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LINEAR q-NONUNIFORM DIFFERENCE EQUATIONS

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Abstract

We introduce basic concepts of q-nonuniform differentiation and integration and study linear q-nonuniform difference equations and systems, as well as their application in q-nonuniform difference linear control systems.

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1 Introduction

Considering the most general divided difference derivative [16, 17],

$$\mathcal{D}f(t(s)) = \frac{f(t(s+\frac{1}{2}))-f(t(s-\frac{1}{2}))}{t(s+\frac{1}{2})-t(s-\frac{1}{2})}, \tag{1}$$

admitting the property that if $f(t) = P_n(t(s))$ is a polynomial of degree n in $t(s)$, then $\mathcal{D}f(t(s)) = \tilde{P}_{n-1}(t(s))$ is a polynomial in $t(s)$ of degree $n-1$, one is led to the following most important canonical forms for $t(s)$ in order of increasing complexity:

$$t(s) = x(0); \tag{2}$$

$$t(s) = s; \tag{3}$$

$$t(s) = q^s; \tag{4}$$

$$t(s) = \frac{q^s + q^{-s}}{2}, \; q \in \mathbf{C}, s \in \mathbf{Z}. \tag{5}$$

When the function $t(s)$ is given by (2)-(4), the divided difference derivative (1) leads to the ordinary differential derivative $Df(x) = \frac{d}{dx}f(x)$, finite difference derivative $\Delta f(t) = f(t+1) - f(t) = (e^{\frac{d}{dt}} - 1)f(t)$ and q-difference derivative (or Jackson derivative [14]) $D_q f(t) = \frac{f(qt)-f(t)}{qt-t} = \frac{q^{\frac{d}{dt}}-1}{qt-t}f(t)$, respectively.

When $x(s)$ is given by (5), the corresponding derivative is usually referred to as the *Askey-Wilson* first order divided difference operator [2] that one can write:

$$\mathcal{D}f(x(z)) = \frac{f(x(q^{\frac{1}{2}}z))-f(x(q^{-\frac{1}{2}}z))}{x(q^{\frac{1}{2}}z)-x(q^{-\frac{1}{2}}z)}, \tag{6}$$

where $x(z) = \frac{z+z^{-1}}{2}$, having in mind that $z = q^s$.

The calculus related to (2), the continuous or differential calculus, is clearly the classical one. The one related to (3)-(5) (difference, q-difference and q-nonuniform difference respectively) is referred to as the *discrete calculus*. Its interest is two fold: On one hand it generalizes the continuous calculus, and on the other, it uses discrete variable. As a consequence, data are finite and the use of computer is potentially possible. The difference calculus (see e.g. [9]) is an independent subject of analysis likewise developed than the continuous one. The q-difference calculus has been a rich source of intensive studies in the last years (see e.g. [11, 15, 10, 3] and references therein).

The q-nonuniform difference calculus is the subject study of this work. In this calculus, the independent variable is the one in (5). Such a variable can arise naturally as any function satisfying a linear second order difference equation whose characteristic equation has mutually inverse roots: $t(s+1) + at(s) + t(s-1) = 0$, $a - const$. There have been intensive applications of this calculus in orthogonal polynomials theory starting in the eighties (see [16, 17, 2, 18] and references therein), and many fundamental results on basic functions and series were obtained (see [7, 8, 12, 13, 19] and references therein). But to the best of our knowledge, study and application of this calculus from purely the point of view of q-nonuniform difference equations is the side somehow missing in the literature. This work aims to contribute to face this case. Clearly, it can be seen as an extension of [1] from the q-uniform lattice (4) to the q-nonuniform one (5).

In the following sections, we first introduce basic concepts of q-nonuniform differentiation and integration necessary for the sequel, then study linear q-nonuniform difference equations and systems as well as their application in q-nonuniform difference linear control systems.

2 q-Nonuniform differentiation and integration

Consider again the derivative in (6). In the sequel, for clarity, we will suppose that $0 \leq |q| \leq 1$. Basing on this derivative, one defines the integration that is the inverse of the differentiation operation as follows [5]:

$$\begin{aligned} &\int_{x(q^N)}^{x(q^s)} g(x(z))d_q x(z) \\ &\stackrel{\text{def}}{=} \sum_{r=s+\frac{1}{2}}^{N-\frac{1}{2}} [x(q^{r-\frac{1}{2}}) - x(q^{r+\frac{1}{2}})]g(x(q^r)) \qquad (7) \\ &= \sum_{x(z)=x(q^{s+\frac{1}{2}})}^{x(q^{N-\frac{1}{2}})} [x(zq^{-\frac{1}{2}}) - x(zq^{+\frac{1}{2}})]g(x(z)). \qquad (8) \end{aligned}$$

Such defined integral and the derivative (6) admit most of the properties of continuous integrals and derivative (see e.g. [5]):

Derivation and integration of a polynomial. Direct calculation shows that [5] if $P_n(x(z))$ is a polynomial in $x(z)$ of degree n, then its (q-nonuniform divided difference) derivative is a polynomial of degree $n-1$ and its (q-nonuniform) integral is a polynomial of degree $n+1$. This calculation is founded on the relation

$$P_n(x(z)) = \sum_{k=0}^{n} a_k (\tfrac{z+z^{-1}}{2})^k = \sum_{m=0}^{n} b_m (z^m + z^{-m}), \qquad (9)$$

where

$$b_0 = 2^{-1} \sum_{r=0}^{[\frac{n}{2}]} a_{2r} 2^{-2r} C_{2r}^r ; b_m = 2^{-m} \sum_{r=0}^{[\frac{n-m}{2}]} a_{m+2r} 2^{-2r} C_{m+2r}^r , n \geq 1, \qquad (10)$$

where $[a]$ stands for the integer part of and C_b^a for the combinatorics.

Derivative of a product.

$$\begin{aligned} \mathcal{D}(fg)(x(z)) &= f(x(q^{\frac{1}{2}}z))\mathcal{D}g(x(z)) + g(x(q^{-\frac{1}{2}}z))\mathcal{D}f(x(z)) \\ &= g(x(q^{\frac{1}{2}}z))\mathcal{D}f(x(z)) + f(x(q^{-\frac{1}{2}}z))\mathcal{D}g(x(z)) \qquad (11) \end{aligned}$$

Derivative of a ratio.

$$\mathcal{D}(f/g)(x(z)) = \frac{g(x(q^{-\frac{1}{2}}z))\mathcal{D}f(x(z)) - f(x(q^{-\frac{1}{2}}z))\mathcal{D}g(x(z))}{g(x(q^{\frac{1}{2}}z))g(x(q^{-\frac{1}{2}}z))} \tag{12}$$

Derivative of a composite function.

$$\mathcal{D}(f(g))(x(z)) = (\mathcal{D}_g f(g))(x(z))\mathcal{D}g(x(z)) \tag{13}$$

where

$$(\mathcal{D}_g f(g))(x(z)) = \frac{f(g(x(zq^{\frac{1}{2}}))) - f(g(x(zq^{-\frac{1}{2}})))}{g(x(zq^{\frac{1}{2}})) - g(x(zq^{-\frac{1}{2}}))}. \tag{14}$$

