Small Energy Compactness for Approximation 
Harmonic Mappings

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Abstract

In this paper, we consider an elliptic system

$$\triangle u = -\Omega \cdot \nabla u + f$$

where $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^K)$ and $f \in L\ln^+ L$, and $\Omega$ belongs to $L^2(\mathbb{R}^2, M_K(R) \otimes \mathbb{R}^2)$ which is antisymmetry. In the first part we prove a compactness theorem for the systems.

As a corollary, we can obtain the compactness theorem for a sequence of mappings from a Riemannian surface with tension fields bounded in $L\ln^+ L$.

In the second part we prove the energy identity for a sequence of mappings from a surface to a sphere with tension fields bounded in $L\ln^+ L$. At last we construct a blow-up sequence of mappings from $B_1$ to $S^2$ with tension fields bounded in $L\ln^+ L$ but there exists neck with positive length during blowing up.

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1 Introduction and main results

At first we recall the definitions of some function spaces. Let
\[ f^*(t) = \inf \{ s : |\{ x : |f(x)| > s \}| \leq t \} \]
be the non-increasing rearrangement function of \( f \). The Lorentz space \( L^{2,1} \) is defined by
\[ \{ f : \| f \|_{L^{2,1}} = \int_0^\infty f^*(t)t^{-\frac{1}{2}}dt < \infty \} \]
The space \( L \ln^+ L \) (Zygmund class) is defined by
\[ \{ f : \| f \|_{L \ln^+ L} = \int_0^\infty f^*(t) \ln(2 + \frac{1}{t})dt < \infty \} \]
It is well-known that the \( L \ln^+ L \)-norm of \( f \) is equivalent to
\[ \int |f(x)| \ln(2 + |f(x)|)dx. \]
Let \( B_1 \) be the unit disc in \( R^2 \). We say that \( f \) belongs to the local Hardy space \( H^1(B_1) \) if
\[ (f - \frac{\int_{B_1} f(y)dy}{|B_1|})\chi_{B_1} \in H^1(R^2) \]
where \( H^1(R^2) \) is the usually Hardy space on \( R^2 \), \( |B_1| \) is the volume of \( B_1 \) and
\[ \| f \|_{H^1(B_1)} = \|(f - \frac{\int_{B_1} f(y)dy}{|B_1|})\chi_{B_1} \|_{H^1} + \| f \|_{L^1(B_1)}. \]
It is well-known (see [17]) that \( L \ln^+ L(B_1) \subset H^1(B_1) \), i.e.
\[ \| f \|_{H^1(B_1)} \leq C\| f \|_{L \ln^+ L(B_1)}. \]
Consider the elliptic systems on \( R^2 \),
\[ \triangle u^i = -\Omega^i_j \cdot \nabla u^j + f^i, i, j = 1, ..., K \]
where \( \Omega^i_j \in L^2(R^2, R^2) \) and \( \Omega^i_j = -\Omega^j_i \).
Some important systems can be written in this form. In [13] this elliptic system has been studied and important results and applications have been obtained.

Here we consider the case that \( f \in L \ln^+ L(B_1) \). Our first main result is the following compactness theorem.

**Theorem 1.1** Assume that a sequence of mappings \( u_n \in W^{1,2}(B_1, R^K) \) solves the following systems
\[ \triangle u_n = -\Omega \cdot \nabla u_n + f_n. \]
Suppose that
\[ \| u_n \|_{W^{1,2}(B_1)} + \| f_n \|_{L \ln^+ L(B_1)} \leq \Lambda \]
and \( \{ \Omega_n \} \) is precompact in \( L^1(B_{\frac{r}{2}}) \), i.e. there is no "oscillation".

There exists a positive constant \( \epsilon_K \) which depends only on \( K \) such that if

\[
\int_{B_1} |\Omega_n|^2 \, dx \leq \epsilon_K^2,
\]

then there exists a subsequence of \( u_n \) (still denoted by \( u_n \)) and \( u \) such that

\[
\lim_{n \to \infty} \| u_n - u \|_{W^{1,2}(B_{\frac{r}{2}})} = 0.
\]

As an application we can obtain the following compactness theorem for the mappings from a compact Riemannian surface to a compact Riemannian manifold with tension fields bounded in \( L \ln^+ L \).

**Theorem 1.2** Let \( N \) be a compact Riemannian manifold. Assume that a sequence of mappings \( u_n \in W^{1,2}(B_1, N) \) satisfies

\[
\| u_n \|_{W^{1,2}(B_1)} + \| \tau(u_n) \|_{L \ln^+ L(B_1)} \leq \Lambda.
\]

There exists a positive constant \( \epsilon_N \) which depends only on the target manifold such that if \( E(u_n, B_1) \leq \epsilon_N^2 \) then there exists a subsequence of \( u_n \) (still denoted by \( u_n \)) and \( u \) such that

\[
\lim_{n \to \infty} \| u_n - u \|_{W^{1,2}(B_{\frac{r}{2}})} = 0.
\]

**Remark 1:** If \( \tau(u_n) \) are bounded in \( L^p \) for some \( p > 1 \), this result was proved in [2].

**Remark 2:** By this theorem, if the strong convergence fails to hold, then we can get a nontrivial harmonic sphere by suitable rescaling. So as a direct corollary, if the target manifold doesn’t admit any harmonic sphere, then the strong convergence holds for the mappings with tension fields bounded in \( L \ln^+ L \).

For a sequence of mappings with tension fields bounded in \( L^2 \) or a heat flow from a surface, the energy identity and neckless has been proved in [2, 7, 11, 12, 16]. In [6] we extended these results for mappings with tension fields bounded in \( L^\frac{6}{5} \).

In the case that the target manifold is a sphere, in [8] they have some observations and claims for the general tension fields. In [6] we proved the energy identity and neckless for a sequence of mappings with tension fields bounded in \( L^p \) for some \( p > 1 \). With the help of Theorem 1.2 and some identities in [5] we can extend the energy identity in place of \( L^{\frac{6}{5}} \) by \( L \ln^+ L \). The energy identity is stated as

**Theorem 1.3** Let \( M \) be a compact Riemannian surface. Suppose that \( u_n \) are maps from \( M \) to \( S^{K-1} \) with

\[
\| u_n \|_{W^{1,2}(M)} + \| \tau(u_n) \|_{L \ln^+ L(M)} \leq \Lambda.
\]
Assume that \( u_n \) tends to \( u \) weakly in \( W^{1,2}(M,S^{K-1}) \) but not strongly in \( W^{1,2}(M,S^{K-1}) \). Then there exists a finite set of harmonic spheres \( \psi_j : S^2(= R^2 \cup \{ \infty \}) \rightarrow S^{K-1} \) \( (j = 1, \cdots, m) \) such that the energy identity holds, i.e.

\[
\lim_{n \to \infty} E(u_n) = E(u) + \sum_{j=1}^{m} E(\psi_j).
\]

In [10] Parker constructed a sequence of mappings from \( S^2 \) to \( S^2 \) with tension fields bounded in \( L^1 \) which satisfies the Palais-Smale condition, but the energy identity doesn’t hold.

