ANALOGUE OF EAKIN-SATHAYE THEOREM
OVER REES ALGEBRA

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Abstract

In this paper we discuss a seminal result of the Eakin-Sathaye theorem in the Rees algebra setting. Some part of this paper is a survey and expository.

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1. Introduction

Let $R$ be a ring and $I$ be an ideal in $R$ and $t$ is a variable over $R$. Then Rees algebra of $I$ is the subring of $R[t]$ defined as $R[I] = \{a_0 + a_1 t + \ldots, a_n t^n | n \in \mathbb{N}, a_i \in I \} = \bigoplus_{n \geq 0} I^n t^n$. Rees algebra of an ideal is a classical object that has been studied by mathematicians (see [2], [3], [4], [5], [6], [7]). The book (see [11]) by I. Swanson and C. Huneke is a landmark for research in this direction. Using the method of moving curves and surfaces, the defining equations of the Rees algebra have been discovered. The Rees algebra of an ideal in a commutative ring is the quotient of polynomial ring by its ideal of defining equations. Blowing up, one of the key operations in birational algebraic geometry, can be replaced by considerations involving Rees algebra. The following Eakin-Sathaye theorem (see [1]) is a seminal result in the theory and this has created new directions for research.

**Theorem 1.1** (Eakin-Sathaye). Let $R$ be a local ring with infinite residue field and let $I$ be an ideal in $R$ such that for some integers $n$ and $r$ with $n \geq 1$ and $r \geq 0$, $I^n$ can be generated by fewer than $\binom{n+r}{r}$ elements. Then there are elements $y_1, \ldots, y_r$ in $I$, such that $(y_1, \ldots, y_r)I^{n-1} = I^n$.

By this result an upper bound on the length of the shortest superficial sequence for $I$ that generates a reduction of $I$ and also an upper bound on reduction number can be found. In other words this theorem relates the number of generators of certain power of an ideal with the existence of a distinguished reduction for that ideal. In the original paper [1] this theorem was proved by adding on indeterminates to $R$ to form a so-called Nagata extension and then acting on this extension by a permutation group. Later, in [10], a purely “internal proof was given that used elements parametrized by points in Zariski-open sets. Hoa and Trung [6] have given a combinatorial proof using the theory of generic initial ideals and Borel-fixed ideals Giulio Caviglia (see [12]) proving that this result can be obtained as a special case of Green’s general hyperplane restriction theorem. L. Carroll (see [9]) proved a generalization of the Eakin-Sathaye theorem on reductions to the case of complete, and so of joint reductions in the sense of Rees. Carroll reinterprets his result as a “multiplicative normalization theorem and Parameswaran, V. Srinivas result (see [13]) as an “additive normalization theorem. The main goal of this paper is to discuss the Eakin-Sathaye theorem in the Rees algebra setting.

One tries to find a possible proof of the Eakin-Sathaye theorem that uses properties of those algebras that appear naturally in this context. A point of view is the following:

Let $F$ be the usual fibre cone of the Rees ring over the closed point. By the graded version of Noether Normalization, $F$ is a finitely generated graded module over a standard graded polynomial ring in $s$ variables over the base field $k$, where $s$ is the analytic spread of the ideal $I$ of the local ring $R$, with $R$ having the infinite residue field $k$. It easily follows that the hypotheses of the Eakin-Sathaye theorem imply that $s$ is at most $r$. Hence the ideal $I$ certainly has a reduction generated by $r$ elements, and we can lift these elements from $r$ generic linear forms in $F$. 

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We had wondered (and this is why we indicate this possible approach to Eakin-Sathaye) whether it might be possible to develop this point of view to yield a proof of the Eakin-Sathaye theorem using the structure of the Rees ring as an algebra over $R$ or of the structure of $F$ as a standard graded affine ring over $k$, rather than working with a mixture of the ideal theory of $R$ and Zariski-open properties connected to $F$ (or to the Rees ring of $I$ or to the associated graded ring of $I$).

