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**PERIODIC ORBITS OF CIRCLE HOMEOMORPHISMS  
WITH A BREAK POINT**

Akhtam Dzhaliilov<sup>1</sup>

*Faculty of Mathematics, Samarkand State University,  
Boulevard st. 15, 703004, Samarkand, Uzbekistan  
and*

*The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy*

and

Abdumajid Begmatov<sup>2</sup>

*Institute of Mathematics and Information Technologies, Uzbek Academy of Sciences,  
Durmon Yuli st. 29, 100125, Tashkent, Uzbekistan.*

**Abstract**

Let  $f_\theta(x) = F_0(x) + \theta \pmod{1}$ ,  $x \in S^1$ ,  $\theta \in [0, 1]$  be a family of preserving orientation circle homeomorphisms with a single break point  $x_b$ , i.e. with a jump in the first derivative  $F_0$  at the point  $x = x_b$ . Suppose that  $F_0'(x)$  is absolutely continuous on  $[x_b, x_b+1]$  and  $F_0''(x) \in L_\alpha([0, 1])$  for some  $\alpha > 1$ . Consider  $f_\theta$  with rational rotation number  $\rho_\theta = \frac{p}{q}$  of rank  $n$ , i.e.  $\frac{p}{q} = [k_1, k_2, \dots, k_n]$ . We prove that for sufficiently large  $n$ , the homeomorphism  $f_\theta$  has either a unique periodic orbit of period  $q$  or two periodic orbits of period  $q$ . Also the renormalization behaviour of  $f_\theta$  with rational rotation number  $\rho_\theta = \frac{p}{q}$  is studied.

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<sup>1</sup>a.dzhalilov@yahoo.com

<sup>2</sup>begmatovms@mail.ru

# 1 Introduction

Circle homeomorphisms constitute one important class of one-dimensional dynamical systems. The investigation of their properties was initiated by Poincaré [17], who came across them in his studies of differential equations more than a century ago. Since then interest in these maps never diminished. Circle maps are also important because of their applications to natural sciences (see for instance [2, 15, 18, 16, 7, 10, 14]).

We identify the unit circle  $S^1 = R^1/Z^1$  with the half open interval  $[0, 1)$ . Consider the one-parameter families of the orientation preserving circle homeomorphisms

$$f_\theta(x) = F_0(x) + \theta \pmod{1}, \quad x \in S^1, \quad \theta \in [0; 1] \quad (1)$$

where the initial lift  $F_0 : R^1 \rightarrow R^1$  satisfies the following conditions:

- (a)  $F_0$  is continuous and strictly increasing on  $R^1$
- (b)  $F_0(0) = 0$ ,  $F_0(x + 1) = F_0(x) + 1$ ,  $x \in R^1$ ;
- (c) there is a point  $x_b \in S^1$  such that the one-sided derivatives  $F'_0(x_b \pm 0)$  exist, are positive and  $F'_0(x_b - 0) \neq F'_0(x_b + 0)$ ;
- (d)  $F_0$  is absolutely continuous on  $[x_b, x_b + 1]$ ;
- (e)  $F''_0 \in L_\alpha([0; 1], d\ell)$  for some  $\alpha > 1$ , where  $\ell$  is Lebesgue measure on the circle.

The conditions (d) and (e) are called the **Kaznelson-Ornsten's smoothness conditions**. The point  $x_b$  is called a **break point** of  $f_\theta$ . The ratio

$$\sigma(x_b) = \sqrt{\frac{F'_0(x_b - 0)}{F'_0(x_b + 0)}}$$

is called the **jump ratio** of  $f_\theta$  at  $x_b$  or, for short,  $f_\theta$ -jump ratio. Notice that the parameter  $\sigma = \sigma(x_b)$  is obviously an invariant under smooth coordinate transformations and characterizes the type of the singularity.

Put  $F_\theta = F_0 + \theta$ ,  $\theta \in [0, 1]$ . The rotation number  $\rho_\theta$  of  $f_\theta$  is defined by (see [6] for details)

$$\rho_\theta = \lim_{n \rightarrow \infty} \frac{F_\theta^n(x)}{n} \pmod{1},$$

where the limit exists for all  $x \in R^1$  and is independent of  $x$ . Here and later,  $F^n$  denotes the  $n$ th iteration of  $F$ .

The families like

$$A_\theta(x) = x + \frac{c}{2\pi} \sin(2\pi x) + \theta \pmod{1}$$

were studied for various constants  $c$  (see for instance [1, 4, 8, 9, 13, 20]). For  $c < 1$  the maps are diffeomorphisms and there is a result in [9], which says that the rotation number is absolutely continuous as a function of  $\theta$ . When  $c > 1$  the maps are non homeomorphisms and have no

rotation number. In this case, both endpoints of rotation interval are rational almost everywhere w.r.t Lebesgue measure. Notice that the results are quite different if the family (1) has singularity points. Swiatek in [20] studied the family (1) with several critical points. It is proved that the set of parameter values corresponding to irrational rotation numbers has Lebesgue measure zero. In other words, the intervals on which frequency-locking occurs fill up the set of full measure. Khanin and Vul in [13] studied renormalizations, periodic points and rational rotation numbers of the family (1) with single break point  $x_b$  such that  $f_\theta \in C^{2+\varepsilon}(S^1 \setminus \{x_b\})$ . On one hand, the set of the parameter values corresponding to irrational rotation numbers has a zero measure, and the dynamics is characterized by nontrivial scaling transformations. On the other hand, similar to the case of circle diffeomorphisms (see [12, 11]), the renormalizations group behavior of such maps is rather simple. In the renormalized coordinates, the iterations of  $f_\theta$  approximated to linear-fractional transformations in the norm  $\|\cdot\|_{C^2(S^1 \setminus \{x_b\})}$  (see [13, 21]). Note that, if the circle homeomorphism  $f$  with a single break point  $x = x_b$  belongs to  $C^{2+\varepsilon}(S^1 \setminus \{x_b\})$  and its rotation number is irrational, then  $f$ -invariant measure is singular w.r.t. Lebesgue measure on the circle [5].

In this paper, our purpose is to study the family (1) with a single point, but with a weaker smoothness condition for  $f_\theta$ .

It is easy to see that  $\rho_\theta$  is the increasing function of  $\theta$ . Note that for each rational number  $a$  the set  $I(a) = \{\theta : \rho_\theta = a\}$  is a nontrivial closed interval and  $I(a)$  consists of only one point if  $a$  is irrational. Consider any rational number  $\frac{p}{q}$  which belongs to interval  $[0, 1]$ . Denote the left and right endpoints of interval  $I(\frac{p}{q})$  by  $\theta_1(\frac{p}{q})$  and  $\theta_2(\frac{p}{q})$ , respectively.

We say that the rational number  $\frac{p}{q}$  is of **rank**  $n$ , if  $\frac{p}{q} = [k_1, k_2, \dots, k_n]$ ,  $k_n > 1$ .

The main purpose of our paper is to prove the following:

**Theorem 1.1.** Let  $\{f_\theta : \theta \in [0, 1]\}$  be the family of circle homeomorphisms defined by (1) with the initial lift  $F_0$  satisfying the conditions (a)-(e). There exists a number  $N = N(F_0) > 0$  such that if  $f_\theta$  belongs to this family and its rotation number  $\rho_\theta$  is rational with rank  $n$  greater than  $N$ , then the following assertions hold:

- (i) if  $\theta \in \{\theta_1(\frac{p}{q}), \theta_2(\frac{p}{q})\}$ , then  $f_\theta$  has a unique periodic orbit of period  $q$ ;
- (ii) if  $\theta \in (\theta_1(\frac{p}{q}), \theta_2(\frac{p}{q}))$ , then  $f_\theta$  has exactly two periodic orbits of period  $q$  : one of them is stable and the other is unstable.