In the last section we construct a sequence of mappings from \( B_1 \) to \( S^2 \) with tension fields bounded in \( L \ln^+ L \) but there exists neck with positive length during blowing up.

Throughout this paper, without illustration the letter \( C \) denotes a positive constant which depends only on \( \Lambda \) and the target manifold \( N \) and may vary in different cases. Furthermore, we do not always distinguish the sequence and its subsequence.

2 Some lemmas

In this section we prove some lemmas.

Lemma 2.1 ([5] P142 Theorem 3.3.10) For each \( m \geq 2 \), the space \( W^{1,1}(R^m) \) is continuously embedded in \( L^{\frac{m}{m-1}}(R^m) \).

As a corollary, when \( m = 2 \) we have

Lemma 2.2 The Riesz potential is bounded from the Hardy space \( H^1(R^2) \) to \( L^{2,1}(R^2) \), i.e.

\[
\| R_i * f \|_{L^{2,1}(R^2)} \leq C \| f \|_{H^1(R^2)}
\]

where \( R_i(x) = \frac{x_i}{|x|^2}, \ i = 1, 2. \)

Remark: The proof is contained in [5] (P141-P142, the proof of Theorem 3.3.8). For reader’s convenience we illustrate the proof here.

Let \( \Phi \) be the Newtonian potential of \( f \). Hélein ([5] P142) showed that \( \Phi \in W^{2,1}(R^2) \) and

\[
\| d\Phi \|_{L^{2,1}(R^2)} \leq C \| f \|_{H^1(R^2)}.
\]

As the Riesz potential of \( f \) is the partial derivative of \( \Phi \), i.e. \( R_i * f = \partial_i (N * f) = \partial_i \Phi \), we obtain the desired result.

In this paper we will use the following local form of Lemma 2.2.

Lemma 2.3 The Riesz potential is bounded from \( H^1(B_1) \) to \( L^{2,1}(B_1) \), i.e.

\[
\| R_i * f \|_{L^{2,1}(B_1)} \leq C \| f \|_{H^1(B_1)}.
\]
Proof: Let $\overline{f}$ be the mean value of $f$ over $B_1$. We obtain that 

$$
\|R_i \ast f\|_{L^2,1(B_1)} \leq \|R_i \ast [(f - \overline{f}) \chi_{B_1}]\|_{L^2,1(B_1)} + \|R_i \ast (\overline{f} \chi_{B_1})\|_{L^2,1(B_1)} \\
\leq C(\|f - \overline{f}\|_{H^1(B_1)} + \|\overline{f}\|_{H^1(B_1)}) \\
\leq C\|f\|_{H^1(B_1)}.
$$

We also need the following result on the Hardy space $H^1(R^2)$.

Lemma 2.4 ([1]) If $f, g \in W^{1,2}(R^2)$, then $\nabla f \nabla^\perp g = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g$ belongs to the Hardy space $H^1(R^2)$ and furthermore 

$$
\|\nabla f \nabla^\perp g\|_{H^1} \leq C\|\nabla f\|_2\|\nabla g\|_2. 
$$

The following lemma due to Rivi`ere is essential in our argument.

Lemma 2.5 ([13], Theorem 1.4) Let $M_K(R)$ be the space of square $K \times K$ real matrices. There exists $\epsilon_K$ such that for every $\Omega \in L^2(B_1, M_K(R) \otimes R^2)$ satisfying 

$$
\Omega_i^j = -\Omega_j^i; \quad \int_{B_1} |\Omega|^2 dx < \epsilon_K^2,
$$

there exist two matrix functions $A \in L^\infty(B_1, M_K(R) \cap W^{1,2}(B_1, M_K(R))$ and $B \in W^{1,2}(B_1, M_K(R))$ satisfying 

$$
\begin{cases}
\nabla A = A\Omega + \nabla^\perp B; \\
\int_{B_1} (|\nabla A|^2 + |\nabla B|^2) dx + \|\text{dist}(A, SO(K))\|_{L^\infty(B_1)}^2 \leq C_K\|\Omega\|_{L^2(B_1)}^2,
\end{cases}
$$

As an application of Lemma 2.5, we can prove that 

Lemma 2.6 Let $u \in W^{1,2}(B_1, R^K)$ be a solution of the elliptic systems 

$$
\Delta u = -\Omega \cdot \nabla u + f
$$

where $f \in L^\ln^+ L(B_1)$ and $\Omega \in L^2(B_1, M_K(R) \otimes R^2)$ is antisymmetry, that is, $\Omega_i^j = -\Omega_j^i$.

There exists a positive constant $\epsilon_K$ which depends only on $K$ such that if 

$$
\int_{B_1} |\Omega|^2 dx \leq \epsilon_K^2,
$$

we have the following estimate 

$$
\|\nabla^2 u\|_{L^1(B_1)} \leq C(\|\nabla u\|_{L^2(B_1)} + \|f\|_{L^\ln^+ L(B_1)}).
$$

As a direct consequence of the fact that $W^{1,1}(B_1) \subset L^{2,1}(B_1)$, there holds 

$$
\|\nabla u\|_{L^{2,1}(B_1)} \leq C(\|\nabla u\|_{L^2(B_1)} + \|\nabla^2 u\|_{L^1(B_1)}) \leq C(\|\nabla u\|_{L^2(B_1)} + \|f\|_{L^\ln^+ L(B_1)}).
$$
Proof: As \( \int_{B_1} |\Omega|^2 dx \) is small enough, by Lemma 2.5 we see that there exist \( A \in L^\infty(B_1, M_K(R)) \cap W^{1,2}(B_1, M_K(R)) \) and \( B \in W^{1,2}(B_1, M_K(R)) \) satisfying

\[
\begin{cases}
\nabla A = A\Omega + \nabla B; \\
\int_{B_1} (|\nabla A|^2 + |\nabla B|^2) dx + \|\text{dist}(A, SO(K))\|_{L^\infty(B_1)}^2 \leq C_K \|\Omega\|_{L^2(B_1)}^2.
\end{cases}
\]

As \( \|\text{dist}(A, SO(K))\|_{L^\infty(B_1)} \leq C_K \) is small, we can almost consider \( A \) as an identity matrix.

