A VECTOR MATRIX APPROACH OF COUNTING CYCLIC QUOTIENTS
OF SOME ABELIAN $p$-GROUPS

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Abstract

We determine in this paper, the precise number of cyclic quotients of Abelian $p$-groups of exponent $p^i$ and rank $r > 1$, $i = 1, 2, \ldots, n$ for all natural numbers $n$.

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1 INTRODUCTION

The mathematical motivation for this paper is as follows:
Let $\pi$ be a finite Abelian group, $R$ a commutative Noetherian ring, $G_*(A)$ the Quillen $K$-theory of the category of finitely-generated $A$-modules, for any ring $A$ with identity. In [2], D. L. Webb established the formula

$$G_n(\mathbb{Z}\pi) \cong \bigoplus_{\rho \in X(\pi)} G_n(\mathbb{Z}<\rho>), \ n \geq 0$$

where $\mathbb{Z}<\rho>$ denotes the ring of fractions $\mathbb{Z}(\rho)[1/|\rho|]$ obtained by inverting $|\rho|$, $\mathbb{Z}(\rho)$ denotes the quotient of the group ring $\mathbb{Z}\rho$ by the $|\rho|^{-1}$ cyclotomic polynomial $\Phi_{|\rho|}$ evaluated at a generator of $\rho$ (the ideal factored out is independent of the choice of generator for $\rho$), $|.|$ denotes cardinality and $X(\pi)$ the set of cyclic quotients of $\pi$.

A natural problem is that of computing $G_n(\mathbb{Z}\pi)$ as explicitly as possible and from the formula above, it is desirable to know the number of cyclic quotients of $\pi$.

The object of this paper is to establish the precise number of cyclic quotients of $\pi$, for

$$\pi := \mathbb{Z}/p^n \oplus \mathbb{Z}/p^n \oplus \cdots \oplus \mathbb{Z}/p^n, \ n \geq 1, \ r > 1$$

The results of the cases $n = 1$ and 2 have been completed and appears in [1]. The organization of the paper is as follows:

Section 2, which is the main body of the work, is devoted to a proof of the following generalized result.

**Theorem M:**

Let

$$\pi := \mathbb{Z}/p^j \oplus \mathbb{Z}/p^j \oplus \cdots \oplus \mathbb{Z}/p^j, \ r > 1, \ j \in \{1, 2, ..., n\}, \ p$$

a prime number and $\gamma$ is a subgroup of $\pi$. Then the number of the cyclic factor groups $\pi/\gamma$ up to isomorphism, such that $|\pi/\gamma| = p^j$ for all $j$ summed to $n$, is $(\frac{p^r-1}{p^j-1})(\frac{p^{n(r-1)}-1}{p^{n-1}})$.

Section 3 is devoted to the conclusion and a proof of the following useful result:

**Lemma E:**

Let

$$\pi := \mathbb{Z}/p^n \oplus \mathbb{Z}/p^n \oplus \cdots \oplus \mathbb{Z}/p^n, \ r > 1, \ n \ a positive integer, \ p$$

a prime number and $\gamma$ is a subgroup of $\pi$. Then the number of the cyclic factor groups $\pi/\gamma$ up to isomorphism, such that $|\pi/\gamma| = p^n$, is $p^{(n-1)(r-1)}(\frac{p^r-1}{p^{n-1}})$. 

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2 MAIN BODY

In this paper, we need the following fundamental definition.

Definition: (Fundamental)

Let

\[ \pi := \mathbb{Z}/p^i \oplus \mathbb{Z}/p^i \oplus \cdots \oplus \mathbb{Z}/p^i, \quad i \geq 1, \quad r > 1, \quad p \]

a prime number and \( \gamma \) a subgroup of \( \pi \) of order \( p^{ir-i} \), then we define a subgroup base for \( \gamma \) as \( (r-i) \), \( r \)-tuples generating \( \gamma \). This can be represented as \( (r-i) \)-rows of an \( r \times r \)-matrix whose rows generate \( \pi \).

In this section, we first establish the following:

Lemma E:

Let

\[ \pi := \mathbb{Z}/p^n \oplus \mathbb{Z}/p^n \oplus \cdots \oplus \mathbb{Z}/p^n, \quad r > 1, \quad n \text{ a positive integer, } p \]

a prime number and \( \gamma \) is a subgroup of \( \pi \). Then the number of the cyclic factor groups \( \pi/\gamma \) up to isomorphism, such that \( |\pi/\gamma| = p^n \), is \( p^{(n-1)(r-1)} \frac{p^r-1}{p-1} \).

