ENERGY IDENTITY FOR THE MAPS FROM A SURFACE WITH TENSION FIELD BOUNDED IN $L^p$

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Abstract

Let $M$ be a closed Riemannian surface and $u_n$ a sequence of maps from $M$ to Riemannian manifold $N$ satisfying

$$\sup_n (\|\nabla u_n\|_{L^2(M)} + \|\tau(u_n)\|_{L^p(M)}) \leq \Lambda$$

for some $p > 1$, where $\tau(u_n)$ is the tension field of the mapping $u_n$.

For the general target manifold $N$, if $p \geq \frac{6}{5}$, we prove the energy identity and neckless during blowing up.

If the target manifold $N$ is a standard sphere $S^{m-1}$, we get the same results for any $p > 1$.  

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1 Introduction

Let \((M, g)\) be a closed Riemannian manifold and \((N, h)\) be a Riemannian manifold without boundary. For a mapping \(u\) from \(M\) to \(N\) in \(W^{1,2}(M, N)\), the energy density of \(u\) is defined by

\[
e(u) = \frac{1}{2}|du|^2 = \text{Trace}_g u^* h
\]

where \(u^* h\) is the pull-back of the metric tensor \(h\).

The energy of the mapping \(u\) is defined as

\[
E(u) = \int_M e(u) dV
\]

where \(dV\) is the volume element of \((M, g)\).

A map \(u \in C^1(M, N)\) is called harmonic if it is a critical point of the energy \(E\).

By Nash embedding theorem we know that \((N, h)\) can be isometrically into an Euclidean space \(R^K\) with some positive integer \(K\). Then \((N, h)\) may be considered as a submanifold of \(R^K\) with the metric induced from the Euclidean metric. Thus a map \(u \in C^1(M, N)\) can be considered as a map of \(C^1(M, R^K)\) whose image lies on \(N\). In this sense we can get the following Euler-Lagrange equation

\[
\Delta u = A(u)(du, du).
\]

The tension field \(\tau(u)\) is defined by

\[
\tau(u) = \Delta_M u - A(u)(du, du)
\]

where \(A(u)(du, du)\) is the second fundamental form of \(N\) in \(R^K\). So \(u\) is harmonic means that \(\tau(u) = 0\).

The harmonic mappings are of special interest when \(M\) is a Riemann surface. Consider a sequence of mappings \(u_n\) from Riemann surface \(M\) to \(N\) with bounded energies. It is clear that \(u_n\) converges weakly to \(u\) in \(W^{1,2}(M, N)\) for some \(u \in W^{1,2}(M, N)\). But in general, it may not converge strongly in \(W^{1,2}(M, N)\). When \(\tau(u_n) = 0\), i.e. \(u_n\) are all harmonic, Parker in [9] proved that the lost energy is exactly the sum of some harmonic spheres which is defined as a harmonic mapping from \(S^2\) to \(N\). This result is called energy identity. Also he proved that the images of these harmonic spheres and \(u(M)\) are connected, i.e. there is no neck during blowing up.

When \(\tau(u_n)\) is bounded in \(L^2\), the energy identity is proved in [10] for the sphere and [2] for the general target manifold. In [11] they proved there is no neck during blowing up. For the heat flow of harmonic mappings, the results can also be found in [15, 16]. When the target manifold is a sphere, there are some good observations in [8] for general tension field.

In this paper we prove the energy identity and neckless during blowing up for a sequence of maps \(u_n\) with \(\tau(u_n)\) bounded in \(L^p\) for some \(p \geq \frac{6}{5}\). Especially, when the target manifold is a sphere, we obtain the same results for any \(p > 1\).
When $\tau(u_n)$ is bounded in $L^p$ for some $p > 1$, the small energy regularity proved in [2] implies that $u_n$ converges strongly in $W^{1,2}(M,N)$ outside a finite set of points. For simplicity in exposition, it is no matter to assume that $M$ is the unit disk $D_1 = D(0,1)$ and there is only one singular point at 0.

In this paper we prove the following two theorems.

**Theorem 1** Let $\{u_n\}$ be a sequence of mappings from $D_1$ to $N$ in $W^{1,2}(D_1,N)$ with tension field $\tau(u_n)$. If

(a) $\|u_n\|_{W^{1,2}(D_1)} + \|\tau(u_n)\|_{L^p(D_1)} \leq \Lambda$ for some $p \geq 6$;
(b) $u_n \rightharpoonup u$ strongly in $W^{1,2}(D_1 \setminus \{0\}, R^K)$ as $n \to \infty$.

Then there exist a subsequence of $\{u_n\}$ (we still denote it by $\{u_n\}$) and some nonnegative integer $k$. For any $i = 1, \ldots, k$, there exist points $x^i_n$, positive numbers $r_n^i$ and a nonconstant harmonic sphere $w^i$ (which we view as a map from $R^2 \cup \{\infty\} \to N$) such that

(1) $x^i_n \to 0, r_n^i \to 0$ as $n \to \infty$;
(2) $\lim_{n \to \infty} (\frac{|x^i_n|}{r_n^i} + \frac{|x^j_n - x^i_n|}{r_n^i + r_n^j}) = \infty$ for any $i \neq j$;
(3) $w^i$ is the weak limit or strong limit of $u_n(x^i_n + r_n^i x)$ in $W^{1,2}_{\text{Loc}}(R^2, N)$;
(4) **Energy identity:**
\[
\lim_{n \to \infty} E(u_n, D_1) = E(u, D_1) + \sum_{i=1}^k E(w^i); \tag{1}
\]
(5) **Neckless:** The image $u(D_1) \cup \bigcup_{i=1}^k w^i(R^2)$ is a connected set.

**Theorem 2** If the target manifold $N$ is a standard sphere $S^{m-1}$, Theorem 1 holds for any $p > 1$.

This paper is organized as follows. In section 2 we state some basic lemmas and some standard arguments in the blow-up analysis.

In section 3 and section 4 we prove Theorem 1. In the proof, we used delicate analysis based on the ideas in [2], [7] and [11]. Energy identity is proved in section 3 and neckless is proved in section 4.

In section 5 and section 6 we introduced new ideas in proving Theorem 2. Similarly, energy identity is proved in section 5 and neckless is proved in section 6.

Throughout this paper, without illustration the letter $C$ denotes a positive constant which depends only on $p, \Lambda$ and the target manifold $N$ and may vary in different cases. Furthermore, we do not always distinguish the sequence and its subsequence.

## 2 Some basic lemmas and standard arguments

The Lorentz space $L^{2,1}$ is defined as
\[
\{f : \|f\|_{L^{2,1}} = \int_0^\infty f^*(t) t^{-\frac{1}{2}} dt < \infty\}
\]
where $f^*(t) = \inf\{s > 0 : |\{x : |f(x)| > s\}| \leq t\}$ is the non-increasing rearrangement function of $f$. The following property is fundamental on $R^2$. 

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Lemma 3 For any $\delta > 0$, we have
$$\|f\|_{L^2(D_\delta)} \leq C\|f\|_{L^{2,1}(D_\delta)};$$
$$\|f\|_{L^{2,1}(D_\delta)} \leq C\delta^{\frac{2-q}{2}}\|f\|_{L^q(R^2)}, \quad q > 2.$$ 

Proof: We only prove the last inequality. By the definition of rearrangement function, when $q > 2$ there holds
$$\|f\|_{L^{2,1}(D_\delta)} = \int_{|D_\delta|}^\infty (f \chi_{D_\delta})^*(t) t^{-\frac{1}{2}} dt 
\leq \int_0^{|D_\delta|} f^*(t) t^{-\frac{1}{2}} dt 
\leq \left( \int_0^\infty (f^*)^q(t) dt \right)^{\frac{1}{q}} \left( \int_0^{\frac{\pi}{2}} t^{-\frac{q}{2(q-1)}} dt \right)^{\frac{q-1}{q}} 
\leq Cq\delta^{1-\frac{2}{q}}\|f\|_q.$$ 

So we complete the proof of this lemma.

Lemma 4 ([5] P142 Theorem 3.3.10) For each $m \geq 2$, the space $W^{1,1}(R^m)$ is continuously embedded in $L^{\frac{m}{m-1}} L^{1,1}(R^m)$.

As a corollary, when $m = 2$ we have

Lemma 5 The Riesz potential is bounded from the Hardy space $H^1(R^2)$ to $L^{2,1}(R^2)$, i.e.
$$\|R_i * f\|_{L^{2,1}(R^2)} \leq C\|f\|_{H^1(R^2)}$$

where $R_i(x) = \frac{x}{|x|^2}$, $i = 1, 2$.

Remark: The proof is contained in [5] P141-P142 (The proof of Theorem 3.3.8). For convenience we illustrate the proof here.