By Hodge decomposition we get that

\[
A\nabla u = \nabla D + \nabla^\perp E
\]

where \( D, E \in W^{1,2}(B_1) \) satisfy

\[
\int_{B_1} D(x) dx = \int_{B_1} E(x) dx = 0; \|D\|_{L^2(B_1)} + \|E\|_{L^2(B_1)} \leq C \|\nabla u\|_{L^2(B_1)}.
\]

Now we have

\[
\triangle D = \text{div}(A\nabla u) = \nabla A\nabla u + A\triangle u = (\nabla A - A\Omega)\nabla u + Af = \nabla^\perp B\nabla u + Af.
\]

Take \( \varphi \in C_0^\infty(B_1) \) and \( \varphi(x) = 1 \) for \( x \in B_1^+ \). We get

\[
\triangle(\varphi D) = \varphi \triangle D + 2\nabla D\nabla \varphi + D\triangle \varphi
\]

\[
= \varphi \nabla^\perp B\nabla u + \varphi Af + 2\nabla D\nabla \varphi + D\triangle \varphi
\]

\[
= \nabla^\perp(\varphi B)\nabla u + h
\]

where

\[
h = -B\nabla^\perp \varphi \nabla u + \varphi Af + 2\nabla D\nabla \varphi + D\triangle \varphi.
\]

It is easy to check that

\[
\|h\|_{L^{\infty}_{ln+}} \leq C(\|B\nabla^\perp \varphi \nabla u\|_2 + \|\varphi Af\|_{L^{\infty}_{ln+}} + \|\nabla D\nabla \varphi + D\triangle \varphi\|_2)
\]

\[
\leq C(\|B\|_{L^p(B_1)} \|\nabla u\|_{L^2(B_1)} + \|f\|_{L^{\infty}_{ln+} L(B_1)} + \|\nabla D\|_{L^2(B_1)} + \|D\|_{L^2(B_1)})
\]

\[
\leq C(\|\nabla B\|_{L^2(B_1)} \|\nabla u\|_{L^2(B_1)} + \|f\|_{L^{\infty}_{ln+} L(B_1)} + \|\nabla D\|_{L^2(B_1)})
\]

\[
\leq C(\|\nabla u\|_{L^2(B_1)} + \|f\|_{L^{\infty}_{ln+} L(B_1)}).
\]

By Theorem 3.2.9 in [5] (or the standard Calderón-Zygmund singular integral theory) and Lemma 2.4 we have

\[
\|\nabla^2 D\|_{L^1(B_1^+)} \leq \|\nabla^2(\varphi D)\|_1 \leq \|\triangle(\varphi D)\|_{H^1(B_1)}
\]

\[
\leq C(\|\nabla^\perp(\varphi B)\nabla u\|_{H^1} + \|h\|_{H^1(B_1)})
\]

\[
\leq C(\|\nabla(\varphi B)\|_2 \|\nabla u\|_{L^2(B_1)} + \|h\|_{L^{\infty}_{ln+}})
\]

\[
\leq C(\|\nabla B\|_{L^2(B_1)} \|\nabla u\|_{L^2(B_1)} + \|h\|_{L^{\infty}_{ln+}})
\]

\[
\leq C(\|\nabla u\|_{L^2(B_1)} + \|f\|_{L^{\infty}_{ln+} L(B_1)}).
\]
On the other hand we have
\[ \Delta E = \text{curl}(A \nabla u) = \nabla^\perp A \nabla u. \]
A similar argument as above for \( D \) shows that
\[ \| \nabla^2 E \|_{L^1(B_2^+)} \leq C \| \nabla u \|_{L^2(B_1)}. \]
So we proved this lemma.

The following lemma is well known.

**Lemma 2.7 (Rellich’s compactness Theorem)** If the sequence \( \{f_n\} \) is bounded in \( W^{1,1}(B_1) \), then \( \{f_n\} \) is precompact in \( L^p(B_1) \) when \( 1 \leq p < 2 \).

The following result was essentially proved in [3] (P12, also see [9]).

**Lemma 2.8** If the sequence \( \{f_n\} \) is bounded in \( W^{1,1}(R^2) \) and satisfies
\[ f_n \to f \text{ strongly in } L^1; f_n \to f \text{ weakly in } L^2. \]
Then there exist at most countable points \( x_j \) and \( a_j > 0 \) such that
\[ f_n^2 \rightharpoonup f^2 + \sum_j a_j \delta_{x_j} \text{ in the sense of measure}. \]

**Proof:** The proof is completely the same as that of Theorem 9 in [3] (P12).

Set \( g_n = f_n - f \), as \( \nabla g_n \) is bounded in \( L^1 \), we see that there exists a finite nonnegative Borel measure \( \mu \) such that
\[ |\nabla g_n| \rightharpoonup \mu \text{ in the sense of measure.} \]
Also we can assume that
\[ g_n^2 \rightharpoonup \nu \geq 0 \text{ in the sense of measure.} \]

Take \( \phi \in C_0^\infty(R^2) \), the Sobolev embedding \( W^{1,1}(R^2) \subset L^2(R^2) \) shows that
\[ \left( \int (\phi g_n)^2 dx \right)^{1/2} \leq C \int |\nabla (\phi g_n)| dx. \]
Letting \( n \to \infty \), by the fact that \( g_n \to 0 \) in \( L^1 \) we obtain that
\[ \left( \int \phi^2 d\nu \right)^{1/2} \leq C \int |\phi| d\mu. \]
By approximation, for any ball \( B(x, r) \) there holds
\[ \nu(B(x, r)) \leq C \mu(B(x, r))^2. \]
(3)
Since \( \mu \) is finite, the inequality (3) implies that for any Borel set \( E \),
\[ \nu(E) = \int_E D_\mu \nu d\mu \]
where
\[ D_\mu \nu(x) = \lim_{r \to 0} \frac{\nu(B(x, r))}{\mu(B(x, r))}, \text{ for } \mu - a.e. \ x \in \mathbb{R}^2. \]

See Federer [4], P152-P169.

As \( \mu \) is finite, there are at most countable points \( x_j \) such that
\[ \mu_j = \mu(\{x_j\}) > 0. \]

If \( \mu(\{x\}) = 0 \), by (3) we get that
\[ D_\mu \nu(x) = \lim_{r \to 0} \frac{\nu(B(x, r))}{\mu(B(x, r))} \leq C \lim_{r \to 0} \mu(B(x, r)) = C \mu(\{x\}) = 0. \]

Denote \( a_j = D_\mu \nu(x_j) \), from (4) it follows that
\[ \nu = \sum_j a_j \delta_{x_j}. \]

In the sense of measure, there holds
\[ \nu = \lim_{n \to \infty} g_n^2 = \lim_{n \to \infty} (f_n^2 + f^2 - 2f_n \cdot f) = \lim_{n \to \infty} (f_n^2 - f^2). \]

So we proved this lemma.