Proof:

Let

\[ \pi := \mathbb{Z}/p^n \oplus \mathbb{Z}/p^n \oplus \cdots \oplus \mathbb{Z}/p^n, \quad r > 1, \quad n \text{ a positive integer and } p \]

a prime number.

Then the required cyclic quotients are realized in \( n \) number of cases as follows:

Case 1:

We define

\[ \mathbb{Z}/p^n \cong \mathbb{Z}_{p^n}^* := < a >, \]

\( \epsilon \in \{ a^l \}, \quad 0 \leq l \leq p^n - 1 \)

and applying the fundamental definition given above, we obtain the following set of subgroup
base representations in \( r \times r \)-matrices:

\[
A = \begin{pmatrix}
   a b^n & 1 & 1 & \ldots & 1 & 1 & 1 \\
   1 & a & 1 & \ldots & 1 & 1 & 1 \\
   1 & 1 & a & \ldots & 1 & 1 & 1 \\
   \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
   1 & 1 & 1 & \ldots & a & 1 & 1 \\
   1 & 1 & 1 & \ldots & 1 & a & 1 \\
   1 & 1 & 1 & \ldots & 1 & 1 & a \\
\end{pmatrix}
\]

Thus applying a counting rule on set \( A \) yields a total sum of cyclic quotients \( \pi/\gamma \) for which

\[|\pi/\gamma| = p^n\]

as:

\[1 + p^n + (p^n)^2 + \ldots + (p^n)^{r-3} + (p^n)^{r-2} + (p^n)^{r-1}\]

That is,

\[\frac{p^{nr} - 1}{p^n - 1},\]

for any prime \( p \) and any integer \( r > 1 \).

Next, consider

**Case 2:**

In this case, we define

\[Z/p^n \cong \langle Z^*_{p^n-1}, Z^*_p \rangle := \langle a \rangle,\]

\[\epsilon_\alpha \in \{a^i\}, \quad 1 \leq i \leq p^{n-1}, \quad g \in d(i, p^{n-1}) = 1,\]

\[\epsilon_\beta \in \{a^i\}, \quad 1 \leq i \leq p, \quad g \in d(i, p) = 1,\]

\[\epsilon_\gamma \in \{a^k\}, \quad 0 \leq k \leq p^{n-1} - 1,\]

\[\epsilon_\kappa \in \{a^l\}, \quad 0 \leq l \leq p - 1\]

and applying our fundamental definition together with a counting rule we form the following sets
of subgroup base representations in $r \times r$ matrices with their respective results:

$$B_1 = \begin{pmatrix}
  a^{p^{n-1}} & \epsilon_\beta & 1 & \ldots & 1 & 1 & 1 \\
  1 & a^{p^{n-1}} & 1 & \ldots & 1 & 1 & 1 \\
  1 & 1 & a & \ldots & 1 & 1 & 1 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  1 & 1 & 1 & \ldots & a & 1 & 1 \\
  1 & 1 & 1 & \ldots & 1 & a & 1 \\
  1 & 1 & 1 & \ldots & 1 & 1 & a
\end{pmatrix}, \quad
B_2 = \begin{pmatrix}
  a^{p^{n-1}} & 1 & 1 & \ldots & 1 & 1 & \epsilon_\alpha \\
  1 & a & 1 & \ldots & 1 & 1 & \epsilon_\gamma \\
  1 & 1 & a & \ldots & 1 & 1 & \epsilon_\gamma \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  1 & 1 & 1 & \ldots & a & 1 & \epsilon_\gamma \\
  1 & 1 & 1 & \ldots & 1 & a & \epsilon_\gamma \\
  1 & 1 & 1 & \ldots & 1 & 1 & a
\end{pmatrix},
$$

This generates a total sum of cyclic quotients:

$$(p - 1) + p(p - 1) + \cdots + p^{r-2}(p - 1),$$

$$B_2 = \begin{pmatrix}
  a^p & 1 & 1 & \ldots & 1 & 1 & \epsilon_\alpha \\
  1 & a & 1 & \ldots & 1 & 1 & \epsilon_\gamma \\
  1 & 1 & a & \ldots & 1 & 1 & \epsilon_\gamma \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  1 & 1 & 1 & \ldots & a & 1 & \epsilon_\gamma \\
  1 & 1 & 1 & \ldots & 1 & a & \epsilon_\gamma \\
  1 & 1 & 1 & \ldots & 1 & 1 & a
\end{pmatrix},
$$