## 2 Dynamical partitions of circle homeomorphisms with rational rotation number

Let  $f$  be an orientation preserving homeomorphism of the circle with rational rotation number  $\rho = \frac{p}{q} = [k_1, k_2, \dots, k_n]$ . Since the rank of  $\frac{p}{q}$  equals  $n$  we put  $p_n := p$  and  $q_n := q$ . For  $1 \leq m \leq n$

denote by  $\frac{p_m}{q_m} = [k_1, k_2, \dots, k_m]$ , the convergent of  $\frac{p}{q}$ . Their denominators  $q_m$  satisfy  $q_{m+1} = k_{m+1}q_m + q_{m-1}$ ,  $1 \leq m \leq n-1$ ,  $q_0 = 1$ ,  $q_1 = k_1$ .

Since the rotation number  $\rho = \frac{p}{q}$  is rational homeomorphism  $f$  has at least one periodic orbit of period  $q$  (see [6]). Let  $O_f(t, q_n) = \{f^i(t), i = 0, 1, \dots, q_n - 1\}$  be a periodic orbit of  $f$  of period  $q$ . For an arbitrary point  $x_0 \in O_f(t, q_n)$ , denote by  $\Delta_0^{(n)}(x_0)$  the closed interval with endpoints  $x_0$  and  $x_{q_n} = f^{q_n}x_0$ . If  $n$  is odd then  $x_{q_n}$  is to the left of  $x_0$ , and to the right of  $x_0$  if  $n$  is even. Denote by  $\Delta_i^{(n)}(x_0)$  the iterates of the interval  $\Delta_0^{(n)}(x_0)$  under  $f$ :  $\Delta_i^{(n)}(x_0) = f^i \Delta_0^{(n)}(x_0)$ ,  $i \geq 1$ . It is well known that each of the following system of intervals

$$\xi_m(x_0) = \left\{ \Delta_i^{(m-1)}(x_0), 0 \leq i < q_m; \Delta_j^{(m)}(x_0), 0 \leq j < q_{m-1} \right\}, 1 \leq m < n,$$

$$\xi_n(x_0) = \left\{ \Delta_i^{(n-1)}(x_0), 0 \leq i < q_n \right\},$$

cover the whole circle and that their interiors are mutually disjoint (see [3, 19]). The partition  $\xi_m(x_0)$  is called the  $m$ th **dynamical partition** of the point  $x_0$ . We briefly recall the structure of the dynamical partitions. The passage from  $\xi_m(x_0)$  to  $\xi_{m+1}(x_0)$ ,  $1 \leq m < n-2$  is simple: namely, all intervals of rank  $m$  are preserved and each of the intervals  $\Delta_i^{(m-1)}(x_0)$ ,  $0 \leq i < q_m$ , is divided into  $(k_{m+1} + 1)$  intervals:  $\Delta_i^{(m-1)}(x_0) = \Delta_i^{(m+1)}(x_0) \cup \bigcup_{s=0}^{k_{m+1}-1} \Delta_{i+q_{m-1}+sq_m}^{(m)}(x_0)$ . Note that the endpoints of intervals  $\Delta_i^{(n-1)}(x_0)$ ,  $0 \leq i < q_n - 1$  are periodic points of  $f$  of period  $q_n$ . Also each interval of partition  $\xi_n(x_0)$  is periodic of period  $q_n$ . The following lemma plays a key role for studying metrical properties of the homeomorphism  $f$ .

**Lemma 2.1.** Let  $f$  be a circle homeomorphism with lift  $F$  and rational rotation number  $\rho_f = \frac{p_n}{q_n}$  of rank  $n$ . Let the finite derivatives  $F'(x_b \pm 0) > 0$  exist and let  $F \in C^1([x_b, x_b + 1])$  and  $\text{var}_{[x_b, x_b+1]} \log F' = \bar{v} < \infty$ . We write

$$v = \bar{v} + |\log F'(x_b - 0) - \log F'(x_b + 0)| = \bar{v} + 2 \log \sigma(x_b).$$

In this case, the inequality

$$e^{-v} \leq \prod_{s=0}^{q_k-1} F'(x_s) \leq e^v \tag{2}$$

holds for any  $1 \leq k \leq n$  and  $x_0 \in S^1$  such that  $x_i \neq x_b, i = 0, 1, 2, \dots, n$ .

The last inequality is called the **Denjoy inequality**. The proof of Lemma 2.1 is just like that of the similar assertion for diffeomorphism (see for instance [12]). Using Lemma 2.1 it can easily be shown that the lengths of the intervals of the dynamical partition  $\xi_n$  are exponentially small.

**Corollary 2.1.** Suppose that  $\Delta^{(k)} \subset \Delta^{(l)} \in \xi_l(x_0)$ ,  $\Delta^{(k)} \in \xi_k(x_0)$ ,  $1 \leq l < k \leq n$ . Then for

some constant  $M_0 > 0$

$$l(\Delta^{(k)}) \leq M_0 \lambda^{k-l} l(\Delta^{(l)}), \quad (3)$$

where  $\lambda = (1 + e^{-v})^{-1/2} < 1$ .

### 3 Renormalizations of circle homeomorphisms with rational rotation number

We discuss renormalizations of circle homeomorphism  $f_\theta$  with rational rotation number. The main idea of the method is to study large time iterates of the original mappings in a rescaled coordinate system corresponding to some neighborhood of a given point. Let  $\frac{p_n}{q_n} \in [0, 1]$ ,  $n \geq 1$  be an arbitrary rational number of rank  $n$ . Let us fix some  $\theta \in I(\frac{p}{q})$  and denote  $F = F_\theta$  and  $f = f_\theta$  (we omit the parameter  $\theta$  in the sequel). Let  $O_f(t, q_n) = \{f^i(t), i = 0, 1, \dots, q_n - 1\}$  be an arbitrary periodic orbit of  $f$  of period  $q_n$ . Denote by  $[y_1, y_2]$  the closed interval formed by two consecutive points (in the counterclockwise direction) of orbit  $O_f(t, q_n)$  and containing the break point  $x_b$  of  $f$ . We introduce the renormalized coordinate  $z$  on  $[y_1, y_2]$  given by the formula  $z = (x - y_2)/(y_1 - y_2)$ . It is clear that the normalized coordinate  $z$  changes from 1 to 0, when  $x$  is moving from  $y_1$  to  $y_2$ . Denote by  $d$  the renormalized coordinate of break point  $x_b$ , i.e.  $d = (x_b - y_2)/(y_1 - y_2)$ .