In the following $\mu(R[I^n t])$ denotes the minimal number of generators of $R[I^n t]$ over $R$. We prove the following:

**Theorem 1.2.** Let $R$ be a local Noetherian ring with infinite residue field. Let $I$ be an ideal in $R$ and $R[It]$ denotes the Rees algebra of $I$. Let $n \geq 1$ and $r \geq 0$ be integers such that, $\mu_R(R[I^n t]) < \binom{n+r}{r}$ and $F$ be the usual fibre cone of the Rees ring over the closed point. Then there exists an ideal $J \subset I$ such that $R[It]$ is a finitely generated module over $R[It]$ i.e. for some choice of $f_1, \ldots, f_r$ in the degree 1 component $F_1$ of $F$, homogeneous generators of $F$ as a finitely generated homogeneous module over the standard graded affine algebra $k[f_1, \ldots, f_r]$ lie in degree at most $n-1$.

### 2. Preliminaries

The aim of this section is to give some definitions, examples and results which are used in the proof of our theorem. We hope that this will improve the readability and understanding of the proof of the theorem.

Let $R$ be a commutative Noetherian ring and $M$ be an $R$-Module. Then the Rees algebra of $M$ is the symmetric algebra $Sym(M)$ of $M$ modulo its $R$-torsions $A$ i.e. modulo elements killed by non-zero zero divisors. We denote the Rees algebra of $M$ by $Rees(M)$. A module $M$ is said to be of linear type if $A = 0$. The symmetric algebra corresponds to polynomials with indeterminates on $M$ defined by $Sym(M) = T(M) / (x \otimes y)$. This approach does not provide a satisfactory definition in all the cases in the sense that it may give wrong answers even for an ideal, if the ring is not a domain. In [2] Rees algebra of a module has been defined in the following way:

**Definition 2.1.** Let $M$ be an $R$-module. Then Rees algebra of $M$ is $Rees(M) = \frac{Sym(M)}{\cap L_g}$, where intersection is taken over all homomorphisms $g : M \to F$, where $F$ is a free $R$-module and $L_g$ denotes the kernel of $Sym(g)$.

This is a functorial definition, for if $h : M \to N$ is a homomorphism of $R$-modules, then for every homomorphism from $g : N \to F$, the map $gh : M \to F$ is a homomorphism, so $Sym(h) : Sym(M) \to Sym(N)$ induces an $R$-algebra homomorphism. As the symmetric algebra functor preserves epimorphism, so does the Rees algebra functor. Rees algebra of a module may be computed in terms of a maximal map $f$ from $M$ to a free module as the image of the map induced by $f$ on symmetric algebras.
Definition 2.2. Let \( M \) be an \( R \)-module and \( F \) be a free module, we say that \( f : M \to F \) is a versal map, if \( f \) is a homomorphism and every homomorphism from \( M \) to a free module factors through \( f \).

It follows from the definition that if \( f : M \to F \) is a versal map from \( M \) to \( F \), then \( \text{Rees}(M) = \text{Rees}(f) \). If \( M \) is finitely generated \( R \)-module, then it easy to find such a map.

Definition 2.3. An element \( x \) in an ideal \( I \) is said be superficial with respect to \( R \)-module \( M \) if there exists a natural number \( c \) such that for every \( n \geq c \), \( (I^{n+1} : x) \cap I^c M = I^n M \).

Definition 2.4. A reduction of \( I \) is an ideal \( J \subseteq I \) such that \( I^{n+1} = JI^n \), for some non-negative integer \( n \). The least such integer \( n \) is called the reduction number of \( I \) relative to \( J \) and the least such \( n \) for all such \( J \) is called the reduction number of \( I \).

Example [10]: Let \( I = \langle X_1^3, X_1X_2, X_2^4 \rangle \subset k[X_1, X_2] \) be an ideal. Then \( I = \langle X_1X_2, X_1^3 + X_2^4 \rangle \) is a reduction of \( I \).

Remark I

(1) Reduction defines a relationship between two ideals which is preserved under homomorphisms and ring extensions.

(2) Reduction process gets rid of superfluous elements of an ideal without disturbing the algebraic multiplicities associated with it.

(3) Reductions play a role in the theory of finite morphisms of the blow-up \( \text{Blow}_{V(I)}(\text{Spec}(R)) \) with the reduction number being a control element. In case \( R \) is a local ring (of infinite residue field), minimal reductions are particularly valuable because they help to control the co-homology of the blow-up.

An excellent exposition on Rees algebra is given by D. A. Cox in [8]. We give some examples here to make this article self-contained. Given the ideal \( I = (f_1, f_2, f_3) \) in \( R = K[X_1, X_2] \), its Rees algebra is a graded \( R \)-algebra \( \text{Rees}(I) = R \oplus I \oplus I^2 \oplus \ldots \), where the elements in the summand \( I^r \) are assigned degree \( r \). The canonical morphism \( \text{Proj}(\text{Rees}(I)) \to \text{Spec}(R) \) is the blow-up of the ideal \( I \). Note that since \( R \) already has grading and \( I \) is homogeneous, the Rees algebra is bigraded. Since \( f_1, f_2, f_3 \) generate \( I \), the Rees algebra \( \text{Rees}(I) \) can be described as the image of \( R \)-algebra homomorphism \( h : R[X_1, X_2, X_3] \to \text{Rees}(I) \) by \( X_i \to f_i \). If \( K \) = kernel of the map \( h \), then \( \text{Rees}(I) = \frac{R[X_1, X_2, X_3]}{K} \), so \( K = \oplus_{r=1}^{\infty} \text{Syz}(I^r) \). The Rees algebra of \( I = (f_1, f_2, \ldots, f_r) \subset R \) is closely related to the symmetric algebra \( \text{Sym}_R(I) = R \oplus I \oplus \text{Sym}^2(I) \oplus \text{Sym}^3(I) \oplus \ldots \) via the canonical surjection \( \alpha : \text{Sym}_R(I) \to \text{Rees}(I) \). If \( K_1 \) is a graded piece of \( K \) in degree 1 with respect to \( X_i \), then \( K_1 = \{ a_1X_1 + a_2X_2 + \ldots + a_rX_r | a_i \in R \text{ and } a_1f_1 + a_2X_2 + \ldots + a_rf_r = 0 \} \) \( \cong \text{Syz}(f_1, f_2, \ldots, f_r) \). Since \( < K_1 > \subset K \), we get a commutative diagram
Expressing \( I \) and taking the determinant of the coefficients gives \( \text{Rees}(f) \) by the columns of 

\[
\begin{array}{ccc}
R[X_1, X_2, X_3] & \rightarrow & R[X_1, X_2, X_3] \\
K_1 & \downarrow & K \\
\text{Sym}(I) & \rightarrow & \text{Rees}(I)
\end{array}
\]

The vertical map on the right is an isomorphism by definition and the vertical map on the left is known to be an isomorphism. Thus syzygies of the \( f_i \) define the symmetric algebra and give the degree 1 relations of the Rees algebra.

**Example 1** (See [8]): Let \( k \) be a field and \( \text{char}(k) = p \) (prime). Let \( R = \frac{k[X_1, X_2, X_3]}{(X_1^p, X_2^p) + (X_1, X_2, X_3)^{p+1}} \). Let \( x_i \) denote the image of \( X_i \) for \( i = 1, 2, 3 \) and \( Rx_3 = (X_1, X_2, X_3)^p \) be an ideal. Write \( f_1 : Rx_3 \rightarrow R \) for the inclusion mapping and \( f_2 : Rx_3 \rightarrow R^2 = \text{Rees}(Rx_3) \) defined by \( f_2(x_3) = x_1t_1 + x_2t_2 \). Note that \( f_2 \) is also an embedding. The algebra \( \text{Rees}(f_2) \) is the same as the classical Rees Algebra \( R = \bigoplus_{n=0}^{\infty}(x_3^n) \) and has \( p \)th graded component \( (x_3)^p \neq 0 \). On the other hand \( \text{Rees}(f_2)_p = R(x_1^p + x_2^p) = 0 \) and it follows that \( \text{Rees}(f_2) \) cannot surject onto the classical \( R(f_2) \) by any graded homomorphism. So \( \text{Rees}(f_2) \not\cong \text{Rees}(Rx_3) \) as a graded ring. Note that \( f_2 \) is not a versal map and dual of \( Rx_3 \) requires three generators and a versal map \( g : Rx_3 \rightarrow R^3 \) is defined by \( g(x_3) = x_1t_1 + x_2t_2 + x_3t_3 \). Then we have \( \text{Rees}(g)_p = x_1t_1^p + x_2t_2^p + x_3t_3^p = \text{Rees}((Rx_3)_p) \).

Now we describe an example (see [8]) to show that the defining equations of the syzygies can be of higher degree if the generators of the ideal fail to be a regular sequence.

**Example 2** (See [8]): Let \( R = k[X_1, X_2] \), where \( k \) is a field, and \( I = (X_1^2, X_1X_2, X_2^2) \). Consider the Hilbert-Burch resolution \( 0 \rightarrow R(-3)^2 \xrightarrow{A} R(-2)^2 \xrightarrow{B} R \rightarrow 0 \), where \( A = \begin{pmatrix} X_2 & 0 \\ -X_1 & X_2 \\ 0 & -X_2 \end{pmatrix} \) and \( B = (X_1^2, X_1X_2, X_2^2) \). Hilbert-Burch resolution shows that \( \text{Syz}(X_1^2, X_1X_2, X_2^2) \) is generated by the columns of \( A \). This shows that \( K_1 \) is generated by \( p = X_2X_3 - X_1X_4, q = X_2X_4 - X_1X_5 \). Expressing \( p, q \) in terms of \( X_1 \) and \( X_2 \) gives \( p = (-X_4)X_1 + (X_3)X_2, q = (-X_4)X_1 + (X_4)X_2 \) and taking the determinant of the coefficients gives \( -X_4^2 + X_3X_5 \). Therefore, the Rees algebra of \( I \) is given by

\[
\text{Rees}(I) = \frac{k[X_1, X_2, X_3, X_4, X_5]}{(X_2X_3 - X_1X_4, X_2X_4 - X_1X_5, -X_4^2 + X_3X_5)}.
\]

See [8] for more details.

3. Eakin-Sathaye Theorem over Rees Algebra

The following result illustrates a connection of reductions with Rees algebra.

**Theorem 3.1.** Let \( I \) and \( J \) be two ideals in a Noetherian ring \( R \). Then \( J \) is a reduction of \( I \) iff \( R[It] \) is a finitely generated module over \( R[Jt] \).
Definition 3.2. 

Proof. Suppose $I$ is a reduction of $I$. Then $\exists$ an integer $n$ such that $JI^n = I^{n+1}$ and $\forall k \geq 1$, $J^kI^n = I^{n+k}$. By definition $R[I] = R \oplus I \oplus \ldots \oplus I^n \oplus \ldots$ and $R[It] = R \oplus It \oplus \ldots \oplus Itn \oplus \ldots$ This implies that $(R[I])_{k+n} = I^n t^n (R[It])_k$. Since $R$ is Noetherian, $I'$ is finitely generated for $i = 0, 1, 2, \ldots$. Let $x_{i1}, \ldots, x_{ik}$ be the generators of the $R$-module $I'$. Then $R[I'] = \sum x_{ij} t^i R[It]$. So that $R[I']$ is a finitely generated module over $R[It]$. 

Conversely, note that both the rings $R[I']$ and $R[It]$ are $\mathbb{N}$-graded, where $\mathbb{N}$ denotes the set of natural numbers. Let $n$ be the largest degree of one of these generators. Then $I^n + I^{n+1} = (R[I'])_{n+1} = \sum_{i=1}^{n+1} (J^i t^i) (I^{n+1-i} t^{n+1-i}) = JI^n t^{n+1}$. So that $I^{n+1} = JI^n$ and $J$ is a reduction of $I$. 

Remark II

(1) The minimum integer $n$ such that $JI^n = I^{n+1}$ is the largest degree of an element in a homogeneous minimal generating set of the ring $R[I']$ over $R[It]$. 

(2) In general there is no unique minimal reduction of an ideal but in Noetherian local rings minimal reductions exist.

Associated graded ring and fibre cone ring of $I$ are closely related to Rees algebra in the following way:

Definition 3.2. Let $(R, m)$ be a Noetherian local ring. Then

(1) associated graded ring of an ideal $I$ in $R$ is

\[
gr_I(R) = \oplus_{n \geq 0} \frac{I^n}{I^{n+1}} = \frac{R[I]}{IR[I]} = \frac{R[It, t^{-1}]}{mR[It, t^{-1}]}.\]

(2) $F_I(R) = \frac{R[It]}{mR[It]} = \frac{R}{m} \oplus \frac{I}{mI} \oplus \frac{I^2}{mI^2} \oplus \ldots$ 

The Krull dimension of $F_I$ is also called analytic spread of $I$ and is denoted by $l(I)$. 

Now we state a theorem of Northcott and Rees [4], which is used in the next result.

Theorem 3.3 (4). Let $(R, m)$ be a Noetherian local ring with infinite residue field and $I$ an ideal of analytic spread at most $l$. Then there exists a non-empty Zariski open subset $U$ of $(I/mI)^l$ such that whenever $x_1, x_2, \ldots, x_l \in I$ with $(x_1 + mI, x_2 + mI, \ldots, x_l + mI) \subset U$, then $(x_1, x_2, \ldots, x_l)$ is a reduction of $I$. 

Now we state our main result which is an analogue of the Eakin-Sathaye theorem for Rees algebra. We prove this theorem by induction. Proof of this theorem is based on the idea of a proof given by Huneke and Swanson (see [11]).

Theorem 3.4. Let $R$ be a local Noetherian ring with infinite residue field. Let $I$ be an ideal in $R$ and $R[It]$ denotes the Rees algebra of $I$. Let $n \geq 1$ and $r \geq 0$ be integers such that, $\mu_R(R[It]) < \binom{n+r}{r}$. Then there exists an ideal $J \subset I$ such that $R[It]$ is a finitely generated module over $R[It]$ i.e. for some choice of $f_1, \ldots, f_r$ in the degree 1 component $F_1$ of $F$, homogeneous generators of
$F$ as a finitely generated homogeneous module over the standard graded affine algebra $k[f_1, \ldots, f_r]$ lie in degree at most $n - 1$.

Proof. The proof is based on double induction. If $r = 0$, then $\binom{n}{0} = 1$, and by the hypothesis $\mu(R[I^n t]) = 0$, this implies that $R[I^n t] = 0$. Therefore take $J = 0$. If $n = 1$, then $\binom{r + 1}{1} = r + 1$ and by the hypothesis of the theorem, $\mu(R[I]) \leq r$. Then there exists $f_1, f_2, \ldots, f_r \in R[I]$ such that $R[I] = (f_1, f_2, \ldots, f_r)$. Since $I^0 = R$, we have $R[I] = (f_1, f_2, \ldots, f_r)R[I^0 t]$. Take $J = (f_1, f_2, \ldots, f_r)$. Hence result is proved for $n = 1$.

Thus we can assume that $r > 0$ and $n > 1$. Suppose the result is not true for some $n, r$. If there exists a counter example to this theorem, we take $r$ minimal and take $n$ minimal for this $r$. Then $\mu(R[I^{n - 1} t]) \geq \binom{n - 1 + r}{r}$ elements.

Suppose $y \in I \setminus mI$ such that vector space $\text{Dim}_{R/m}(yI^{n - 1} + mI^n/mI^n)$ is at least $\binom{n - 1 + r}{r}$. For $r = 1$ by assumption $R[I^n t]$ is generated by at most $n$ elements. So by Nakayama lemma, $yR[I^{n - 1} t] = R[I^n t]$. Thus for $r = 1$ result is proved. Therefore in this case we can assume that $r > 1$. Now consider $R' = R/(yI^{n - 1} + mI^n)$. Since $\binom{n + r - 1}{r - 1} + \binom{n + r - 1}{r} = \binom{n + r}{r}$, it follows that $\mu(R'[I^n t]) < \binom{n + r - 1}{r - 1}$. Then by induction on $r$, $\exists f_2, \ldots, f_r \in R[I]$ such that $(f_2, \ldots, f_r)R'[I^{n - 1} t] = R'[I^n t]$. Hence $R[I^n t] \subseteq (f_2, \ldots, f_r)R[I^{n - 1} t] + yR[I^{n - 1} t] + mR[I^n t]$. Now apply Nakayama lemma to get the required for this case.

Now suppose $\forall y \in I \setminus mI$. Then as a vector space $\text{dim}_{R/m}(yI^{n - 1} + mI^n/mI^n) < \binom{n - 1 + r}{r}$. Set $R' = R/(mI^n : y)$. Then $\mu(I^{n - 1} R') < \binom{n - 1 + r}{r}$. Now by induction on $n$, $\exists f_1, \ldots, f_r \in I$ such that $(f_1, \ldots, f_r)I^{n - 2} R' = I^{n - 1} R'$. By Theorem 3.3, $\exists$ a non-empty Zariski open subset $U$ of $(I/mI)^r$ such that whenever $(f_1 + mI, \ldots, f_r + mI) \subseteq U$, then $(f_1, \ldots, f_r)R'$ is a reduction of $R'$ with reduction number at most $n - 2$. Let $(x_1, x_2, \ldots, x_s)$, with each $x_i \in I \setminus mI$. Then, by construction, for each $x_i$ there exists a non-empty Zariski open subset $U_i$ in $(I/mI)^r$ such that whenever $(f_1 + mI, \ldots, f_r + mI) \subseteq U_i$, then $(f_1, \ldots, f_r)R/(mI^n : x_i)$ is a reduction of $I(R/(mI^n : x_i))$, with reduction number at most $n - 2$. Let $U = \cap_i U_i$. Choose $f_1, \ldots, f_r \in I$ such that $(f_1 + mI, f_2 + mI, \ldots, f_r + mI) \subseteq U$. Then for all $i = 1, \ldots, s$, $I^{n - 1} \subseteq (f_1, \ldots, f_r)I^{n - 2} + (mI^n : x_i)$, so that $x_i I^{n - 1} \subseteq (f_1, \ldots, f_r)I^{n - 1} + mI^n$. It follows that $I^n \subseteq (f_1, \ldots, f_r)I^{n - 1} + mI^n$, then by Nakayama lemma we get $I^n = (f_1, \ldots, f_r)I^{n - 1}$. Then $R[I^n t] = (f_1, \ldots, f_r)R[I^{n - 1} t]$. Take $J = (f_1, \ldots, f_r) \subseteq I$. Hence by Proposition 3.1, $R[I]$ is a finitely generated module over $R[J]$. Then the result follows. □

The following propositions are independent but we give them here because these may be useful for the development of the theory for modules.

Proposition 3.5. Let $(R, m)$ be Noetherian local ring and $J \subseteq I$ is a reduction. Suppose $B$ is the sub algebra of $F_I(R)$ generated by $(J + mI)/mI$ over $R/m$. Then $\mu(J) \leq l(I)$. 7
Proof. Since \( J \subseteq I \) is a reduction, by Theorem 3.1, \( R[I] \) is a finitely generated module over \( R[J] \). Hence \( mR[I] \cap R[J] \) is also a finitely generated module and \( \frac{R[I]}{mR[I] \cap R[J]} \cong B \). Since \( B \subseteq F_I \) and Krull dimension of \( F_I = l(I) \), it follows that \( \mu(J) \leq l(I) \). □

Proposition 3.6. Let \((R, m)\) be a local Noetherian ring with infinite residue field and let \( I \) be an ideal in \( R \) such that for associated Rees algebra \( R[I] \) of \( I \), \( mR[I^{n-1}] = 0 \). Let \( y \in R[I] \) and \( K = \{0 : y|_R[I^{n-1}] = \{z \in R[I^{n-1}] | yz = 0\} \}. Then, for any \( J \subseteq I^{n-1} \), \( \frac{R[I]}{(K \cap R[I])} \cong yR[J] \).

Proof. Consider the homomorphism \( f_y : R[J] \to R[J] \) defined by multiplication by \( y \). Then \( Ker f_y = K \cap R[J] \) and by the fundamental theorem of homomorphism, we get \( \frac{R[I]}{(K \cap R[I])} \cong yR[J] \). □

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