This generates a total sum of cyclic quotients:

$$(p^{n-1})^{r-2}(p^{n-1} - p^{-2}) + (p^{n-1})^{r-3}(p^{n-1} - p^{-2}) + \cdots + (p^{n-1} - p^{-2}).$$
Continuing with this rule we finally consider the following set of subgroup bases:

\[
B_t = \left\{ \begin{pmatrix} a & 1 & \epsilon_\kappa & \cdots & \epsilon_\gamma & 1 & 1 \\ 1 & a & \epsilon_\kappa & \cdots & \epsilon_\gamma & 1 & 1 \\ 1 & 1 & a^p & \cdots & \epsilon_\alpha & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & a^{p^{n-1}} & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & a & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & a \end{pmatrix} \right\}
\]

and obtain a sum of number of cyclic quotients as:

\[
(p^{n-1})^{r-3}(p^{n-1} - p^{n-2})p^2 + (p^{n-1})(p^{n-1} - p^{n-2})p + \cdots + p(p - 1)(p^{n-1})
\]

where

\[
|B_1| + |B_2| + \cdots + |B_t| = r(r - 1)
\]

Continuing in this way with the other cases, we next consider, the following last case.

Case \(n - 1\):

In this case, we define

\[
\mathbb{Z}/p^n \cong \left\{ \mathbb{Z}_{p^{r-2}}, \mathbb{Z}_{p^n}, \cdots, \mathbb{Z}_{p^r} \right\} := \langle a \rangle,
\]

\(\text{for } (r-1)-\text{terms}\)

\(\epsilon_\alpha \in \{a^i\}, \quad 1 \leq i \leq p^{n-r+2}, \quad g \mid d(i, p^{n-r+2}) = 1,\)

\(\epsilon_\beta \in \{a^i\}, \quad 1 \leq i \leq p, \quad g \mid d(i, p) = 1,\)

\(\epsilon_\gamma \in \{a^k\}, \quad 0 \leq k \leq p^{n-r+2} - 1,\)

\(\epsilon_\kappa \in \{a^l\}, \quad 0 \leq l \leq p - 1,\)

and similarly, applying our fundamental definition together with counting rule we form the following sets of subgroup base representations in \(r \times r\) matrices with their respective results:
\[ D_1 = \begin{pmatrix}
\epsilon_k & \epsilon_k & \ldots & \epsilon_k & \epsilon_\beta & 1 \\
1 & a^p & \epsilon_k & \ldots & \epsilon_k & \epsilon_\beta \\
1 & 1 & a^p & \ldots & \epsilon_k & \epsilon_\beta \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & a^p & \epsilon_\beta \\
1 & 1 & 1 & \ldots & 1 & a^p \\
1 & 1 & 1 & \ldots & 1 & 1 & a^p
\end{pmatrix}, \]

and obtain a sum of number of cyclic quotients for the first set above in this case as:

\[ (p-1)^{r-2} p^{r-3} \ldots p^2 p + (p-1)^{r-2} p^{r-2} p^{r-3} \ldots p^2 + \ldots + (p-1)^{r-2} p^{r-2} p^{r-3} \ldots p \]

And next the above set in this case, is:

\[ D_2 = \begin{pmatrix}
a & \epsilon_k & \ldots & \epsilon_k & \epsilon_\gamma & \epsilon_k \\
1 & a^p & \epsilon_k & \ldots & \epsilon_k & \epsilon_\gamma \\
1 & 1 & a^p & \ldots & \epsilon_k & \epsilon_\gamma \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & 1 & \ldots & a^p & \epsilon_\gamma \\
1 & 1 & 1 & \ldots & 1 & a^p^{n-r+2} \\
1 & 1 & 1 & \ldots & 1 & 1 & a^p^{n-r+2}
\end{pmatrix} \]

and we obtain a sum of number of cyclic quotients for the above set in this case as:

\[ (p^{n-r+2} - p^{n-r+1})^{r-2} p^{n-r+2} p^{r-2} p^{r-3} \ldots p^2 p + \]
And finally, for the proof of Lemma 6.5, where

and obtain a sum of number of cyclic quotients for this set as:

\[
D_v = \begin{pmatrix}
\begin{pmatrix} a^p & \epsilon_\kappa & 1 & \ldots & \epsilon_\gamma & \epsilon_\kappa & \epsilon_\beta \\ 1 & a^p & 1 & \ldots & \epsilon_\gamma & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & a & \ldots & \epsilon_\gamma & \epsilon_\kappa & \epsilon_\kappa \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & a^{p^{n-r+2}} & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \ldots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \ldots & 1 & 1 & a^p
\end{pmatrix} \\
\begin{pmatrix} a^p & 1 & \epsilon_\gamma & \ldots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & a & \epsilon_\gamma & \ldots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & a^{p^n} & \ldots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & a^{p^{n-r+2}} & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \ldots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \ldots & 1 & 1 & a^p
\end{pmatrix}
\end{pmatrix}
\]

and obtain a sum of number of cyclic quotients for this set as:

\[
(p - 1)^{r-2} p^{r-2} (p^{n-r+2})^{r-3} \ldots p + (p - 1)^{r-2} p^{r-2} p^{n-r+2} (p^{n-r+2})^2
\]

\[
+ \ldots + (p - 1)^{r-2} p^{r-2} p^{n-r+2},
\]

where

\[
|D_1| + |D_2| + \ldots + |D_v| = r(r - 1)
\]