Now, we define the function  $\bar{f}_{\rho, n}$  corresponding to  $F^{q_n}$  in this new coordinate by :

$$\bar{f}_{\frac{p_n}{q_n}, n}(z) = \frac{F^{q_n}(y_2 + z(y_1 - y_2)) - y_2 - p_n}{y_1 - y_2}, \quad z \in [0, 1]. \quad (4)$$

This map is called *n*th **renormalization** of  $f$  on the interval  $[y_1, y_2]$ . Next, we define the piecewise-linear function  $G_{d, n}$  on  $[0, 1]$  by the formula:

$$G_{d, n}(z) = \begin{cases} \frac{\sigma z}{(\sigma-1)z+d(1-\sigma^2)+\sigma^2}, & \text{if } z \in [0, d], \\ \frac{\sigma^2 z + d(1-\sigma^2)}{\sigma(\sigma-1)z+d(1-\sigma^2)+\sigma}, & \text{if } z \in (d, 1]. \end{cases} \quad (5)$$

Furthermore, we formulate and prove the theorem on renormalization behavior of the circle homeomorphism  $f$  with rational rotation number. Thus, we have the following theorem.

**Theorem 3.1.** Let  $\{f_\theta : \theta \in [0, 1]\}$  be the family of circle homeomorphisms defined by (1) with the initial lift  $F_0$  satisfying the conditions (a)-(e). Then, for any  $\varepsilon > 0$  there exists  $N = N(\varepsilon, F_0) > 0$  (which doesn't depend on choice of periodic orbit), such that if  $f$  belongs to this family and its rotation number  $\rho = \frac{p_n}{q_n}$  is rational with rank  $n$ ,  $n > N$  the following estimates hold:

$$\begin{aligned} \|\bar{f}_{\rho, n}(z) - G_d(z)\|_{C([0, 1] \setminus \{d\})} &\leq C_1 \varepsilon, \\ \|\bar{f}'_{\rho, n}(z) - G'_d(z)\|_{L^1([0, 1], d\ell)} &\leq C_1 \varepsilon, \\ \|\bar{f}''_{\rho, n}(z) - G''_d(z)\|_{L^1([0, 1], d\ell)} &\leq C_1 \varepsilon, \end{aligned}$$

where the constant  $C_1 > 0$  depends only on initial lift  $F_0$ .

**Proof of Theorem 3.1.** Consider the dynamical partition generated by periodic orbit  $O_f(t, q_n)$ . We put  $x_0 = y_1$ . Consider the partition  $\xi_n(x_0)$ . It is clear that  $\Delta_0^{(n-1)}(x_0) = [y_1, y_2]$  and  $f^{q_n} \Delta_0^{(n-1)}(x_0) = \Delta_0^{(n-1)}(x_0)$ . It follows from Corollary 2.1, that the intervals of the dynamical partition  $\xi_n(x_0)$  have exponentially small length, i.e.  $l(\Delta_0^{(n-1)}(x_0)) \leq \text{const} \lambda^n$ ,  $\lambda \in (0, 1)$ . Note that the function  $\bar{f}_{\rho, n}(z)$  can be represented as the superposition of two functions,  $\bar{f}_1$  and  $\bar{f}_2$ , which correspond to the mappings  $f : \Delta_0^{(n-1)}(x_0) \rightarrow \Delta_1^{(n-1)}(x_0)$ ,  $f^{q_n-1} : \Delta_1^{(n-1)}(x_0) \rightarrow \Delta_{q_n}^{(n-1)}(x_0) = \Delta_0^{(n-1)}(x_0)$ , respectively. We introduce relative coordinates  $z_i$ ,  $i = 0, 1, \dots, q_n - 1$ , in the intervals  $\Delta_i^{(n-1)}(x_0)$ :

$$z_i = (f^i(x) - f^i(y_2)) / (f^i(y_1) - f^i(y_2)), \quad x \in \Delta_0^{(n-1)}(x_0).$$

Then the functions  $\bar{f}_1$  and  $\bar{f}_2$  can be written as

$$\bar{f}_1(z_0) = \frac{f(y_2 + (y_1 - y_2)z_0) - f(y_2)}{f(y_1) - f(y_2)}, \quad (6)$$

$$\bar{f}_2(z_1) = \frac{f^{q_n-1}(f(y_2)) + (f(y_1) - f(y_2))z_1 - y_2}{y_1 - y_2}. \quad (7)$$

It is clear that  $\bar{f}_{\rho, n}(z) = \bar{f}_2(\bar{f}_1(z))$ . Define the following functions:

$$g(z_1) = \frac{\sigma z_1}{1 + z_1(\sigma - 1)}, \quad R_d(z_0) = \begin{cases} \frac{z_0}{\sigma^2(1-d)+d}, & \text{if } z_0 \in [0, d], \\ \frac{\sigma^2 z_0 + d(1-\sigma^2)}{\sigma^2(1-d)+d}, & \text{if } z_0 \in (d, 1]. \end{cases} \quad (8)$$

We put  $M_n = \exp\left\{\sum_{i=1}^{q_n-2} \int_{\Delta_i^{(n-1)}} \frac{f''(y)}{2f'(y)} dy\right\}$ . From now on we shall denote by  $K$  constants that depend only on the original family  $f_\theta$ . Next, we formulate two necessary lemmas.

**Lemma 3.1.** For any  $\varepsilon > 0$ , the following relation holds for sufficiently large  $n$

$$z_{q_n-1}(z_1) = \frac{z_1 M_n \exp \tau_n(z_1)}{1 + z_1(M_n \exp \tau_n(z_1) - 1)}, \quad (9)$$

where the function  $\tau_n(z_1)$  and its derivatives satisfies the following inequalities:

$$\max_{0 \leq z_1 \leq 1} |\tau_n(z_1)| \leq K\varepsilon, \quad \max_{0 \leq z_1 \leq 1} |(z_1 - z_1^2) \tau_n'(z_1)| \leq K\varepsilon, \quad (10)$$

$$\|\tau_n'(z_1)\|_{L_1([0,1], dl)} \leq K\varepsilon, \quad \|(z_1 - z_1^2) \tau_n''(z_1)\|_{L_1([0,1], dl)} \leq K\varepsilon. \quad (11)$$

**Lemma 3.2.** The following estimates hold for sufficiently large  $n$

$$\|\bar{f}_1(z_0) - R_d(z_0)\|_{C^1([0,1] \setminus \{d\})} \leq K \lambda^{\frac{n}{\beta}}, \quad (12)$$

$$\|\bar{f}'_1(z_0) - R'_d(z_0)\|_{L_1([0,1], d\ell)} \leq K\lambda^{\frac{n}{\beta}}, \quad (13)$$

$$\|\bar{f}''_1(z_0) - R''_d(z_0)\|_{L_1([0,1], d\ell)} \leq K\lambda^{\frac{n}{\beta}},$$

where  $\lambda \in (0, 1)$  and  $\beta = \frac{\alpha}{\alpha-1}$ .