Derivative of the inverse function. Let $y = f(x)$, and $x = f^{-1}(y)$ where f^{-1} is the inverse to f function. We have

$$\mathcal{D}_y f^{-1} = \frac{1}{\mathcal{D}f}, \tag{15}$$

where

$$(\mathcal{D}_y f^{-1}(y))(x(z)) = \frac{f^{-1}(y(x(zq^{\frac{1}{2}}))) - f^{-1}(y(x(zq^{-\frac{1}{2}})))}{y(x(zq^{\frac{1}{2}})) - y(x(zq^{-\frac{1}{2}}))}. \tag{16}$$

Fundamental principles of analysis.

$$(i) \qquad \mathcal{D}\left[\int_{x(q^N)}^{x(z)} g(x(z))d_q x(z)\right] = g(x(z)), \tag{17}$$

$$(ii) \quad \int_{x(q^N)}^{x(z)} (\mathcal{D}F)(x(z))d_q x(z) = F(x(z)) - F(x(q^N)). \tag{18}$$

Integration by parts. Using (11), one obtains

$$\begin{aligned} &\int_{x(q^N)}^{x(z)} f(x(q^{\frac{1}{2}}z))\mathcal{D}g(x(z))d_q x(z) \\ &= [fg]_{x(q^N)}^{x(z)} - \int_{x(q^N)}^{x(z)} g(x(q^{-\frac{1}{2}}z))\mathcal{D}f(x(z))d_q x(z). \end{aligned} \tag{19}$$

Change of integration variables. Letting

$$d_q(x(z)) \stackrel{\text{def}}{=} x(q^{-\frac{1}{2}}z) - x(q^{\frac{1}{2}}z); \; d_q(z) \stackrel{\text{def}}{=} (1-q)z, \tag{20}$$

one makes the change of integration variables from the q-nonuniform to the uniform one

$$\int_{x(q^N)}^{x(z)} f(x(z))d_q(x(z)) = \frac{1}{2\sqrt{q}} \int_{q^N}^{z} (1 - z^{-2}) f(x(z)) d_q(z) \tag{21}$$

3 q-Nonuniform linear difference equations of first order

Consider the first order q-nonuniform difference equation

$$\mathcal{D}y(x(z)) = a(x(z))y(x(q^{-\frac{1}{2}}z)) + b(x(z)). \tag{22}$$

As in differential or difference calculus, to solve it, we first solve the corresponding homogeneous equation

$$\mathcal{D}y(x(z)) = a(x(z))y(x(q^{-\frac{1}{2}}z)). \tag{23}$$

For that, we write it under a recursive form

$$y(x(q^{\frac{1}{2}}z)) = p(z)y(x(q^{-\frac{1}{2}}z)) \tag{24}$$

where

$$p(z) = 1 + \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{2}(z - z^{-1})a(x(z)). \tag{25}$$

If $y_0(x(z))$ is the solution of the homogeneous equation (23) we have

$$y_0(x(q^{\frac{1}{2}}z)) = p(z)y_0(x(q^{-\frac{1}{2}}z)) \tag{26}$$

and using the recursion

$$y_0(x(z)) = p(zq^{-\frac{1}{2}})y_0(x(q^{-1}z)) \tag{27}$$

we get

$$y_0(x(z)) = [\prod_{i=0}^{N-1} p(zq^{-\frac{1}{2}-i})]y_0(x(q^{-N}z)), \tag{28}$$

while using the recursion

$$y_0(x(z)) = [p(zq^{\frac{1}{2}})]^{-1}y_0(x(qz)) \tag{29}$$

one gets

$$y_0(x(z)) = [\prod_{i=0}^{N-1} p(zq^{\frac{1}{2}+i})]^{-1}y_0(x(q^{N}z)). \tag{30}$$

Consider next the associated to (22) first order q-nonuniform difference equation

$$\mathcal{D}\tilde{y}(x(z)) = \tilde{a}(x(z))\tilde{y}(x(q^{\frac{1}{2}}z)) + \tilde{b}(x(z)). \tag{31}$$

Again, the corresponding homogeneous equation

$$\mathcal{D}\tilde{y}(x(z)) = \tilde{a}(x(z))y(x(q^{\frac{1}{2}}z)) \tag{32}$$

can be written as

$$\tilde{y}(x(q^{\frac{1}{2}}z))\tilde{p}(z) = \tilde{y}(x(q^{-\frac{1}{2}}z)) \tag{33}$$

where

$$\tilde{p}(z) = 1 - \tfrac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2}(z - z^{-1})\tilde{a}(x(z)). \tag{34}$$

Using the recursion

$$\tilde{y}_0(x(z)) = \tilde{p}(zq^{\frac{1}{2}})\tilde{y}_0(x(qz)) \tag{35}$$

we get

$$\tilde{y}_0(x(z)) = [\prod_{i=0}^{N-1} \tilde{p}(zq^{\frac{1}{2}+i})]\tilde{y}_0(x(q^N z)). \tag{36}$$

while using the recursion

$$\tilde{y}_0(x(z)) = [\tilde{p}(zq^{-\frac{1}{2}})]^{-1}\tilde{y}_0(x(q^{-1}z)) \tag{37}$$

one obtains

$$\tilde{y}_0(x(z)) = [\prod_{i=0}^{N-1} \tilde{p}(zq^{-\frac{1}{2}-i})]^{-1}\tilde{y}_0(x(q^{-N}z)). \tag{38}$$

Hence the following

Lemma 3.1 *Equations (28) and (36) or (30) and (38), can be obtained from each other by replacing q by q^{-1} and $\tilde{a}(x)$ by $a(x)$. That is, to obtain a solution of (32) with $\tilde{a}(x) = a(x)$, it suffices to replace q by q^{-1} in a solution of (23) and vice-versa.*

Next, it is clear that we can write (30) and (36) under the forms

$$y_0(x(z)) = [\prod_{t=z_0q^{-1}}^{z} p(tq^{\frac{1}{2}})]^{-1}y_0(x(z_0)) \tag{39}$$

and

$$\tilde{y}_0(x(z)) = [\prod_{t=z_0q^{-1}}^{z} \tilde{p}(tq^{\frac{1}{2}})]\tilde{y}_0(x(z_0)). \tag{40}$$

for $z = q^{-k_z}z_0$, for a some positive integer k_z.

Now, if $\tilde{a}(x) = -a(x)$, then $\tilde{p}(x) = p(x)$ and multiplying (39) and (40) gives

$$y_0(x(z))\tilde{y}_0(x(z)) = y_0(x(z_0))\tilde{y}_0(x(z_0)),\ z = q^{k_z}z_0. \tag{41}$$

In particular, if $y_0(x(z_0))\tilde{y}_0(x(z_0) = 1$, then $y_0(x(z))\tilde{y}_0(x(z)) = 1$, $\forall z = q^{-k_z}x_0$. Using (28) and (38), one obtains that $y_0(x(z))\tilde{y}_0(x(z)) = 1$, $\forall z = q^{k_z}z_0$, i.e. $y_0(x(z))$ and $\tilde{y}_0(x(z))$ are mutually inverse, which proves the following

Theorem 3.1 *If $y(x(z))$ and $\tilde{y}(x(z))$ are respective solutions of the associated equations*

$$\mathcal{D}y(x(z)) = a(x(z))y(x(q^{-\frac{1}{2}}z)) \tag{42}$$

$$\mathcal{D}\tilde{y}(x(z)) = -a(x(z))\tilde{y}(x(q^{\frac{1}{2}}z)) \tag{43}$$

satisfying the conditions

$$\tilde{y}(x(z_0))y(x(z_0)) = 1 \tag{44}$$

then these functions are mutually inverse.