3 Proof of Theorem 1.1 and Theorem 1.2

Lemma 2.6 shows that the sequence \( u_n \) is bounded in \( W^{2,1}(B_{\frac{3}{2}}) \), so we can find \( u \) such that
\[ u_n \rightharpoonup u \text{ strongly in } W^{1,1}(B_{\frac{3}{2}}); u_n \rightharpoonup u \text{ weakly in } W^{1,2}(B_{\frac{3}{2}}). \]

Lemma 2.8 implies that there exist at most countable points \( x_i \) and \( a_i > 0 \) such that
\[ |\nabla u_n|^2 \to |\nabla u|^2 + \sum_i a_i \delta_{x_i} \text{ in } M(B_{\frac{3}{2}}). \]

Denote \( \mu = \sum_i a_i \delta_{x_i} \). If \( \mu(B_{\frac{3}{2}}) = 0 \), then by the fact \( \nabla u_n \to \nabla u \) weakly in \( L^2(B_{\frac{3}{2}}) \) and
\[ \lim_{n \to \infty} \|\nabla u_n\|_{L^2(B_{\frac{3}{2}})}^2 = \|\nabla u\|_{L^2(B_{\frac{3}{2}})}^2 \]
there holds
\[ \lim_{n \to \infty} \|\nabla u_n - \nabla u\|_{L^2(B_{\frac{3}{2}})} = 0. \]

So if Theorem 1.1 wasn’t true, then there would exist a point \( x_0 \in B_{\frac{3}{2}} \) such that
\[ \mu(\{x_0\}) = a > 0. \]

By Lemma 2.5, we see that, for any \( n \), there exist \( A_n \in \mathcal{L}^{\infty}(B_1, M_K(R)) \cap W^{1,2}(B_1, M_K(R)) \) and \( B_n \in W^{1,2}(B_1, M_K(R)) \) satisfying
\[
\begin{align*}
\nabla A_n &= A_n \Omega_n + \nabla^\perp B_n; \\
\int_{B_1} (|\nabla A_n|^2 + |\nabla B_n|^2) dx + \|\text{dist}(A_n, SO(K))\|_{L^\infty(B_1)}^2 &\leq C_K \|\Omega_n\|_{L^2(B_1)}.
\end{align*}
\]
As $\|dist(A_n, SO(K))\|_{L^\infty(B_1)} \leq C_K \epsilon_K$ is small, we can almost consider $A_n$ as an identity matrix.

By Lemma 2.6 we can check that

$$\|\nabla (A_n \nabla u_n)\|_{L^1(B_{3 \over 4})} \leq \|\nabla A_n \nabla u_n\|_{L^1(B_{3 \over 4})} + \|A_n \nabla^2 u_n\|_{L^1(B_{3 \over 4})} \leq \|\nabla A_n\|_{L^2(B_1)} \|\nabla u_n\|_{L^2(B_1)} + C \|\nabla^2 u_n\|_{L^1(B_{3 \over 4})} \leq C(\Omega_n) \|\nabla u_n\|_{L^2(B_1)} + \|\nabla u_n\|_{L^2(B_1)} + \|f_n\|_{L^1(B_1)} \leq C.$$ 

As $W^{1,1}(B_1)$ compactly embedded in $L^1(B_1)$, passing to a subsequence there exists a $v_1 \in L^1(B_{3 \over 4})$ such that

$$\lim_{n \to \infty} \|2 A_n \nabla u_n - v_1\|_{L^1(B_{3 \over 4})} = 0 \tag{6}$$

and

$$\|v_1\|_{L^2(B_{3 \over 4})} \leq 2 \liminf_{n \to \infty} \|A_n \nabla u_n\|_{L^2(B_{3 \over 4})} \leq C.$$ 

Similarly, we can find a $v_2 \in L^2(B_{3 \over 4})$ such that

$$\lim_{n \to \infty} \|A_n u_n - v_2\|_{L^2(B_{3 \over 4})} = 0 \tag{7}$$

and furthermore

$$\|v_2\|_{L^\infty(B_{3 \over 4})} \leq \lim_{n \to \infty} \|A_n u_n\|_{L^\infty(B_{3 \over 4})} \leq C.$$ 

By the assumption we can choose a subsequence of $\Omega_n$ such that

$$\lim_{n \to \infty} \|\Omega_n - \Omega\|_{L^1(B_{3 \over 4})} = 0. \tag{8}$$

Let $r$ be small enough so that

$$\int_{B(x_0, 2r)} |\nabla u_n|^2 dx < 2a = 2 \mu(\{x_0\})$$

for any $n$.

Take $\varphi_r \in C_0^\infty(B(x_0, 2r))$ with $\varphi_r(x) = 1$ for $x \in B(x_0, r)$. For simplicity, assume that $\int_{B(x_0, 2r)} u_n dx = 0$, we get

$$div(A_n \nabla (\varphi_r u_n))$$

$$= \nabla A_n \nabla (\varphi_r u_n) + A_n \Delta (\varphi_r u_n)$$

$$= \nabla A_n \nabla (\varphi_r u_n) - A_n \varphi_r \nabla u_n + A_n (\varphi_r f_n + 2 \nabla u_n \varphi_r + u_n \Delta \varphi_r)$$

$$= (\nabla A_n - A_n \nabla) (\varphi_r u_n) + A_n (\nabla u_n \varphi_r + 2 \nabla u_n \varphi_r + f_n + u_n \Delta \varphi_r)$$

$$= \nabla^T B_n \nabla (\varphi_r u_n) + \varphi_r A_n f_n + h_{n,r} \tag{9}$$

where

$$h_{n,r} = A_n (\nabla u_n + 2 \varphi_r \nabla \varphi_r + u_n \Delta \varphi_r).$$
Set \( h_r = (v_2 \Omega + v_1) \nabla \varphi_r + v_2 \triangle \varphi_r \), by (6), (7) and (8) we have
\[
\lim_{n \to \infty} \| h_{n,r} - h_r \|_1 
\leq C r^{-2} \lim_{n \to \infty} (\| A_n (\Omega_n u_n + 2 \nabla u_n) - v_2 \Omega - v_1 \|_{L^1(B(x_0,2r))} + \| A_n u_n - v_2 \|_{L^1(B(x_0,2r))})
\leq C r^{-2} \lim_{n \to \infty} (\| A_n u_n - v_2 \|_{L^1(B_{3r})} + \| v_2 (\Omega_n - \Omega) \|_{L^1(B_{3r})})
+ 2 A_n \nabla u_n - v_1 \|_{L^1(B_{3r})} + \| A_n u_n - v_2 \|_{L^1(B_{3r})})
\leq C r^{-2} \lim_{n \to \infty} (\| A_n u_n - v_2 \|_{L^2(B_{3r})} \| \Omega_n \|_{L^2(B_1)} + \| \Omega_n - \Omega \|_{L^1(B_{3r})})
= 0. \tag{10}
\]