And finally, for the proof of Lemma 6 to be complete, we consider the next case:

Case n :

In this case, we define

\[
Z/p^n \cong \left\{ Z_{p^{n-r+1}}^{*}, Z_p^{*}, \ldots, Z_p^{*} \right\} := \langle a^i \rangle, \quad 1 \leq i \leq p^{n-r+1}, \quad g c d(i, p^{n-r+1}) = 1,
\]

\[
\epsilon_\beta \in \{ a^i \}, \quad 1 \leq i \leq p, \quad g c d(i, p) = 1,
\]

\[
\epsilon_\gamma \in \{ a^k \}, \quad 0 \leq k \leq p^{n-r+1} - 1,
\]

\[
\epsilon_\kappa \in \{ a^l \}, \quad 0 \leq l \leq p - 1,
\]

and similarly, applying our fundamental definition together with the counting rule we form the
Finally, we give the proof of theorem \( M \).

Following set of subgroup base representations in \( r \times r \) matrices with their respective results:

\[
\mathcal{F} = \left\{ \begin{pmatrix} a^{n-r+1} & \epsilon_\gamma & \ldots & \epsilon_\gamma & \epsilon_\delta \\ 1 & a^p & \epsilon_\gamma & \ldots & \epsilon_\gamma & \epsilon_\delta \\ 1 & 1 & a^p & \ldots & \epsilon_\gamma & \epsilon_\delta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & a^p & \epsilon_\delta \\ 1 & 1 & 1 & \ldots & 1 & a^p \end{pmatrix}, \right\}
\]

\[
= \left\{ \begin{pmatrix} a^p & \epsilon_\gamma & \ldots & \epsilon_\gamma & \epsilon_\delta \\ 1 & a^p & \epsilon_\gamma & \ldots & \epsilon_\gamma & \epsilon_\delta \\ 1 & 1 & a^{n-r+1} & \ldots & \epsilon_\gamma & \epsilon_\delta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & a^p & \epsilon_\delta \\ 1 & 1 & 1 & \ldots & 1 & a^p \end{pmatrix}, \right\}
\]

\[
= \left\{ \begin{pmatrix} a^p & \epsilon_\gamma & \ldots & \epsilon_\gamma & \epsilon_\delta \\ 1 & a^p & \epsilon_\gamma & \ldots & \epsilon_\gamma & \epsilon_\delta \\ 1 & 1 & a^p & \ldots & \epsilon_\gamma & \epsilon_\delta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & a^p & \epsilon_\delta \\ 1 & 1 & 1 & \ldots & 1 & a^p \end{pmatrix}, \right\}
\]

and we obtain a sum of number of cyclic quotients for the above set in the last case as:

\[
(p - 1)^{r-1}p^{-2}p^{-3}\ldots p^{2}p + (p - 1)^{r-1}p^{-2}p^{-3}\ldots p^{2}p^{n-r+1} + \cdots + (p - 1)^{r-1}p^{-2}p^{-3} \cdots (p^{n-r+1})^{2}p + \cdots + (p - 1)^{r-1}p^{-2}p^{-3} \cdots p^{2}p +
\]

\[
(p - 1)^{r-1}(p^{n-r+1})^{r-2}p^{-3}\ldots p^{2}p + (p^{n-r+1})^{r-1}p^{-2}p^{-3}\ldots p^{2}p
\]

where \( |\mathcal{F}| = r \).

Therefore total sums of results obtained in Cases 1, 2, ..., to the last yields the formula:

\[
p^{(n-1)(r-1)}\left(\frac{p^r - 1}{p - 1}\right) \quad \square
\]

Finally, we give the proof of theorem \( M \).
Theorem M:

Summing from \( j = 1 \) to \( n \), the number of cyclic quotients up to isomorphism of

\[
\mathbb{Z}/p^j \oplus \mathbb{Z}/p^j \oplus \cdots \oplus \mathbb{Z}/p^j, \quad r > 1, \quad p
\]

a prime is then \( \left( \frac{p^r - 1}{p-1} \right) \left( \frac{p^r (r-1) - 1}{p^{r-1} - 1} \right) \).

Proof:

This follows from [1] and Lemma E. \( \square \)

3 CONCLUSION

This paper solves a very special case of a well-motivated general problem.

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References