For an easy flow of our presentation, we shall prove these two Lemmas at the end of this section. So we continue our proof of Theorem 3.1. It is not hard to show that  $\bar{f}_2(z_1) = z_{q_n-1}(z_1)$ . Using the last relation and Lemma 3.1, we obtain

$$\|\bar{f}_2(z_1) - \frac{M_n z_1}{1 + z_1(M_n - 1)}\|_{C^1([0,1])} \leq K\varepsilon, \quad (14)$$

$$\|\bar{f}''_2(z_1) - \frac{2M_n(1 - M_n)}{(1 + z_1(M_n - 1))^3}\|_{L_1([0,1], d\ell)} \leq K\varepsilon. \quad (15)$$

It is clear that

$$\ln M_n = \sum_{i=1}^{q_n-2} \int_{\Delta_i^{(n-1)}(x_0)} \frac{f''(y)}{2f'(y)} dy = \ln \sigma - \int_{\Delta_0^{(n-1)}(x_0)} \frac{f''(y)}{2f'(y)} dy - \int_{\Delta_{q_n-1}^{(n-1)}(x_0)} \frac{f''(y)}{2f'(y)} dy. \quad (16)$$

Thus, we have

$$\left| \int_{\Delta_k^{(n-1)}(x_0)} \frac{f''(y)}{2f'(y)} dy \right| \leq K \|f''\|_{\alpha} \lambda^{n/\beta}, \quad \text{for } k = 0, q_n - 1,$$

Together with relations (14)-(16) this implies that

$$\|\bar{f}_2(z_1) - g(z_1)\|_{C^1([0,1])} \leq K\varepsilon,$$

$$\|\bar{f}''_2(z_1) - g''(z_1)\|_{L_1([0,1], d\ell)} \leq K\varepsilon.$$

So, the relation  $\bar{f}_{\rho, n}(z) = \bar{f}_2(\bar{f}_1(z))$  and Lemma 3.2 imply the proof of Theorem 3.1.

**Proof of Lemma 3.1.** Denote  $a_i = f^i(y_1)$ ,  $b_i = f^i(y_2)$ ,  $c_i = f^i(x)$ ,  $i = 1, 2, \dots, q_n - 1$ .

Then we get

$$z_{i+1} = (x_{i+1} - b_{i+1}) / (a_{i+1} - b_{i+1}). \quad (17)$$

It is easy to check that

$$x_{i+1} = f(x_i) = f(a_i) + f'(a_i)(x_i - a_i) + \int_{a_i}^{x_i} f''(y)(x_i - y) dy,$$

$$b_{i+1} = f(b_i) = f(a_i) + f'(a_i)(b_i - a_i) + \int_{a_i}^{b_i} f''(y)(b_i - y) dy.$$

By definition  $a_{i+1} = f(a_i)$ . Substituting this into (17) we get

$$z_{i+1} = z_i(1 + A_i(z_i - 1)), \quad i = 1, 2, \dots, q_n - 1, \quad (18)$$

where

$$A_i = - \frac{\frac{1}{f'(a_i)(x_i - a_i)} \int_{a_i}^{x_i} f''(y)(y - a_i) dy + \frac{1}{f'(a_i)(b_i - x_i)} \int_{x_i}^{b_i} f''(y)(b_i - y) dy}{1 + \frac{1}{f'(a_i)(b_i - a_i)} \int_{a_i}^{b_i} f''(y)(b_i - y) dy}. \quad (19)$$

We denote

$$\tau_n(z_1) = \sum_{i=1}^{q_n-2} \psi_i.$$

where

$$\chi_i = \int_{a_i}^{b_i} \frac{f''(y)}{2f'(y)} dy, \quad \psi_i = -\chi_i - \ln \left( \frac{1 + A_i z_i}{1 + A_i(z_i - 1)} \right), \quad i = 1, 2, \dots, q_n - 1,$$

Using (18) we obtain

$$\frac{1 - z_{i+1}}{z_{i+1}} = \frac{1 - z_i}{z_i} \frac{1 + A_i z_i}{1 + A_i(z_i - 1)} = \frac{1 - z_i}{z_i} \exp\{-\chi_i\} \exp\{-\psi_i\}. \quad (20)$$

Taking iteration of (20) we get

$$\frac{1 - z_{q_n-1}}{z_{q_n-1}} = \frac{1 - z_1}{z_1} \exp\left\{-\sum_{i=1}^{q_n-2} \chi_i\right\} \exp\left\{-\sum_{i=1}^{q_n-2} \psi_i\right\} = \frac{1 - z_1}{z_1} \frac{1}{M_n \exp \tau_n(z_1)}. \quad (21)$$

Solving equation (21) with respect to  $z_{q_n-1}$  we obtain the relation (9).

Let us estimate  $\tau_n(z_1)$ . First we estimate  $A_i$ . Denote by  $V_i$  the second term of the denominator of (19). Since  $f''(x) \in L_\alpha([0, 1], dl)$  applying the Holder inequality we obtain

$$|V_i| \leq \frac{1}{f'(a_i)(b_i - a_i)} \int_{a_i}^{b_i} |f''(y)|(y - a_i) dy \leq \frac{\|f''\|_\alpha (b_i - a_i)^{1+\frac{1}{\beta}}}{f'(a_i)(b_i - a_i)(1 + \beta)} \leq K(b_i - a_i)^{\frac{1}{\beta}}. \quad (22)$$

Analogously, it can be shown that the moduli of both terms in (19) are not greater than  $K(b_i - a_i)^{\frac{1}{\beta}}$ . Let us recall that  $[a_i, b_i] \in \xi_n(x_0)$  and  $\ell([a_i, b_i]) \leq K\lambda^n$ ,  $i = 0, 1, \dots, q_n - 2$ . This, together with the expression for  $A_i$  imply that  $|A_i| \leq Const\lambda^{\frac{n}{\beta}}$ . Next, we rewrite  $\tau_n(z_1)$  in the form

$$\tau_n(z_1) = - \sum_{i=0}^{q_n-2} \chi_i - \sum_{i=0}^{q_n-2} \ln \left( \frac{1 + A_i z_i}{1 + A_i(z_i - 1)} \right) = - \ln M_n - \sum_{i=0}^{q_n-2} A_i - \sum_{i=0}^{q_n-2} O(A_i^2). \quad (23)$$

We estimate the last sum in (23). Note that each term of (19) containing an integral is not greater than  $\int_{a_i}^{b_i} |f''(y)| dy$ . Using the estimate for  $A_i$ , it can easily be shown that

$$\sum_{i=1}^{q_n-2} O(A_i^2) \leq K\lambda^{\frac{n}{\beta}}. \quad (24)$$



We rewrite the second to the last sum in (23) in the following form

$$\begin{aligned}
\sum_{i=1}^{q_n-2} A_i &= - \sum_{i=1}^{q_n-2} \int_{a_i}^{b_i} \frac{f''(y)}{2f'(y)} dy - \sum_{i=1}^{q_n-2} \left[ \frac{1}{f'(a_i)(x_i - a_i)} \int_{a_i}^{x_i} f''(y)(y - a_i) dy - \frac{1}{2} \int_{a_i}^{x_i} \frac{f''(y)}{2f'(y)} dy \right] - \\
&\quad - \sum_{i=1}^{q_n-2} \left[ \frac{1}{f'(a_i)(b_i - x_i)} \int_{x_i}^{b_i} f''(y)(b_i - y) dy - \frac{1}{2} \int_{x_i}^{b_i} \frac{f''(y)}{2f'(y)} dy \right] + \\
&\quad + \sum_{i=1}^{q_n-2} \frac{V_i}{1 + V_i} \left[ \frac{1}{f'(a_i)(x_i - a_i)} \int_{a_i}^{x_i} f''(y)(y - a_i) dy + \frac{1}{f'(a_i)(b_i - x_i)} \int_{x_i}^{b_i} f''(y)(b_i - y) dy \right].
\end{aligned} \tag{25}$$