Let \( D_{n,r} \) be the Newtonian potential of \( \text{div}(A_n \nabla (\varphi_r u_n)) \) and \( E_{n,r} \) be the Newtonian potential of \( \text{curl}(A_n \nabla (\varphi_r u_n)) \). By (9) we obtain that
\[
\nabla D_{n,r} = R \ast (\nabla^\perp B_n \nabla (\varphi_r u_n) + \varphi_r A_n f_n + h_{n,r}).
\]
where \( R = \nabla N \) is the Riesz potential.

Similarly we get that
\[
\nabla E_{n,r} = R \ast (\nabla^\perp A_n \nabla (\varphi_r u_n)).
\]

We can see that \( A_n \nabla (\varphi_r u_n) - \nabla D_{n,r} - \nabla^\perp E_{n,r} \) is a harmonic function on \( R^2 \). On the other hand, it is easy to check that when \(|x| > 2\) there holds
\[
|\nabla D_{n,r}(x)| + |\nabla E_{n,r}(x)| \leq \frac{C}{|x|}.
\]
As there is no nonzero harmonic function which vanishes at \( \infty \) in \( R^2 \), there must be
\[
A_n \nabla (\varphi_r u_n) = \nabla D_{n,r} + \nabla^\perp E_{n,r}. \tag{11}
\]

Set
\[
g_{n,r} = \nabla N \ast (\nabla^\perp A_n \nabla (\varphi_r u_n) + \varphi_r A_n f_n) + \nabla^\perp N \ast (\nabla^\perp B_n \nabla (\varphi_r u_n)),
\]
then by (11) we get that for \( x \in B(x_0, r) \) there holds
\[
A_n \nabla u_n(x) = A_n \nabla (\varphi_r u_n)(x)
= \nabla D_{n,r} + \nabla^\perp E_{n,r}
= g_{n,r} + R \ast h_{n,r}. \tag{12}
\]
As the Riesz potential \( R \) is bounded from \( L^1(R^2) \) to \( L^{2,\infty}(R^2) \), by (10) we obtain that
\[
\lim_{n \to \infty} \| A_n \nabla u_n - g_{n,r} - R \ast h_r \|_{L^{2,\infty}(B(x_0, r))} \leq \lim_{n \to \infty} \| R \ast (h_{n,r} - h_r) \|_{L^{2,\infty}}
\leq C \lim_{n \to \infty} \| h_{n,r} - h_r \|_1
= 0. \tag{13}
\]
By Lemma 2.3 and Lemma 2.4 we have
\[ \| \nabla N (\nabla^\perp A_n \nabla (\varphi_r u_n)) + \nabla^\perp N (\nabla^\perp B_n \nabla (\varphi_r u_n)) \|_{L^2,1} \]
\[ \leq C (\| \nabla^\perp A_n \|_{L^2 (B_1)} + \| \nabla B_n \|_{L^2 (B_1)} \| \nabla (\varphi_r u_n) \|_2 ) \]
\[ \leq C \| \Omega_n \|_{L^2 (B_1)} \| \nabla u_n \|_{L^2 (B(x_0,2r))} \]
\[ \leq C \varepsilon K \sqrt{a}. \tag{14} \]

Similarly we can also get
\[ \| R^\ast (\varphi_r A_n f_n) \|_{L^2,1(B_1)} \leq C \| \varphi_r A_n f_n \|_{H^1 (B_1)} \]
\[ \leq C \| \varphi_r A_n f_n \|_{L^1 \ln L^2,1(B_1)} \]
\[ \leq C \| f_n \|_{L^1 \ln L^2,1(B_1)} \]
\[ \leq C. \tag{15} \]

On the other hand, we have
\[ \| h_r \|_{L^2(B_1)} \leq C r^{-2} (\| |v_2\Omega| + |v_1| + |v_2| \|_{L^2(B_{\frac{3}{4}})}) \]
\[ \leq C r^{-2} (\| \Omega \|_{L^2(B_1)} + \| v_1 \|_{L^2(B_{\frac{3}{4}})} + 1) \]
\[ \leq C r^{-2}. \]

So we have
\[ \| R^\ast h_r \|_{L^2,1(B_1)} \leq C \| h_r \|_{L^2(B_1)} \leq C r^{-2}. \tag{16} \]

From Lemma 2.6, (14), (15) and (16) it follows that
\[ \| A_n \nabla u_n - g_{n,r} - R^\ast h_r \|_{L^2,1(B_1)} \leq C r^{-2}. \]

By (13) and the duality between $L^{2,1}$ and $L^{2,\infty}$ we have
\[ \lim_{n \to \infty} \| A_n \nabla u_n - g_{n,r} - R^\ast h_r \|_{L^2(B(x_0,r))} \]
\[ \leq \lim_{n \to \infty} \| A_n \nabla u_n - g_{n,r} - R^\ast h_r \|_{L^{2,1}(B_{\frac{3}{4}})} \| A_n \nabla u_n - g_{n,r} - R^\ast h_r \|_{L^{2,\infty}} \]
\[ = 0. \tag{17} \]

It can be shown that
\[ \| R^\ast (\varphi_r A_n f_n) \|_{L^2,\infty} \leq C \| \varphi_r A_n f_n \|_1 \]
\[ \leq C \| f_n \|_{L^1(B(x_0,2r))} \]
\[ \leq C (\ln \frac{1}{r^2})^{-1} \| f_n \|_{L^1 \ln L^2(B_1)} \]
\[ \leq C (\ln \frac{1}{r})^{-1}. \tag{18} \]
With (15) we get that
\[ \|R \ast (\varphi_r A_n f_n)\|_{L^2(B_1)} \leq 2 \|R \ast (\varphi_r A_n f_n)\|_{L^2,1(B_1)} \|R \ast (\varphi_r A_n f_n)\|_{L^2,\infty} \leq C(\ln \frac{1}{r})^{-1}. \]  

(19)

It is clear that (14) and (19) imply that
\[ \|g_{n,r}\|_{L^2(B_1)}^2 \leq C(\epsilon_K^2 a + (\ln \frac{1}{r})^{-1}). \]