Let $\Phi$ be the Newtonian potential of $f$, then Hélein in [5] P142 showed that $\Phi \in W^{2,1}(R^2)$ and
$$\|d\Phi\|_{L^{2,1}(R^2)} \leq C\|f\|_{H^1(R^2)}.$$ 

As the Riesz potential of $f$ is the partial derivative of $\Phi$, i.e. $R_i * f = \partial_i (N * f) = \partial_i \Phi$, we obtain the desired result.

Lemma 6 ([5], P137 Theorem 3.3.4) If $\nabla f \in L^{2,1}(R^2)$, then $f$ is continuous and
$$\|f - f(0)\|_{C^0(D_1)} \leq C\|\nabla f\|_{L^{2,1}(D_1)}.$$ 

We also need the following result on the Hardy space $H^1(R^2)$.

Lemma 7 ([1]) If $f, g \in W^{1,2}(R^2)$, then $\nabla f \nabla^\perp g = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g$ belongs to the Hardy space $H^1(R^2)$ and furthermore
$$\|\nabla f \nabla^\perp g\|_{H^1} \leq C\|\nabla f\|_2 \|\nabla g\|_2.$$
Now we recall the regular theory for the mapping with small energy on the unit disk and the tension field in $L^p$ ($p > 1$).

**Lemma 8** Let $\bar{u}$ be the mean value of $u$ on the disk $D_{1/2}$. There exists a positive constant $\epsilon_N$ that depends only on the target manifold such that if $E(u, D_1) \leq \epsilon_N^2$ then

$$\|u - \bar{u}\|_{W^{2,p}(D_{1/2})} \leq C(\|
abla u\|_{L^2(D_1)} + \|\tau(u)\|_p)$$

(3)

where $p > 1$.

As a direct consequence of (3) and the Sobolev embedding $W^{2,p}(\mathbb{R}^2) \subset C^0(\mathbb{R}^2)$, we have

$$\|u\|_{\text{Osc}(D_{1/2})} = \sup_{x,y \in D_{1/2}} |u(x) - u(y)| \leq C(\|\nabla u\|_{L^2(D_1)} + \|\tau(u)\|_p).$$

(4)

The lemma has been proved in [2].

**Remark 1.** In [2] they proved this lemma for the mean value of $u$ on the unit disk. Note that

$$\left| \int_{D_{1/2}} u(x) \, dx \right|_{D_1} - \left| \int_{D_{1/2}} u(x) \, dx \right|_{D_{1/2}} \leq C\|\nabla u\|_{L^2(D_1)}.$$

So we can use the mean value of $u$ on $D_{1/2}$ in this lemma.

**Remark 2.** Suppose we have a sequence of mappings $u_n$ from the unit disk $D_1$ to $N$ with $\|u_n\|_{W^{1,2}(D_1)} + \|\tau(u_n)\|_{L^p(D_1)} \leq \Lambda$ for some $p > 1$.

A point $x \in D_1$ is called an energy concentration point (blow-up point) if for any $r, D(x, r) \subset D_1$,

$$\sup_n E(u_n, D(x, r)) > \epsilon_N^2$$

where $\epsilon_N$ is given in this lemma.

If $x \in D_1$ isn’t an energy concentration point, then we can find a positive number $\delta$ such that

$$E(u_n, D(x, \delta)) \leq \epsilon_N^2, \forall n.$$

Then it follows from Lemma 8 that we have a uniformly $W^{2,p}(D(x, \delta/2))$-bound for $u_n$. Because $W^{2,p}$ is compactly embedded into $W^{1,2}$, there is a subsequence of $u_n$ (denoted by $u_n$) and $u \in W^{2,p}(D(x, \delta/2))$ such that

$$\lim_{n \to \infty} u_n = u \text{ in } W^{1,2}(D(x, \delta/2)).$$

So $u_n$ converges to $u$ strongly in $W^{1,2}(D_1)$ outside a finite set of points.

Under the assumptions in the theorems, by the standard blow-up argument, i.e. rescaling $u_n$ suitable and repeated, we can obtain some nonnegative integer $k$. For any $i = 1, \ldots, k$, there exist a point $x_i^n$, a positive number $r_i^n$ and a nonconstant harmonic sphere $w_i$ satisfying (1), (2) and (3) of Theorem 1. By the standard induction argument in [2] we only need to prove the theorems in the case that there is only one bubble.
In this case we may assume that $w$ is the strong limit of the sequence $u_n(x_n + r_n x)$ in $W^{1,2}_{\text{Loc}}(R^2)$.

It does nothing to assume that $x_n = 0$. Set $w_n(x) = u_n(r_n x)$.

As
\[
\lim_{\delta \to 0} \lim_{n \to \infty} E(u_n, D_1 \setminus D_{\delta}) = E(u, D_1),
\]
the energy identity is equivalent to
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \lim_{R \to \infty} E(u_n, D_\delta \setminus D_{rn} R) = 0. \tag{5}
\]

To prove the set $u(D_1)$ and $w(R^2 \cup \infty)$ is connected, it is enough to show that
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \lim_{R \to \infty} \sup_{x,y \in D_\delta \setminus D_{rn} R} |u_n(x) - u_n(y)| = 0. \tag{6}
\]

### 3 Energy identity for the general target manifold

In this section, we prove the energy identity for the general target manifold when $p \geq \frac{6}{5}$.

Assume that there is only one bubble $w$ which is the strong limit of $u_n(r_n \cdot)$ in $W^{1,2}_{\text{Loc}}(R^2)$. Let $\epsilon_N$ be the constant in Lemma 8. Furthermore, by the standard argument of blow-up analysis we can assume that for any $n$,
\[
E(u_n, D_{rn}) = \sup_{r \leq r_n, D(x, r) \subseteq D_1} E(u_n, D(x, r)) = \frac{\epsilon_N^2}{4}. \tag{7}
\]

By the argument in [2], we can show

**Lemma 9** ([2]) If $\tau(u_n)$ is bounded in $L^p$ for some $p > 1$, then the tangential energy on the neck domain equals to zero, i.e.
\[
\lim_{\delta \to 0} \lim_{R \to \infty} \lim_{n \to \infty} \int_{D_\delta \setminus D_{rn} R} |x|^{-2} |\partial_\theta u|^2 dx = 0. \tag{8}
\]

**Proof:** The proof is the same as that in [2], we sketch it.

For any $\epsilon > 0$, take $\delta, R$ such that for any $n$,
\[
E(u, D_{4\delta}) + E(w, R^2 \setminus D_R) + \delta \frac{2^{(p-1)} - 1}{p} < \epsilon^2.
\]

It is no matter to suppose that $r_n R = 2^{-j_n} \delta = 2^{-j_0}$. When $n$ is big enough, for any $j_0 \leq j \leq j_n$, there holds (see [2])
\[
E(u_n, D_{2^{j-1}} \setminus D_{2^{-j}}) < \epsilon^2.
\]

For any $j$, set $h_n(2^{-j}) = \frac{1}{2\pi} \int_{S^1} u_n(2^{-j}, \theta) d\theta$ and
\[
h_n(t) = h_n(2^{-j}) + (h_n(2^{1-j}) - h_n(2^{-j})) \frac{\ln(2^j t)}{\ln 2}, t \in [2^{-j}, 2^{1-j}].
\]

It is easy to check that
\[
\frac{d^2 h_n(t)}{dt^2} + \frac{1}{t} \frac{dh_n(t)}{dt} = 0, t \in [2^{-j}, 2^{1-j}].
\]
Consider $h_n(x) = h_n(|x|)$ as a map from $R^2$ to $R^K$, then $\triangle h_n = 0$ in $R^2$. Set $P_j = D_{2^{j-1}} \setminus D_{2^{-j}}$, we have
\[
\triangle(u_n - h_n) = \triangle u_n - \triangle h_n = \triangle u_n = A(u_n) + \tau(u_n), x \in P_j.
\] (9)

Taking the inner product of this equation with $u_n - h_n$ and integrating over $P_j$, we get that
\[
\int_{P_j} |\nabla(u_n - h_n)|^2 dx = -\int_{P_j} (u_n - h_n)(A(u_n) + \tau(u_n)) dx + \int_{\partial P_j} (u_n - h_n)(u_n - h_n)_r ds.
\]
Note that by the definition, $h_n(2^{-j})$ is the mean value of $\{2^{-j}\} \times S^1$ and $(h_n)_r$ is independent of $\theta$. So the integral of $(u_n - h_n)(h_n)_r$ on $\partial P_j$ vanishes.