The first after the sign of equality sum equals to  $(-\ln M_n)$ . Since  $|V_i| \leq K\lambda^{\frac{n}{\beta}}$ , the last sum is not greater than  $K\lambda^{\frac{n}{\beta}}$ . Together with relations (23)-(25), the last inequality implies that

$$\begin{aligned}
\tau_n(z_1) &= - \sum_{i=1}^{q_n-2} \left[ \frac{1}{f'(a_i)(x_i - a_i)} \int_{a_i}^{x_i} f''(y)(y - a_i) dy - \frac{1}{2} \int_{a_i}^{x_i} \frac{f''(y)}{2f'(y)} dy \right] - \\
&\quad - \sum_{i=1}^{q_n-2} \left[ \frac{1}{f'(a_i)(b_i - x_i)} \int_{x_i}^{b_i} f''(y)(b_i - y) dy - \frac{1}{2} \int_{x_i}^{b_i} \frac{f''(y)}{2f'(y)} dy \right] + O(\lambda^{\frac{n}{\beta}})
\end{aligned} \tag{26}$$

Denote by  $S_n$  and  $\bar{S}_n$  the last two sums in (26) respectively. Then, we show that for any  $\varepsilon > 0$ , the following estimates hold for sufficiently large  $n$  :

$$|S_n|, |\bar{S}_n| \leq K\varepsilon. \tag{27}$$

We prove only the estimate for  $S_n$ , the one for  $\bar{S}_n$  is quite similar. Rewrite the sum  $S_n$  as

$$\begin{aligned}
S_n &= \sum_{i=1}^{q_n-2} \int_{a_i}^{x_i} \frac{f''(y)}{f'(a_i)} \left( \frac{y - a_i}{x_i - a_i} - \frac{1}{2} \right) dy + \\
&\quad + \sum_{i=1}^{q_n-2} \int_{a_i}^{x_i} \left( \frac{f''(y)}{f'(a_i)f'(y)} \int_{a_i}^y f''(t) dt \right) dy \equiv S_n^{(1)} + S_n^{(2)}.
\end{aligned} \tag{28}$$

Using the condition  $f''(x) \in L_\alpha(S^1, d\ell)$ ,  $\alpha > 1$ , and the Hölder inequality, it can easily be shown that

$$|S_n^{(2)}| \leq K \sum_{i=1}^{q_n-2} \left( \int_{a_i}^{x_i} |f''(y)| dy \right)^2 \leq K\lambda^{\frac{n}{\beta}}. \tag{29}$$

Let us estimate the sum  $S_n^{(1)}$ . Fix an arbitrary  $\varepsilon > 0$ . Since  $f''(x) \in L_\alpha(S^1, d\ell)$ , it can be written in the form

$$f''(x) = g_\varepsilon(x) + s_\varepsilon(x), \quad x \in S^1, \tag{30}$$

where  $g_\varepsilon(x)$  is a continuous function on  $S^1$  and  $\|s_\varepsilon\|_{L^1} < \varepsilon$ . Substituting (30) in expression for  $S_n^{(1)}$ , we obtain

$$\begin{aligned} |S_n^{(1)}| &\leq \left| \sum_{i=1}^{q_n-2} \frac{1}{f'(a_i)} \int_{a_i}^{x_i} h_\varepsilon(y) \left( \frac{y-a_i}{x_i-a_i} - \frac{1}{2} \right) dy \right| + \\ &+ \left| \sum_{i=1}^{q_n-2} \frac{1}{f'(a_i)} \int_{a_i}^{x_i} s_\varepsilon(y) \left( \frac{y-a_i}{x_i-a_i} - \frac{1}{2} \right) dy \right| \equiv P_n + Q_n. \end{aligned} \quad (31)$$

First, we estimate the sum  $P_n$ . Denote by  $t_i$  the middle of the interval  $[a_i, x_i]$  i.e.  $t_i = \frac{x_i+a_i}{2}$ . We rewrite the sum  $P_n$  in the following form

$$P_n = \left| \sum_{i=1}^{q_n-2} \frac{1}{f'(a_i)} \int_{a_i}^{t_i} h_\varepsilon(y) \left( \frac{y-a_i}{x_i-a_i} - \frac{1}{2} \right) dy + \frac{1}{f'(a_i)} \int_{t_i}^{x_i} h_\varepsilon(y) \left( \frac{y-a_i}{x_i-a_i} - \frac{1}{2} \right) dy \right|.$$

Applying the Mean Value Theorem we obtain

$$\begin{aligned} P_n &= \left| \sum_{i=1}^{q_n-2} \frac{h_\varepsilon(\xi_1^i)}{f'(a_i)} \int_{a_i}^{t_i} \left( \frac{y-a_i}{x_i-a_i} - \frac{1}{2} \right) dy + \frac{h_\varepsilon(\xi_2^i)}{f'(a_i)} \int_{t_i}^{x_i} \left( \frac{y-a_i}{x_i-a_i} - \frac{1}{2} \right) dy \right| = \sum_{i=1}^{q_n-2} \frac{x_i-a_i}{16f'(a_i)} |h_\varepsilon(\xi_2^i) - h_\varepsilon(\xi_1^i)| \leq \\ &\leq \sum_{i=1}^{q_n-2} \frac{x_i-a_i}{16f'(a_i)} \sup_{|\xi_1^i - \xi_2^i| < \lambda^n} |h_\varepsilon(\xi_2^i) - h_\varepsilon(\xi_1^i)| \leq K \max_{1 \leq i \leq q_n-2} \omega(\lambda^n, h_\varepsilon), \end{aligned} \quad (32)$$

where  $\omega(\lambda^n, h_\varepsilon)$  is the modulus of continuity of  $h_\varepsilon$ . Since  $\lambda \in (0, 1)$ , we have  $\omega(\lambda^n, h_\varepsilon) \rightarrow 0$ , as  $n \rightarrow \infty$ . Next, we estimate the sum  $Q_n$ . It is easy to see that

$$Q_n \leq K \sum_{i=1}^{q_n-2} \int_{a_i}^{x_i} |s_\varepsilon(y)| dy \leq K \int_{S^1} |s_\varepsilon(y)| dy \leq K\varepsilon.$$

Hence, the relations in (27) are proved. Then, summing (23)-(27), we obtain the first relation in (10).

Let us prove the second relation in (10). Note that there exists a constant  $C_2 > 0$  such that the following inequalities hold for all  $i$ ,  $i = 1, 2, \dots, q_n - 2$ ,

$$\frac{1}{C_2} \leq \frac{z_1(1-z_1)}{z_i(1-z_i)} \leq C_2, \quad \frac{1}{C_2} \leq \frac{dz_i}{dz_1} \leq C_2. \quad (33)$$

Notice that the function  $\frac{d\psi_i}{dz_i}$  is defined almost everywhere. Using (19), we calculate the derivative of  $\psi_i$  by  $z_i$ :

$$\frac{d\psi_i}{dz_i} = \frac{A_i^2 - A_i'}{(1 + A_i z_i)(1 + A_i(z_i - 1))} \quad (34)$$

where

$$\begin{aligned} A_i' &= \frac{dA_i}{dz_i} = \frac{dA_i}{dx_i} \frac{dx_i}{dz_i} = (b_i - a_i) \frac{dA_i}{dx_i}, \\ \frac{dA_i}{dx_i} &= \frac{\frac{1}{f'(a_i)(x_i-a_i)^2} \int_{a_i}^{x_i} f''(y)(y-a_i) dy - \frac{1}{f'(a_i)(b_i-x_i)^2} \int_{x_i}^{b_i} f''(y)(b_i-y) dy}{1 + \frac{1}{f'(a_i)(b_i-a_i)} \int_{a_i}^{b_i} f''(y)(b_i-y) dy}. \end{aligned} \quad (35)$$