For any \( \epsilon > 0 \), we can take \( t < r \) small enough so that
\[ \int_{B(x_0,t)} |R \ast h_r|^2 dx < \epsilon a. \]

By (17) we can get
\[ \|\nabla u_n\|_{L^2(B(x_0,t))}^2 \leq 2 \|A_n \nabla u_n\|_{L^2(B(x_0,t))}^2 \leq C(\|g_{n,r}\|_{L^2(B_1)}^2 + \|R \ast h_r\|_{L^2(B(x_0,t))}^2) \leq C(\epsilon_K^2 a + (\ln \frac{1}{r})^{-1} + \epsilon a). \]

Take \( r, \epsilon_N, \epsilon \) small enough, we can obtain that
\[ a \leq \lim_{n \to \infty} \|\nabla u_n\|_{L^2(B(x_0,t))}^2 \leq \frac{a}{2} \]
which contracts to the fact \( a > 0 \), so we proved Theorem 1.1.

Now we prove Theorem 1.2. At first we rewrite the equation as
\[ \Delta u_n = A(u_n) (du_n, du_n) + \tau(u_n) = -\sum_{l=1}^{K-m} \langle \nabla \nu_l, \nabla u_n \rangle \nu_l + \tau(u_n) \]
where \( \nu_l \) \( (1 \leq l \leq K - m) \) is the orthogonal frame field for the normal bundle to \( N \). Here we didn’t distinguish \( \nu_l \) and \( \nu_l \circ u_n \). By the fact that \( \langle \nabla u_n, \nu_l \rangle = 0 \), we obtain that
\[ \Delta u_n^i = -\sum_{l=1}^{K-m} \langle \nabla \nu_l, \nabla u_n \rangle \nu_l^j + \tau^j(u_n) = -\sum_{l=1}^{K-m} \sum_{j=1}^{K} (\nu_l^j \nabla \nu_l^i - \nu_l^j \nabla \nu_l^i) \nabla u_n^j + \tau^j(u_n). \]

Set \( (\Omega_n)^i_j = \sum_{l=1}^{K-m} \nu_l^j \nabla \nu_l^i - \nu_l^j \nabla \nu_l^i \), there holds
\[ (\Omega_n)^i_j = -(\Omega_n)^i_j; \quad |\Omega_n| \leq C|\nabla u_n|. \]

Now the equation can be rewritten as
\[ \Delta u_n = -\Omega_n \nabla u_n + \tau(u_n). \]  

(20)

As the energy of the map \( u_n \) is small enough in \( B_1 \), we see that \( \|\Omega_n\|_{L^2(B_1)} \leq C \epsilon_N \) is small enough. It is easy to see that
\[ |\nabla \Omega_n| \leq C(|\nabla u_n|^2 + |\nabla^2 u_n|). \]

By Lemma 2.6 we know that \( \{\Omega_n\} \) is bounded in \( W^{1,1}(B_2) \). From Lemma 2.7 it follows that the set \( \{\Omega_n\} \) is precompact in \( L^1(B_2) \). So we can apply Theorem 1.1 to obtain Theorem 1.2.
4 Energy identity for the sphere

In this section, we only prove Theorem 1.3 for the case \( M = B_1 \). For a general Riemann surface the proof is of little difference.

By Theorem 1.2, we can see that \( u_n \) converges strongly to \( u \) away from a finite set of points. For simplicity, we assume that there is only one energy concentration point 0 and there is only one bubble. The general case can be derived by the induction argument in [2].

Suppose that the bubble \( \psi \) is the strong limit of \( u_n(r_n x) \) in \( W^{1,2}_{\text{loc}}(B_1, S^{K-1}) \). Then it is easy to see that the energy identity is equivalent to

\[
\lim_{\delta \to 0} \lim_{R \to \infty} \lim_{n \to \infty} E(u_n, B_\delta \setminus B_{r_n R}) = 0.
\]

For any \( \epsilon > 0 \), choose \( R > 0, \delta > 0 \) such that

\[
E(u, B_\delta) + E(\psi, B_R) < \epsilon^2.
\]

The standard blow-up analysis (see [2]) shows that there exists \( N_0 > 0 \) such that if \( n \geq N_0 \) then for any \( r_n R < t < \delta \)

\[
\int_{B_{2t} \setminus B_t} |\nabla u_n|^2 < \epsilon^2.
\]

Now we estimate the \( L^{2,\infty} \)-norm of \( \nabla u_n \) on the neck domain.

**Lemma 4.1** Suppose that \( u \) is a map from the unit disk \( B_1 \) to \( N \) in \( W^{1,2}(B_1, N) \) with \( \tau(u) \in L \ln^+ L(B_1) \). Assume that \( \epsilon > 0 \) is small enough, if \( r, \delta > 0 \) satisfies that

\[
\int_{B_{2t} \setminus B_t} |\nabla u|^2 < \epsilon^2
\]

for any \( r < t < 4\delta \). Then we have

\[
\|\nabla u\|_{L^{2,\infty}(B_t \setminus B_{rt})} \leq C(\epsilon + (\ln \frac{1}{\delta})^{-1})
\]

where \( C \) only depends on \( \|\nabla u\|_2, \|\tau(u)\|_{L \ln^+ L} \) and the target manifold.

**Proof:** Suppose that \( \delta = 2^N r \), then for any \( 1 \leq i \leq N + 2 \) there holds

\[
E(u, B_{2^i r} \setminus B_{2^{i-1} r}) < \epsilon^2.
\]

Take \( \psi \in C_0^\infty(B_2) \) satisfying that \( \psi = 1 \) in \( B_1 \). Set \( \theta_i(x) = \psi(\frac{x}{2^{i+2} r}) - \psi(\frac{x}{2^{i-1} r}) \) and \( \overline{u}_i = \int_{B_{2^{i+2} r} \setminus B_{2^{i-1} r}} u(x) dx \), then

\[
\Delta(\theta_i(u - \overline{u}_i)) = \theta_i \Delta u + 2 \nabla \theta_i \nabla u + (u - \overline{u}_i) \Delta \theta_i
\]

\[
= \theta_i A(u) + \theta_i \tau(u) + 2 \nabla \theta_i \nabla u + (u - \overline{u}_i) \Delta \theta_i.
\]
For \( x \in B_{2i+1} \setminus B_{2ir} \), there holds

\[
|\nabla u(x)| = |\nabla (\theta_i(u - \overline{u_i}))(x)| \\
= |R* (\theta_iA(u) + \theta_i\tau(u) + 2\nabla \theta_i u + (u - \overline{u_i})\Delta \theta_i)(x)| \\
\leq |R* (\theta_iA(u))(x)| + |R* (\theta_i\tau(u))(x)| + |R* (2\nabla \theta_i u + (u - \overline{u_i})\Delta \theta_i)(x)| \\
= I_1(x) + I_2(x) + I_3(x).
\]