When $j_0 < j < j_n$, by Lemma 8 we have
\[
\|u_n - h_n\|_{C^0(P_j)} \leq \|u_n - h_n(2^{-j})\|_{C^0(P_j)} + \|u_n - h_n(2^{1-j})\|_{C^0(P_j)} \\
\leq 2\|u_n\|_{Osc(P_j)} \\
\leq C(\|\nabla u_n\|_{L^2(P_{j-1} \cup P_{j} \cup P_{j+1})} + 2^{2(1-p)/p})\|\tau(u_n)\|_{p}) \\
\leq C(\epsilon + 2^{-2(\rho-1)/p}) \\
\leq C(\epsilon + \delta^{2(\rho-1)/p}) \leq C\epsilon.
\]

Summing $j$ for $j_0 < j < j_n$, we have
\[
\int_{D_{2^j} \setminus D_{2^{j_n}}} |\nabla(u_n - h_n)|^2 dx = \sum_{j_0 < j < j_n} \int_{P_j} |\nabla(u_n - h_n)|^2 dx \\
\leq \sum_{j_0 < j < j_n} \int_{P_j} |u_n - h_n|(\|A(u_n)\| + \|\tau(u_n)\|) dx \\
+ \sum_{j_0 < j < j_n} \int_{\partial P_j} (u_n - h_n)(u_n - h_n)_r ds \\
\leq C\epsilon(\int_{D_{2^j} \setminus D_{2^{j_n}}} (\|\nabla u_n\|^2 + \|\tau(u_n)\|) dx + \int_{\partial D_{2^j} \setminus \partial D_{2^{j_n}}} |\nabla u_n| ds) \\
\leq C\epsilon(\int_{D_{2^j} \setminus D_{2^{j_n}}} |\nabla u_n|^2 dx + \delta^{2(\rho-1)/p} + \epsilon) \\
\leq C\epsilon.
\] (10)

Here we use the inequality $\int_{\partial D_{2^j} \setminus \partial D_{2^{j_n}}} |\nabla u_n| ds \leq C\epsilon$, which can be derived from the Sobolev trace embedding theorem.

As $h_n(x)$ is independent of $\theta$, it can be shown that
\[
\int_{D_{2^j} \setminus D_{2^{j_n}}} |x|^{-2}|\partial_\theta u_n|^2 dx \leq \int_{D_{2^j} \setminus D_{2^{j_n}}} |\nabla(u_n - h_n)|^2 dx \leq C\epsilon.
\]
So this lemma is proved.

It is left to show that the normal energy on the neck domain also equals to zero. We need the following Pohozaev equality which was first proved by Lin-Wang [7].
Lemma 10 (Pohozaev equality, [7], lemma 2.4, P374)

Let $u$ be a solution to

$$\triangle u + A(u)(du, du) = \tau(u),$$

then there holds

$$\int_{\partial D_t} (|\partial_r u|^2 - r^{-2}|\partial_y u|^2)ds = \frac{2}{t} \int_{D_t} \tau \cdot (x \nabla u)dx. \quad (11)$$

As a direct corollary, integrating it over $[0, \delta]$ we have

$$\int_{D_\delta} (|\partial_r u|^2 - r^{-2}|\partial_y u|^2)dx = \int_0^\delta \frac{2}{t} \int_{D_t} \tau \cdot (x \nabla u)dxdt. \quad (12)$$

Proof: Multiplying both sides of the equation by $x \nabla u$ and integrating it over $D_t$, we get

$$\int_{D_t} |\nabla u|^2 dx - t \int_{\partial D_t} |\partial_r u|^2 ds + \frac{1}{2} \int_{D_t} x \nabla |\nabla u|^2 dx = - \int_{D_t} \tau \cdot (x \nabla u)dx.$$

Note that

$$\frac{1}{2} \int_{D_t} x \nabla |\nabla u|^2 dx = - \int_{D_t} |\nabla u|^2 dx + \frac{t}{2} \int_{\partial D_t} |\nabla u|^2 ds.$$

Hence,

$$\int_{\partial D_t} (|\partial_r u|^2 - \frac{1}{2} |\nabla u|^2)ds = \frac{1}{t} \int_{D_t} \tau \cdot (x \nabla u)dx.$$

As $|\nabla u|^2 = |\partial_r u|^2 + r^{-2}|\partial_y u|^2$, we proved this lemma.

Now we use this equality to estimate the normal energy on the neck domain. We prove the following lemma.

Lemma 11 If $\tau(u_n)$ is bounded in $L^p$ for some $p \geq \frac{6}{5}$, then for $\delta$ small enough, there holds

$$|\int_{D_\delta} (|\partial_r u_n|^2 - |x|^{-2}|\partial_y u_n|^2)dx| \leq C\delta^{-\frac{4p-1}{p}}$$

where $C$ depends on $p$, $\Lambda$, the target manifold $N$ and the bubble $w$.

Proof: Take $\psi \in C_0^\infty(D_1)$ satisfying that $\psi = 1$ in $D_1$, then

$$\triangle(\psi u_n) = \psi A(u_n)(du_n, du_n) + \psi \tau_n + 2\nabla \psi \nabla u_n + u_n \triangle \psi.$$

Set $g_n = \psi A(u_n)(du_n, du_n) + \psi \tau_n + 2\nabla \psi \nabla u_n + u_n \triangle \psi$. When $|x| < 1$,

$$\partial_i u_n(x) = R_i \ast g_n(x) = \int \frac{x_i - y_i}{|x - y|^2} g_n(y)dy.$$

Let $\Phi_n$ be the Newtonian potential of $\psi \tau_n$, then $\triangle \Phi_n = \psi \tau_n$. The corresponding Pohozaev equality is

$$\int_{D_\delta} (|\partial_r \Phi_n|^2 - r^{-2}|\partial_y \Phi_n|^2)dx = \int_0^\delta \frac{2}{t} \int_{D_t} \psi \tau_n \cdot (x \nabla \Phi_n)dxdt. \quad (13)$$

Here $\partial_i \Phi_n(x) = R_i \ast (\psi \tau_n)(x) = \int \frac{x_i - y_i}{|x - y|^p} (\psi \tau_n)(y)dy$.

As $\tau_n$ is bounded in $L^p$ ($p > 1$), there holds

$$\int_{D_\delta} |\nabla \Phi_n|^2 dx \leq C\delta^{\frac{4p-1}{p}} \|\nabla \Phi_n\|^2_{2p} \leq C\delta^{\frac{4p-1}{p}} \|\tau_n\|^2 \leq C\delta^{\frac{4p-1}{p}}.$$
By (13), it can be shown that for any $\delta > 0$,

$$|\int_0^\delta \frac{1}{t} \int_{D_t} \psi \tau_n \cdot (x \nabla \Phi_n) dx dt| \leq \int_{D_\delta} |\nabla \Phi_n|^2 dx \leq C\delta^{\frac{d(p-1)}{p}}. \quad (14)$$

For $\delta$ small enough, we have

$$|\int_{D_\delta} (|\partial_r u_n|^2 - r^{p-2} |\partial_\theta u_n|^2) dx| = |\int_0^\delta \frac{2}{t} \int_{D_t} \tau_n \cdot (x \nabla u_n) dx dt|$$

\[
\leq 2 |\int_0^\delta \frac{1}{t} \int_{D_t} \tau_n \cdot (x \nabla \Phi_n) dx dt| + 2 \int_0^\delta \frac{1}{t} \int_{D_t} |x \tau_n| |\nabla(u_n - \Phi_n)(x)| dx dt \\
\leq C\delta^{\frac{d(p-1)}{p}} + 2 \int_{D_\delta} |x \tau_n| |\nabla(u_n - \Phi_n)(x)||x| |\frac{1}{x}| dx. \quad (15)
\]

For any $j > 0$, set $\varphi_j(x) = \psi(\frac{x}{2^{j-1}r}) - \psi(\frac{x}{2^{j-1}r})$. When $2^{-j}\delta \leq |x| < 2^{1-j}\delta$, we have

$$|\partial_t(u_n - \Phi_n)(x)| = |\int \frac{x - y}{|x - y|^2} (g_n(y) - \psi \tau_n(y)) dy|$$

\[
\leq \int \frac{|\psi A(u_n)(d_n, d_n) + 2\nabla \psi \nabla u_n + u_n \triangle \psi|(y)}{|x - y|} dy \\
\leq \int \frac{|\psi A(u_n)(y)|}{|x - y|} dy + C \int_{1 < |y| < 2} (|\nabla u_n| + |u_n|)(y) dy \\
\leq \int \frac{||\varphi_j A(u_n)(y)||}{|x - y|} dy + \int \frac{||\psi - \varphi_j A(u_n)(y)||}{|x - y|} dy + C \\
\leq \int \frac{||\varphi_j A(u_n)(y)||}{|x - y|} dy + \int \frac{||A(u_n)(y)||}{|x|} dy + C \\
\leq \int \frac{||\varphi_j A(u_n)(y)||}{|x - y|} dy + \frac{C}{|x|}. \quad (16)
\]

When $\delta > 0$ is small enough and $n$ is big enough, for any $j > 0$ we claim that

$$\|\varphi_j A(u_n)\|_{L^p} \leq C(2^{-j}\delta)^{-\frac{d(p-1)}{p}} \quad (17)$$

where the constant $C$ depends only on $p$, $\Lambda$, the bubble $w$ and the target manifold $N$.