It is, however, possible to give a direct proof of the theorem: Applying the derivative to the product $\tilde{y}(x(z))y(x(z))$, we get

$$\begin{aligned} \mathcal{D}(\tilde{y}y) &= \mathcal{D}\tilde{y}.y(x(zq^{-\frac{1}{2}})) + \tilde{y}(x(zq^{\frac{1}{2}})).\mathcal{D}y \\ &= -\tilde{y}(x(zq^{\frac{1}{2}}))a(x(z))y(x(zq^{-\frac{1}{2}})) + \tilde{y}(x(zq^{\frac{1}{2}}))a(x(z))y(x(q^{-\frac{1}{2}})) = 0. \end{aligned} \tag{45}$$

This means that $\tilde{y}(x(z))y(x(z)) = const$, and the condition (44) brings the desired result.

From this theorem and lemma 3.1, one easily deduces the following

Corollary 3.1 *The inverse of the solution of*

$$\mathcal{D}y(x(z)) = a(x(z))y(x(q^{-\frac{1}{2}}z)) \tag{46}$$

is the solution of

$$\mathcal{D}y(x(z)) = -a(x(z))y(x(q^{-\frac{1}{2}}z)) \tag{47}$$

with q replaced by q^{-1}.

To ensure the exponential shape of the solution of (23), we solve it *heuristically* as follows. First, we write it in the form of a q-difference equation in the z-variable:

$$D_q y(x(z)) = \gamma(z)y(x(z)), \tag{48}$$

where

$$\gamma(z) = \frac{1}{2}a(x(q^{\frac{1}{2}}z))(1 - \frac{1}{qz^2}). \tag{49}$$

To deal with (48), define a q-version of the logarithm function (of function as variable) by setting

$$(\widetilde{\ln}_q f)(z) =^{def} \int_{z_0}^{z} \frac{D_q f(z)}{f(z)} d_q z, \tag{50}$$

where $z_0 = f^{-1}(1)$; and a q-version of the exponential function as the inverse of the q-logarithm

$$\tilde{e}_q =^{def} \widetilde{\ln}_q^{-1}. \tag{51}$$

This means that

$$f(z)=\tilde{e}_q^{\int_{z_0}^{z}\frac{D_qf(z)}{f(z)}d_qz}. \tag{52}$$

Examples.

1) By (50)

$$\widetilde{\ln}_q z^b=\int_1^z\frac{D_qz^b}{z^b}d_qz=\frac{q^b-1}{q-1}\int_1^z\frac{1}{z}d_qz=\frac{q^b-1}{\ln q}\ln z=\log_q z^{q^b-1} \tag{53}$$

2) If $e_q^z=\sum_{k=0}^{\infty}\frac{z^k}{[k]_q!}$ is the usual q-exponential function, then by (52) we have that

$$D_qe_q^{az}=ae_q^{az}\Rightarrow e_q^{az}=\tilde{e}_q^{\int_0^z ad_qz}=\tilde{e}_q^{az}, \tag{54}$$

and

$$D_qe_{q^2}^{z^2}=(q+1)ze_{q^2}^{z^2}\Rightarrow e_{q^2}^{z^2}=\tilde{e}_q^{\int_0^z(q+1)zd_qz}=\tilde{e}_q^{z^2}. \tag{55}$$

More generally

$$D_qe_{q^b}^{z^b}=(D_qz^b)e_{q^b}^{z^b}\Rightarrow e_{q^b}^{z^b}=\tilde{e}_q^{\int_0^z(D_qz^b)d_qz}=\tilde{e}_q^{z^b}. \tag{56}$$

On the other side by (54), (55) and (56), we have that

$$\widetilde{\ln}_q(e_q^{az})=az=\ln_q(e_q^{az}), \tag{57}$$
$$\widetilde{\ln}_q(e_{q^2}^{z^2})=z^2\neq\ln_q(e_{q^2}^{z^2}), \tag{58}$$
$$\widetilde{\ln}_q(e_{q^b}^{z^b})=z^b\neq\ln_q(e_{q^b}^{z^b}), \tag{59}$$

where $\ln_q$ is the inverse to usual q-exponential e_q.

It is also worth noting that unlike the usual q-exponential and q-logarithm cases, from (50) and (52), follow clearly the formulae

$$D_q\widetilde{\ln}_qu(z)=\frac{D_qu(z)}{u(z)},\ D_q\tilde{e}_q^{v(z)}=\tilde{e}_q^{v(z)}D_qv(z) \tag{60}$$

similarly to the differential situations.

We can now write *formally* the solution of (48)-(49) as

$$y(x(z))=\tilde{e}_q^{\frac{1}{2}\int_{z_0}^{z}a(x(zq^{\frac{1}{2}}))(1-\frac{1}{qz^2})d_qz}=\tilde{e}_q^{\int_{z_0}^{z}a(x(zq^{\frac{1}{2}}))(D_qx(z))d_qz}. \tag{61}$$

To compare (28) and (30) with (61), let us write them first as

$$y_0(x(z))=[\prod_{t=qz_0}^{z}p(tq^{-\frac{1}{2}})]y_0(x(z_0)),\ z=z_0.q^{k_z} \tag{62}$$

and

$$y_0(x(z)) = [\prod_{t=q^{-1}z_0}^{z} p(tq^{\frac{1}{2}})]^{-1} y_0(x(z_0)),\ z = z_0.q^{-k_z} \tag{63}$$

for some positive integer k_z and complex $z_0 \neq 0, \infty$. Hence one can write

$$\tilde{e}_q^{\frac{1}{2}\int_{z_0}^{z} a(x(zq^{\frac{1}{2}}))(1-\frac{1}{qz^2})d_q z} = \begin{cases} \prod_{t=qz_0}^{z} p(tq^{-\frac{1}{2}}), & z = z_0.q^{k_z} \\ (\prod_{t=q^{-1}z_0}^{z} p(tq^{\frac{1}{2}}))^{-1}, & z = z_0.q^{-k_z}, \end{cases} \tag{64}$$

where we deleted the factor $y_0(x(z_0))$, since in the definition (50), z_0 was chosen so that $y(x(z_0)) = 1$.

