$C_3 - iC_4 = \alpha(e^{-ip} - 1), \qquad C_3 + iC_4 = \alpha^*(e^{ip} - 1), \tag{2.15}$$
$$C_1 - C_2 = (L_0 - J^3) - (\tilde{L} - \tilde{J}^3) = 1,$$

where α is a function of the coupling of the strength of the marginal deformation of the $\mathbb{Z}_2$ twist operator. Now from the BPS condition in (2.13) we obtain the following dispersion relation

$$\begin{aligned} C_1 + C_2 &= (L_0 + \tilde{L}_0) + (J^3 + \tilde{J}^3), \\ &= \Delta - J, \\ &= \sqrt{(C_1 - C_2)^2 + 4C_1C_2}, \\ &= \sqrt{1 + 16g^2\sin^2(\frac{p}{2})}, \qquad g = |\alpha|. \end{aligned} \tag{2.16}$$

g will depend on the parameters of the D1-D5 system, namely the charges Q_1, Q_5 and the six dimensional string coupling g_6. In the next subsection we will mention on how it depends on these parameters.

For the purposes of this paper it will be useful to parametrize the dispersion relation using spectral parameters as follows: introduce

$$\frac{x^+}{x^-} = \exp(ip) \tag{2.17}$$

subject to the constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{i}{g}. \tag{2.18}$$

We can use this constraint to evaluate the following

$$\begin{aligned} x^- &= \frac{i}{2g(e^{ip} - 1)}\left(1 + \sqrt{1 + 16g^2\sin^2\frac{p}{2}}\right), \\ c &= -i(x^+ - x^-) = \frac{1}{2g}\left(1 + \sqrt{1 + 16g^2\sin^2\frac{p}{2}}\right). \end{aligned} \tag{2.19}$$

here we have taken the positive branch in the square root. We can therefore identify the central charges

$$C_1 + C_2 = 2gc - 1, \qquad C_3 - iC_4 = \alpha(\frac{x^-}{x^+} - 1), \tag{2.20}$$
$$C_3 + iC_4 = \alpha^*(\frac{x^+}{x^-} - 1), \qquad C_1 = gc, \qquad C_2 = gc - 1.$$

There are other magnon like states which satisfy the BPS relation. They carry the central charges

$$C_3 - iC_4 = \alpha(e^{ip} - 1), \qquad C_3 + iC_4 = \alpha^*(e^{-ip} - 1), \qquad C_1 - C_2 = Q. \tag{2.21}$$

These can be thought of bound states of Q elementary magnons and are called dyonic magnons. Similar to the case of the elementary giant magnons, their dispersion relation is given by

$$C_1 + C_2 = \sqrt{Q^2 + 16g^2 \sin^2(\frac{p}{2})}. \tag{2.22}$$

One can use spectral parameters to parametrize dispersion relation as

$$\frac{x^+}{x^-} = \exp(ip), \tag{2.23}$$

with the constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = i\frac{Q}{g}. \tag{2.24}$$

Then the dispersion relation is given by

$$C_1 + C_2 = 2gc - Q, \tag{2.25}$$

where c is defined as $c = -i(x^+ - x^-)$.

2.2 Magnons at strong coupling

We have seen that magnons are BPS states and they carry large J charge in the CFT, therefore we should expect to find them as classical solutions to the string sigma model on $AdS_3 \times S^3 \times T^4$. Since the magnons have angular momentum along S^3 these solutions are rotating along a direction in S^3. Magnon solutions found for the case of $AdS_5 \times S^5$ which have non-trivial configurations along time and a S^3 within S^5 serve as magnon solutions for the $AdS_3 \times S^3$ case. The description of magnons as semi-classical solutions makes sense when the radius of curvature in comparison with string length $R^2/\alpha' = \sqrt{g_6^2 Q_1 Q_5} >> 1$. There are three interesting limits of these solutions.

1. **Plane wave limit**

   This limit was discovered by [19] and is given by

   $$g \to \infty, \qquad k = 2gp \ \text{ fixed}, \qquad Q \ \text{ fixed}. \tag{2.26}$$

   The dispersion relation (2.22) then reduces to

   $$\Delta - J = \sqrt{Q^2 + k^2}. \tag{2.27}$$

   For a single magnon $Q = 1$, from (2.19) we see that spectral parameters in this limit are given by

   $$\begin{aligned} x^+ \sim x^- &= r + O(1/g), \\ &= \frac{1}{k}\left(1 + \sqrt{1+k^2}\right) + O(1/g). \end{aligned} \tag{2.28}$$

   Thus the spectral parameters are real in this limit. From the above equation one can re-write the momentum k and the frequency $\omega = \sqrt{1+k^2}$ of the plane wave in terms of the spectral parameter. This is given by

   $$k(r) = \frac{2r}{r^2 - 1}, \qquad \omega(r) = \frac{r^2+1}{r^2-1}. \tag{2.29}$$

These solutions are quanta associated with linearised fluctuations of the world sheet fields around a string which orbits the equator of S^3 [19] The fluctuations are of the form of plane waves and solve the linearised equations of motion of the world sheet theory. They have wave number $k(r)$ and frequency $\omega(r)$ given by (2.29). States with $Q > 1$ are bound states of plane waves.

2. **Giant magnon limit**

   These solutions are obtained in the following limit

$$g \to \infty, \qquad p \ \text{fixed}, \qquad Q \ \text{fixed}. \tag{2.30}$$

From (2.18) and (2.24) we see that that in this limit the spectral parameters have the property

$$x^+ \sim \frac{1}{x^-} \sim \exp(ip/2) + O(1/g). \tag{2.31}$$

Giant magnons solutions of the string sigma model for $AdS_3 \times S^3 \times T^4$ were studied in [8] by using the giant magnon solution for the case of $AdS_5 \times S^5$ found by [20]. They were shown to be BPS solutions in $AdS_3 \times S^3 \times T^4$ and their dispersion relation is given by

$$\Delta - J = \frac{R^2}{\pi\alpha'} \left|\sin\frac{p}{2}\right| + O((\frac{R^2}{\pi\alpha'})^0). \tag{2.32}$$

From comparison with the exact dispersion relation in (2.16) and the strong coupling dispersion relation (2.32) we see that g is a function of the parameters of the D1-D5 system g_6^2, Q_1, Q_5 such that

$$16g^2 = \frac{R^2}{\pi\alpha'} = \frac{g_6^2 Q_1 Q_5}{\pi^2}, \quad \text{for} \quad g_6^2 Q_1 Q_5 >> 1. \tag{2.33}$$

Since these are classical solutions of the sigma model, corrections to the dispersion relation will be organized as inverse powers of $\frac{R^2}{\pi\alpha'}$. We perform a one loop calculation and show that there is no term of $O((\frac{R^2}{\pi\alpha'})^0)$ It is important to note that though the plane wave excitation and the giant magnon classical solution looks different, they are representatives of the same state in different regions of momentum space [21].

3. **Dyonic giant magnon limit**

   This limit is given by

$$g \to \infty, \qquad Q \to \infty, \qquad \frac{Q}{g} = \text{fixed}, \quad p = \text{fixed}. \tag{2.34}$$

These solutions were found by [22, 23, 24, 25]. They have non-trivial field configurations in the S^3, details of these solutions are given in Appendix A. Again from the constraint (2.24) obeyed by the spectral parameters we find that

$$x^+ = \bar{x}^- \sim O(g^0). \tag{2.35}$$

The spectral parameters in this case depends on the parameter Q/g which can be tuned to any value, this fact will play an important role in our analysis. The dispersion relation obeyed by the giant dyonic magnons is given by

$$\Delta - J = \sqrt{Q^2 + \frac{R^2}{\alpha'} \sin^2\left(\frac{p}{2}\right)} + O((\frac{R^2}{\pi\alpha'})^0). \tag{2.36}$$

The dyonic giant magnon dispersion relation can be obtained from the exact dispersion relation (2.22) by performing the giant magnon limit. On comparing the result with the dyonic giant magnon dispersion relation in (2.36) we obtain the identification (2.33). We will show that the correction to the dispersion relation (2.36) at one loop in the sigma model vanishes. Both the giant magnon solution as well as the plane wave excitations can be obtained as a further limit of the dyonic giant magnon. $Q \to 0$ limit reproduces the giant magnon, and taking the limit $x^+ \sim x^- = r$ on the spectral parameters of the dyonic giant magnon reduces it to the plane wave excitation [22].

Studying the dispersion relation of magnons at strong coupling helped us to make the identification (2.33) between the parameter g and the parameters of the D1-D5 system. A similar comparison can be done at weak coupling. In [11] magnons were studied in the limit of small momentum p and in first order in the $\mathbb{Z}_2$ blow up mode. In this limit the dispersion relation of a single magnon is shown to be

$$\Delta - J = 1 + \frac{1}{2\pi^2} g_6^2 (Q_1 Q_5) \frac{p^2}{4}, \tag{2.37}$$

where g_6 is the coupling of the marginal $\mathbb{Z}_2$ blow up mode in the symmetric product. Thus comparing this to the exact dispersion relation (2.16) we see that again we have

$$16 g^2 = \frac{g_6^2 Q_1 Q_5}{\pi^2}, \quad \text{for} \quad g_6^2 Q_1 Q_5 << 1. \tag{2.38}$$

A simple conjecture for the full dependence of the coupling g on the parameters of the system is then

$$16 g^2 = \frac{g_6^2 Q_1 Q_5}{\pi^2}. \tag{2.39}$$

We will test the relation (2.39) to one loop in the strong coupling expansion. Since perturbation theory of the string sigma model is controlled by the parameter $\frac{R^2}{\pi\alpha'}$ it is clear that possible corrections to the relation (2.39) are of the form

$$4g = (\frac{R^2}{\pi\alpha'}) + g_0 + g_1 (\frac{R^2}{\pi\alpha'})^{-1} + \cdots . \tag{2.40}$$

Substituting this in the exact form of the dispersion relation we obtain

$$E - J = \frac{R^2}{\pi\alpha'} \left|\sin\frac{p}{2}\right| + g_0 \left|\sin\frac{p}{2}\right| + f_1(p) (\frac{R^2}{\pi\alpha'})^{-1} + \cdots . \tag{2.41}$$

We will evaluate the one loop correction to the dispersion relation at strong coupling and show that $g_0 = 0$, which is consistent with the proposed relation (2.39).

3 The $SU(1|1)$ invariant S-matrix for magnons

In this section we show that demanding $SU(1|1)$ symmetry, the S-matrix for the scattering of magnons in the D1-D5 system is determined up to a phase. Recall that the symmetries preserved by the chiral primaries of the D1-D5 system are two copies of $SU(1|1) \times SU(1|1)$ with common central charges. We demand that the S-matrix is symmetric under one of the $SU(1|1)$ in each copy of $SU(1|1) \times SU(1|1)$ using the conventional co-product between two states. Therefore let us focus on the $SU(1|1)$ generated by Q_1, S_1 with central charge C_1. The algebra is given by

$$\{Q_1, S_1\} = C_1. \tag{3.1}$$

To write down the $SU(1|1)$ invariant S-matrix we follow [26]. Introduce the generator B which has the following commutation relations with the $SU(1|1)$ generators

$$[B, Q_1] = -2Q_1, \qquad [B, S_1] = 2S_1. \tag{3.2}$$

Basically B is a $U(1)$ outer automorphism under which Q_1 and S_1 have charges -2 and $+2$ respectively. This algebra has a quadratic Casimir given by

$$\mathcal{J} = 2[Q_1, S_1] + \{B, C_1\}. \tag{3.3}$$

Let us write down the action of these generators of one of the $SU(1|1)$ super algebra on a single magnon which is in the fundamental short multiplet of $SU(1|1) \times SU(1|1)$.

$$\begin{aligned}
B|\phi_p\rangle \otimes |0\rangle &= (\beta+1)|\phi_p\rangle \otimes |0\rangle, \qquad & B|\psi_p\rangle \otimes |0\rangle &= (\beta-1)|\psi_p\rangle \otimes |0\rangle, \\
Q_1|\phi_p\rangle \otimes |0\rangle &= a|\psi_p\rangle \otimes |0\rangle, & Q_1|\psi_p\rangle \otimes |0\rangle &= 0, \\
S_1|\phi_p\rangle \otimes |0\rangle &= 0, & S_1|\psi_p\rangle \otimes |0\rangle &= b|\phi_p \otimes |0\rangle, \\
C_1|\phi_p\rangle \otimes |0\rangle &= ab|\phi_p\rangle \otimes |0\rangle, & C_1|\psi_p\rangle \otimes |0\rangle &= ab|\psi_p\rangle \otimes |0\rangle, \\
\mathcal{J}^2|\phi_p\rangle \otimes |0\rangle &= 2\beta ab|\phi_p\rangle \otimes |0\rangle, & \mathcal{J}|\psi_p\rangle \otimes |0\rangle &= 2\beta ab|\psi_p\rangle \otimes |0\rangle.
\end{aligned} \tag{3.4}$$

From the definition of the central charge C_1 in (2.20) we obtain

$$ab = gc. \tag{3.5}$$

Following the notation of [26] we denote the above representation as $(\mathbf{1}|\mathbf{1})_{c,\beta}$. The tensor product of two fundamental representations of $SU(1|1)$ decomposes into a direct sum of two fundamental representations. We write this as

$$(\mathbf{1}|\mathbf{1})_{c_1,\beta_1} \otimes (\mathbf{1},|\mathbf{1})_{c_2,\beta_2} = (\mathbf{1}|\mathbf{1})_{(c_1+c_2),\beta_1+\beta_2+1} \oplus (\mathbf{1}|\mathbf{1})_{(c_1+c_2),\beta_1+\beta_2-1}. \tag{3.6}$$

Since there are only two multiplets occurring in the decomposition, one only needs the identity and the quadratic Casimir on the tensor product to write down any operator which is invariant under the sum of the generators. The quadratic Casimir on the tensor product is given by

$$\mathcal{J}^{(12)} = 2g\beta_1 c_1 + 2g\beta_2 c_2 + 2B^{(1)}C_1^{(2)} + 2C_1^{(1)}B^{(2)} + 4Q_1^{(1)}S_1^{(2)} - 4S_1^{(1)}Q_1^{(2)}. \tag{3.7}$$

We also need the square of the quadratic Casimir which is given by

$$(\mathcal{J}^{(12)})^2 = 4(\beta_1+\beta_2)(gc_1+gc_2)\mathcal{J}^{(12)} - 4(\beta_1+\beta_2+1)(\beta_1+\beta_2-1)(gc_1+gc_2)^2\mathcal{I}^{(12)}. \quad (3.8)$$

The $SU(1|1)$ S-matrix is written as the product of the graded permutation operator and the R-matrix.

$$\mathcal{S}_{12} = \mathcal{P}_{12}\mathcal{R}_{12}(\alpha_1, \alpha_2), \quad (3.9)$$

where α_1, α_2 are spectral parameters corresponding to the two states. Since the R-matrix is also an invariant under the sum of the generators corresponding to the two sates, we can write it as

$$\mathcal{R}_{12}(\alpha_1, \alpha_2) = R_{12,1}(\alpha_1, \alpha_2)\mathcal{I}^{(12)} + R_{12,2}(\alpha_1, \alpha_2)\mathcal{J}^{(12)}. \quad (3.10)$$

We now demand that the R-matrix satisfies the unitarity constraint and the Yang-Baxter relations given by the equations

$$\mathcal{R}_{12}\mathcal{R}_{21} = \mathcal{I}_{12}, \qquad \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \quad (3.11)$$

These conditions determine the scalars $R_{12,1}, R_{12,2}$ to be of the form

$$\begin{aligned} R_{12,1}(a_1, a_2) &= \frac{\alpha_2 - \alpha_1 - \frac{i}{2}(\beta_1+\beta_2)(gc_1+gc_2)}{\alpha_2 - \alpha_1 - \frac{i}{2}(gc_1+gc_2)} R_{12,0}(\alpha_1, \alpha_2), \\ R_{12,2} &= \frac{\frac{i}{4}}{\alpha_2 - \alpha_1 - \frac{i}{2}(gc_1+gc_2)} R_{12,0}(\alpha_1, \alpha_2). \end{aligned} \quad (3.12)$$

From the unitarity condition it is easy to see that the undermined scalar $R_{12,0}$ satisfies the condition

$$R_{12,0}(\alpha_1, \alpha_2)R_{12,0}(\alpha_2, \alpha_1) = 1. \quad (3.13)$$

Since the S-matrix is written in terms of the quadratic casimir of the first $SU(1|1)$, it is has the following invariance

$$\begin{aligned} &[Q_1^{(1)} \otimes 1 + (-1)^F \otimes Q_1^{(2)}, \mathcal{S}_{12}] = 0, \\ &[S_1^{(1)} \otimes 1 + (-1)^F \otimes S_1^{(2)}, \mathcal{S}_{12}] = 0. \end{aligned} \quad (3.14)$$

Thus this $SU(1|1)$ invariance is realized under the conventional co-product. The symmetry corresponding to the other $SU(1|1)$ in each copy is possibly realized using a non-trivial co-product. In appendix A we write down a non-trivial co-product under which the S-matrix written down in (3.9), (3.10) (3.12) is invariant with respect to the second $SU(1|1)$ in each copy.

Therefore we have shown that the requirement of invariance under one of the $SU(1|1)$ determines the S-matrix up to a phase. Let us write down the action of the S-matrix explicitly on the two particle states. We first identify the parameters α in terms of the spectral parameters $x^{\pm}$ as

$$\alpha = \frac{g}{2}(x^+ + x^-), \qquad \text{and} \qquad c = -i(x^+ - x^-). \quad (3.15)$$

where the spectral parameters x^+, x^- are related to the momentum p_1 of the first magnon by (2.17) and (2.18). To write down the explicit action of the S-matrix we first define the its action on two particle states as

$$\mathcal{S}_{12}|i\rangle_{(1)} \otimes |j\rangle_{(2)} = S(p_2, p_1)_{ij}^{kl} |k\rangle_{(2)} \otimes |l\rangle_{(1)}. \tag{3.16}$$

Now given (3.9), (3.10) and (3.12), we see that explicitly the action of the S-matrix on the two magnon states is given by

$$\begin{aligned}
\mathcal{S}_{12}|\phi_{p_1}\phi_{p_2}\rangle \otimes |0\rangle &= \frac{y^+ - x^-}{y^- - x^+}|\phi_{p_2}\phi_{p_1}\rangle \otimes |0\rangle, \\
\mathcal{S}_{12}|\phi_{p_1}\psi_{p_2}\rangle \otimes |0\rangle &= \frac{y^+ - x^+}{y^- - x^+}|\psi_{p_2}\phi_{p_1}\rangle \otimes |0\rangle + \frac{y^+ - y^-}{y^- - x^+}\frac{a^{(1)}}{a^{(2)}}|\phi_{p_2}\psi_{p_1}\rangle \otimes |0\rangle, \\
\mathcal{S}_{12}|\psi_{p_1}\phi_{p_2}\rangle \otimes |0\rangle &= \frac{y^- - x^-}{y^- - x^+}|\phi_{p_2}\psi_{p_1}\rangle \otimes |0\rangle + \frac{x^+ - x^-}{y^- - x^+}\frac{a^{(2)}}{a^{(1)}}|\psi_{p_2}\phi_{p_1}\rangle \otimes |0\rangle, \\
\mathcal{S}_{12}|\psi_{p_1}\psi_{p_2}\rangle \otimes |0\rangle &= -|\psi_{p_2}\psi_{p_1}\rangle \otimes |0\rangle.
\end{aligned} \tag{3.17}$$

Here $y^\pm$ refer to the spectral parameter of the second magnon and we have suppressed the overall phase factor for convenience of notation. $a^{(1)}, a^{(2)}$ refer to the parameter a in (3.4) for the two states. Note that since there are two copies of the extended $SU(1|1) \times SU(1|1)$ which share the same central charge, the full S-matrix is a tensor product of the both given by $\mathcal{S}_{12} \otimes \mathcal{S}_{12}$. In this paper we will be interested in magnons only in the $SU(2)$ sector. We can read out the scattering amplitude of these magnons from the action of the S-matrix on the bosonic state $|\phi_1\phi_2\rangle$. From the first line of (3.17) we see that the amplitude for scattering of two magnons in this sector is given by

$$S(x^\pm, y^\pm)_{SU(2)} = S_0(x^\pm, y^\pm)\left(\frac{x^+ - y^-}{x^- - y^+}\right)^2, \tag{3.18}$$

where $S_0(x^\pm, y^\pm)$ is the undetermined phase factor. $x^\pm, y^\pm$ implicitly depend on momenta p_1, p_2 through the equations (2.17) and (2.18). We have also squared the amplitude due to the existence of two copies of the extended $SU(1|1) \times SU(1|1)$.

3.1 Leading contribution to the phase factor

The phase factor $S_0(x^\pm, y^\pm)$ in general cannot be determined by symmetry considerations alone. In this paper we will evaluate the phase factor to the leading and the first sub-leading terms in the semi-classical expansion. For this it is convenient to parametrize the phase factor $S_0(x^\pm, y^\pm)$ as follows:

$$\begin{aligned}
S_0(x^\pm, y^\pm) &= \sigma_{\rm BDS} \times \sigma^2(x^\pm, y^\pm), \\
&= \frac{x^- - y^+}{x^+ - y^-}\frac{1 - \frac{1}{x^+y^-}}{1 - \frac{1}{x^-y^+}} \times \sigma^2(x^\pm, y^\pm),
\end{aligned} \tag{3.19}$$

where

$$\sigma_{\rm BDS} = \frac{x^- - y^+}{x^+ - y^-}\frac{1 - \frac{1}{x^+y^-}}{1 - \frac{1}{x^-y^+}}. \tag{3.20}$$

This parametrization of the undetermined phase is used so that it is easy to see the difference from the scattering amplitude of two magnons in the $SU(2)$ sector of $\mathcal{N}=4$ Yang-Mills. In that theory, the scattering amplitude of the $SU(2)$ subsector is of the same form as in (3.18) and (3.19) with the same pre-factor σ_{BDS} [27]. the possible difference between the two theories is therefore parametrized by differences in σ. Let us write σ as

$$\sigma(x^{\pm}, y^{\pm}) = \exp(i\theta(x^{\pm}, y^{\pm})). \tag{3.21}$$

Unitarity demands that $\theta(x, y)$ is anti-symmetric in its arguments. We further write θ in terms of the strong coupling expansion as

$$\theta(x^{\pm}, y^{\pm}) = g\left(\theta_0(x^{\pm}, y^{\pm}) + \frac{1}{g}\theta_1(x^{\pm}, y^{\pm}) + \frac{1}{g^2}\theta_2(x^{\pm}, y^{\pm}) + \cdots\right). \tag{3.22}$$

This form of the expansion is easily motivated by examining the sigma model expansion which is organized in terms of inverse powers of $\frac{R^2}{\pi\alpha'}$, with the leading term proportional to $\frac{R^2}{\pi\alpha'}$ and the fact that at strong coupling $4g = \frac{R^2}{\pi\alpha'}$.

The leading semi-classical contribution $\theta_0(x^{\pm}, y^{\pm})$ can be evaluated using the relation between phase shift and time delay for the scattering of two magnons given by [28]

$$\frac{\partial\theta_0(p_1, p_2)}{\partial E_{p_1}} = \Delta T_{12}, \tag{3.23}$$

where E_{p_1} is the energy of the first magnon. This leading term in the S-matrix for the scattering of two magnons has been evaluated in [20]. Though this was evaluated for giant magnons in the $AdS_5 \times S^5$ geometry, the answer just depends on the classical solution of the magnons. In fact the crucial ingredient in the calculation was just the map of classical string theory on $R \times S^2$ to the Sine-Gordan model. In the strong coupling limit, the magnon solution of the D1-D5 system is identical to that of $\mathcal{N}=4$ Yang-Mills. It is just a solution of classical string theory on $R \times S^2$. Therefore the leading term in the strong coupling expansion for the S-matrix for scattering of two giant magnons with momentum p_1 and p_2 is the same as that evaluated in [20]. This is given by

$$\begin{aligned} &S(p_1, p_2) = \exp(i\delta), \\ &\text{where } \delta = -\frac{\sqrt{g_6^2 Q_1 Q_5}}{\pi}\left(\cos\frac{p_2}{2} - \cos\frac{p_1}{2}\right)\log\left(\frac{\sin^2\frac{p_1-p_2}{4}}{\sin^2\frac{p_1+p_2}{4}}\right), \end{aligned} \tag{3.24}$$

with $\text{sign}(\sin(\frac{p_1}{2}) > 0$ and $\text{sign}(\sin(\frac{p_2}{2}) > 0$. Thus in the leading semi-classical limit, the factor $\sigma(x, y)$ is identical to that evaluated for the case of magnons in the $\mathcal{N}=4$ Yang-Mills. This is given by

$$\theta_0(x^{\pm}, y^{\pm}) = k(x^+, y^+) - k(x^+, y^-) - k(x^-, y^+) + k(x^-, y^-), \tag{3.25}$$

where

$$k(x, y) = \left[\left(y + \frac{1}{y}\right) - \left(x + \frac{1}{x}\right)\right]\log\left(1 - \frac{1}{xy}\right). \tag{3.26}$$

This form of the leading contribution of the phase for the scattering of magnons in the $SU(2)$ sector for $\mathcal{N}=4$ Yang-Mills in terms of the spectral parameters was first proposed in [29]. To show that this phase factor agrees with that computed using the semi-classical solution of giant magnons in $R\times S^2$, all we have to do is to evaluate (3.18) with S_0 given by (3.19) in terms of the momenta, rather than the spectral parameters. In the giant magnon limit (2.30) we can expand the equations (2.19) to the leading order. For sign$(\sin\frac{p_1}{2})>0$, sign$(\sin\frac{p_2}{2})>0$ we obtain

$$\begin{aligned} x^- &= \exp(\frac{-ip_1}{2}) + O(1/g), \quad x^+ = \exp(\frac{ip_1}{2}) + O(1/g), \\ y^- &= \exp(\frac{-ip_2}{2}) + O(1/g), \quad y^+ = \exp(\frac{ip_2}{2}) + O(1/g). \end{aligned} \tag{3.27}$$

Substituting this in the expression for the scattering amplitude 3.18) with S_0 given by (3.19)and retaining only the leading order contribution from the phase (3.26) we obtain

$$\begin{aligned} S(p_1,p_2) &= \exp(i\delta), \\ \delta &= -\frac{\sqrt{g_6^2 Q_1 Q_5}}{\pi}\left(\cos\frac{p_2}{2} - \cos\frac{p_1}{2}\right)\log\left(\frac{\sin^2\frac{p_1-p_2}{4}}{\sin^2\frac{p_1+p_2}{4}}\right). \end{aligned} \tag{3.28}$$

Here we have substituted the relation $4g = R^2/\pi\alpha'$ which is valid in the strong coupling limit. Note that this agrees precisely with (3.24), the leading amplitude obtained from using the classical solution of the magnons in the $R\times S^2$ geometry.

4 One-loop corrections

Our goal in this section is to determine the one-loop correction to the phase factor $\theta_1(p_1,p_2)$ as well as the one-loop correction to the dispersion relation at strong coupling. We follow the method developed by [9]. Let us now summarize their approach: Let $\delta(k;p)$ be the phase shift corresponding to the scattering of a plane wave off the either a giant or a dyonic magnon. Where k is the momentum of the plane wave of charge Q and p the momentum carried by the magnon. Then the one-loop correction to the dispersion relation is given by [9]

$$\Delta E(p) = \frac{1}{2\pi}\sum_{I=1}^{N_F}(-1)^{F_I}\int_{-\infty}^{\infty} dk\frac{\partial\delta_I(k;p)}{\partial k}\sqrt{k^2+Q^2}. \tag{4.1}$$

Here I labels the fluctuations of the magnons with Bose/Fermi statistics depending on the sign of $(-1)^{F_I}$. While the one loop correction to the scattering phase is given by

$$\Delta\Theta(p_1,p_2) = \frac{1}{4\pi}\sum_{I=1}^{N_F}(-1)^{F_I}\int_{-\infty}^{\infty} dk\left(\frac{\partial\delta_I(k;p_1)}{\partial k}\delta_I(k,p_2) - \frac{\partial\delta_I(k;p_2)}{\partial k}\delta_I(k,p_1)\right). \tag{4.2}$$

The above formula is explicitly anti-symmetric in p_1,p_2, but the second term is equal to the first one up to a total divergence [3]. Integrability of the world sheet sigma model is an important

[3] In [9] the formula was written without the explicit anti-symmetrization.

ingredient which goes into the derivation of the one loop phase shifts for the scattering of two solitons. [9].

Both (4.1) and (4.2) are functions of the momentum p_1, p_2 of the magnon, however in the expressions for the S-matrix given in (3.18) and (3.19), the expansion of the phase factor in (3.22) are written in terms of the spectral parameters $x^\pm, y^\pm$ of the magnon. The spectral parameters for the elementary magnon are related to the momentum by equations (2.19) which involve the coupling g. In general one might expect that on substitution of the spectral parameters of the magnon in terms of their momenta there might be mixing between what we have organized as the BDS factor, θ_0 and θ_1. If this occurs then the formula in (4.2) would only give the contribution of what occurs as the coefficient of g^0 term and would not directly give θ_1. The way to avoid this difficulty is to use the expression (4.2) for dyonic magnons [9]. Note that for dyonic magnons in the classical limit (2.34), the spectral parameters are of order one and due to the presence of the additional parameter Q (2.22). This ensures that in the expansion organized as (3.22) and there is no mixing of orders. There is a correction to θ_0 from the $\sigma_{\rm BDS}$ factor due to the fact that dyonic magnons are bound states, however there is no change in the one loop term θ_1 [30, 9]. Though this was found for $SU(2)$ giant magnons in the case of AdS_5 the same arguments go through for the the case of $SU(2)$ magnons in AdS_3 since this relies only on the $\sigma_{\rm BDS}$ factor and θ_0 which is identical for both. In summary to read out the one loop correction θ_1 we can use the classical solutions of dyonic magnons and read out the plane wave phase shifts shifts $\delta(k, x^\pm)$ directly in terms of the spectral parameters of the dyonic magnons and substitute them in (4.2) to read out θ_1. Thus we have

$$2\theta_1(x^\pm, y^\pm) = \Delta\Theta(p_1(x^\pm), p_2(y^\pm)). \tag{4.3}$$

Note that there is a factor of 2 on the LHS of (4.3) since this is total one loop phase contribution in $\sigma^2(x^\pm, y^\pm)$.

Finally we mention that in near horizon geometry of the D1-D5 system plane waves can be excited along directions in $AdS_3 \times S^3$ as well as along T^4. Both the bosonic and fermionic co-ordinates for T^4 couple trivially to the co-ordinates of $AdS_3 \times S^3$. Thus plane wave excitations along T^4 do not suffer any phase shifts and therefore do not contribute to the one loop energy shift of the magnon or the scattering phase. One might suspect that one can apply this argument to the case of plane wave excitations along AdS_3. Indeed the bosonic co-ordinates couple trivially to that of S^3 and we will explicitly show that the phase shifts of bosonic plane wave excitations along AdS_3 vanish. But, the fermionic co-ordinates of AdS_3 couple non-trivially to that of S^3 due to the presence of the Ramond-Ramond flux. Thus the fermionic plane wave excitations do suffer phase shifts due to the presence of a magnon in S^3. For the rest of the paper we ignore the T^4 directions and deal with only $AdS^3 \times S^3$.

The rest of this section is organized as follows: We first evaluate the phase shifts suffered by the plane waves which are bosonic when they scatter off the dyonic magnon solution by using the dressing method of [24]. We then show that the phase shifts of the fermionic plane wave excitations are 1/2 of the bosonic ones using the finite gap method of [31]. Finally we use (4.1) and (4.2) to obtain the one loop corrections to the dispersion relation and the one loop correction to the phase factor in the S-matrix.

4.1 Bosonic phase shifts: The dressing method

As we have discussed earlier, the dyonic giant magnon solution reduces to the plane wave excitation on taking the limit $x^+ \sim x^- = r$. The corresponding plane wave excitation is a solution of the linearised equation of motion with the following wave number and frequency

$$k(r) = \frac{2r}{r^2 - 1}, \qquad \omega(r) = \frac{r^2+1}{r^2-1}. \tag{4.4}$$

This observation facilitates in extracting out the phase shift suffered by a plane wave scattering off the dyonic giant magnon solution. The strategy is a follows: The dressing method of [24] provides a construction of a solution of N dyonic giant magnons (DGMs) which have non-trivial configurations in S^3. They are parametrized by their spectral parameters $x_i^\pm$ and $i = 1, \cdots N$. We then use the dressing method again to obtain a solution with $N+1$ DGMs and take the limit where this new DGM reduces to a plane wave excitation. Thus we would have obtained a linearised fluctuation around the N DGM's. This provides an easy method to extract out its spectrum as well as the phase shift suffered by the plane wave scattering in the N DGM background.

Since we are interested in a DGM in the S^3 of $AdS^3 \times S^3$ we can ignore the AdS^3 directions. The S^3 directions are parametrized as follows

$$\{(Z_1, Z_2) : |Z_1|^2 + |Z_2|^2 = 1\} \quad \leftrightarrow \quad g = \begin{pmatrix} Z_1 & -iZ_2 \\ -i\bar{Z}_2 & \bar{Z}_1 \end{pmatrix} \in SU(2), \tag{4.5}$$

where $g(z, \bar{z})$ is a 2×2 matrix valued valued field satisfying the equations of motion

$$\bar{\partial}\left(\partial g \;\; g^{-1}\right) + \partial\left(\bar{\partial} g \;\; g^{-1}\right) = 0. \tag{4.6}$$

The dressing method provides a easy construction of the solution corresponding to DGM's. Then by taking the plane wave limit on one of the DGM we can obtain the bosonic phase shifts suffered by this plane wave as it scatters off the rest of the DGMs. The details of this calculation is given in the appendix B, we quote the final results. From (B.19) we can read out the phase shifts for the perturbation associated with the S^3 fluctuations around a N-DGM background. This is given by

$$\begin{aligned} \delta_{Z_1} \equiv -i\ln(\delta Z_1(\infty)) + i\ln(\delta Z_1(-\infty)) &= 0, \\ \delta_{Z_2} \equiv -i\ln(\delta Z_2(\infty)) + i\ln(\delta Z_2(-\infty)) &= -\sum_{i=1}^{N}\left[2i\ln\left(\frac{r - x_i^+}{r - x_i^-}\right) - i\ln\left(\frac{x_i^+}{x_i^-}\right)\right]. \end{aligned} \tag{4.7}$$

where r is the spectral parameter associated with the plane wave which is related to the frequency ω and wave number k by (4.4). As is shown in (B.20) the phase shifts for the complex conjugate fields are given by

$$\begin{aligned}\delta_{\bar{Z}_1}(1/r) \equiv -i\ln(\delta\bar{Z}_1(\infty)) + i\ln(\delta\bar{Z}_1(-\infty)) &= 0, \\ \delta_{\bar{Z}_2}(1/r) \equiv -i\ln(\delta\bar{Z}_2(\infty)) + i\ln(\delta\bar{Z}_2(-\infty)) &= \sum_{i=1}^{N}\left[2i\ln\left(\frac{r-x_i^+}{r-x_i^-}\right) - i\ln\left(\frac{x_i^+}{x_i^-}\right)\right].\end{aligned} \tag{4.8}$$

A simple consistency check of these results for phase shifts is that if the spectral parameter of any of the DGM reduces to a real number which means that the DGM is in fact a plane wave excitation, then the phase shift must vanish. This is because there is no scattering between two plane wave excitations. This property can be seen to be true form the expressions in (4.7) and (4.8). As we have discussed before, since the bosonic co-ordinates of AdS_3 couple trivially to that of the S^3 we expect the plane wave fluctuations along the AdS_3 directions to suffer no phase shifts. On the other hand the fermionic co-ordinates of S^3 and that of AdS_3 couple with each other, therefore we expect plane wave fluctuations along all the fermionic co-ordinates to suffer phase shifts. We will show this explicitly in the next section using the finite gap approach.

4.2 Fermionic phase shifts: Finite gap method

To obtain the phase shifts corresponding to the fermionic plane wave fluctuations around the dyonic magnon and to prove that there are no phase shifts for bosonic plane wave fluctuations in AdS_3 we use the description of classical solutions with periodic boundary conditions developed in [31] for the case of the sigma model on $AdS_3 \times S^3$. Our discussion closely follows that of $AdS_5 \times S^5$ in [9]. Classical string propagation on $AdS_3 \times S^3$ with Ramond-Ramond flux can be described as a non-linear sigma model on the supergroup $SU'(1,1|2)$ [32]. The sigma model is given by

$$S = -\frac{R^2}{2\pi\alpha'}\int d^2z \mathrm{Tr}'[\partial^\mu g^{-1}\partial_\mu g], \tag{4.9}$$

where g takes values in the supergroup $SU'(1,1|2)$ and Tr' is the non-degenerate bi-invariant metric. The metric has the signature $(-1,1,1,1,1,1)$[4]. We will not require the detail structure of the sigma model but only the properties of group element g. An element of the supergroup $SU'(1,1|2)$ satisfies the following

$$g = \exp(ix), \tag{4.10}$$

where x is a 4×4 supermatrix given by

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{4.11}$$

with a and d being bosonic Hermitian 2×2 matrices and b and c fermionic 2×2 matrices such that $b = c^\dagger$. They also satisfy

$$\mathrm{Tr}\, a = \mathrm{Tr}\, b = 0. \tag{4.12}$$

[4]See [32] for details.

The bosonic part of the supergroup $SU'(1,1|2)$ is given by $SU(1,1)\times SU(2)$. Classical solutions of the sigma model are described in terms of the monodromy matrix constructed using the one form j given by

$$j = -dgg^{-1}. \tag{4.13}$$

The equations of motion of the sigma model is given by

$$\partial_+ j_- + \partial_- j_+ = 0, \tag{4.14}$$

where the $\pm$ subscripts refer to the light cone directions of the world sheet. The one form j also satisfies the identity

$$\partial_+ j_- - \partial_- j_+ + [j_+, j_-] = 0. \tag{4.15}$$

It can then be shown using (4.14) and (4.15) the connection

$$J_\pm(x) = \frac{j_\pm(\sigma,\tau)}{1\mp x}, \tag{4.16}$$

with $x \in C$, is a family of flat connections. That is it obeys the zero curvature condition

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0. \tag{4.17}$$

Using this flat connection we can construct the monodromy matrix

$$\Omega(x) = P\exp\left[\int_0^{2\pi} d\sigma \frac{1}{2}\left(\frac{j_+}{1-x} - \frac{j_-}{1+x}\right)\right], \tag{4.18}$$

where P refers to path ordering. Note that here x is the spectral parameter which characterizes the flat connection. Since the monodromy matrix takes values in the supergroup $SU'(1,1|2)$ its eigenvalues are of the form

$$\{e^{i\hat{p}_1}, e^{i\hat{p}_2}|e^{i\tilde{p}_1}, e^{i\tilde{p}_2}\}, \tag{4.19}$$

with

$$\hat{p}_1 = -\hat{p}_2, \qquad \tilde{p}_1 = -\tilde{p}_2. \tag{4.20}$$

The above condition arises from the definitions in (4.10), (4.11) and the traceless property (4.12). Classical strings propagating on the sigma model are classified by the analytical properties of the eigenvalues of the monodromy matrix. The p's are the quasi-momentum, they are meromorphic functions of the spectral parameter x. We will label them p_i with $i = 1,2,3,4$ and $p_1 = \hat{p}_1, p_2 = \hat{p}_2, p_3 = \tilde{p}_1, p_4 = \tilde{p}_2$. They have the following properties [31]:

1. $p(x)$ has poles with equal residue $-\frac{l}{2}$ at points $x = \pm 1$ where l is the length of the string.

2. $p(x)$ can have branch cuts in the complex plane, its discontinuity across each cut is fixed by the equation

$$p_i(x+i\epsilon) + p_j(x-i\epsilon) = 2\pi n_{ij}, \tag{4.21}$$

where $n_{ij} \in \mathbb{Z}$.

3. Using properties (1) and (2) we can write the quasi-momentum as

$$p_i(x) = G_i(x) - \frac{l}{2}\left(\frac{1}{x-1} + \frac{1}{x+1}\right), \tag{4.22}$$

where $G(x)$ is called the resolvant. Substituting this form for the resolvant in (4.21) we obtain the fundamental equation for the resolvant given by

$$G_i(x+i\epsilon) + G_j(x-i\epsilon) = 2\pi n_{ij} + l\left(\frac{1}{x-1} + \frac{1}{x+1}\right). \tag{4.23}$$

4. The various different classical solutions satisfying the Virasoro conditions are given by different solutions for the resolvant.

5. Plane wave excitations about the classical solutions are described by introducing a pole at a position r with unit residue on the real line. The position of the pole is constrained by the equation

$$G_i(r+i\epsilon) + G_j(r-i\epsilon) = 2\pi n_{ij} + l\left(\frac{1}{r-1} + \frac{1}{r+1}\right), \tag{4.24}$$

where G_i is the resolvant of the corresponding classical solution.

6. From (2.29) we can now identify $k(r) = \frac{2r}{r^2-1}$ to be the wave number of the plane wave excitation and write the above equation as

$$G_i(r+i\epsilon) + G_j(r-i\epsilon) - k(r)l = 2\pi n_{ij}, \qquad n_{ij} \in \mathbb{Z} \tag{4.25}$$

This equation is the quantization condition for the plane waves and it determines the phase shifts suffered by the plane waves on scattering off the soliton described by the resolvant G_i. The phase shifts corresponding to plane wave excitation with (ij) polarization is given by $-(G_i(r+i\epsilon) + G_j(r-i\epsilon))$

We can now apply these results to the case of the dyonic giant magnon and determine the relation of the fermionic plane wave to the bosonic ones. Let us first embed the $SU(2)$ dyonic magnon solution in $AdS_3 \times S^3$. As we have seen in the previous section, this is a purely bosonic configuration of fields for which only the bosons corresponding to the $SU(2)$ are turned on. Thus the resolvant corresponding to the $\tilde{p}_1$ and $\tilde{p}_2$ quasi momentum is non-zero. Using (4.20) we have

$$\tilde{p}_1(x) = -\tilde{p}_2(x) = G(x) - l\frac{x}{x^2-1}. \tag{4.26}$$

Since there is no non-trivial configurations of fields turned on in the other directions, the corresponding resolvants vanish and the quasi momenta have only the poles at $x = \pm 1$. Again using (4.20) we have

$$\hat{p}_1(x) = -\hat{p}_2(x) = -l\frac{x}{x^2-1}. \tag{4.27}$$

We now can determine the phase shifts for various polarizations of the plane wave excitations about the dyonic giant magnon using the relation (4.25).

1. For fluctuations along the co-ordinate Z_2 in S^3, the plane wave momenta is quantized as

$$\tilde{p}_1(r) - \tilde{p}_2(r) = 2G(r) - k(r)l = 2\pi n_{\tilde{1}\tilde{2}}, \tag{4.28}$$

Thus the phase shift along the co-ordinate Z_2 is given by $-2G(r)$ where $G(r)$ is the resolvant of the dyonic giant magnon solution.

$$\delta_{Z_2}(r) = -2G(r). \tag{4.29}$$

2. The phase shift for plane wave fluctuations along the conjugate co-ordinate $\bar{Z}_2$ using this method is given by

$$\tilde{p}_2(r) - \tilde{p}_1(r) = -2G(r) + k(r)l = 2\pi n_{\tilde{2}\tilde{1}}, \tag{4.30}$$

To read out the phase shift one need to cast it same form as in (4.25). We do this by using the relation $k(r) = -k(1/r)$, then the above equation becomes

$$\tilde{p}_2(1/r) - \tilde{p}_1(1/r) = -2G(1/r) - k(r)l = 2\pi n_{\tilde{2}\tilde{1}}, \tag{4.31}$$

Thus the phase shifts along the $\bar{Z}_2$ direction is given by $2G(1/r)$.

$$\delta_{\bar{Z}_2}(r) = 2G(1/r). \tag{4.32}$$

3. Plane wave phase shifts along the complex conjugate pairs of AdS_3 directions vanish, and that can be seen from the quantization condition

$$\hat{p}_1(r) - \hat{p}_2(r) = k(r)l = 2\pi n_{\hat{1}\hat{2}}. \tag{4.33}$$

Reading out the phase shift we have

$$\delta_{Y_2}(r) = \delta_{\bar{Y}_2}(r) = 0. \tag{4.34}$$

4. There are 2 complex fermions say θ and η which are partners of the co-ordinates Z_2 in S^3 and Y_2 in AdS_3. Phase shifts along these directions are given by the quantization conditions

$$\tilde{p}_1 - \hat{p}_2 = G(r) - k(r)l = 2\pi n_{\tilde{1}\hat{2}}, \qquad \tilde{p}_2 - \hat{p}_1 = G(r) - k(r)l = 2\pi n_{\tilde{2}\hat{1}}. \tag{4.35}$$

Thus the phase shifts along the fermionic directions are given by

$$\delta_\theta(r) = \delta_\eta(r) = -G(r). \tag{4.36}$$

5. Going through a similar argument for the case of the bosons we obtain the plane wave phase shifts along the complex conjugate fermionic directions to be

$$\delta_{\bar{\theta}}(r) = \delta_{\bar{\eta}}(r) = G(1/r). \tag{4.37}$$

We now have the phase shifts for plane wave fluctuations along all the 4 transverse directions of $AdS_3 \times S^3$ in terms of the resolvant $G(r)$ of the dyonic magnon solution. From the explicit calculation of the bosonic phase shifts along the Z_2 and $\bar{Z}_2$ directions in (4.7) and (4.8) we can identify $G(r)$ to be

$$G(r) = G(r, x^{\pm}) = -\left[\frac{1}{i}\ln\left(\frac{r - x^+}{r - x^-}\right) - \frac{1}{2i}\ln\left(\frac{x^+}{x^-}\right)\right]. \tag{4.38}$$

Here we have taken the value of the phase shift suffered by the plane wave on scattering off a single giant magnon. To summarize we have the following results for the plane phase wave shifts along various polarizations

$$\begin{aligned} S^3 \quad &: \quad \delta_{Z_2} = -2G(r, x^{\pm}), \quad \delta_{\bar{Z}_2} = 2G(1/r, x^{\pm}), \\ AdS_3 \quad &: \quad \delta_{Y_2} = \delta_{\bar{Y}_2} = 0, \\ \text{fermionic} \quad &: \quad \delta_\theta = \delta_\eta = -G(r, x^{\pm}), \\ &: \quad \delta_{\bar{\theta}} = \delta_{\bar{\eta}} = G(1/r, x^{\pm}). \end{aligned} \tag{4.39}$$

4.3 One loop energy shift

To evaluate the one loop energy shift we can substitute the phase shifts given in (4.39) into the expression for the one loop correction for the energy of a magnon with momentum p given in (4.1). We obtain the following

$$\begin{aligned} 2\pi\Delta E(x^{\pm}) &= \int_{-1}^{1} dr\sqrt{k(r)^2 + m^2}\frac{\partial}{\partial r}\left[\delta_{Z_2} + \delta_{\bar{Z}_2} - (\delta_\theta + \delta_\eta + \delta_{\bar{\theta}} + \delta_{\bar{\eta}})\right], \\ &= \int_{-1}^{1} dr\sqrt{k(r)^2 + m^2}\frac{\partial}{\partial r}\left[-2G(r; x^{\pm}) + 2G(1/r; x^{\pm})\right. \\ &\qquad \left. -2G(1/r, x^{\pm}) + G(1/r, x^{\pm})\right], \\ &= 0. \end{aligned} \tag{4.40}$$

We have changed the variable of integration from the momentum of the plane wave to r. Thus the one loop correction to the energy of the dyonic magnon vanishes. Since the giant magnon can be obtained as a smooth limit of the dyonic giant magnon, this also implies that the one loop correction to the energy of the giant magnon vanishes. From the discussion in section 2.2 we see that this implies the coefficient of one loop correction g_0 in (2.41) vanishes. Thus the relation $4g = R^2/\pi\alpha'$ is true to one loop in the strong coupling expansion. Applying this result to the dyonic giant magnon dispersion relation in (2.36) we see that possible one loop corrections to the relation does not exist and the dispersion is one loop exact. This must be the case since the dispersion relation arises as a result of a BPS condition.

4.4 One loop scattering phase

Using (4.2) the one loop correction to the scattering phase of two dyonic magnons can be found, but then using the relation (4.3) and the explicit values of the phase shifts evaluated in (4.39) we can evaluate the one loop contribution to the phase factor in $\sigma(x^\pm, y^\pm)$. This is given by

$$
\begin{aligned}
& 2\theta_1(x^\pm, y^\pm) \qquad (4.41)\\
= \; & \frac{1}{4\pi}\int_{-\infty}^{\infty} dr \left(\left[\frac{\partial \delta_{Z_2}(r,x^\pm)}{\partial r}\delta_{Z_2}(r,y^\pm) - \frac{\partial \delta_{\theta}(r,x^\pm)}{\partial r}\delta_{\theta}(r,y^\pm) - \frac{\partial \delta_{\eta}(r,x^\pm)}{\partial r}\delta_{\eta}(r,y^\pm)\right]\right. \\
& \left. + \left[\frac{\partial \delta_{\bar{Z}_2}(r,x^\pm)}{\partial r}\delta_{\bar{Z}_2}(r,y^\pm) - \frac{\partial \delta_{\bar{\theta}}(r,x^\pm)}{\partial r}\delta_{\bar{\theta}}(r,y^\pm) - \frac{\partial \delta_{\bar{\eta}}(r,x^\pm)}{\partial r}\delta_{\bar{\eta}}(r,y^\pm)\right]\right) \\
& -(x^\pm \leftrightarrow y^\pm), \\
= \; & \frac{1}{2\pi}\left[\int_{-1}^{+1} dr \; \frac{\partial G(r,x^\pm)}{\partial r} G(r,y^\pm) \; + \int_{-1}^{+1} dr \; \frac{\partial G(\frac{1}{r},x^\pm)}{\partial r} G(\frac{1}{r},y^\pm)\right] \\
& -(x^\pm \leftrightarrow y^\pm),
\end{aligned}
$$

where

$$
G(r,x^\pm) = -\left(\frac{1}{i}\ln\left(\frac{r-x^+}{r-x^-}\right) - \frac{1}{2i}\ln\frac{x^+}{x^-}\right). \qquad (4.42)
$$

Hence

$$
\begin{aligned}
\frac{\partial G(r,x^\pm)}{\partial r} &= i\left(\frac{1}{r-x^+} - \frac{1}{r-x^-}\right), \\
\frac{\partial G(\frac{1}{r},x^\pm)}{\partial r} &= i\left(\frac{1}{r-\frac{1}{x^+}} - \frac{1}{r-\frac{1}{x^-}}\right). \qquad (4.43)
\end{aligned}
$$

Using the above expressions, the one loop correction to the dressing phase can be arranged as

$$
2\theta_1(x^\pm, y^\pm) = \chi_1(x^+,y^+) - \chi_1(x^+,y^-) - \chi_1(x^-,y^+) + \chi_1(x^-,y^-), \qquad (4.44)
$$

where

$$
\begin{aligned}
\chi_1(x,y) &= -\frac{1}{2\pi}\left[\int_{-1}^{+1} \frac{dr}{r-x}\left(\ln(r-y) - \frac{1}{2}\ln y\right)\right. \qquad (4.45)\\
& \left. + \int_{-1}^{+1} \frac{dr}{r-\frac{1}{x}}\left(\ln(1/r-y) - \frac{1}{2}\log y\right) - (x \leftrightarrow y)\right], \\
&\equiv -\frac{1}{2\pi}\left[I_1(x,y) - I_1(y,x) + I_2(x,y) - I_2(y,x)\right]. \qquad (4.46)
\end{aligned}
$$

The above integrals can be integrated using

$$
-\int dt \; \frac{\ln(1-t)}{t} = \mathrm{Li}_2(t),
$$

where $\mathrm{Li}_2(t)$ is a dilogarithmic function of argument t. We obtain the result

$$
\begin{aligned}
I_1(x,y) &= \left(\ln(x-y) - \frac{1}{2}\ln(y)\right)\ln\left(\frac{x-1}{x+1}\right) + \left[\mathrm{Li}_2\left(\frac{x+1}{x-y}\right) - \mathrm{Li}_2\left(\frac{x-1}{x-y}\right)\right], \\
I_2(x,y) &= \left(\ln\left(1-\frac{y}{x}\right) - \frac{1}{2}\ln(y)\right)\ln\left(\frac{1-x}{1+x}\right) \\
& + \left[\mathrm{Li}_2\left[(x+1)\frac{y}{y-x}\right] - \mathrm{Li}_2\left[(x-1)\frac{y}{x-y}\right]\right]. \qquad (4.47)
\end{aligned}
$$

5 Unitarity check on the crossing relations

S-matrices obey crossing relations. These are conditions which are obtained when one of the Hilbert space the S-matrix acts on is replaced by its anti-particle. Formally the conditions can be written as the following

$$\begin{aligned} \mathcal{C}^{-1} \otimes I \mathcal{S}_{12}^{T_1}(-p_1, p_2) \mathcal{C} \otimes I \mathcal{S}_{12}(p_1, p_2) &= I, \\ I \otimes \mathcal{C}^{-1} \mathcal{S}_{12}^{T_2}(p_1, -p_2) I \otimes \mathcal{C} \mathcal{S}_{12}(p_1, p_2) &= I. \end{aligned} \tag{5.1}$$

Where $\mathcal{C}$ is the charge conjugation operation, T_1, T_2 refer to the transpose operations on the first and second Hilbert space respectively. As we have seen symmetries constrain the S-matrix to a form given by

$$\mathcal{S}_{12}(p_1, p_2) = S_0(p_1, p_2) \hat{\mathcal{S}}_{12}(p_1, p_2), \tag{5.2}$$

where $\hat{\mathcal{S}}_{12}(p_1, p_2)$ is completely determined by symmetries and $S_0(p_1, p_2)$ is the scalar function which cannot be determined by symmetries alone. It satisfies the unitarity condition

$$S_0(p_1, p_2) S_0(p_2, p_1) = 1. \tag{5.3}$$

Then substituting the above form for the S-matrix in (5.1) one obtains the following conditions on the scalar function

$$\begin{aligned} S_0(-p_1, p_2) S_0(p_1, p_2) &= f(p_1, p_2), \\ S_0(p_1, -p_2) S_0(p_1, p_2) &= g(p_1, p_2). \end{aligned} \tag{5.4}$$

These are part of the consistency conditions which are necessary for the S-matrix to satisfy the crossing symmetry relations in (5.1). The unitarity condition in (5.3) then implies the following constraint on the function f.

$$f(p_1, p_2) g(p_2, p_1) = 1. \tag{5.5}$$

The function $f(p_1, p_2)$ can be evaluated if one can implement the transformation to the anti-particle by means of the antipode operation as it was done for the case of $AdS_5 \times S^5$ in [33]. The validity of the function $f(p_1, p_2)$ for this case was tested to one loop in the strong coupling expansion in [34]. The energy and momentum of the anti-particle is equal in magnitude but opposite in sign to the particle. Translating this to the spectral parameters $x^{\pm}$ it can be seen that the spectral parameters of the anti-particle is related to the particle by the following

$$\bar{x}^{\pm} = \frac{1}{x^{\pm}}, \tag{5.6}$$

where the bar denotes the parameters for the anti-particle. From (2.17) it is easy to see that the above transformation reverses the sign of the momentum. To see that it also changes the sign of the energy given by $C_1 + C_2 = 2gc - 1$, first use (2.19) to show

$$\frac{1}{x^-} = \frac{i}{2g(e^{-ip} - 1)} \left(1 - \sqrt{1 + 16g^2 \sin^2 \frac{p}{2}} \right). \tag{5.7}$$

Using this one can evaluate

$$\begin{aligned}\bar{C}_1 + \bar{C}_2 &= 2g\bar{c} - 1, \\ &= -i2g\left(\frac{1}{x^+} - \frac{1}{x^-}\right) - 1, \\ &= -\sqrt{1 + 16g^2 \sin^2 \frac{p}{2}}.\end{aligned} \tag{5.8}$$

However it is shown in the appendix C, the antipode operation for the $SU(1|1)$ invariant S-matrix does not implement the above transformation on the spectral parameters. Instead, it changes $x^+ \leftrightarrow x^-$. This certainly reverses the sign of the momentum, but does not change the sign of the energy. Thus we are unable to determine the function $f(p_1, p_2)$ by the means of the antipode operation. However we can still test the whether the unitarity condition on the function f given in (5.5) holds. In terms of spectral parameters this condition is given by

$$f(x, y)g(y, \frac{1}{x}) = 1. \tag{5.9}$$

Here and in the rest of this section x, y refer to the variables $x^\pm, y^\pm$ respectively. Note the inversion of the variable in the second term, this is due to the fact the crossing conditions are formulated in the universal cover of the spectral parameter plane [33]. Translating the equations in (5.4) in terms of the spectral parameters we have

$$S_0(x, y)S_0(\frac{1}{x}, y) = f(x, y), \qquad S_0(x, y)S_0(x, \frac{1}{y}) = g(x, y). \tag{5.10}$$

From (5.9) and the form for S_0 given in (3.19) we see that we obtain the following constraint on the dressing factor $\sigma(x, y) = \exp(i\theta(x, y))$

$$\ln(\frac{y^+}{y^-}) + i\left[\theta(x, y) + \theta\left(\frac{1}{x}, y\right)\right] = -\ln(\frac{x^-}{x^+}) - i\left[\theta\left(y, \frac{1}{x}\right) + \theta\left(\frac{1}{y}, \frac{1}{x}\right)\right]. \tag{5.11}$$

The above equation is obtained by taking the logarithm of (5.9) and substituting for $f(x, y), g(x, y)$ from (5.10). We then use the antisymmetry property of $\theta(x, y)$ to arrive the following constraint on the dressing factor

$$\ln\frac{y^+}{y^-} + i\theta(x, y) - \ln\frac{x^+}{x^-} - i\theta\left(\frac{1}{x}, \frac{1}{y}\right) = 0. \tag{5.12}$$

We now show that phase θ_0 determined up to one loop in the sigma model coupling satisfies the constraint (5.12). As we have seen the dressing phase $\theta(x, y)$ is expanded as follows.

$$\theta(x, y) = g\theta_0(x, y) + \theta_1(x, y) + \cdots. \tag{5.13}$$

Using the form for $\theta_0(x, y)$ in (3.25) and (3.26), one can deduce the following relation for θ_0, which is

$$\begin{aligned}\theta_0\left(\frac{1}{x}, \frac{1}{y}\right) &= \theta_0(x, y) - \ln\left(\frac{x^+}{x^-}\right)\left(y^- + \frac{1}{y^-} - y^+ - \frac{1}{y^+}\right) \\ &\quad + \ln\left(\frac{y^+}{y^-}\right)\left(x^- + \frac{1}{x^-} - x^+ - \frac{1}{x^+}\right).\end{aligned} \tag{5.14}$$

Now substituting the constraint satisfied by the spectral parameters in (2.18) we get

$$g\theta_0\left(\frac{1}{x},\frac{1}{y}\right) = g\theta_0(x,y) + i\ln\left(\frac{x^+}{x^-}\right) - i\ln\left(\frac{y^+}{y^-}\right). \tag{5.15}$$

Let us now examine the one loop correction to the phase $\theta_1(x,y)$ given in (4.41). From (4.42), we see that $G(r,\frac{1}{x^{\pm}}) = G(\frac{1}{r},x^{\pm})$. Using this property, it is easy to see that the one loop dressing phase given in (4.41) satisfies the relation

$$\theta_1\left(\frac{1}{x},\frac{1}{y}\right) = \theta_1(x,y). \tag{5.16}$$

Combining the results of (5.15) and (5.16), we see that the constraint on the dressing factor (5.12) is satisfied. Note that we have not resorted to any expansion of the spectral parameters in terms of the momentum and the coupling to verify this constraint [5].

6 Conclusions

We have seen that $SU(1|1)$ symmetries constrain the S-matrix of the magnon excitations of this system up to a phase. We have then determined the phase in the sigma model expansion to one loop. We then show that that phase satisfies the constraint of unitarity implied by crossing symmetry. Using the semi-classical methods we also showed that the one loop correction to the dispersion relation vanishes at strong coupling.

In [7] a proposal for the quantum Bethe equations which are valid at all values of coupling was made from the symmetries of the coset description of the Green-Schwarz string on $AdS_3 \times S^3$. The Bethe equations were undetermined up to a phase, our work provide the value of this phase to one loop at strong coupling. To make this more explicit one should identify the magnons described here as a sub-sector of the full theory and show that Bethe equations proposed in [7] in this sub-sector reduce to the Bethe equations obtained from the S-matrix given in this paper. The S-matrix in this paper is invariant under one of $SU(1|1)$ symmetries using the ordinary co-product while the second $SU(1|1)$ is realized as a non-trivial co-product. Using this fact it is perhaps possible to implement crossing symmetry with some modifications of the approach in [33] and determine the function $f(x,y), g(x,y)$ in the crossing equations given in (5.10). This will enable one to make more consistency checks on the phase $\theta(x,y)$ and perhaps determine it completely.

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[5] We have verified that the dressing phase obtained for $\mathcal{N}=4$ SYM at one loop satisfies a similar constraint.

A The extended $SU(1|1) \times SU(1|1)$ algebra

It was shown in [8], that that magnons in the D1-D5 system are BPS states in the extended $SU(1|1) \times SU(1|1)$ algebra given in (2.11). To make the paper self contained we derive their dispersion relation from the extended algebra. The derivation presented here is simplified compared to the one in [8].

We write down the simplest irreducible representation of the extended algebra (2.11) relevant for the magnons. The ground state is $|\phi_p\rangle \otimes |0\rangle$. The action of the charges on the ground state is given by

$$\begin{aligned} Q_1|\phi_p\rangle| \otimes |0\rangle &= a(p)|\psi_p\rangle \otimes ||0\rangle, \qquad & S_1|\phi_p\rangle \otimes |0\rangle &= 0 = 0, \\ Q_2|\phi_p\rangle \otimes |0\rangle &= 0, & S_2|\phi_p\rangle \otimes |0\rangle &= b'(p)|\phi_p\rangle \otimes |\psi^-\rangle. \end{aligned} \tag{A.1}$$

Now from analysis of first order perturbation theory in the $\mathbb{Z}_2$ twisted operator in the symmetric product of the D1-D5 system the following identification of the excited states can be inferred (See eq (2.9) of [8], use the map in (2.10) to translate the notations for the charges)

$$|\psi_p\rangle \otimes ||0\rangle = |\phi_p\rangle \otimes |\psi^-\rangle. \tag{A.2}$$

This identification cuts down the number of states in the representation to 2, which is the right number for a half BPS state in the extended algebra (2.11). Due to this identification we can restrict ourselves to the single excited state $|\psi_p\rangle \otimes |0\rangle$ and we can rewrite the last equation in (A.1) as

$$S_2|\phi_p\rangle \otimes |0\rangle = b'(p)|\psi_p\rangle \otimes |0\rangle. \tag{A.3}$$

The action of the charges on the excited state are given by

$$\begin{aligned} Q_1|\psi_p\rangle \otimes |0\rangle &= 0, \qquad & S_1|\psi_p\rangle \otimes |0\rangle &= b(p)|\phi_p\rangle \otimes 0\rangle, \\ Q_2|\psi_p\rangle \otimes |0\rangle &= a'(p)|\phi_p\rangle \otimes |0\rangle, & S_2|\phi\psi^+\rangle \otimes |0\rangle &= 0. \end{aligned} \tag{A.4}$$

It is easy to see that the above actions of the charges on states are consistent with the nilpotent relations of the algebra that is $Q_1^2 = Q_2^2 = S_1^2 = S_2^2 = 0$. From (A.1) and (A.4) we also see that for the states in this multiplet, the action of Q_1 is proportional to the action of S_2 and the action of S_1 is proportional to the action of Q_2. We can write this as

$$b'(p)Q_1 = a(p)S_2, \qquad a'(p)S_1 = b(p)Q_2. \tag{A.5}$$

We can now evaluate the central charges on the various states of this irreducible multiplet. From the algebra is is clear that the various central charge of the entire multiplet must be same.

$$\begin{aligned} \{Q_1, Q_2\}|\phi_p\rangle \otimes 0\rangle &= a(p)a'(p)|\phi_p\rangle \otimes 0\rangle, \\ \{Q_1, Q_2\}|\psi_p\rangle \otimes |0\rangle &= a(p)a'(p)|\psi_p\rangle \otimes 0\rangle. \end{aligned} \tag{A.6}$$

From the definition of the extended algebra in (2.11) we see that this results in the following value of the central charge for a single magnon with momentum p

$$C_3 - iC_4 = a(p)a'(p). \tag{A.7}$$

Similarly we can show that the other central charges are

$$C_3 + iC_4 = b(p)b'(p), \qquad C_1 = a(p)b(p), \qquad C_2 = a'(p)b'(p). \tag{A.8}$$

We can also show the relation $\{Q_1, S_2\} = \{Q_2, S_1\} = 0$ holds on the states of the short multiplet using the relations in (A.1), (A.4).

We now wish to find the dependence of the central charges $C_3 \pm iC_4$ on the momentum of the magnon. For this it is useful to examine the action of these central charges on a two magnon state. From the spin chain description of the magnons introduced in [8] it can be seen that the action of these central charges on a 2 magnon state is given by

$$(C_3 - iC_4)|\phi_{p_1}\phi_{p_2}\rangle \otimes |0\rangle = [a(p_1)a'(p_1)\exp(-ip_2) + a(p_2)a'(p_2)]|\phi_{p_1}\phi_{p_2}\rangle \otimes |0\rangle. \tag{A.9}$$

The reason the phase $\exp(-ip_2)$ occurs in the first term is because the the central charge $C_3 \pm iC_4$ changes the length the states, which translates to a insertion of additional momentum factors [8]. Another way of thinking of this action is that that the tensor product of the two magnons involves a non-trivial co-product [35]. Therefore on multi-magnon states the action of the central charge is given by

$$C_3 - iC_4|\phi_{p_1}\phi_{p_2}\cdots\phi_{p_j}\rangle \otimes |0\rangle = \sum_{k=1}^{j} a(p_j)a'(p_j) \prod_{l=k+1}^{j} \exp(-ip_l)|\phi_{p_1}\phi_{p_2}\cdots\phi_{p_j}\rangle \otimes |0\rangle. \tag{A.10}$$

Now on physical states the total central charge should vanish and the extended algebra (2.11) should reduce to the usual $SU(1|1) \times SU(1|1)$ algebra. This fixes the form of $a(p_j)a'(p_j)$ to be

$$a(p_j)a'(p_j) = \alpha(\exp(-ip_j) - 1), \tag{A.11}$$

where α is a constant independent of the momentum. Thus the total central charge on the multi-magnon state of j magnons is given by

$$C_3 - iC_4 = \alpha \sum_{k=1}^{j} (\exp(-ip_k - 1) \prod_{l=k+1}^{j} \exp(-ip_l)) = \alpha(\prod_{k=1}^{j} \exp(-ip_k) - 1). \tag{A.12}$$

Therefore on states which satisfy this physical state condition $\sum_{k=1}^{j} p_k = 0$, the additional central charge $C_3 - iC_4$ vanishes. A similar argument for the central charge $C_3 + iC_4$ results in the equation

$$b(p_j)b'(p_j) = \alpha^*(\exp(ip_j) - 1). \tag{A.13}$$

We are now in a position to derive the dispersion relation for a single magnon with momentum p. We have

$$a(p)a'(p) = \alpha(e^{-ip} - 1), \qquad b(p)b'(p) = \alpha^*(e^{-ip} - 1), \tag{A.14}$$
$$C_1 + C_2 = a(p)b(p) - a'(p)b'(p), \qquad C_1 - C_2 = 1 = a(p)b(p) - a'(p)b'(p).$$

The reason $C_1 - C_2 = 1$ is because, we are dealing with a single magnon. Note that the values of the central charges in (A.14) satisfies the BPS condition (2.13). Now as a result of these equations we have

$$C_1 + C_2 = \sqrt{1 + 16|\alpha|^2 \sin^2(\frac{p}{2})}. \tag{A.15}$$

This completes our derivation of the magnon dispersion relation.

A.1 Invariance of the S-matrix

Due to the fact that the S-matrix is constructed out of the Casimirs of the first $SU(1|1)$, that is the algebra generated by the charges Q_1, S_2, C_1 it is invariant under the trivial co-product of this algebra as is seen in (3.14). From the relations in (A.5) which hold on states of the short multiplet, we can show that the invariance with respect to the second $SU(1|1)$ generators is of the following form

$$[\frac{a(p)^{(1)}}{b'(p)^{(1)}} S_2^{(1)} \otimes 1 + (-1)^F \otimes \frac{a(p)^{(2)}}{b'(p)^{(2)}} S_2^{(1)}, \mathcal{S}_{12}] = 0, \tag{A.16}$$
$$[\frac{b(p)^{(1)}}{a'(p)^{(1)}} Q_2^{(1)} \otimes 1 + (-1)^F \otimes \frac{b(p)^{(2)}}{a'(p)^{(2)}} S_2^{(1)}, \mathcal{S}_{12}] = 0.$$

Here the superscripts refer to the two states in the tensor product. The above relations can also be written in terms of the central charges $C_3 \pm iC_4$ and C_1, C_2 as the follows: let us define

$$C_3 + iC_4 = C, \qquad C_3 - iC_4 = \bar{C}. \tag{A.17}$$

Re-writing the above equations in terms of the central charges we obtain

$$[C^{(1)} S_2^{(1)} \otimes C_2^{(2)} + (-1)^F C_2^{(1)} \otimes C^{(2)} S_2^{(2)}, \mathcal{S}_{12}] = 0, \tag{A.18}$$
$$[\bar{C}^{(1)} Q_2^{(1)} \otimes C_2^{(2)} + (-1)^F C_2^{(1)} \otimes \mathcal{C}^{(2)} Q_2^{(2)}, \mathcal{S}_{12}] = 0.$$

Note that the factors which arise in the invariance of the S-matrix in (A.18) are momentum dependent factors involving the central charges. Thus the invariance of the S-matrix under the second $SU(1|1)$ is realized under the above non-trivial co-product.

B Phase shifts by the dressing method

As discussed in section 4.1, the dressing phase method devised in [24] is a powerful method to obtain a $N + 1$-soliton solution from a N soliton solution. We can use this method to find the phase shifts suffered by a plane wave scattering in an N-DGM background taking the plane wave limit on one of the dyonic giant magnon. We will first outline the main points of the dressing method , for details see [24]:

1. We begin by introducing an auxiliary complex parameter x and a set of three matrices ($\psi(x)$, A and B) which satisfies the system of equations

$$i\bar{\partial}\psi = \frac{A\psi}{1+x}, \quad i\partial\psi = \frac{B\psi}{1-x}, \tag{B.1}$$

with A and B independent of x, the spectral parameter. It is then easy to verify that $\psi(0)$ satisfies the equation of motion for g given in (4.6). We also need to impose the $SU(2)$ condition on the matrix $\psi(x)$.

2. Now consider the transformation

$$\begin{aligned} \psi &\rightarrow \psi' = \chi\psi \\ A &\rightarrow A' = \chi A\chi^{-1} + i\,(1+x)\,\bar{\partial}\chi\chi^{-1}, \\ B &\rightarrow B' = \chi B\chi^{-1} + i\,(1-x)\,\bar{\partial}\chi\chi^{-1}. \end{aligned} \tag{B.2}$$

If we can choose χ in such a way that the new A' and B' are independent of λ, then the set ($\psi'(\lambda)$, A' and B') will be a new solution to (B.1) and hence provide a new solution $g' = \psi'(0)$to the principal chiral model (4.6). The dressing function χ is completely fixed by the above requirements to be [24]

$$\chi_N(x) = \left(1 + \frac{x_N - \bar{x}_N}{x - x_N} P_N\right)\sqrt{\frac{x_N}{\bar{x}_N}}. \tag{B.3}$$

Where

$$P_N = \frac{\psi_{N-1}(\bar{x}_N) e e^\dagger \psi^\dagger_{N-1}(\bar{x}_N)}{e^\dagger \psi^\dagger_{N-1}(\bar{x}_N)\psi(\bar{x}_N) e}, \tag{B.4}$$

and e is the constant column vector $(1,1)$. We have put a subscript N in χ to signify that one can successively dress starting from a solution and N refers to the number of the step in the iteration.

3. The vacuum solution for ψ is given by

$$\psi_0 = \begin{pmatrix} e^{iZ(x)} & 0 \\ 0 & e^{-iZ(x)} \end{pmatrix}, \qquad \text{where } Z(x) = \frac{z_-}{x-1} + \frac{z_+}{x+1}, \quad z_\pm = \frac{1}{2}(\sigma \pm t). \tag{B.5}$$

$g_0 = \psi_0(0)$ will then be a point like string rotating around the equator of the S^3.

4. We now dress the solution using the dressing factor (B.3) on the vacuum solution to get multi-solition solutions. Given a N-soliton solution multiplying it by the dressing factor (B.3) we can dress it to a $N+1$ soliton solution.

$$\psi_{N+1} = \chi_{N+1}\psi_N. \tag{B.6}$$

Thus by starting from the vacuum solution ψ_0 we can dress it to get 1-soliton, 2-soliton and so on. At the end we will identify the parameters, appearing in the dressing factor χ_N, with the spectral parameters of the solution as $x_N \leftrightarrow x_N^+, \bar{x}_N \leftrightarrow x_N^-$ the spectral parameters of the N soliton solution.

We can now apply the dressing method to calculate the phase shift suffered by a plane wave scattering off the background of the N-DGM solution. To obtain the phase shift we have to dress ψ_N to get ψ_{N+1} and take the plane wave limit on the last $N+1$th magnon. This essentially means that we need to take the spectral parameter of the last $N+1$ magnon to be real. Let $x_{N+1} = r\exp(i\frac{q}{2})$, it will turn out we will need to examine the solutions up to terms linear in q. We have

$$\psi_{N+1} = \chi_{N+1}\psi_N = \left(1 + \frac{x_{N+1} - \bar{x}_{N+1}}{x - x_{N+1}} P_{N+1}\right)\sqrt{\frac{x_{N+1}}{\bar{x}_{N+1}}}\psi_N, \tag{B.7}$$

let us define $g_N = \psi_N(0)$, then it follows from the above equation

$$\begin{aligned} g_{N+1} &= \sqrt{\frac{x_{N+1}}{\bar{x}_{N+1}}} g_N - \frac{x_{N+1} - \bar{x}_{N+1}}{x_{N+1}} P_{N+1}\sqrt{\frac{x_{N+1}}{\bar{x}_{N+1}}} g_N, \\ &= e^{i\frac{q}{2}} g_N - 2i(\sin\frac{q}{2})P_{N+1} g_N. \end{aligned} \tag{B.8}$$

Now we need the asymptotic behavior i.e $\sigma \to \pm\infty$ up to terms linear in q. So we have [6]

$$\begin{aligned} & g_{N+1}(\pm\infty) = g_N(\pm\infty) + i(\sin\frac{q}{2})g_N(\pm\infty) - 2i(\sin\frac{q}{2})P_{N+1}(r,\pm\infty)g_N(\pm\infty), \\ \Rightarrow \quad & \delta g_N(\pm\infty) \equiv g_{N+1}(\pm\infty) - g_N(\pm\infty) = i(\sin\frac{q}{2})(1 - 2P_{N+1}(r,\pm\infty))g_N(\pm\infty). \end{aligned} \tag{B.9}$$

So in order to determine $\delta g_N(\pm\infty)$, we need to know $g_N(\pm\infty)$ and $P_{N+1}(r,\pm\infty)$. To determine $g_N(\pm\infty)$ we first note that

$$\begin{aligned} P_i(x_i \neq \bar{x}_i, \sigma \to \infty) &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\ P_i(x_i \neq \bar{x}_i, \sigma \to -\infty) &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{B.10}$$

Using the dressing prescription we get

$$\psi_N(x,\infty) = \sqrt{\frac{x_1 x_2 \cdots x_N}{\bar{x}_1 \bar{x}_2 \cdots \bar{x}_N}} \begin{pmatrix} 1 & 0 \\ 0 & \prod_{i=1}^N \left(\frac{x-\bar{x}_i}{x-x_i}\right) \end{pmatrix} \begin{pmatrix} e^{iZ(x)} & 0 \\ 0 & e^{-iZ(x)} \end{pmatrix}. \tag{B.11}$$

After substituting $x_i \equiv r_i\, e^{\frac{ip_i}{2}}$, we obtain

$$\psi_N(x,\infty) = e^{\frac{iP}{2}} \begin{pmatrix} e^{iZ(x)} & 0 \\ 0 & \prod_{i=1}^N \left(\frac{x-\bar{x}_i}{x-x_i}\right) e^{-iZ(x)} \end{pmatrix}, \quad P \equiv \sum_{i=1}^N p_i, \tag{B.12}$$

[6] We use the shorthand $P_{N+1}(r,\pm\infty)$ to denote $P_{N+1}(x_N = \bar{x}_N = r, \sigma \to \pm\infty)$ and $g_N(\pm\infty)$ to denote $g_N(\sigma \to \pm\infty)$

and similarly

$$\psi_N(x,-\infty) = e^{\frac{iP}{2}} \begin{pmatrix} \prod_{i=1}^N \left(\frac{x-\bar{x}_i}{x-x_i}\right) e^{iZ(x)} & 0 \\ 0 & e^{-iZ(x)} \end{pmatrix}. \tag{B.13}$$

As a consequence of these equations we get

$$g_N(-\infty) = \begin{pmatrix} e^{i(t-\frac{P}{2})} & 0 \\ 0 & e^{-i(t-\frac{P}{2})} \end{pmatrix}, \quad g_N(\infty) = \begin{pmatrix} e^{i(t+\frac{P}{2})} & 0 \\ 0 & e^{-i(t+\frac{P}{2})} \end{pmatrix}. \tag{B.14}$$

Now our next task is to determine $P_{N+1}(r,\pm\infty)$ for which we shall use (B.4) . This requires the expression for $\psi_N(r,\pm\infty)$, which we read off from (B.12) and (B.13). The result is given by

$$\begin{aligned} P_{N+1}(r,\infty) &= \frac{1}{2}\begin{pmatrix} 1 & \prod_{i=1}^N \bar{\Sigma}_i \; e^{2iZ(r)} \\ \prod_{i=1}^N \Sigma_i \; e^{-2iZ(r)} & 1 \end{pmatrix}, \\ P_{N+1}(r,-\infty) &= \frac{1}{2}\begin{pmatrix} 1 & \prod_{i=1}^N \Sigma_i \; e^{2iZ(r)} \\ \prod_{i=1}^N \bar{\Sigma}_i \; e^{-2iZ(r)} & 1 \end{pmatrix}, \quad \Sigma_i \equiv \frac{r-\bar{x}_i}{r-x_i}. \end{aligned} \tag{B.15}$$

Finally using (B.9), (B.14), and (B.15), we obtain

$$\begin{aligned} \delta g_N(+\infty) &= -i\sin\frac{q}{2}\begin{pmatrix} 0 & \prod_{i=1}^N \bar{\Sigma}_i \; e^{i(2Z(r)-t-\frac{P}{2})} \\ \prod_{i=1}^N \Sigma_i \; e^{-i(2Z(r)-t-\frac{P}{2})} & 0 \end{pmatrix}, \\ \delta g_N(-\infty) &= -i\sin\frac{q}{2}\begin{pmatrix} 0 & \prod_{i=1}^N \Sigma_i \; e^{i(2Z(r)-t+\frac{P}{2})} \\ \prod_{i=1}^N \bar{\Sigma}_i \; e^{-i(2Z(r)-t+\frac{P}{2})} & 0 \end{pmatrix}. \end{aligned} \tag{B.16}$$

From (B.5) one can check that

$$2Z(r) - t = \omega t - kx, \quad \omega = \frac{1+r^2}{1-r^2}, \quad k = \frac{2r}{1-r^2}. \tag{B.17}$$

Thus the perturbation is a plane wave of wave number k and frequency $\omega = \sqrt{1+k^2}$. Now from (B.16), we get

$$\begin{aligned} \delta Z_1(\pm\infty) &= 0 \\ \delta Z_2(\infty) &= \sin\frac{q}{2}\prod_{i=1}^N \bar{\Sigma}_i \; e^{i(\omega t - kx - \frac{P}{2})} \\ \delta Z_2(-\infty) &= \sin\frac{q}{2}\prod_{i=1}^N \Sigma_i \; e^{i(\omega t - kx + \frac{P}{2})} \end{aligned} \tag{B.18}$$

Thus the phase shifts for the perturbation associated with the S^3 fluctuations around a N-DGM background are

$$\begin{aligned} \delta_{Z_1} &\equiv -i\ln(\delta Z_1(\infty)) + i\ln(\delta Z_1(-\infty)) = 0, \\ \delta_{Z_2} &\equiv -i\ln(\delta Z_2(\infty)) + i\ln(\delta Z_2(-\infty)) = 2i\ln(\prod_{i=1}^N \Sigma_i) - P, \\ &= -i\sum_{i=1}^N \left[2\ln\left(\frac{r-x_i^+}{r-x_i^-}\right) - \ln\left(\frac{x_i^+}{x_i^-}\right)\right]. \end{aligned} \tag{B.19}$$

Here we have used the fact that $\bar{\Sigma}_i = \frac{1}{\Sigma_i}$, which can be easily verified from (B.15). To obtain the last line in (B.19) we have reinstated the definition of the spectral parameters $x_i^+ = x_i$ and $x^- = \bar{x}_i$. One can also show that the phase shifts for the complex conjugate fields are given by

$$\begin{aligned}
\delta_{\bar{Z}_1}(1/r) &\equiv -i\ln(\delta\bar{Z}_1(\infty)) + i\ln(\delta\bar{Z}_1(-\infty)) = 0, \\
\delta_{\bar{Z}_2}(1/r) &\equiv -i\ln(\delta\bar{Z}_2(\infty)) + i\ln(\delta\bar{Z}_2(-\infty)) = -2i\ln(\prod_{i=1}^{N}\Sigma_i) + P, \\
&= i\sum_{i=1}^{N}\left[2\ln\left(\frac{r-x_i^+}{r-x_i^-}\right) - \ln\left(\frac{x_i^+}{x_i^-}\right)\right].
\end{aligned} \tag{B.20}$$

The reason that the difference in phases at ∞ and $-\infty$ gives the value of the phase shift at the momentum $1/r$ is because the momentum and frequency of the complex conjugate field is of the opposite sign compared to the field Z_2.

C Crossing equations using the antipode operation

In this section we derive the crossing relations for the $SU(1|1)$ matrix in (3.9), (3.10) and (3.12) using an algebraic formulation of the crossing relations in terms of an antipode as it was done for the case of the $SU(1|2)$ matrix in $\mathcal{N}=4$ super Yang-Mills by [33]. To simplify our notations we will introduce the fundamental representation of $SU(1|1)$ again and some notations below.

$$\begin{aligned}
Q|\phi\rangle = a|\psi\rangle, \qquad S|\psi\rangle = b|\phi\rangle, \\
B|\phi\rangle = (\beta+1)|\phi\rangle, \qquad B|\psi\rangle = (\beta-1)|\psi\rangle.
\end{aligned} \tag{C.1}$$

We can write the operators of the algebra in terms of the states as

$$\begin{aligned}
Q = a|\psi\rangle\langle\phi|, \qquad S = b|\phi\rangle\langle\psi|, \\
B = (\beta+1)|\phi\rangle\langle\phi| + (\beta-1)|\psi\rangle\langle\psi|.
\end{aligned} \tag{C.2}$$

We will need to take the super-transpose of these operators. The super-transpose defined by

$$M_{ij}^{st} = (-1)^{d(i)d(j)+d(j)} M_{ji}. \tag{C.3}$$

where $d(\phi) = 0, d(\psi) = 1$. For the generators of $SU(1|1)$ the super-transpose works out to be given by

$$Q^{st} = \frac{a}{b}S, \qquad S^{st} = -\frac{b}{a}Q, \qquad B^{st} = B \tag{C.4}$$

We also need to take the charge conjugate of these operators. For this purpose, let us define the charge conjugate generators as

$$\begin{aligned}
\bar{Q} = \bar{a}|\psi\rangle\langle\phi|, \qquad \bar{S} = \bar{b}|\phi\rangle\langle\psi|, \\
\bar{B} = (\bar{\beta}+1)|\phi\rangle\langle\phi| + (\bar{\beta}-1)|\psi\rangle\langle\psi|.
\end{aligned} \tag{C.5}$$

For the above charge conjugate representation, the super-transpose is given by

$$\bar{Q}^{st} = \frac{\bar{a}}{\bar{b}}\bar{S}, \qquad \bar{S}^{st} = -\frac{\bar{b}}{\bar{a}}\bar{Q}, \qquad \bar{B}^{st} = \bar{B}. \tag{C.6}$$

The coefficients $\bar{a}, \bar{c}, \bar{b}, \bar{\beta}$ must satisfy the following. We now need to construct the charge conjugate operation. This operation must satisfy

$$\mathcal{C}\mathcal{O} + \bar{\mathcal{O}}^{st}\mathcal{O} = 0, \tag{C.7}$$

for any operator $\mathcal{O}$ of the algebra. Here we have implemented the antipode operation as in [33] using $\mathcal{S}\mathcal{A} = -\mathcal{A}$ where $\mathcal{S}$ is the antipode operation. Writing out the above condition for the generators in $SU(1|1)$ we obtain

$$\mathcal{C}Q + \bar{Q}^{st}\mathcal{C} = 0, \quad \mathcal{C}S + \bar{S}^{st}\mathcal{C} = 0, \quad \mathcal{C}B + \bar{B}^{st}\mathcal{C} = 0. \tag{C.8}$$

The operator $\mathcal{C}$, the charge conjugation operator must be obtained from the above equations. The equations in (C.8) also constrain the coefficients $\bar{a}, \bar{c}, \bar{b}, \bar{\beta}$. Substituting the form of the generators in terms of states given in (C.2) and (C.5) we obtain

$$\mathcal{C} \;=\; -\frac{\bar{a}}{a}|\phi\rangle\langle\psi| + |\psi\rangle\langle\phi| = -\frac{\bar{a}}{ab}S + \frac{1}{a}Q, \tag{C.9}$$

with the condition

$$ab + \bar{a}\bar{b} = 0, \qquad c = -\bar{c}, \qquad \beta = -\bar{\beta}. \tag{C.10}$$

We can also write down the inverse of C as the following

$$C^{-1} \;=\; -\frac{a}{\bar{a}}|\psi\rangle\langle\phi| + |\phi\rangle\langle\psi| = -\frac{1}{\bar{a}}Q + \frac{1}{b}S. \tag{C.11}$$

The crossing constraint which the full R matrix must satisfy according to [33] is the following

$$(C^{-1}\otimes I)\mathcal{R}_{\bar{1}2}(\bar{\alpha}_1, \alpha_2)^{st_1}(C\otimes I)\mathcal{R}_{12}(\alpha_1, \alpha_2) = I. \tag{C.12}$$

where super-transpose acts on the first Hilbert space. From the structure of the R matrix in (3.10) and (3.12) we see that we need the following relation to implement the crossing condition

$$\begin{aligned} & (C^{-1}\otimes I)(J_{\bar{1}2}^2)^{st}(C\otimes I) \qquad\qquad (\text{C.13})\\ = \;& 2\beta_1 gc_1 + 2\beta_2 gc_2 - 2B^{(1)}C^{(2)} - 2C^{(1)}B^{(2)} - 4Q^{(1)}S^{(1)} + 4S^{(1)}Q^{(2)}, \\ = \;& 4\beta_1 c_1 + 4\beta_2 c_2 - J_{12}^2. \end{aligned}$$

Note that here we have used the equations in (C.8) as well as the relations $\bar{\beta}_1 = -\beta_1, \bar{c}_1 = -\bar{c}_1$. To obtain the second line we have used the definition of the Casimir given in (3.7). Now using (C.13) and (3.8) in the condition (C.12) we see that the we must require that the coefficient of J_{12}^2 should vanish on the LHS. This leads to the following equation

$$\begin{aligned} & [R_{\bar{1}2,1}(\bar{\alpha}_1, \alpha_2) + (4b_1 gc_1 + 4b_2 gc_2)R_{\bar{1}2;2}(\bar{\alpha}_1, \alpha_2)]R_{12,2}(\alpha_1, \alpha_2) \qquad (\text{C.14})\\ & -R_{\bar{1}2,2}(\bar{\alpha}_1, \alpha_2)R_{12,1}(\alpha_1, \alpha_2) - 4R_{\bar{1}2,2}(\bar{\alpha}_1, \alpha_2)R_{12,2}(\alpha_1, \alpha_2)(b_1 + b_2)(gc_1 + gc_2) = 0. \end{aligned}$$

On substituting the form for the coefficients of the R matrix given in (3.12) we obtain the condition

$$\bar{\alpha}_1 = \alpha_1, \tag{C.15}$$

The terms proportional to g vanish due to the following identity

$$\frac{1}{2}(-\beta_1+\beta_2)(-c_1+c_2) - (\beta_1 c_1 + \beta_2 c_2) - \frac{1}{2}(\beta_1+\beta_2)(c_1+c_2) + (\beta_1\beta_2)(c_1+c_2) = 0. \tag{C.16}$$

This is an important consistency check of our equations and the definition of the conjugation operation. Now the term in (C.12) which is proportional to identity leads to the following equation

$$\begin{aligned} & R_{\bar{1}2,0}(\alpha_1,\alpha_2)R_{12,0}(\alpha_1,\alpha_2) \\ & \times \frac{(\alpha_2-\alpha_1)^2 + \frac{1}{4}(gc_1+gc_2)^2}{((\alpha_2-\alpha_1) - \frac{i}{2}(-gc_1+gc_2))((\alpha_2-\alpha_1)-\frac{i}{2}(gc_1+gc_2))} = 1. \end{aligned} \tag{C.17}$$

One of the important consistency checks of the above equation is that the constraint should not depend on the β's since these are the charges of the generator B which is an outer automorphism. An identity similar to (C.16) ensures this. Thus the phase factor has to satisfy the following constraints

$$\begin{aligned} R_{12,0}(\alpha_1,\alpha_2)R_{12,0}(\alpha_2,\alpha_1) &= 1, \\ R_{\bar{1}2,0}(\alpha_1,\alpha_2)R_{12,0}(\alpha_1,\alpha_2) &= f(1,2), \\ &= \frac{(\alpha_2-\alpha_1) - \frac{i}{2}(-gc_1+gc_2)}{(\alpha_2-\alpha_1)+\frac{i}{2}(gc_1+gc_2)}. \end{aligned} \tag{C.18}$$

α and c are related to the spectral parameters by

$$\alpha = \frac{g}{2}(x^+ + x^-), \qquad c = -i(x^+ - x^-). \tag{C.19}$$

Substituting these expression for α and c in terms of the spectral parameters we obtain

$$f(x,y) = \frac{y^- - x^-}{y^+ - x^-}. \tag{C.20}$$

For completeness we mention that carrying out a similar analysis with the conjugation operation on the particle 2 leads to the function

$$g(x,y) = \frac{y^+ - x^+}{y^+ - x^-}. \tag{C.21}$$

Note that here the functions f and g satisfy the unitarity constraint

$$f(x,y)g(y,\bar{x}) = 1 \tag{C.22}$$

with $\bar{x}$ defined as $\bar{x}^+ = x^-, \bar{x}^- = x^+$.

The conjugation operation obtained using the antipode results in

$$\bar{\alpha} = \alpha, \qquad \bar{c} = -c \tag{C.23}$$

From (C.19) we see that the spectral parameters for the conjugate particle is therefore

$$\bar{x}^{+} = x^{-}, \qquad \bar{x}^{-} = x^{+} \tag{C.24}$$

As discussed in section 5, this reverses the sign of the momentum but does not change the sign of the energy. Thus the algebraic method of using the antipode does not result in the required conjugation which transforms a particle to an anti-particle.

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