Take $\delta > 0$ and $R(w)$ which depends on $w$ such that

$$E(u, D_{\delta_\delta}) \leq \frac{\epsilon^2}{8}; \quad E(w, R^2 \setminus D_{R(w)}) \leq \frac{\epsilon^2}{8}.$$  

The standard blow-up analysis (see [2]) show that for any $j$ with $8r_n R(w) \leq 2^{-j}\delta$ and $n$ big enough, there holds

$$E(u_n, D_{2^{j-\delta}} \setminus D_{2^{-3-\delta}}) \leq \frac{\epsilon^2}{3}. \quad (18)$$

By (7), when $2^{-j}\delta < \frac{\epsilon^2}{8}$, there holds

$$E(u_n, D_{2^{j-\delta}} \setminus D_{2^{-3-\delta}}) \leq \frac{\epsilon^2}{4}. \quad (19)$$
So when \(2^{-j}\delta < \frac{r_n}{10}\) or \(2^{-j}\delta \geq 8r_nR(w)\), by Lemma 8, we have

\[
\|\varphi_j A(u_n)\|_{2^p} \leq C\|\nabla u_n\|_{L^{2(p)}}^2 (D_{2^j-2\delta}) \leq C\|u_n - \bar{u}_{n,j}\|_{W^{2,p}}^2 (D_{2^j-2\delta}) \\
\leq C\left(\sum_{i=1}^{\infty} \|\nabla u_n\|_{L^{2(p)}}^2 (B_i) \right)^{\frac{2}{2p}} \\
\leq C\sum_{i=1}^{m} \|\nabla u_n\|_{L^{2(p)}}^2 (B_i) \\
\leq C\sum_{i=1}^{m} \|u_n - \bar{u}_{n,i}\|_{W^{2,p}(B_i)}^2 \\
\leq C\sum_{i=1}^{m} ((r_n)^{-\frac{4(p-1)}{p}} \|\nabla u_n\|^2_{L^2(B_i)} + \|\tau(u_n)\|^2_p) \\
\leq Cm((2^{-j}δ)^{\frac{4(p-1)}{p}} + 1) \\
\leq C(2^{-j}δ)^{\frac{4(p-1)}{p}}
\]

where \(\bar{u}_{n,i}\) is the mean of \(u_n\) on \(D_{2^j-2\delta}\).

On the other hand, when \(\frac{r_n}{10} \leq 2^{-j}δ \leq 8r_nR(w)\), we can find no more than \(CR(w)^2\) balls with radius \(\frac{r_n}{2}\) to cover \(D_{2^j-2\delta}\), i.e.

\[
D_{2^j-2\delta} \subset \bigcup_{i=1}^{m} D(y_i, \frac{r_n}{2}).
\]

Denote \(B_i = D(y_i, \frac{r_n}{2})\) and \(2B_i = D(y_i, r_n)\). By (7), for any \(i\) with \(i \leq m\) there holds

\[
E(u_n, 2B_i) \leq \frac{cN}{4}.
\]

Using Lemma 8 we have

\[
\|\varphi_j A(u_n)\|_{2^p} \leq C\|\nabla u_n\|_{L^{2(p)}}^2 (D_{2^j-2\delta}) \leq C\left(\sum_{i=1}^{m} \|\nabla u_n\|_{L^{2(p)}}^2 (B_i) \right)^{\frac{2}{2p}} \\
\leq C\sum_{i=1}^{m} \|\nabla u_n\|_{L^{2(p)}}^2 (B_i) \\
\leq C\sum_{i=1}^{m} \|u_n - \bar{u}_{n,i}\|_{W^{2,p}(B_i)}^2 \\
\leq C\sum_{i=1}^{m} ((r_n)^{-\frac{4(p-1)}{p}} \|\nabla u_n\|^2_{L^2(B_i)} + \|\tau(u_n)\|^2_p) \\
\leq Cm((2^{-j}δ)^{\frac{4(p-1)}{p}} + 1) \\
\leq C(2^{-j}δ)^{\frac{4(p-1)}{p}}
\]

where \(\bar{u}_{n,i}\) is the mean of \(u_n\) over \(B_i\) and the constant \(C\) depends only on \(p, \Lambda\), the bubble \(w\) and the target manifold \(N\). So we proved the claim (17).

By (16) and (17), when \(p > 1\) we get that

\[
\int_{D_\delta} |\tau_n| |\nabla (u_n - \Phi_n)(x)||x| \ln \frac{1}{|x|} dx \\
\leq \sum_{j=1}^{\infty} \int_{2^{-j}\delta < |x| < 2^{j+1}\delta} |\tau_n| |\nabla (u_n - \Phi_n)(x)||x| \ln \frac{1}{|x|} dx \\
\leq C\sum_{j=1}^{\infty} \int_{2^{-j}\delta < |x| < 2^{j+1}\delta} |\tau_n| \left(\frac{1}{|x|} + \int |\varphi_j A(u_n)(y)| \frac{1}{|x - y|} dy\right) |x| \ln \frac{1}{|x|} dx
\]
\[ \begin{align*}
&\leq C \left( \int_{D_{\delta}} |\tau_n| |\nabla(u_n - \Phi_n)| ||ln \frac{1}{|x|}||dx + \sum_{j=1}^{\infty} \int_{2^{-j}\delta < |x| < 2^{1-j}\delta} |\tau_n| \left( \int \frac{|\varphi_j A(u_n)(y)|}{|x-y|} dy \right) |ln \frac{1}{|x|}| dx \right) \\
&\leq C \left( \frac{1}{\delta} \left( \int_{D_{\delta}} \frac{1}{|x|} \right)^{\frac{1}{\delta-1}} + \sum_{j=1}^{\infty} 2^{-j} \delta \left( \frac{2j}{\delta} \right) ||\varphi_j A(u_n)||_{L^p} \right) \\
&\leq C(\delta + \sum_{j=1}^{\infty} 2^{-j} \delta \ln \frac{2j}{\delta} (2^{-j}\delta)^{-\frac{2-p}{p}}) \\
&\leq C(\delta + \delta^{\frac{2(a-1)}{p}} \ln \frac{1}{\delta}) \\
&\leq C\delta^{\frac{1}{p}}. \quad (20)
\end{align*} \]

It is clear that (15) and (20) imply that
\[ |\int_{D_{\delta}} (|\partial_x u_n|^2 - r^{-2} |\partial_y u_n|^2) dx| \leq C\delta^{\frac{1}{p}}. \quad (21) \]

Now we use these lemmas to prove the energy identity. Note that \( w \) is harmonic, from Lemma 10 we see that \( \int_{D_R} (|\partial_x w|^2 - r^{-2} |\partial_y w|^2) dx = 0 \) for any \( R > 0 \). It is easy to see that
\[ \lim_{R \to \infty} \lim_{n \to \infty} \int_{D_{\delta \cap D_{rn}}} (|\partial_x u_n|^2 - r^{-2} |\partial_y u_n|^2) dx = \lim_{R \to \infty} \int_{D_R} (|\partial_x w|^2 - r^{-2} |\partial_y w|^2) dx = 0. \]

Letting \( \delta \to 0 \) in (21), we obtain
\[ \lim_{\delta \to 0} \lim_{R \to \infty} \lim_{n \to \infty} \int_{D_{\delta \cap D_{rn}}} (|\partial_x u_n|^2 - r^{-2} |\partial_y u_n|^2) dx \]
\[ \leq \lim_{\delta \to 0} \lim_{n \to \infty} \int_{D_{\delta}} (|\partial_x u_n|^2 - r^{-2} |\partial_y u_n|^2) dx + \lim_{R \to \infty} \lim_{n \to \infty} \int_{D_{rn}} (|\partial_x u_n|^2 - r^{-2} |\partial_y u_n|^2) dx \]
\[ = 0. \]

Using Lemma 9 we obtain that the normal energy also vanishes on the neck domain. So the energy identity is proved.
4 Neckless for the general target manifold

In this section we use the method in [11] to prove the neckless during blowing up.

For any $\epsilon > 0$, take $\delta, R$ such that

$$ E(u, D_{4\delta}) + E(w, R^2 \setminus D_R) + \delta^{2(p-1)} r < \epsilon^2. $$

Suppose $r_nR = 2^{-j_0}\delta = 2^{-j_0}$. When $n$ is big enough, the standard blow-up analysis show that for any $j_0 \leq j \leq j_n$,

$$ E(u_n, D_{2^{2-j}} \setminus D_{2^{-j}}) < \epsilon^2. $$

For any $j_0 < j < j_n$, set $L_j = \min \{ j - j_0, j_n - j \}$. Now we estimate the norm $\| \nabla u_n \|_{L^2(P)}$.