By Sobolev inequality it can be shown that when \( x \in B_{2i+1} \setminus B_{2ir} \), with \( i \geq 2 \),

\[
I_3(x) \leq \int \left| R(x - y)(2\nabla \theta_i u + (u - \overline{u_i})\Delta \theta_i)(y) \right| dy \\
\leq C \int_{(B_{2i+3r} \setminus B_{2i+2r}) \cup (B_{2i-1r} \setminus B_{2i-2r})} \frac{1}{|x - y|}(2^{-i}r^{-1}|\nabla u| + 2^{-2i}r^{-2}|u - \overline{u_i}|)(y) dy \\
\leq C \int_{B_{2i+3r} \setminus B_{2i-2r}} (2^i)^{-1}(2^i)^{-1}|\nabla u| + (2^i)^{-2}|u - \overline{u_i}|)(y) dy \\
\leq C(2^i)^{-2} \int_{B_{2i+3r} \setminus B_{2i-2r}} |\nabla u|(y) dy \\
\leq C\varepsilon \frac{1}{|x|}.
\]

As Riesz potential is bounded from \( L^1(R^2) \) to \( L^{2;\infty}(R^2) \), we get that for any \( a > 0 \),

\[
|\{I_1(x) + I_2(x) > a\}| \leq Ca^{-2}||I_1 + I_2||_{L^{2;\infty}(R^2)}^2 \\
\leq Ca^{-2}||\theta_i(|A(u)| + |\tau(u)|)||_1^2 \\
\leq Ca^{-2}(\int \theta_i(|\nabla u|^2 + |\tau(u)|)(x)dx)^2 \\
\leq Ca^{-2}(\int \theta_i|\nabla u(x)|^2 dx)^2 + (\int \theta_i|\tau(u)(x)|dx)^2).
\]

So there holds

\[
|\{x \in B_{2i} \setminus B_{4r} : |\nabla u(x)| > 2a\}| \\
= \sum_{i=2}^{N-1} |\{x \in B_{2i+1} \setminus B_{2ir} : |\nabla u(x)| > 2a\}| \\
\leq \sum_{i=2}^{N-1} |\{x \in B_{2i+1} \setminus B_{2ir} : I_1(x) + I_2(x) > a\}| \\
+ \sum_{i=2}^{N-1} |\{x \in B_{2i+1} \setminus B_{2ir} : I_3(x) > a\}| \\
\leq \sum_{i=2}^{N-1} |\{I_1(x) + I_2(x) > a\}| + |\{x \in B_1 : \frac{C\varepsilon}{|x|} > a\}| \\
\leq Ca^{-2}\sum_{i=2}^{N-1} (\varepsilon^2 \int \theta_i|\nabla u|^2(x)dx + (\int \theta_i|\tau(u)(x)|dx)^2) + \frac{C\varepsilon^2}{a^2} \\
\leq Ca^{-2}(\varepsilon^2 \int (\sum_{i=2}^{N-1} \theta_i)|\nabla u|^2(x)dx + (\int (\sum_{i=2}^{N-1} \theta_i)|\tau(u)(x)|dx)^2 + \varepsilon^2).
\]
\[ \leq Ca^{-2}(\epsilon^2 E(u) + (\int_{B_{4\delta}} |\tau(u)(x)|dx)^2 + \epsilon^2) \]
\[ \leq Ca^{-2}(\epsilon^2 + (\|\tau(u)\|_{L^\ln+L})^2) \]
\[ \leq Ca^{-2}(\epsilon^2 + \ln^{-2} \frac{1}{\delta}). \]

So we proved this lemma.

Now we estimate the $L^2,1$-norm of $\nabla u_n$ on $B_1$. We recall some identities in ([5], P132-134). Embed $S^{K-1}$ into $R^K$, then we have
\[ \tau(u_n) = \Delta u_n + u_n|\nabla u_n|^2 = f_n. \]
Set
\[ \beta^i_n = u^i_n du^j_n - u^j_n du^i_n, \]
then
\[ \delta \beta^i_n = f^j_n u^i_n - f^i_n u^j_n, \]
and
\[ \Delta \beta^i_n = d(f^j_n u^i_n - f^i_n u^j_n) + 2\delta (du^i_n \wedge du^j_n). \tag{23} \]
Consider
\[ \left\{ \begin{array}{ll}
\Delta \Phi^i_n = 2du^i_n \wedge du^j_n & \text{in } B_1 \\
\Phi^i_n = 0 & \text{on } \partial B_1,
\end{array} \right. \]
and
\[ \left\{ \begin{array}{ll}
\Delta \Psi^i_n = f^j_n u^i_n - f^i_n u^j_n & \text{in } B_1 \\
\Psi^i_n = 0 & \text{on } \partial B_1.
\end{array} \right. \]
Let
\[ H^i_n = \beta^i_n - \delta \Phi^i_n - d\Psi^i_n. \]
By Theorem 3.3.8 in [5] (also see [15], Theorem 1.100), we have
\[ \|\nabla \Phi^i_n\|_{L^2(B_1)} \leq C\|du^i_n \wedge du^j_n\|_{H^1} \leq C\|\nabla u_n\|_{L^2(B_1)}^2 \leq C; \]
\[ \|\nabla \Psi^i_n\|_{L^2(B_1)} \leq C\|f^j_n u^i_n - f^i_n u^j_n\|_{H^1(B_1)} \leq C\|\tau(u_n)\|_{L^\ln+L} \leq C. \]
It is clear that $H^i_n$ is a harmonic 1-form in $B_1$ and
\[ \int_{B_1} |H_n|^2 dx \leq \int_{B_1} |\beta_n|^2 dx + \|\nabla \Phi_n\|_{L^2(B_1)}^2 + \|\nabla \Psi_n\|_{L^2(B_1)}^2 \]
\[ \leq C(\int_{B_1} |\nabla u_n|^2 dx + \|\nabla \Phi_n\|_{L^2(B_1)}^2 + \|\nabla \Psi_n\|_{L^2(B_1)}^2) \]
\[ \leq C. \]

By the basic property of the harmonic function, we know that
\[ \|H_n\|_{C^0(B_{1/2})} \leq C \]
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which implies that
\[ \|\beta_n\|_{L^2(B_{\frac{1}{2}})} \leq \|H_n\|_{C^0(B_{\frac{1}{2}})} + \|\nabla \Phi_n\|_{L^2(B_1)} + \|\nabla \Psi_n\|_{L^2(B_1)} \leq C. \] (24)

By Lemma 4.1 we can get that
\[ \|\nabla u_n\|_{L^2,\infty(B_{\frac{1}{2}})} \leq C(\epsilon + (\ln \frac{1}{\delta})^{-1}). \]

So using the duality between $L^{2,1}$ and $L^{2,\infty}$ one shows that
\[ \|\beta_n\|^2_{L^2(B_{\frac{1}{2}} \setminus B_{\frac{1}{3n}} R)} \leq \|\beta_n\|_{L^2,\infty(B_{\frac{1}{2}} \setminus B_{\frac{1}{3n}} R)} \|\nabla u_n\|_{L^2,\infty(B_{\frac{1}{2}} \setminus B_{\frac{1}{3n}} R)} \leq C(\epsilon + (\ln \frac{1}{\delta})^{-1}). \]

On the other hand, as $u_n(B_1) \subset S^{K-1}$ we have the following equality
\[ 2|\nabla u_n|^2 = \sum_{i,j=1}^K |\beta_n^{ij}|^2. \]

So we can get that
\[ \|\nabla u_n\|^2_{L^2(B_{\frac{1}{2}} \setminus B_{\frac{1}{3n}} R)} \leq C(\epsilon + (\ln \frac{1}{\delta})^{-1}) \]
which implies (21). Thus we proved Theorem 1.3.

5 An example

In this section we construct a sequence of mappings from $B_1$ to $S^2$ with tension fields bounded in $L \ln^+ L$ but there exists neck with positive length during blowing up.

Let $\pi$ be the stereographic projection from $R^2$ to $S^2 \subset R^3$ which is a conformal harmonic mapping given by
\[ \pi(r, \theta) = \left( \frac{2r \cos \theta}{1 + r^2}, \frac{2r \sin \theta}{1 + r^2}, \frac{1 - r^2}{1 + r^2} \right). \]

Set
\[ f_n(t) = \begin{cases} 0, & t < e^{-2n}; \\ t^{-1} (\ln \frac{1}{t})^{-2}, & e^{-2n} \leq t \leq e^{-\frac{4n}{3}}; \\ -t^{-1} (\ln \frac{1}{t})^{-2}, & e^{-\frac{4n}{3}} < t \leq e^{-n}; \\ 0, & t \geq e^{-n}. \end{cases} \]

One can check that
\[ \int |f_n(t)| \ln(2 + |t^{-1} f_n(t)|) dt = \int_{e^{-2n}}^{e^{-n}} t^{-1} (\ln \frac{1}{t})^{-2} \ln(2 + t^{-2} (\ln \frac{1}{t})^{-2}) dt \leq \int_{e^{-2n}}^{e^{-n}} t^{-1} (\ln \frac{1}{t})^{-2} \ln t^{-3} dt = 3 \int_{e^{-2n}}^{e^{-n}} t^{-1} (\ln \frac{1}{t})^{-1} dt = 3 \int_n^{2n} s^{-1} ds = 3 \ln 2. \] (25)
Take $\varphi \in C_0^\infty(B_1)$, $\varphi(x) \equiv 1$ in $B_{\frac{1}{2}}$. Set $h_n(r) = \int_{e^{-2n}}^{e^{-2n}t^{-1}} f_n(s) ds dt$ and

$$u_n(r, \theta) = \begin{cases} \pi(\frac{\varphi(e^{2n}r)}{e^{2n}r}, \theta), & r < e^{-2n}; \\ (\sin h_n(r), 0, \cos h_n(r)), & r \geq e^{-2n}. \end{cases}$$

It is easy to see that $u_n \rightharpoonup 0$ weakly in $W^{1,2}$ and there is only one bubble $\pi$ which was produced by the sequence $u_n(e^{-2n})$. Now we check that $\tau(u_n)$ are bounded in $L \ln^+ L$.

Because $h_n'(r) = 0$ for $r > e^{-n}$ and the map $\pi(\frac{1}{dr}, \theta)$ is harmonic for any constant $a > 0$, one checks that when $|x| < \frac{e^{-2n}}{2}$ or $|x| > e^{-n}$, there holds $\tau(u_n)(x) = 0$. Also we can check that

$$|\tau(u_n)(x)| \leq C(\|\nabla u_n\|^2(x) + |\nabla^2 u_n|(x)) \leq C e^{2n}$$

for $\frac{e^{-2n}}{2} \leq |x| \leq e^{-2n}$.

In the case that $e^{-2n} < |x| \leq e^{-n}$, a simple computation shows that

$$|\tau(u_n)(r, \theta)| = |h_n''(r) + \frac{h_n'(r)}{r}| = |f(r)|.$$

By (25) we have

$$\int_{B_1} |\tau(u_n)(x)| \ln(2 + |\tau(u_n)(x)|)$$

$$\leq \int_{\frac{e^{-2n}}{2} \leq |x| \leq e^{-2n}} e^{2n} \ln(2 + e^{2n}) dx + \int_{e^{-2n} < |x| \leq e^{-n}} |\frac{f(|x|)}{|x|}| \ln(2 + |\frac{|x|}{|x|}|) dx$$

$$\leq 3 e^{-4n} n e^{2n} + \int_{e^{-2n}}^{e^{-n}} |f_n(t)| \ln(2 + |t^{-1} f_n(t)|) dt$$

$$\leq 3 e^{-2} + 3 \ln 2$$

$$\leq 6.$$

So it follows from Theorem 1.3 that the energy identity holds, i.e.

$$\lim_{n \to \infty} E(u_n) = E(\pi).$$

In fact we can obtain this equality easily by direct computations.

Now we prove that there exists neck with positive length. Some direct computations show that

$$\|u_n\|_{Qsc} \geq \int_{e^{-2n}}^{e^{-n}} h_n'(r) dr$$

$$= \int_{e^{-2n}}^{e^{-\frac{4n}{3}}} r^{-1}(\frac{1}{n} - (\ln \frac{1}{r})^{-1}) dr + \int_{e^{-2n}}^{e^{-\frac{4n}{3}}} r^{-1}(\ln \frac{1}{r})^{-1} - \frac{1}{2n} dr$$

$$= \int_{e^{-2n}}^{e^{-\frac{4n}{3}}} r^{-1}(\ln \frac{1}{r})^{-1} dr - \int_{e^{-2n}}^{e^{-\frac{4n}{3}}} r^{-1}(\ln \frac{1}{r})^{-1} dr$$

$$= \ln \frac{9}{8} > 0.$$
References