Using (24), (29), (33)-(35) we obtain

$$\begin{aligned} |(z_1 - z_1^2)\tau_n'(z_1)| &= \left| (z_1 - z_1^2) \sum_{i=1}^{q_n-2} \frac{d\psi_i}{dz_i} \frac{dz_i}{dz_1} \right| \leq \\ &\leq K \left| \sum_{i=1}^{q_n-2} (z_i - z_i^2)(b_i - a_i) \left[ \int_{a_i}^{x_i} f''(y) \frac{y - a_i}{(x_i - a_i)^2} dy - \int_{x_i}^{b_i} f''(y) \frac{b_i - y}{(b_i - x_i)^2} dy \right] \right| + O(\lambda^{\frac{n}{\beta}}). \end{aligned} \quad (36)$$

Denote by  $E_n$  the last sum in (36). Using relation (30) we rewrite  $E_n$  in the following form

$$\begin{aligned} E_n &= \left| \sum_{i=1}^{q_n-2} (z_i - z_i^2)(b_i - a_i) \left[ \int_{a_i}^{x_i} h_\varepsilon(y) \frac{y - a_i}{(x_i - a_i)^2} dy - \int_{x_i}^{b_i} h_\varepsilon(y) \frac{b_i - y}{(b_i - x_i)^2} dy \right] \right| + \\ &+ \left| \sum_{i=1}^{q_n-2} \left[ z_i \int_{a_i}^{x_i} s_\varepsilon(y) \frac{y - a_i}{x_i - a_i} dy - (1 - z_i) \int_{x_i}^{b_i} s_\varepsilon(y) \frac{b_i - y}{b_i - x_i} dy \right] \right| \equiv E_n^{(1)} + E_n^{(2)}. \end{aligned} \quad (37)$$

First, we estimate the sum  $E_n^{(1)}$ . Applying the Mean Value Theorem again we get

$$E_n^{(1)} \leq K \sum_{i=1}^{q_n-2} (b_i - a_i) |h_\varepsilon(\xi_1^i) - h_\varepsilon(\xi_2^i)| \leq \quad (38)$$

$$\leq K \sum_{i=1}^{q_n-2} (b_i - a_i) \omega(\lambda^n, h_\varepsilon) \leq K \max_{1 \leq i \leq q_n-2} \omega(\lambda^n, h_\varepsilon),$$

where  $\omega(\lambda^n, h_\varepsilon) = \sup_{|\xi_1^i - \xi_2^i| < \lambda^n} |h_\varepsilon(\xi_2^i) - h_\varepsilon(\xi_1^i)|$ . Let us estimate  $E_n^{(2)}$ . It is easy to see that

$$E_n^{(2)} \leq \frac{1}{2} \sum_{i=1}^{q_n-2} \left[ \int_{a_i}^{x_i} |s_\varepsilon(y)| dy + \int_{x_i}^{b_i} |s_\varepsilon(y)| dy \right] \leq \frac{1}{2} \int_{S^1} |s_\varepsilon(y)| dy < \frac{\varepsilon}{2}.$$

This, together with (36)-(38) imply the second relation in (10). Now, we prove the first relation in (11). Using the same arguments as in (36), we can show that

$$\begin{aligned} &\int_0^1 |\tau_n'(z_1)| dz_1 \leq \\ &\leq K \int_0^1 \left| \sum_{i=1}^{q_n-2} (b_i - a_i) \left[ \int_{a_i}^{x_i} \frac{f''(y)}{f'(a_i)} \frac{y - a_i}{(x_i - a_i)^2} dy - \int_{x_i}^{b_i} \frac{f''(y)}{f'(a_i)} \frac{b_i - y}{(b_i - x_i)^2} dy \right] \right| dz_1 + O(\lambda^{\frac{n}{\beta}}). \end{aligned} \quad (39)$$

Using relations (33), it is easy to see that

$$\begin{aligned} &\int_0^1 |\tau_n'(z_1)| dz_1 \leq \\ &K \sum_{i=1}^{q_n-2} \int_{a_i}^{b_i} \left| \int_{a_i}^{x_i} f''(y) \frac{y - a_i}{(x_i - a_i)^2} dy - \int_{x_i}^{b_i} f''(y) \frac{b_i - y}{(b_i - x_i)^2} dy \right| dx_i + O(\lambda^{\frac{n}{\beta}}). \end{aligned} \quad (40)$$

We denote by  $I_n$  the last sum in (40) and estimate it. Using the representation (30), we get

$$I_n \leq \sum_{i=1}^{q_n-2} \int_{a_i}^{b_i} \left| \int_{a_i}^{x_i} h_\varepsilon(y) \frac{y-a_i}{(x_i-a_i)^2} dy - \int_{x_i}^{b_i} h_\varepsilon(y) \frac{b_i-y}{(b_i-x_i)^2} dy \right| dx_i + \quad (41)$$

$$+ \sum_{i=1}^{q_n-2} \int_{a_i}^{b_i} \left| \int_{a_i}^{x_i} s_\varepsilon(y) \frac{y-a_i}{(x_i-a_i)^2} dy \right| dx_i + \sum_{i=1}^{q_n-2} \int_{a_i}^{b_i} \left| \int_{x_i}^{b_i} s_\varepsilon(y) \frac{b_i-y}{(b_i-x_i)^2} dy \right| dx_i.$$

It can easily be shown that the first sum in (41) is not greater than

$$\max_{1 \leq i \leq q_n-2} \omega(\lambda^n, h_\varepsilon). \quad (42)$$

Denote by  $I_n^{(1)}$  the second to the last sum in (41). Applying the Hölder inequality we obtain

$$I_n^{(1)} = \sum_{i=1}^{q_n-2} \int_{a_i}^{b_i} \left| \frac{1}{(x_i-a_i)^2} \int_{a_i}^{x_i} s_\varepsilon(y)(y-a_i) dy \right| dx_i \leq$$

$$\leq K \sum_{i=1}^{q_n-2} \int_{a_i}^{b_i} (x_i-a_i)^{\frac{1}{\beta}-1} \left( \int_{a_i}^{x_i} |s_\varepsilon(y)|^\alpha dy \right)^{\frac{1}{\alpha}} dx_i \leq$$

$$\leq K \sum_{i=1}^{q_n-2} \left( \int_{a_i}^{b_i} |s_\varepsilon(y)|^\alpha dy \right)^{\frac{1}{\alpha}} (b_i-a_i)^{\frac{1}{\beta}} \leq K \left[ \sum_{i=1}^{q_n-2} \int_{a_i}^{b_i} |s_\varepsilon(y)|^\alpha dy \right]^{\frac{1}{\alpha}} \leq K \varepsilon^{\frac{1}{\alpha}}.$$

Analogously, it can be shown that the last sum in (41), is also not greater than  $K \varepsilon^{\frac{1}{\alpha}}$ . Together with (39)-(42), this implies the first relation in (11).