Denote $P_{j,t} = D_{2^{-t}} \setminus D_{2^{-t-j}}$ and take $h_{n,j,t}$ similar to $h_n$ in the last section but $h_{n,j,t}(2^{t-j}) = \frac{1}{2\pi} \int_{S^1} u_n(2^{t-j}, \theta) d\theta$. By an argument similar to the one used in deriving (10), we have, for any $0 < t < L_j$,

$$ \int_{P_{j,t}} r^{-2} |\partial_\theta u_n|^2 dx \leq C\epsilon \left( \int_{P_{j,t}} |\nabla u_n|^2 dx + (2^{-j})^{2(p-1)} \right) + \int_{\partial P_{j,t}} |u_n - h_{n,j,t}| |\nabla u_n| ds. \hspace{1cm} (22) $$

Set $f_j(t) = \int_{P_{j,t}} |\nabla u_n|^2 dx$, a simple computation shows that

$$ f_j'(t) = \ln 2(2^{-j}) \int_{[2^{-j}]} |\nabla u_n|^2 ds + 2^{-j} \int_{\{2^{-j}\times S^1\}} |\nabla u_n|^2 ds. $$

As $h_{n,j,t}$ is independent of $\theta$ and $h_{n,j,t}$ is the mean value of $u_n$ at the two components of $\partial P_{j,t}$, by Poincaré inequality we get

$$ \int_{\partial P_{j,t}} |u_n - h_{n,j,t}| |\nabla u_n| ds = \int_{\{2^{-j}\times S^1\}} |u_n - h_{n,j,t}| |\nabla u_n| ds 
+ \int_{\{2^{-j}\times S^1\}} |u_n - h_{n,j,t}| |\nabla u_n| ds
\leq \left( \int_{\{2^{-j}\times S^1\}} |u_n - h_{n,j,t}|^2 ds \right)^{\frac{1}{2}} \left( \int_{\{2^{-j}\times S^1\}} |\nabla u_n|^2 ds \right)^{\frac{1}{2}} 
+ \left( \int_{\{2^{-j}\times S^1\}} |u_n - h_{n,j,t}|^2 ds \right)^{\frac{1}{2}} \left( \int_{\{2^{-j}\times S^1\}} |\nabla u_n|^2 ds \right)^{\frac{1}{2}} 
\leq C(2^{-j}) \int_{\{2^{-j}\times S^1\}} |\nabla u_n|^2 ds + 2^{-j} \int_{\{2^{-j}\times S^1\}} |\nabla u_n|^2 ds 
\leq C f_j'(t). $$

On the other hand, by a similar argument as we did in obtaining (21), we have

$$ |\int_{P_{j,t}} (|\partial_\nu u_n|^2 - r^{-2} |\partial_\theta u_n|^2) dx | \leq C \left( 2^{-j} \frac{p-1}{p} + (2^{-j-1} \frac{p-1}{p} \right) \leq C(2^{-j}) \frac{p-1}{p}. \hspace{1cm} (23) $$

Since $|\nabla u|^2 = |\partial_\nu u|^2 + r^{-2} |\partial_\theta u|^2 = 2 r^{-2} |\partial_\theta u|^2 + (|\partial_\nu u|^2 - r^{-2} |\partial_\theta u|^2)$, by (22) and (23) we have

$$ f_j(t) \leq 2 \int_{P_{j,t}} r^{-2} |\partial_\theta u_n|^2 dx + \int_{P_{j,t}} (|\partial_\nu u_n|^2 - r^{-2} |\partial_\theta u_n|^2) dx 
\leq C\epsilon f_j(t) + (2^{-j})^{\frac{2(p-1)}{p}} + C f_j'(t) + C(2^{-j}) \frac{p-1}{p} 
\leq C(\epsilon f_j(t) + 2 \frac{f_j(t)}{p} f_j'(t)). $$
Take $\epsilon$ small enough and denote $\epsilon_p = \frac{e^2}{p} \ln 2$, then for some positive constant $C$ big enough there holds

$$f_j'(t) - \frac{1}{C} f_j(t) + C e^{-\epsilon_p t} e^{\epsilon_p t} \geq 0.$$  

It is no matter to assume that $\epsilon_p > \frac{1}{C}$, then we have

$$(e^{-\frac{t}{C}} f_j(t))' + C e^{-\epsilon_p t} e^{(\epsilon_p - \frac{1}{C}) t} \geq 0.$$  

Integrating this inequality over $[2, L_j]$, we get

$$f_j(2) \leq C (e^{-\frac{L_j}{C}} f_j(L_j) + e^{-\epsilon_p} \int_1^{L_j} e^{(\epsilon_p - \frac{1}{C}) t} dt) \leq C (e^{-\frac{L_j}{C}} f_j(L_j) + e^{-\epsilon_p} e^{(\epsilon_p - \frac{1}{C}) L_j}).$$  

Note that $j \geq L_j$, there holds

$$f_j(2) \leq C (e^{-\frac{L_j}{C}} f_j(L_j) + e^{-\epsilon_p}).$$  

Since the energy identity has been proved in the last section, we can take $\delta$ small such that the energy on the neck domain is less than $\epsilon^2$ which implies that $f_j(L_j) < \epsilon^2$. So we get

$$f_j(2) \leq C (e^{-\frac{L_j}{C}} \epsilon^2 + e^{-\epsilon_p}).$$  

Using Lemma 8 on the domain $P_j = D_{2rt_j} \setminus D_{2r_j}$ when $j < j_n$, we obtain

$$\|u_n\|_{Osc(P_j)} \leq C (\|\nabla u_n\|_{L^2(P_{2r_j} \cup P_j \cup P_{r_{j+1}})} + 2 \frac{r_{j+1} - r_j}{p} \|\tau(u_n)\|_p) \leq C (f_j(2) + e^{-2\epsilon_p}).$$  

Summing $j$ from $j_0$ to $j_n$ we obtain that

$$\|u_n\|_{Osc(D_{2rn} \setminus D_{2r_n R})} \leq \sum_{j=j_0}^{j_n} \|u_n\|_{Osc(P_j)} \leq C \sum_{j=j_0}^{j_n} (f_j(2) + e^{-2\epsilon_p}) \leq C \sum_{j=j_0}^{j_n} (e^{-\frac{L_j}{C}} \epsilon^2 + e^{-\epsilon_p} + e^{-2\epsilon_p}) \leq C (\sum_{i=0}^{\infty} e^{-\frac{i}{C}} \epsilon^2 + \sum_{j=j_0}^{\infty} e^{-\frac{1}{C}}) \leq C (\epsilon^2 + e^{-\epsilon_p}) \leq C (\epsilon^2 + \delta^\frac{1}{p}).$$  

Here we use the assumption that $\epsilon_p > \frac{1}{C}$. So we proved that there is no neck during the blowing up.
5 Energy identity for the sphere

The notations are the same as before. At first we give the estimate of $L^{2,\infty}$ norm of $\nabla u_i$ on the neck domain.

**Lemma 12** Suppose that $u$ is a map from the unit disk $D_1$ to $N$ in $W^{1,2}(D_1,N)$ with $\tau(u) \in L^p$ for some $p > 1$. Assume that $\epsilon > 0$ is small enough, if $r, \delta$ satisfies

$$\int_{D_{2r}\setminus D_1} |\nabla u|^2 < \epsilon^2$$

for any $r < t < 4\delta$. Then we have

$$\|\nabla u\|_{L^{2,\infty}(D_\delta \setminus D_{4\tau})} = \sup_{a > 0} \{ \int_{D_\delta \setminus D_{4\tau}} \|\nabla u(x)\| > a \} \leq C(\epsilon + \delta^{\frac{2(p-1)}{p}}) \tag{24}$$

where $C$ only depends on $\|\nabla u\|_2$, $\|\tau(u)\|_p$ and the target manifold.

**Proof:** Assume $\delta = 2^N r$. Then for any $1 \leq i \leq N + 2$ there holds

$$E(u, D_{2^i r} \setminus D_{2^{i-1} r}) < \epsilon.$$

Take $\psi \in C^\infty_0(D_2)$ satisfying that $\psi = 1$ in $D_1$. Set $\theta_i(x) = \psi(\frac{x}{2^{i-1} r}) - \psi(\frac{x}{2^{i+1} r})$ and $\overline{\theta_i} = \int_{D_{2^{i+3} r} \setminus D_{2^{i+2} r}} \frac{u(x) dx}{|D_{2^{i+3} r} \setminus D_{2^{i+2} r}|}$, then

$$\Delta(\theta_i(u - \overline{\theta_i})) = \theta_i \Delta u + 2 \nabla \theta_i \nabla u + (u - \overline{\theta_i}) \Delta \theta_i$$

$$= \theta_i A(u) + \theta_i \tau(u) + 2 \nabla \theta_i \nabla u + (u - \overline{\theta_i}) \Delta \theta_i.$$

For $x \in D_{2^{i+1} r} \setminus D_{2^i r}$, there holds

$$|\nabla u(x)| = |\nabla (\theta_i(u - \overline{\theta_i}))(x)|$$

$$= |R * (\theta_i A(u) + \theta_i \tau(u) + 2 \nabla \theta_i \nabla u + (u - \overline{\theta_i}) \Delta \theta_i)(x)|$$

$$\leq |R * (\theta_i A(u))(x)| + |R * (\theta_i \tau(u))(x)| + |R * (2 \nabla \theta_i \nabla u + (u - \overline{\theta_i}) \Delta \theta_i)(x)|$$

$$= I_1(x) + I_2(x) + I_3(x).$$

By Sobolev inequality it can be shown that when $x \in D_{2^{i+1} r} \setminus D_{2^i r}$ with $i \geq 2$,

$$I_3(x) \leq \int |R(x - y)(2 \nabla \theta_i \nabla u + (u - \overline{\theta_i}) \Delta \theta_i)(y)| dy$$

$$\leq C \int_{D_{2^{i+2} r} \cup D_{2^{i-1} r} \setminus D_{2^{i-2} r}} \frac{1}{|x - y|} (2^{-i} r^{-1} |\nabla u| + 2^{-2i} r^{-2} |u - \overline{\theta_i}|)(y) dy$$

$$\leq C \int_{D_{2^{i+3} r} \setminus D_{2^{i+2} r}} (2^{i} r^{-1})(2^{i} r^{-1}) |\nabla u| + (2^{i} r^{-2}) |u - \overline{\theta_i}|)(y) dy$$

$$\leq C(2^{i} r)^{-2} \int_{D_{2^{i+3} r} \setminus D_{2^{i+2} r}} |\nabla u|(y) dy$$

$$\leq C|x|^{-1} \sqrt{E(u, D_{2^{i+3} r} \setminus D_{2^{i+2} r})}$$

$$\leq C \epsilon$$

(25)
As Riesz potential is bounded from $L^1(R^2)$ to $L^{2,\infty}(R^2)$, we get that for any $a > 0$,
\[
|\{ I_1(x) + I_2(x) > a \}| \leq C a^{-2} \| I_1 + I_2 \|_{L^{2,\infty}(R^2)}^2 \\
\leq C a^{-2} \| \theta_A(\|u\|) + |\tau (u)| \|_1^2 \\
\leq C a^{-2} \left( \int \theta_A(\|\nabla u\|^2 + |\tau (u)|)(dx) \right)^2 \\
\leq C a^{-2} \left( \int \theta_A(\|\nabla u(x)\|^2)(dx) \right)^2 + \left( \int \theta_A(\|\tau (u)\|(dx) \right)^2 \\
\leq C a^{-2} \left( \int \theta_A(\|\nabla u\|^2 + (2^r)^{\frac{4(p-1)}{r}}) \right).
\]

So there holds
\[
|\{ x \in D_i \setminus D_{4r} : |\nabla u(x)| > 2a \}| \\
= \sum_{i=2}^{N-1} |\{ x \in D_{2i+1} \setminus D_{2i} : |\nabla u(x)| > 2a \}| \\
\leq \sum_{i=2}^{N-1} |\{ x \in D_{2i+1} \setminus D_{2i} : I_1(x) + I_2(x) > a \}| + \sum_{i=2}^{N-1} |\{ x \in D_{2i+1} \setminus D_{2i} : I_3(x) > a \}| \\
\leq \sum_{i=2}^{N-1} |\{ I_1(x) + I_2(x) > a \}| + \{ x \in D_1 : C \theta \geq a \}| \\
\leq C a^{-2} \sum_{i=2}^{N-1} \left( \int \theta_A(\|\nabla u\|^2)(dx) \right)^2 + \left( \int \theta_A(\|\tau (u)\|(dx) \right)^2 \\
\leq C a^{-2} \left( \int \theta_A(\|\nabla u\|^2(x)dx + (2^r)^{\frac{4(p-1)}{r}} + C a^2 \right)
\]
which implies (24).

Now we estimate the norm $\|\nabla u_n\|_{L^{2,1}(D_1)}$. Take $\psi \in C_0^\infty(D_2)$ and $\psi = 1$ in $D_1$, then there holds
\[
\Delta (\psi u_n) = \psi \Delta u_n + 2 \nabla u_n \nabla \psi + u_n \Delta \psi.
\]

Using the fact $|u_n| = 1$ we can rewrite the equation as
\[
\Delta u_n = -u_n \sum_{j=1}^{m} |\nabla u_n|^2 + \tau^j(u_n) \\
= \sum_{j=1}^{m} (u_n^j \nabla u_n^j - u_n^j \nabla u_n^j) \nabla u_n^j + \tau^j(u_n).
\]

Set $(F_n)_j = \psi(u_n^j \nabla u_n^j - u_n^j \nabla u_n^j)$ and $\chi(x) = \psi(\frac{x}{\delta})$, then we have
\[
\Delta (\psi u_n) = F_n \nabla u_n + \psi \tau (u_n) + 2 \nabla u_n \nabla \psi + u_n \Delta \psi = F_n \nabla (\chi u_n) + \psi \tau (u_n) + 2 \nabla u_n \nabla \psi + u_n \Delta \psi.
\]

Some direct computations show that
\[
|\text{div} (F_n)_j | = |\psi(u_n^j \Delta u_n^j - u_n^j \Delta u_n^j) + \nabla \psi(u_n^j \nabla u_n^j - u_n^j \nabla u_n^j)| \\
= |\psi(u_n^j \tau^j(u_n) - u_n^j \tau^j(u_n)) + \nabla \psi(u_n^j \nabla u_n^j - u_n^j \nabla u_n^j)| \\
\leq C (|\tau(u_n)| + |\nabla u_n|).
\]

(26)
Let $G_n$ be the Newtonian potential of $\nabla(div F_n)$. As $div F_n \in C_0^\infty(D_2)$, we have
\[ G_n(x) = N * \nabla(div F_n) = \nabla N * div F_n, \]

In $R^2$ there holds
\[ div(F_n - G_n) = div F_n - div(\nabla N * div F_n) = div F_n - div \nabla(N * div F_n) = 0. \]

So there exists $H_n$ such that
\[ F_n - G_n = \nabla^\perp H_n = (-\partial_2 H_n, \partial_1 H_n) \]

The operator $\nabla N$ is the Riesz potential which is bounded from $L^p(R^2)$ to $L^{2-p}(R^2)$, so by (26) when $1 < p < 2$, it can be shown that
\[ \|G_n\|_{2-p} = \|\nabla N * div F_n\|_{2-p} \leq C\|div F_n\|_p \leq C(||\tau(u_n)||_{L^p(D_2)} + \|\nabla u_n\|_{L^2(D_2)}). \]

Set $\phi(x) = \psi(\frac{x}{2})$. It is no matter to assume that $\int_{D_4} H_n = 0$, then we have
\[ \|\nabla(\phi H_n)\|_2 \leq C\|\nabla H_n\|_{L^2(D_4)} \leq C(||F_n||_{L^2(D_4)} + ||G_n||_{L^2(D_4)}) \leq C(\|\nabla u_n\|_2 + \|G_n\|_{\frac{2p}{2-p}}) \leq C(||\nabla u_n\|_2 + ||\tau(u_n)||_p) \leq C. \]

Set $f_n = \psi \tau(u_n) + 2\nabla u_n \nabla \psi + u_n \Delta \psi + G_n \nabla(\chi u_n)$, now we can obtain that
\[ \Delta(\psi u_n) = F_n \nabla(\chi u_n) - G_n \nabla(\chi u_n) + f_n = \nabla^\perp H_n \nabla(\chi u_n) + f_n \]
\[ = \nabla^\perp(\phi H_n) \nabla(\chi u_n) + f_n. \]

By (28) and Lemma 7 we get that
\[ \|\nabla^\perp(\phi H_n) \nabla(\chi u_n)\|_{H^1} \leq C\|\nabla(\phi H_n)\|_2 \|\nabla(\chi u_n)\|_2 \leq C. \]

On the other hand, there holds
\[ \|f_n\|_p = \|\psi \tau(u_n) + 2\nabla u_n \nabla \psi + u_n \Delta \psi + G_n \nabla(\chi u_n)\|_p \leq C(||\tau(u_n)||_{L^p(D_2)} + \|\nabla u_n\|_{L^p(D_2)} + 1 + ||G_n \nabla(\chi u_n)||_p) \leq C(1 + ||G_n\|_{\frac{2p}{2-p}} \|\nabla(\chi u_n)\|_2) \leq C. \]
Now from (29), (30) and (31) it follows that
\[
\|\nabla u_n\|_{L^{2,1}(D_4)} \leq \|\nabla (\psi u_n)\|_{L^{2,1}} \leq \|\nabla^2 (\phi H_n) \nabla (\chi u_n)\|_{H^1} + \|f_n\| \leq C. \tag{32}
\]

For any \(\epsilon > 0\), take \(\delta\) small such that
\[
E(u, D_{d\delta}) + \delta^{\frac{4(n-1)}{p}} < \epsilon^2
\]
where \(u\) is the weak limit of \(u_n\) in \(W^{1,2}_{\text{Loc}}(R^2)\).

The standard blow-up arguments show that (see [2]) when \(n, R\) are big enough there holds
\[
\int_{D_{2\delta}\setminus D_4} |\nabla u_n|^2 < C\epsilon^2
\]
for any \(r_n R < t < 4\delta\). By Lemma 12 we have
\[
\|\nabla u_n\|_{L^{2,\infty}(D_4\setminus D_{4r_n R})} \leq C(\| \nabla u_n \|_{L^{2,1}} |\nabla u_n|_{L^{2,\infty}(D_4\setminus D_{4r_n R})} )^\frac{1}{2} \leq C\sqrt{\epsilon}
\]
which yields the energy identity.

6 Neckless for the sphere

As the energy identity is true for this case, for any \(\epsilon > 0\), there exists \(\delta\) such that
\[
E(u_n, D_{d\delta} \setminus D_{r_n R}) < \epsilon^2
\]
when \(n, R\) are big enough. For simplicity suppose that \(\int_{D_{4r_n R}\setminus D_{2r_n R}} u_n(x) = 0\).

Take \(\psi \in C_0^\infty(D_2)\) satisfying that \(\psi = 1\) in \(D_1\). Set
\[
\psi_r(x) = \psi(\frac{x}{r}); \varphi_n = \psi_\delta - \psi_{2r_n R}.
\]

Using the similar arguments in the last section, it can be shown that
\[
\triangle (\varphi_n u_n^i) = \varphi_n \triangle u_n^i + 2\nabla u_n^i \nabla \varphi_n + u_n^i \triangle \varphi_n
\]
\[
= \varphi_n (-u_n^i \sum_{j=1}^m \|\nabla u_n^j\|^2 + \tau^i(u_n)) + 2\nabla u_n^i \nabla \varphi_n + u_n^i \triangle \varphi_n
\]
\[
= \varphi_n \sum_{j=1}^m (u_n^j \nabla u_n^i - u_n^j \nabla u_n^i) \nabla u_n^j + 2\nabla u_n^i \nabla \varphi_n + u_n^i \triangle \varphi_n + \varphi_n \tau^i(u_n))
\]
\[
= \sum_{j=1}^m (u_n^j \nabla u_n^i - u_n^j \nabla u_n^i) \nabla (\varphi_n u_n^j) - \sum_{j=1}^m (u_n^j \nabla u_n^i - u_n^j \nabla u_n^i) u_n^j \nabla \varphi_n
\]
\[
+ 2\nabla u_n^i \nabla \varphi_n + u_n^i \triangle \varphi_n + \varphi_n \tau^i(u_n)
\]

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\[ \psi \]

By the same argument as we derived (27), we have, for \( 1 \leq i \leq m \),
\[
\sum_{j=1}^{m} (u^j_n \nabla u^j_n - u^j_n \nabla u^j_n) \nabla (\varphi_n u^j_n) - \sum_{j=1}^{m} (u^j_n)^2 \nabla u^j_n \nabla \varphi_n
\]
\[ + u^i_n \nabla \varphi_n \nabla \left( \sum_{j=1}^{m} (u^j_n)^2 + 2 \nabla u^i_n \nabla \varphi_n + u^i_n \nabla \varphi_n + \varphi_n \tau^i(u_n) \right) \]
\[ = \sum_{j=1}^{m} (u^j_n \nabla u^j_n - u^j_n \nabla u^j_n) \nabla (\varphi_n u^j_n) - \nabla u^i_n \nabla \varphi_n
\]
\[ + 2 \nabla u^i_n \nabla \varphi_n + u^i_n \nabla \varphi_n + \varphi_n \tau^i(u_n) \]
\[ = \psi_{25} \sum_{j=1}^{m} (u^j_n \nabla u^j_n - u^j_n \nabla u^j_n) \nabla (\varphi_n u^j_n) + \text{div}(u^i_n \nabla \varphi_n) + \varphi_n \tau^i(u_n). \] (33)

In the last equality we use the fact \( \psi_{25} = 1 \) in the support of \( \varphi_n \).

Set \( (F_n)_j = \psi_{25} (u^j_n \nabla u^j_n - u^j_n \nabla u^j_n) \), some simple computations show that
\[
|\text{div}(F_n)_j| = |\psi_{25} (u^j_n \Delta u^j_n - u^j_n \Delta u^j_n) + \nabla \psi_{25} (u^j_n \nabla u^j_n - u^j_n \nabla u^j_n)|
\]
\[ = |\psi_{25} (u^j_n \tau^j(u_n) - u^j_n \tau^j(u_n)) + \nabla \psi_{25} (u^j_n \nabla u^j_n - u^j_n \nabla u^j_n)|
\]
\[ \leq C(|\tau(u_n)| \chi_{D_{4 \delta}} + |\nabla u_n| \delta \chi_{D_{4 \delta} \setminus D_{2 \delta}}). \] (34)

It is easy to check that
\[
|\text{div}F_n|_p \leq C(|\tau(u_n)|)_p + \delta \frac{2(p-1)}{p} \left( \int_{D_{4 \delta} \setminus D_{2 \delta}} |\nabla u_n|^2 dx \right)^{\frac{1}{2}} \leq C(1 + \delta^{\frac{2(p-1)}{p}} \epsilon).
\]

Let \( G_n \) be the Newtonian potential of \( \nabla(\text{div}F_n) \), then
\[
\text{div}(F_n - G_n) = \text{div}F_n - \text{div}N \ast \nabla(\text{div}F_n) = \text{div}F_n - \text{div}\nabla(N \ast \text{div}F_n) = 0.
\]

So there exists \( H_n \) such that
\[
F_n - G_n = \nabla \perp H_n = (-\partial_2 H_n, \partial_1 H_n).
\]

By the same argument as we derived (27), we have, for \( 1 < p < 2 \),
\[
\|G_n\|_{\frac{2p}{2-p}} = \|\nabla N \ast \text{div}F_n\|_{\frac{2p}{2-p}} \leq C\|\text{div}F_n\|_p \leq C(1 + \delta^{\frac{2(p-1)}{p}} \epsilon). \] (35)

As a direct consequence there holds
\[
\|G_n\|_{L^2(D_{4 \delta})} \leq C \delta^{\frac{2(p-1)}{p}} \|G_n\|_{\frac{2p}{2-p}} \leq C \delta^{\frac{2(p-1)}{p}} (1 + \delta^{\frac{2(p-1)}{p}} \epsilon) \leq C(\delta^{\frac{2(p-1)}{p}} + \epsilon). \] (36)

Set
\[
a_n = \frac{\int_{D_{2 \delta} \setminus D_{\tau_n R}} H_n(x) dx}{|D_{2 \delta} \setminus D_{\tau_n R}|},
\]
\[
b_n = \frac{\int_{D_{4 \delta} \setminus D_{2 \delta}} H_n(x) dx}{|D_{4 \delta} \setminus D_{2 \delta}|},
\]
\[
L_n = \psi_{25}(1 - \psi_{\tau_n R})(H_n - a_n) - \psi_{25}(b_n - a_n).
\]

As \( \psi_{25}(1 - \psi_{\tau_n R}) = 1, \nabla \psi_{25} = 0 \) in the support of \( \varphi_n \), there holds
\[
\nabla \perp L_n \nabla (\varphi_n u_n) = -\nabla L_n \nabla (\varphi_n u_n)
\]
\[ = -\nabla (\psi_{25}(1 - \psi_{\tau_n R})(H_n - a_n) - \psi_{25}(b_n - a_n)) \nabla (\varphi_n u_n)
\]
\[ = -((H_n - a_n) \nabla (\psi_{25}(1 - \psi_{\tau_n R})) + (\psi_{25}(1 - \psi_{\tau_n R})) \nabla (H_n - a_n)
\]
\[ - (b_n - a_n) \nabla \psi_{25}) \nabla (\varphi_n u_n)
\]
\[ = -\nabla H_n \nabla (\varphi_n u_n) = \nabla \perp H_n \nabla (\varphi_n u_n).
\]
So from (33) we can get that
\[
\Delta (\varphi_n u_n) = F_n \nabla (\varphi_n u_n) + \text{div}(u_n \nabla \varphi_n) + \varphi_n \tau (u_n)
\]
\[
= (F_n - G_n) \nabla (\varphi_n u_n) + G_n \nabla (\varphi_n u_n) + \text{div}(u_n \nabla \varphi_n) + \varphi_n \tau (u_n)
\]
\[
= \nabla H_n \nabla (\varphi_n u_n) + G_n \nabla (\varphi_n u_n) + \text{div}(u_n \nabla \varphi_n) + \varphi_n \tau (u_n)
\]
\[
= \nabla H_n \nabla (\varphi_n u_n) + G_n \nabla (\varphi_n u_n) + \text{div}(u_n \nabla \varphi_n) + \varphi_n \tau (u_n).
\] (37)

By the definition of $L_n, a_n, b_n, (36)$ and Poincaré inequality, there holds
\[
\|\nabla L_n\|_2 = \|\nabla [\psi_{2\delta}(1 - \psi_{R_n})(H_n - a_n) - \psi_{2\delta}(b_n - a_n)]\|_2
\]
\[
= \|\psi_{2\delta}(1 - \psi_{R_n})\nabla H_n + (H_n - a_n)(\nabla \psi_{2\delta} - \nabla \psi_{R_n}) - (b_n - a_n)\nabla \psi_{2\delta}\|_2
\]
\[
= \|\psi_{2\delta}(1 - \psi_{R_n})\nabla H_n + (H_n - b_n)\nabla \psi_{2\delta} - (H_n - a_n)\nabla \psi_{R_n}\|_2
\]
\[
\leq C(\|\nabla H_n\|_{L^2(D_{4\delta}\setminus D_{R_n})} + \|\nabla H_n\|_{L^2(D_{4\delta})} + \|\nabla H_n\|_{L^2(D_{8r_n} R \setminus D_{R_n})})
\]
\[
\leq C(\|\nabla H_n\|_{L^2(D_{4\delta})})
\]
\[
\leq C(\|F_n\|_{L^2(D_{4\delta})} + G_n)_{L^2(D_{4\delta})})
\]
\[
\leq C(\|\nabla u_n\|_{L^2(D_{4\delta})} + \epsilon + \delta^{\frac{2(p-1)}{p}})
\]
\[
\leq C(\epsilon + \delta^{\frac{2(p-1)}{p}}).
\] (38)

Take $t$ small enough, we estimate the $L^{2,1}$-norm of $\nabla u_n$ in $D_{t6} \setminus D_{8r_n} R$. From (37) we know that
\[
\partial_t (\varphi_n u_n) = R_i * \Delta (\varphi_n u_n)
\]
\[
= R_i * (\nabla H_n \nabla (\varphi_n u_n)) + R_i * (G_n \nabla (\varphi_n u_n) + \varphi_h \tau (u_n)) + R_i * \text{div}(u_n \nabla \varphi_n)
\]
\[
= R_i * (\nabla H_n \nabla (\varphi_n u_n)) + R_i * (G_n \nabla (\varphi_n u_n) + \varphi_n \tau (u_n))
\]
\[
+ R_i * \text{div}(u_n \nabla \psi_{3\delta}) - R_i * \text{div}(u_n \nabla \psi_{2r_n R}).
\] (39)

Here $R_i$ is the Riesz potential.

For the first term, by (38), Lemma 5 and Lemma 7, we have
\[
\|R_i * (\nabla H_n \nabla (\varphi_n u_n))\|_{L^{2,1}}
\]
\[
\leq C\|\nabla H_n \nabla (\varphi_n u_n)||_{H^1}
\]
\[
\leq C\|\nabla H_n\|_2 \|\nabla (\varphi_n u_n)\|_2
\]
\[
\leq C(\epsilon + \delta^{\frac{2(p-1)}{p}}).
\] (40)

Here we use the fact $\|\nabla (\varphi_n u_n)\|_2 \leq C$.

For the second term, as Riesz potential is bounded from $L^p(R^2)$ to $L^{\frac{2p}{p-1}}(R^2)$, by Lemma 3 and (35) we can get that
So it can be shown that
\[ \| R_i \ast (G_n \nabla (\varphi_n u_n) + \varphi_n \tau (u_n)) \|_{L^{2,1}(D_\delta)} \]
\[ \leq C \delta^{2(p-1)/p} \| R_i \ast (G_n \nabla (\varphi_n u_n) + \varphi_n \tau (u_n)) \|_{L^{2p/p}} \]
\[ \leq C \delta^{2(p-1)/p} \| G_n \nabla (\varphi_n u_n) + \varphi_n \tau (u_n) \|_p \]
\[ \leq C \delta^{2(p-1)/p} (\| G_n \|_{\frac{2p}{p}} \| \nabla (\varphi_n u_n) \|_2 + \| \tau (u_n) \|_p) \]
\[ \leq C \delta^{2(p-1)/p} (1 + \delta^{-2(p-1)/p} \epsilon + 1) \]
\[ \leq C (\epsilon + \delta^{2(p-1)/p}). \]  

(41)

For the third term, when \( |x| < \frac{\delta}{2} \), direct computations show that
\[ |R_i \ast \text{div}(u_n \nabla \psi_0)(x)| \]
\[ = \left| \int_{R^2} \frac{x_i - y_i}{|x - y|^2} \text{div}(u_n \nabla \psi_0)(y) \, dy \right| \]
\[ \leq C \int_{D_{2\delta} \setminus D_\delta} \frac{|\text{div}(u_n \nabla \psi_0)(y)|}{|x - y|} \, dy \]
\[ \leq \frac{C}{\delta} \]

which implies that
\[ \| R_i \ast \text{div}(u_n \nabla \psi_0) \|_{L^{2,1}(D_\delta)} \leq C \epsilon. \]  

(42)

Using the assumption that \( \int_{D_{4r_n R} \setminus D_{2r_n R}} u_n(x) = 0 \), by Sobolev inequality, we have
\[ \int |\text{div}(u_n \nabla \psi_{2r_n R})(y)| \, dy \leq C \int_{D_{4r_n R} \setminus D_{2r_n R}} ((r_n R)^{-1} |\nabla u_n| + (r_n R)^{-2} |u_n|) \, dx \]
\[ \leq C \int_{D_{4r_n R} \setminus D_{2r_n R}} |\nabla u_n|^2 \, dx \frac{1}{2} \leq C \epsilon. \]

As \( \int_{R^2} \text{div}(u_n \nabla \psi_{2r_n R}) \, dy = 0 \), when \( |x| > 8r_n R \), the last term can be estimated as
\[ |R_i \ast \text{div}(u_n \nabla \psi_{2r_n R})(x)| \]
\[ = \left| \int_{R^2} \frac{x_i - y_i}{|x - y|^2} \text{div}(u_n \nabla \psi_{2r_n R})(y) \, dy \right| \]
\[ = \left| \int \left( \frac{x_i - y_i}{|x - y|^2} - \frac{x_i}{|x|^2} \right) \text{div}(u_n \nabla \psi_{2r_n R})(y) \, dy \right| \]
\[ \leq \int_{D_{4r_n R} \setminus D_{2r_n R}} \frac{|x_i - y_i|}{|x - y|^2} \left| \frac{x_i}{|x|^2} \right| \left| \text{div}(u_n \nabla \psi_{2r_n R})(y) \right| \, dy \]
\[ \leq \frac{C r_n R}{|x|^2} \int_{D_{4r_n R} \setminus D_{2r_n R}} \left| \text{div}(u_n \nabla \psi_{2r_n R})(y) \right| \, dy \]
\[ \leq \frac{C r_n R}{|x|^2}. \]

So it can be shown that
\[ \| R_i \ast \text{div}(u_n \nabla \psi_{2r_n R}) \|_{L^{2,1}(R^2 \setminus D_{8r_n R})} \leq C r_n R \int_{|x| > 8r_n R} \frac{1}{|x|^2} (|x|^2 - (8r_n R)^2)^{-\frac{1}{2}} \, dx \]
\[ \leq C r_n R \int_{8r_n R}^{\infty} t^{-\frac{3}{2}} (t - 8r_n R)^{-\frac{1}{2}} \, dt \]
\[ \leq C \epsilon. \]  

(43)
From (40), (41), (42) and (43) we can obtain that
\[
\left\| \nabla u_n \right\|_{L^{2,1}(D_t \setminus D_{brn} \cap \mathbb{R})} = \left\| \nabla (\varphi_n u_n) \right\|_{L^{2,1}(D_t \setminus D_{brn} \cap \mathbb{R})} \leq C(\epsilon \delta + \delta^2 \frac{2(p-1)}{p} + t).
\]
(44)

Take \( t, \delta \) small enough, we can show that there is no neck during blowing up.

References