If $z_0 = 0$ then

$$\tilde{e}_q^{\frac{1}{2}\int_{0}^{z} a(x(zq^{\frac{1}{2}}))(1-\frac{1}{qz^2})d_q z} = [\prod_{i=0}^{\infty} p(q^{\frac{1}{2}+i}z)]^{-1} \tag{65}$$

and for $z_0 = \infty$

$$\tilde{e}_q^{\frac{1}{2}\int_{\infty}^{z} a(x(zq^{\frac{1}{2}}))(1-\frac{1}{qz^2})d_q z} = \prod_{i=0}^{\infty} p(q^{-\frac{1}{2}-i}z) \tag{66}$$

provided the convergence of the involved integrals and products. It is not difficult to see that the convergence of any of the products is linked to that of the corresponding q-integral in lhs. Indeed, if $h(z) = \frac{1}{2}a(x(zq^{\frac{1}{2}}))(1 - \frac{1}{qz^2})$ is the integrated function in lhs of (65) and (66), then $p(q^{\frac{1}{2}}z) = (q-1)zh(z) + 1$ and consequently, the convergence of the product in (65) is equivalent to that of the series $\sum_{i=0}^{\infty}(q-1)zq^i h(zq^i) = -\int_0^z h(z)d_q z$. Similarly, since $p(q^{-\frac{1}{2}}z) = (q-1)q^{-1}zh(q^{-1}z) + 1$, then the convergence of the product in (66) is equivalent to that of the series $\sum_{i=0}^{\infty}(q-1)zq^{-1-i}h(zq^{-1-i}) = \int_{\infty}^{z} h(z)d_q z$.

Consider now the non homogeneous equation (22). $y_0(x(z))$ being the solution of the corresponding homogeneous equation (23), we are led to search its solution by the *method of variation of constants* under the form

$$y(x(z)) = c(x(z))y_0(x(z)), \tag{67}$$

for unknown $c(x(z))$. Loading (67) in (22) gives

$$c(x(z)) = c + \int_{x(x_0)}^{x(z)} y_0^{-1}(x(q^{\frac{1}{2}}t))b(x(t))d_q x(t). \tag{68}$$

By (67) we then get the general form of the solution of (22):

$$y(x(z)) = \phi(x(z), x(x_0))[y(x(x_0)) + \int_{x(x_0)}^{x(z)} \phi(x(x_0), x(q^{\frac{1}{2}}t))b(x(t))d_q x(t)], \tag{69}$$

where

$$\phi(a, b) = y_0(a)y_0^{-1}(b). \tag{70}$$

For the associated equation (31), the general solution reads

$$\tilde{y}(x(z)) = \tilde{\phi}(x(z), x(x_0))[\tilde{y}(x(x_0)) + \int_{x(x_0)}^{x(z)} \tilde{\phi}(x(x_0), x(q^{-\frac{1}{2}}t))\tilde{b}(x(t))d_q x(t)], \tag{71}$$

where now

$$\tilde{\phi}(a, b) = \tilde{y}_0(a)\tilde{y}_0^{-1}(b). \tag{72}$$

We will now consider some special cases of equation (22). Since by (69), the essential of the solution of (22) consists in solving (23), we can consider here only homogeneous cases:

Case 1.

$$\mathcal{D}y(x(z)) = ay(x(q^{-\frac{1}{2}}z)), \tag{73}$$

a-const. We can solve this equation in two ways: First *formally*, we simply use (61) to get

$$y(x(z)) = \tilde{e}_q^{\frac{a}{2}\int_{z_0}^{z}(1-\frac{1}{qz^2})d_q z} = \tilde{e}_q^{a\int_{z_0}^{z}(D_q x(z))d_q z} = \tilde{e}_q^{\frac{a}{2}(z+z^{-1})} = \tilde{e}_q^{ax(z)} \tag{74}$$

as a solution of (73), for $z_0 = \pm i$.

To solve (73) *explicitly*, we use (29). Factorizing $p(q^{1/2}z)$ with $a(x(z)) = a$, (29) gives

$$y_0(x(z)) = c\frac{1}{(1 - z_1 z^{-1})(1 - z z_2^{-1})} y_0(x(qz)) \tag{75}$$

where

$$c = \frac{2}{(1-q)az_2};\ z_{1,2} = \frac{-q \pm [q^2 + q(q-1)^2 a^2]^{\frac{1}{2}}}{aq(q-1)} \tag{76}$$

and whose solution reads

$$y_0(x(z)) = z^{\frac{\ln c}{\ln q}} \frac{(z_1 q z^{-1}; q)_\infty}{(z_2^{-1} z; q)_\infty}, \tag{77}$$

where $(\alpha; q)_n = \prod_{i=0}^{n-1}(1 - q^i\alpha)$ and $(\alpha; q)_\infty = \lim_{n\to\infty}(\alpha; q)_n$.

Case 2.

$$\mathcal{D}y(x(z)) = a.x(z)y(x(q^{-\frac{1}{2}}z)). \tag{78}$$

As in the previous case, to solve it in the formal way, we first write it in the form of a q-difference equation in the z-variable:

$$D_q y(x(z)) = \beta(z)y(x(z)), \tag{79}$$

where

$$\beta(z) = \frac{a}{4}(q^{\frac{1}{2}}z - q^{\frac{-3}{2}}z^{-3}) = a\frac{q^{\frac{1}{2}}}{q+1}D_q x^2(z), \tag{80}$$

and we find

$$y(x(z) = \tilde{e}_q^{\int_{z_0}^{z} \beta(z) d_q z} = \tilde{e}_q^{\frac{a}{4} \frac{q^{\frac{1}{2}}}{q+1}(z+z^{-1})^2} = \tilde{e}_q^{a \frac{q^{\frac{1}{2}}}{q+1} x^2(z)}, \tag{81}$$

again for $z_0 = \pm i$, as the solution of (78).

To solve (78) explicitly, we also use (29). Factorizing $p(q^{1/2}z)$ with $a(x(z)) = a.x(z)$, (29) gives

$$y_0(x(z)) = c \frac{1}{(1 - z_1 z^{-2})(1 - z^2 z_2^{-1})} y_0(x(qz)) \tag{82}$$

where

$$c = \frac{2}{(1-q) q^{\frac{1}{2}} a z_2}; \; z_{1,2} = \frac{-q^{\frac{1}{2}} \pm [q + (q-1)^2 a^2]^{\frac{1}{2}}}{aq(q-1)} \tag{83}$$

and whose solution reads

$$y_0(x(z)) = z^{\frac{\ln c}{\ln q}} \frac{(z_1 q z^{-2}; q^2)_\infty}{(z_2^{-1} z^2; q^2)_\infty}. \tag{84}$$

Case 3.

$$\mathcal{D} y(x(z)) = \frac{1}{x(zq^{-\frac{1}{2}}) + \nu} y(x(q^{-\frac{1}{2}} z)). \tag{85}$$

Using (61), we get

$$y(x(z)) = \tilde{e}_q^{\int_{z_0}^{z} \frac{D_q(x(z)+\nu)}{x(z)+\nu} d_q z} = \tilde{e}_q^{\widetilde{\ln}_q (x(z)+\nu)} = x(z) + \nu. \tag{86}$$

Case 4.

$$\mathcal{D} y(x(z)) = \frac{q+1}{q^{\frac{1}{2}}} \frac{x(z)}{x^2(zq^{-\frac{1}{2}}) + \mu} y(x(q^{-\frac{1}{2}} z)). \tag{87}$$

Again, using (61), we get

$$y(x(z)) = \tilde{e}_q^{\int_{z_0}^{z} \frac{D_q(x^2(z)+\mu)}{x^2(z)+\mu} d_q z} = \tilde{e}_q^{\widetilde{\ln}_q (x^2(z)+\mu)} = x^2(z) + \mu. \tag{88}$$

Case 5.

$$\mathcal{D} y(x(z)) = \frac{a}{x(bz)} y(x(q^{-\frac{1}{2}} z)). \tag{89}$$

Here we search the series solution $y(x(z)) = \sum_{n=-\infty}^{+\infty} c_n z^n$. Loading the latter in (89) and comparing the coefficients, one gets the easily solvable first order difference equation

$$\tilde{c}_n = \frac{ba(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) q^{-n+1} - 2b^2(q^{n-1} - q^{-n+1})}{2(q^n - q^{-n}) + ba(q^{\frac{1}{2}} - q^{-\frac{1}{2}}) q^{-n}} \tilde{c}_{n-1}; \; \tilde{c}_0, \tag{90}$$

where $\tilde{c}_n = c_{2n}$.