Let us prove the second inequality in (11). It is not too hard to show that there exists a constant  $C_3 > 0$  such that for all  $i, 1 \leq i \leq q_n - 2$

$$\frac{1}{C_3} \leq \int_0^1 \left| \frac{d^2 z_i}{dz_1^2} \right| dz_1 \leq C_3. \quad (43)$$

Note that the function  $\frac{d^2 \psi_i}{dz_i^2}$  is defined almost everywhere. By differentiating (34) we get

$$\frac{d^2 \psi_i}{dz_i^2} = \frac{2A_i A_i' - A_i''}{(1+A_i z_i)(1+A_i(z_i-1))} - \frac{2(A_i' z_i + a_i)}{1+A_i z_i} \cdot \frac{d\psi_i}{dz_i} - \left( \frac{d\psi_i}{dz_i} \right)^2, \quad (44)$$

where

$$A_i'' = \frac{d^2 A_i}{dz_i^2} = (b_i - a_i)^2 \frac{d^2 A_i}{dx_i^2}. \quad (45)$$

Finally, differentiating (35) gives

$$\frac{d^2 A_i}{dx_i^2} = \frac{\frac{2}{f'(a_i)(x_i-a_i)^2} \int_{a_i}^{x_i} (f''(x_i) - f''(y))(y-a_i) dy + \frac{2}{f'(a_i)(b_i-x_i)^2} \int_{x_i}^{b_i} (f''(x_i) - f''(y))(b_i-y) dy}{1 + \frac{1}{f'(a_i)(b_i-a_i)} \int_{a_i}^{b_i} f''(y)(y-a_i) dy}, \quad (46)$$

Using the relations (10), (11), (44) and (46) it can easily be shown that

$$\begin{aligned}
\int_0^1 |(z_1 - z_1^2)\tau_n''(z_1)|dz_1 &= \int_0^1 |(z_1 - z_1^2) \sum_{i=1}^{q_n-2} \left[ \frac{d^2\psi_i}{dz_i^2} \left( \frac{dz_i}{dz_1} \right)^2 + \frac{d\psi_i}{dz_i} \frac{d^2z_i}{dz_1^2} \right]| dz_1 \leq \\
&\leq K \int_0^1 \left| (z_1 - z_1^2) \sum_{i=1}^{q_n-2} (b_i - a_i)^2 \frac{d^2A_i}{dx_i^2} \right| + K\varepsilon \leq \\
&\leq K \int_0^1 \left| (z_1 - z_1^2) \sum_{i=1}^{q_n-2} \left( \frac{b_i - a_i}{x_i - a_i} \right)^2 \int_{a_i}^{x_i} [f''(x_i) - f''(y)] \frac{y - a_i}{x_i - a_i} dy \right| dz_1 + \\
&+ K \int_0^1 \left| (z_1 - z_1^2) \sum_{i=1}^{q_n-2} \left( \frac{b_i - a_i}{x_i - a_i} \right)^2 \int_{x_i}^{b_i} [f''(x_i) - f''(y)] \frac{b_i - y}{b_i - x_i} dy \right| dz_1 + K\varepsilon.
\end{aligned}$$

The proof of the second relation in (11) proceeds now exactly as in the previous case. This concludes the proof of Lemma 3.1.

**Proof of Lemma 3.2.** It is easy to check, that

$$\begin{aligned}
f(x) - f(y_2) &= f'(x_b + 0)(x - y_2) + \int_x^{y_2} f''(y)(y - y_2)dy, \quad x_b < x < y_2, \\
f(x) - f(x_b) &= f'(x_b - 0)(x - x_b) + \int_x^{x_b} f''(y)(y - x_b)dy, \quad y_1 < x < x_b, \\
f(y_1) - f(x_b) &= f'(x_b - 0)(y_1 - x_b) + \int_{y_1}^{x_b} f''(y)(y - x_b)dy, \\
f(x_b) - f(y_2) &= f'(x_b + 0)(x_b - y_2) + \int_{x_b}^{y_2} f''(y)(y - y_2)dy.
\end{aligned}$$

This together with (6) imply that

$$\bar{f}_1(z_0) = \begin{cases} \frac{z_0 + H_1(x)}{\sigma^2(1-d) + d + H_3 + H_4}, & z_0 \in [0, d], \\ \frac{\sigma^2 z_0 + d(1-\sigma^2) + H_2(x) + H_4}{\sigma^2(1-d) + d + H_3 + H_4}, & z_0 \in (d, 1], \end{cases} \quad (47)$$

where

$$\begin{aligned}
H_1(x) &= \frac{1}{f'(x_b + 0)(y_1 - y_2)} \int_x^{y_2} f''(y)(y - y_2)dy, \quad x \in [x_b, y_2], \\
H_2(x) &= \frac{1}{f'(x_b + 0)(y_1 - y_2)} \int_x^{x_b} f''(y)(y - x_b)dy, \quad x \in [y_1, x_b], \\
H_3(x) &= \frac{1}{f'(x_b + 0)(y_1 - y_2)} \int_{y_1}^{x_b} f''(y)(y - x_b)dy,
\end{aligned}$$

$$H_4(x) = \frac{1}{f'(x_b + 0)(y_1 - y_2)} \int_{x_b}^{y_2} f''(y)(y - y_2)dy.$$

Because  $\ell(\Delta_0^{(n-1)}) \leq \lambda^n$ , using the condition (d) and Hölder inequality we find that the relation

$$|H_1(x)| \leq \frac{1}{f'(x_b + 0)(y_1 - y_2)} \int_x^{y_2} |f''(y)(y - y_2)|dy \leq K\lambda^{\frac{n}{\beta}}. \quad (48)$$

holds for all  $x \in [y_1, x_b]$ . Analogously, it can be shown that the following inequalities

$$|H_2(x)|, |H_3|, |H_4| \leq K\lambda^{\frac{n}{\beta}}. \quad (49)$$

for all  $x \in (x_b, y_2]$ . Note that the functions  $\frac{dH_1(x)}{dx}$ ,  $\frac{dH_2(x)}{dx}$  are defined almost everywhere. It is not hard to establish that the following inequalities hold:

$$\int_0^d \left| \frac{dH_1(x)}{dz_0} \right| dz_0, \quad \int_d^1 \left| \frac{dH_2(x)}{dz_0} \right| dz_0 \leq K\lambda^{\frac{n}{\beta}}. \quad (50)$$

Summing (47)-(50), we get relation (12) and the first relation in (13). We have

$$\bar{f}''_1(z_0) = \frac{f''(y_2 + z_0(y_1 - y_2))(y_1 - y_2)^2}{f(y_1) - f(y_2)} = \frac{1}{f'(x_b + 0)} \frac{f''(y_2 + z_0(y_1 - y_2))(y_1 - y_2)}{\sigma^2(1 - d) + d + H_3 + H_4}.$$

for almost all  $z_0$ . Since the inequalities

$$\int_0^d |F_1''(z_0)| dz_0, \quad \int_d^1 |F_1''(z_0)| dz_0 \leq K\lambda^{\frac{n}{\beta}}$$

hold, also in the case (48) this proves the second relation in (13). Lemma 3.2 is completely proved.

Theorem 3.1 implies that  $\bar{f}_{\rho,n}$  is a convex function for  $0 < \sigma < 1$ , while  $\bar{f}_{\rho,n}$  is concave for  $\sigma > 1$ . Indeed, direct computation easily gives

$$\frac{d^2}{dz^2} G_d(z) \geq 2\sigma^2(1 - \sigma), \quad z \neq d, \text{ if } 0 < \sigma < 1, \quad (51)$$

$$\frac{d^2}{dz^2} G_d(z) \leq -\frac{2}{\sigma^3}(\sigma - 1), \quad z \neq d, \text{ if } \sigma > 1, \quad (52)$$

Put  $N = N(\varepsilon, F)$ , such that  $C_1\varepsilon \leq |\sigma - 1| \min\{\sigma^{-3}, \sigma^2\}$ .