Case 6. Similarly to the previous case, one can solve the following equation

$$\mathcal{D}y(x(z)) = \frac{a.x(z)}{x^2(bz)-2}y(x(q^{-\frac{1}{2}}z)). \tag{91}$$

It is useful to note that from the preceding, lemma (3.1) and corollary (3.1), one deduces directly that the function

$$y(x(z)) = \tilde{e}_{q^{-1}}^{-ax(z)} = z^{-\frac{\ln c}{\ln q}}\frac{(z_2^{-1}z;q)_\infty}{(z_1qz^{-1};q)_\infty}, \tag{92}$$

which is the inverse of $\tilde{e}_q^{ax(z)} = z^{\frac{\ln c}{\ln q}}\frac{(z_1qz^{-1};q)_\infty}{(z_2^{-1}z;q)_\infty}$, is the solution of the equation

$$\mathcal{D}y(x(z)) = -ay(x(q^{\frac{1}{2}}z)). \tag{93}$$

Similarly the function $y(x(z)) = \tilde{e}_{q^{-1}}^{-a\frac{q^{\frac{1}{2}}}{q+1}x^2(z)} = z^{-\frac{\ln c}{\ln q}}\frac{(z_2^{-1}z^2;q^2)_\infty}{(z_1qz^{-2};q^2)_\infty}$ is the solution of

$$\mathcal{D}y(x(z)) = -ax(z)y(x(q^{\frac{1}{2}}z)), \tag{94}$$

while $y(x(z)) = \frac{1}{x(z)+\nu}$ and $y(x(z)) = \frac{1}{x^2(z)+\mu}$ solve the equations

$$\mathcal{D}y(x(z)) = -\frac{1}{x(zq^{-\frac{1}{2}})+\nu}y(x(q^{\frac{1}{2}}z)), \tag{95}$$

and

$$\mathcal{D}y(x(z)) = -\frac{q+1}{q^{\frac{1}{2}}}\frac{x(z)}{x^2(zq^{-\frac{1}{2}})+\mu}y(x(q^{\frac{1}{2}}z)) \tag{96}$$

respectively.

To conclude the section, let us note that if instead of (23), one considers the equation

$$\mathcal{D}y(x(z)) = a(x(z))y(x(z)), \tag{97}$$

then the constant coefficient case, i.e. $a(x(z)) = const$, also admits an explicit solution (in series) which is naturally a q-nonuniform grid version of the exponential function e^{ax} [12] and whose interesting properties were proved in [12, 19]. It is, however, clear that although equations (23) and (97) are both q-nonuniform versions of the differential equation $y'(x(z)) = a(x(z))y(x(z))$, most of the properties and structures of the solutions of (23), obtained in this section and the subsequent ones, cannot be easily established when solving the equation (97).

4 q-Nonuniform linear difference equations of second order

Consider now the q-nonuniform linear difference equations of second order. It is clear from the solvability of the first order equation that the derivative to be considered here is not $\mathcal{D}$ but the derivatives $\mathcal{D}^+$ and $\mathcal{D}^-$ defined by

$$\mathcal{D}^+y(x(z)) = \frac{f(x(qz))-f(x(z))}{x(qz)-x(z)};\ \mathcal{D}^-y(x(z)) = \frac{f(x(z))-f(x(z/q))}{x(z)-x(z/q)} \tag{98}$$

Thus, we study q-nonuniform linear difference equations of second order of the form

$$a_0(x(z))[\mathcal{D}^+]^2 y(x(z)) + a_1(x(z))\mathcal{D}^+ y(x(z)) + a_2(x(z))y(x(z)) = 0, \tag{99}$$

and these of the form

$$h_0(x(z))\mathcal{D}^-\mathcal{D}^+ y(x(z)) + h_1(x(z))\mathcal{D}^+ y(x(z)) + h_2(x(z))y(x(z)) = 0, \quad a_0, h_0 \neq 0, \tag{100}$$

when we are interested in the self-adjoint property.

4.1 Solvability

As in the differential or difference cases, there is no general way of solving type (99) or (100) in quadratures for general coefficients $h_0(x(z))$, $h_1(x(z))$, $h_2(x(z))$, $a_0(x(z))$, $a_1(x(z))$ and $a_2(x(z))$. However, some particular cases can be solved explicitly.

(i) Consider equation (99) and suppose that one of its two independent solutions, say $y_1(x(z)), y_2(x(z))$, is known. So the other solution can be found as in differential or difference calculus using the *q-nonuniform version of the Liouville formula*. We find the latter by comparing equation (99) with the following

$$\det\begin{pmatrix} y(x(z)) & y_1(x(z)) & y_2(x(z)) \\ \mathcal{D}^+ y(x(z)) & \mathcal{D}^+ y_1(x(z)) & \mathcal{D}^+ y_2(x(z)) \\ (\mathcal{D}^+)^2 y(x(z)) & (\mathcal{D}^+)^2 y_1(x(z)) & (\mathcal{D}^+)^2 y_2(x(z)) \end{pmatrix} = 0, \tag{101}$$

and simple computations show that we have

$$\mathcal{D}^+ W(y_1, y_2) = [-\tilde{a}_1(x(z)) + \tilde{a}_2(x(z))(x(qz) - x(z))]W(y_1, y_2), \tag{102}$$

where $W(u,v) = u\mathcal{D}^+ v - v\mathcal{D}^+ u$ is the Vronskian, and $\tilde{a}_1 = a_1/a_0, \tilde{a}_2 = a_2/a_0$.

Equation (102) is equivalent to the first order q-nonuniform equation for $W(x(z)) = W(y_1, y_2)(x(z))$

$$\mathcal{D}W(x(z)) = a(x(z))W(x(zq^{-\frac{1}{2}})), \tag{103}$$

where $a(x(z)) = -\tilde{a}_1(x(zq^{-\frac{1}{2}})) + \tilde{a}_2(x(zq^{-\frac{1}{2}}))(x(zq^{\frac{1}{2}}) - x(zq^{-\frac{1}{2}}))$, whose solution gives the required q-nonuniform Liouville formula:

$$W(x(z)) = \{ \begin{array}{ll} \prod_{t=qz_0}^{z} p(tq^{-\frac{1}{2}})W(x(z_0)), & z = z_0.q^{k_z} \\ (\prod_{t=q^{-1}z_0}^{z} p(tq^{\frac{1}{2}}))^{-1}W(x(z_0)), & z = z_0.q^{-k_z}, \end{array} \tag{104}$$

where $p(z)$ is given as in (25), or more compactly

$$W(x(z)) = W(x(z_0))\tilde{e}_q^{\frac{1}{2}\int_{z_0}^{z} a(x(zq^{\frac{1}{2}}))(1-\frac{1}{qz^2})d_q z}. \tag{105}$$

(ii) The next example is when the *coefficients* in equation (99) are *constant*. We can solve it by first factorizing it as

$$[(\mathcal{D}^+ - k_2)(\mathcal{D}^+ - k_1)]y(x(z)) = 0 \tag{106}$$

where k_1, k_2 are the roots of the characteristic algebraic equation

$$a_0 k^2 + a_1 k + a_2 = 0 \tag{107}$$

and then solve the two resulting first order equations. If $k_1 \neq k_2$, one obtains the two required independent solutions: $y_1(x(z)) = \tilde{e}_q^{k_1 x(z)}$ and $y_2(x(z)) = \tilde{e}_q^{k_2 x(z)}$. For example, the trigonometric functions $\widetilde{\cos}_q x(z) = \frac{\tilde{e}_q^{ix(z)} + \tilde{e}_q^{-ix(z)}}{2}$, $\widetilde{\sin}_q x(z) = \frac{\tilde{e}_q^{ix(z)} - \tilde{e}_q^{-ix(z)}}{2i}$ and the hyperbolic ones $\widetilde{\cosh}_q x(z) = \frac{\tilde{e}_q^{x(z)} + \tilde{e}_q^{-x(z)}}{2}$, $\widetilde{\sinh}_q x(z) = \frac{\tilde{e}_q^{x(z)} - \tilde{e}_q^{-x(z)}}{2}$ are solutions of the equations

$$(\mathcal{D}^+)^2 y(x(z)) + y(x(z)) = 0 \tag{108}$$

and

$$(\mathcal{D}^+)^2 y(x(z)) - y(x(z)) = 0 \tag{109}$$

respectively. On the other hand, if (107) admits a double root, then one uses the Liouville formula in (105) to find the second independent solution.

(iii) A particular case of (100) is well known. If (see [18])

$$h_0(x(z)) = \frac{x(qz) - x(z)}{x(zq^{\frac{1}{2}}) - x(zq^{-\frac{1}{2}})}[\sigma(x(z)) - \frac{1}{2}\tau(x(z))(x(zq^{\frac{1}{2}}) - x(zq^{-\frac{1}{2}}))] \tag{110}$$

and

$$h_1(x(z)) = \tau(x(z)); \;\; h_2(x(z)) = const, \tag{111}$$

where σ and τ are polynomials of degree ≤ 2 and 1 respectively, then (100) admits a system of polynomial solutions, the famous Askey-Wilson polynomials [2].

4.2 Orthogonality

Consider the eigenvalue equation

$$a(x(z))\mathcal{D}^-\mathcal{D}^+ y(x(z)) + b(x(z))\mathcal{D}^+ y(x(z)) = \lambda y(x(z)). \tag{112}$$

We can write it under the self-adjoint form

$$\mathcal{D}^-[a(x(qz))\varrho(x(qz))\mathcal{D}^+ y(x(z))] = \lambda \varrho(x(z)) y(x(z)). \tag{113}$$

where

$$\mathcal{D}^-[a(x(qz))\varrho(x(qz))] = \varrho(x(z)) b(x(z)). \tag{114}$$

Suppose that we also have

$$\mathcal{D}^-[a(x(qz))\varrho(x(qz))\mathcal{D}^+\tilde{y}(x(z))] = \tilde{\lambda}\varrho(x(z))\tilde{y}(x(z)), \tag{115}$$

for distinct $\tilde{\lambda}$ and λ. Next, multiply equation (113) by $\tilde{y}((z))$ and equation (115) by $y((z))$ and subtract member by member. We get

$$\begin{aligned} &\tilde{y}(x(z))\mathcal{D}^-[a(x(qz))\varrho(x(qz))\mathcal{D}^+y(x(z))] \\ &-y(x(z))\mathcal{D}^-[a(x(qz))\varrho(x(qz))\mathcal{D}^+\tilde{y}(x(z))] \\ &\qquad = (\lambda-\tilde{\lambda})\varrho(x(z))y(x(z))\tilde{y}(x(z)). \end{aligned} \tag{116}$$

Next, by the easily verified identity

$$\mathcal{D}^-\left[p(u\mathcal{D}^+v - v\mathcal{D}^+u)\right] = u\mathcal{D}^-(p\mathcal{D}^+v) - v\mathcal{D}^-(p\mathcal{D}^+u) \tag{117}$$

we obtain

$$\begin{aligned} &\mathcal{D}^-\left[a(x(qz))\varrho(x(qz))[\tilde{y}\mathcal{D}^+y - y\mathcal{D}^+\tilde{y}]\right] \\ &\qquad = (\lambda-\tilde{\lambda})\varrho(x(z))y(x(z))\tilde{y}(x(z)), \end{aligned} \tag{118}$$

or

$$\mathcal{D}^-\left[a(x(qz))\varrho(x(qz))W(\tilde{y},y)\right] = (\lambda-\tilde{\lambda})\varrho(x(z))y(x(z))\tilde{y}(x(z)). \tag{119}$$

Integrating both sides of the equality from say, $x(z_0)$ to $x(z_1)$, noting that $(\mathcal{D}^-F)(z) = (\mathcal{D}F)(q^{-\frac{1}{2}}z)$ and using finally (18), we get

$$\begin{aligned} &(\lambda-\tilde{\lambda})\int_{x(z_0)}^{x(z_1)}\varrho(x(z))y(x(z))\tilde{y}(x(z))d_qx(z) \\ &= \int_{x(z_0)}^{x(z_1)}\mathcal{D}^-\left[a(x(qz))\varrho(x(qz))W(\tilde{y},y)\right]d_q(x(z)) \\ &\qquad = a(x(qz))\varrho(x(qz))W(\tilde{y},y)\,\Big|_{x(z_0q^{-\frac{1}{2}})}^{x(z_1q^{-\frac{1}{2}})}. \end{aligned} \tag{120}$$

The latter gives the *orthogonality condition* for two eigenfunctions $y(x(z))$ and $\tilde{y}(x(z))$ of (112) corresponding to distinct eigenvalues λ and $\tilde{\lambda}$, on the interval $[x(z_0), x(z_1)]$, with a q-nonuniform discrete weight $\varrho(x(z))$ given by (114):

$$a(x(qz))\varrho(x(qz))W(\tilde{y}(x(z)),y(x(z)))\,\Big|_{x(z_0q^{-\frac{1}{2}})}^{x(z_1q^{-\frac{1}{2}})} = 0. \tag{121}$$

5 q-Nonuniform linear difference systems

We now consider the associated linear matrix systems

$$\mathcal{D}Y(x(z)) = A(x(z))Y(x(q^{-\frac{1}{2}}z)) + B(x(z)), \tag{122}$$

and

$$\mathcal{D}\tilde{Y}(x(z)) = \tilde{Y}(x(q^{\frac{1}{2}}z))\tilde{A}(x(z)) + \tilde{B}(x(z)), \tag{123}$$

where now $A = (a_{i,j})_{i,j=1}^{k}$, $\tilde{A} = (\tilde{a}_{i,j})_{i,j=1}^{k}$, $Y = (y_{i,j})_{i,j=1}^{k}$, $\tilde{Y} = (\tilde{y}_{i,j})_{i,j=1}^{k}$, $B = (b_{i,j})_{i,j=1}^{k}$, and $\tilde{B} = (\tilde{b}_{i,j})_{i,j=1}^{k}$ are kxk-matrices. The matrices A and $\tilde{A}$ must be such that the matrices P and $\tilde{P}$ in (128) and (129) be non singular.

The scalar case studied in section 3 corresponds to $k = 1$ here, and most of the results obtained in the cited section, can be reported here, with little modifications linked mainly to the fact that we are dealing now with matrices and their multiplication is not always commutative.

Consider first the corresponding homogeneous matrix systems

$$\mathcal{D}Y(x(z)) = A(x(z))Y(x(q^{-\frac{1}{2}}z)), \tag{124}$$

and

$$\mathcal{D}\tilde{Y}(x(z)) = \tilde{Y}(x(q^{\frac{1}{2}}z))\tilde{A}(x(z)). \tag{125}$$

In recursive forms they read

$$Y(x(q^{\frac{1}{2}}z)) = P(z)Y(x(q^{-\frac{1}{2}}z)) \tag{126}$$

and

$$\tilde{Y}(x(q^{\frac{1}{2}}z))\tilde{P}(z) = \tilde{Y}(x(q^{-\frac{1}{2}}z)) \tag{127}$$

where

$$P(z) = I + \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2}(z - z^{-1})A(x(z)) \tag{128}$$

and

$$\tilde{P}(z) = I - \frac{q^{\frac{1}{2}}-q^{-\frac{1}{2}}}{2}(z - z^{-1})\tilde{A}(x(z)). \tag{129}$$

The solution of (124) verifies

$$Y(x(z)) = Y(x(q^{N}z))[\prod_{i=N-1}^{z} P(q^{\frac{1}{2}+i}z)]^{-1} \tag{130}$$

$$\tag{131}$$

and

$$Y(x(z)) = [\prod_{i=0}^{N-1} P(q^{-\frac{1}{2}-i}z)]Y(x(q^{-N}z)) \tag{132}$$

while that of (125) verifies

$$\tilde{Y}(x(z)) = [\prod_{i=0}^{N-1} \tilde{P}(q^{\frac{1}{2}+i}z)]\tilde{Y}(x(q^N z)) \tag{133}$$

and

$$\tilde{Y}(x(z)) = \tilde{Y}(x(q^{-N}z))[\prod_{i=N-1}^{z} \tilde{P}(q^{-\frac{1}{2}-i}z)]^{-1}. \tag{134}$$

Comparing (130) with (134), and (132) with (133), we obtain an equivalent to lemma 3.1. Moreover, comparing (130) with (133), and (132) with (134), one obtains an equivalent to theorem 3.1 (and consequently an equivalent to corollary 3.1) which, however, one now formulates as follows

Theorem 5.1 *If $Y(x(z))$ and $\tilde{Y}(x(z))$ are respective solutions of the associated matrix systems*

$$\mathcal{D}Y(x(z)) = A(x(z))Y(x(q^{-\frac{1}{2}}z)) \tag{135}$$

$$\mathcal{D}\tilde{Y}(x(z)) = -\tilde{Y}(x(q^{\frac{1}{2}}z))A(x(z)) \tag{136}$$

satisfying the conditions

$$\tilde{Y}(x(x_0))Y(x(x_0)) = I \tag{137}$$

then these matrix functions are mutually inverse.

A direct proof of this theorem, similar to the one for theorem 3.1, can also be performed.

In the case of constant matrices A and $\tilde{A} = -A$, by the arguments above, the solutions of (124) and (125) are clearly mutually inverse and read

$$Y_0(x(z)) = \tilde{e}_q^{Ax(z))};\ \tilde{Y}_0(x(z)) = Y_0^{-1}(x(z)) = \tilde{e}_{q^{-1}}^{-Ax(z))}. \tag{138}$$

Supposing that Y_0 and $\tilde{Y}_0$ are nonsingular matrix solutions (it is easily seen that such solutions exist) of (124) and (125) respectively, then $y = Y_0C$ and $\tilde{y} = \tilde{C}\tilde{Y}_0$ for constant kxk-matrices $C, \tilde{C}$ will obviously give the general solutions of the systems respectively.

Consider next the non homogeneous equations (122) and (123). Using the *method of variation of constants*, one is led respectively to

$$Y(x(z)) = \Phi(x(z), x(x_0)[Y(x(x_0)) + \int_{x(x_0)}^{x(z)} \Phi(x(x_0), x(q^{\frac{1}{2}}z))B(x(z))d_qx(z)], \tag{139}$$

where

$$\Phi(a, b) = Y_0(a)Y_0^{-1}(b), \tag{140}$$

and

$$\tilde{Y}(x(z)) = [\tilde{Y}(x(x_0)) + \int_{x(x_0)}^{x(z)} \tilde{\Phi}(x(x_0), x(q^{-\frac{1}{2}}z))\tilde{B}(x(z))d_q x(z)]\tilde{\Phi}(x(z), x(x_0)), \tag{141}$$

where now

$$\tilde{\Phi}(a, b) = \tilde{Y}_0^{-1}(b)\tilde{Y}_0(a). \tag{142}$$

Consider next the linear system

$$\mathcal{D}y(x(z)) = A(x(z))y(x(q^{-\frac{1}{2}}z)) + b(x(z)), \tag{143}$$

that is similar to (122) but with now $y = (y_1, \ldots, y_k)^T$ and $b = (b_1, \ldots, b_k)$ k-vectors (similar system for (124) has no sense). If the nonsingular matrix $Y_0(x(z))$ solves (124), it also solves the homogeneous system corresponding to (143), and the general solution of (143) reads

$$y(x(z)) = \Phi(x(z), x(x_0))[y(x(x_0)) + \int_{x(x_0)}^{x(z)} \Phi(x(x_0), x(q^{\frac{1}{2}}z))b(x(z))d_q x(z)]. \tag{144}$$

where $\Phi(a, b)$ is the one given in (140).

The next section is devoted to the controllability of type (143) systems with $b(x(z)) = B(x(z))u(x(z))$, B-a kxm matrix, and u-a $mx1$ control function. As we will see, in controllability theory, the form (144) for the solution of (143) is the most adequate for the analysis.

Finally note that as in the differential case, linear q-nonuniform difference equations of higher order

$$[a_0(x(z))(\mathcal{D}^+)^n + a_1(x(z))(\mathcal{D}^+)^{n-1} + \ldots + a_n(x(z)]y(x(z)) = g(x(z)) \tag{145}$$

can be handled as a particular case of the system (143). Indeed, by the change of variables $z_1(x(z)) = y(x(z))$, $z_2(x(z)) = \mathcal{D}^+ y(x(z))$, ..., $z_n(x(z)) = (\mathcal{D}^+)^{n-1} y(x(z))$, simple operations transform (145) in type (143) system.

6 q-Nonuniform linear difference control systems

The linear control systems theory consists in the study of controllability of linear systems, that is, a set of well defined interconnected objects which interactions can be mathematically modeled by linear systems of divided difference functional equations. Thus a q-nonuniform difference linear control system can be modeled as

$$\mathcal{D}y(x(z)) = A(x(z))y(x(zq^{-\frac{1}{2}})) + B(x(z))u(x(z)) \tag{146}$$

where y is a k-vector, A a kxk-matrix, B a kxm-matrix, and u, a m-vector. The vector y stands for the *state variable* of the system, describing the state of the system at a given time s ($z = q^s$), while u stands for the input or the external force constraining the system that is the resulting trajectory to adopt a predetermined behavior. Thus, u controls the system, which is why we speak of control systems. The matrices A and B are intrinsic characterizations or descriptions of the system. In (146), the state of the system is described by k variables and the external forces act with m inputs.

In practice, it is often difficult, even impossible, to determine the state of a system itself because it is generally characterized by very numerous variables. Instead, one observes the output of the system $v(x)$, characterized by a small number of variables. Hence, a mathematical model more suitable than (146) for the study of the systems controllability reads

$$\begin{aligned} \mathcal{D}y(x(z)) &= A(x(z))y(x(zq^{-\frac{1}{2}})) + B(x(z))u(x(z)) \\ v(x(z)) &= C(x(z))y(x(z)). \end{aligned} \tag{147}$$

with C, a rxk-matrix and v, a r-vector.

6.1 Controllability

There are many versions of definition of the concept of controllability in mathematical control theory: The controllability of the state, controllability of the output, controllability at the origin, complete controllability and so on. The following definition adopted in this work, consists in the *complete controllability* of the state system.

Definition 6.1 *The system (147) is said to be completely controllable (c.c.) if for any given value of $x = x_0 = x(z_0)$, and any initial value of $y = y_0 = y(x_0)$, and any final value of $y = y_f$, there exists a finite value $x = x_1 = x(z_1)$, and a control function $u(x)$, $x_0 \leq x \leq x_1$ such that $y(x_1) = y_f$.*

According to (144), the solution of (147) reads

$$y(x(z)) = \Phi(x(z), x_0)[y_0 + \int_{x_0}^{x(z)} \Phi(x_0, x(zq^{\frac{1}{2}}))B(x(z))u(x(z))d_qx(z)] \tag{148}$$

where $\Phi(a,b) = Y_0(a)Y_0^{-1}(b)$, $Y_0(x(z))$ is the nonsingular matrix solution of the homogeneous system corresponding to the first equation in equations (147).

Hence, the system is c.c. if for any value $x_0 = x(z_0)$ and any values y_0 and y_f, there exists a finite value x_1 and a q-nonuniform discrete function $u(x)$, $x_0 \leq x \leq x_1$, such that

$$y_f = y(x_1) = \Phi(x_1, x_0)[y_0 + \int_{x_0}^{x_1} \Phi(x_0, x(zq^{\frac{1}{2}}))B(x(z))u(x(z))d_qx(z)] \tag{149}$$

It is not difficult to imagine an example of a non c.c. system. The classical one is similar to the following system

$$\begin{aligned} D_qy_1(x(z)) &= a_{11}y_1(x(zq^{-\frac{1}{2}})) + a_{12}y_2(x(zq^{-\frac{1}{2}})) + u(x(z)) \\ D_qy_2(x(z)) &= a_{22}y_2(x(zq^{-\frac{1}{2}})). \end{aligned} \tag{150}$$

This system is not c.c. since $y_2(x(z)) = \tilde{e}_{q^{-1}}^{a_{22}x(z)}$ and consequently $u(x(z))$ has no control over it.

The following controllability criterion is valid not only for constant systems but also for varying ones. Moreover, it gives an explicit expression for the control function $u(x)$. We cite it without proof since it can be proved in similar ways than in the differential or q-difference situations (see e.g. [6, 4]).

Theorem 6.1 *The system (147) is c.c. iff the k.k symmetric matrix*

$$U(x_0, x_1) = \int_{x_0}^{x_1} \Phi(x_0, x(zq^{\frac{1}{2}}))B(x(z))B^T(x(z))\Phi(x_0, x(zq^{\frac{1}{2}}))^T d_q x(z) \tag{151}$$

is nonsingular. In the latter case, the control function is given by

$$u(x(z)) = -B^T(x(z))\Phi(x_0, x(zq^{\frac{1}{2}}))^T U^{-1}(x_0, x_1)[y_0 - \Phi(x_0, x_1)y_f]$$
$$x_0 \leq x \leq x_1 \tag{152}$$

and transfers $y_0 = y(x_0)$ to $y_f = y(x_1)$.

6.2 Observability

The concept of observability is closely related to that of controllability. Generally speaking, a system is completely observable iff the knowledge of the input and output suffices to determine the state of the system.

Definition 6.2 *The system (147) is completely observable (c.o.) if for any x_0, there exists a finite x_1 such that the knowledge of $z(x)$ and $u(x)$ for $x_0 \leq x \leq x_1$ suffice to determine $y_0 = y(x_0)$.*

Similar to theorem 6.1, the basic observability criterion for time varying systems reads

Theorem 6.2 *The system (147) is c.o. iff the symmetric matrix*

$$V(x_0, x_1) = \int_{x_0}^{x_1} \Phi^T(x(zq^{\frac{1}{2}}), x_0)C^T(x(z))C(x(z))\Phi(x(zq^{\frac{1}{2}}), x_0)d_q x(z) \tag{153}$$

is nonsingular. In the latter case, we have

$$y_0 = V^{-1}(x_0, x_1) \int_{x_0}^{x_1} \Phi^T(x(zq^{\frac{1}{2}}), x_0)C^T(t)v(x(z))d_q x(z). \tag{154}$$

The proof of the theorem is similar to the differential or q-difference case as well.

The controllability and observability are two concepts with physically distinct meanings but that are mathematically equivalent as shows the following *q-duality theorem:*

Theorem 6.3 *The system (147) is c.c. iff the dual system*

$$D_q y(x(z)) = -A^T(x(z))y(x(zq^{\frac{1}{2}})) + C^T(x(z))u(x(z))$$
$$v(x(z)) = B^T(x(z))y(x(z)) \qquad (155)$$

is c.o. and conversely.

Proof. Considering (147),(151),(153), and (155), we remark that to prove the *necessity*, it suffices to prove that if $\mathcal{D}\Phi(x(z), x_0) = A(x(z))\Phi(x(zq^{-\frac{1}{2}}), x_0)$ then $\mathcal{D}\Phi^T(x_0, x(z)) = -A^T(x(z))\Phi^T(x_0, x(zq^{\frac{1}{2}}))$. Indeed from theorem 5.1, it follows that if $\Phi(x(z), x_0)$ satisfies $\mathcal{D}Y(x(z)) = A(x(z))Y(x(z))$ then its inverse, that is, $\Phi(x_0, x(z))$ satisfies $\mathcal{D}Z(x) = -Z(x(zq^{\frac{1}{2}}))A(x(z))$. Carrying out the transpose on both sides, one gets the required equality. The *sufficiency* is proved similarly.

This duality clearly allows to relate results in controllability and observability theories.

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