**Lemma 3.3.** The following estimates hold for all  $n > N$

$$\frac{d^2}{dz^2} \bar{f}_{\rho,n}(z) \geq \sigma(1 - \sigma), \quad z \in [0, 1], \quad z \neq d, \text{ if } 0 < \sigma < 1,$$

$$\frac{d^2}{dz^2} \bar{f}_{\rho,n}(z) \leq -\frac{1}{\sigma^3}(\sigma - 1), \quad z \in [0, 1], \quad z \neq d, \text{ if } \sigma > 1,$$

The proof of Lemma 3.3 easily follows from Theorem 3.1 and inequalities (51), (52).

## 4 Proof of Theorem 1.1

Denote by  $\theta_b = \theta_b(\frac{p_n}{q_n})$  the value of parameter such that the break point  $x_b$  belongs to the periodic orbit. Note that the value  $\theta_b$  of parameter  $\theta$  always exists, unique and can be defined by equation  $f_{\theta_b}^q(x) = x + p$ . Suppose that for some  $\theta_0 \in I(\frac{p_n}{q_n})$  there exists a neutral periodic orbit  $O_f(t_0, q)$ , i.e.  $\prod_{i=0}^{q-1} f'_{\theta_0}(t_i) = 1$ .

**Lemma 4.1** The following assertions

- (1)  $\theta_0 = \theta_2(\frac{p}{q})$ ,  $\theta_b = \theta_1(\frac{p}{q})$ , if  $\sigma > 1$ ,
- (2)  $\theta_0 = \theta_1(\frac{p}{q})$ ,  $\theta_b = \theta_2(\frac{p}{q})$ , if  $0 < \sigma < 1$ ,
- (3)  $|\frac{d}{dx} f_{\theta_b}^{q_n}(x_b - 0) - \sigma| \leq C_1 \varepsilon$ ,  $|\frac{d}{dx} f_{\theta_b}^{q_n}(x_b + 0) - \frac{1}{\sigma}| \leq C_1 \varepsilon$

hold for all  $n > N$ .

**Proof of Lemma 4.1.** To be definite, let us consider the case  $\sigma > 1$ . It is clear that  $d = 0$  if  $\theta = \theta_b$ . Because the function  $\bar{f}_{\rho, n}(z)$  is concave, its graph has the form shown in Figure 1, i.e. the whole graph of  $\bar{f}_{\rho, n}$  lies completely over the diagonal. Then we have  $\bar{f}_{\rho, n}(z) \geq z$ ,  $z \in [0, 1]$ . This, and the coordinate change reverses the orientation of the circle, imply

$$f_{\theta_b}^{q_n}(x) \leq x, \quad x \in S^1.$$

Note that the function  $f_{\theta}$  is increasing w.r.t.  $\theta$ . Consequently,

$$f_{\theta_b}^{q_n}(x) < x, \quad x \in S^1, \quad \theta < \theta_b. \quad (53)$$

Then, we have  $\theta_b = \theta_1(\frac{p}{q})$ . Let us prove that  $\theta_0 = \theta_2(\frac{p}{q})$ . By assumption  $\sigma > 1$ . The graph of the function  $\bar{f}_{\rho, n}(z)$  has the form shown in Figure 2, i.e. it lies completely below the diagonal. Indeed, the diagonal is tangent to the graphs at the points  $z = 0$ ,  $z = 1$ , and the function  $\bar{f}_{\rho, n}$  is strictly concave. Using the same argument, as above, we get

$$f_{\theta_b}^{q_n}(x) \geq x, \quad x \in S^1, \quad (54)$$

which implies that  $\theta_0 = \theta_2(\frac{p}{q})$ .

Note that the converse statement is also true. Consider the parameter value corresponding to the end of the interval  $I(\frac{p_n}{q_n})$ , which is not equal to  $\theta_b$ . Then, for this value of the parameter, there exists a periodic orbit which does not contain  $x_b$ . Clearly, this periodic orbit is neutral. Lemma 4.1 is proved.

**Proof of Theorem 1.1.** Note that the number of periodic orbits coincides with the number of solutions of the equation

$$\bar{f}_{\rho, n}(z) = z, \quad 0 \leq z < 1. \quad (55)$$

The assertion (i) was proved above. Using that the function  $\bar{f}_{\rho,n}(z)$  is strictly convex for  $0 < \sigma < 1$  and strictly concave for  $\sigma > 1$ , we conclude that assertion (ii) can easily be obtained as well. We must only use, that since  $\theta \in (\theta_1(\frac{p}{q}), \theta_2(\frac{p}{q}))$ , the periodic orbit can neither contain a break point, nor be neutral. A typical graph of the function  $\bar{f}_{\rho,n}(z)$  for  $\sigma > 1$ ,  $\bar{f}'_{\rho,n}(0) = \bar{f}'_{\rho,n}(1) < 1$  is shown in Figure 3. Theorem 1.1 is proved.

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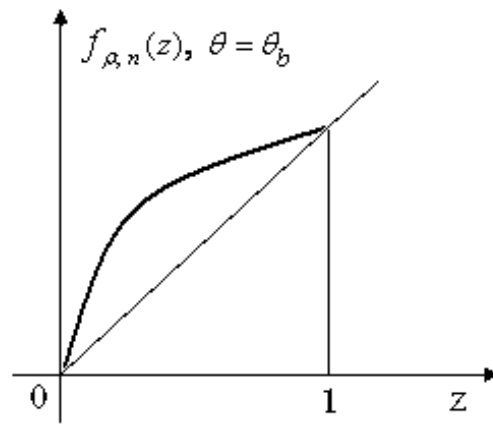


Figure 1:

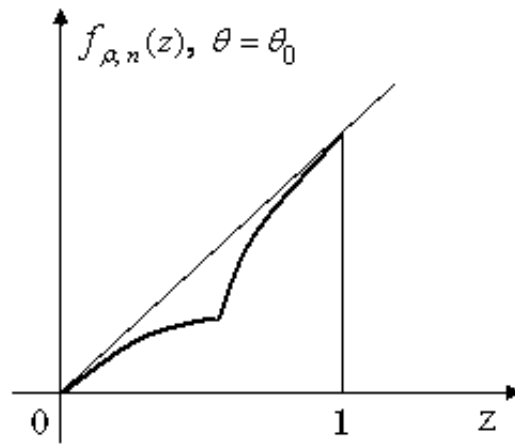


Figure 2:

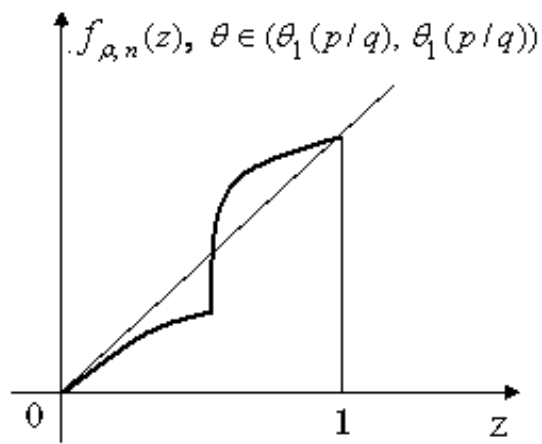


Figure 3: