MODELLING AND DYNAMICS OF A SELF-SUSTAINED ELECTROSTATIC MICRO ELECTROMECHANICAL SYSTEM

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Abstract

This paper deals with the study of a model of self-sustained electrostatic micro Electromechanical system (MEMS). The electrical part contains two nonlinear components: a nonlinear resistance with a negative slope in the current-voltage characteristics and a capacitor having a cubic form as the charge-voltage characteristics. The modal approximation and the finite differences numerical scheme are used to analyze the dynamical behavior of the system: resonant oscillations and bifurcation diagram leading to chaos are observed for some values of the polarization voltage. Hints of applications of the device are given.

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1 Introduction

Since the talk given by Richard Feynmann entitled “There’s plenty of room at the bottom” at the Californian Institute of Technology in December 1959 [1], micro science world has seen a growing interest. Nowadays, technological advances have enabled the fabrication of engineering systems down to micrometer and nanometer scales. They are made of mechanical branches (beams, plates, gears and membranes), and microelectronic circuits for electrical branches. For this reason, these systems are generally refereed to as Micro Electro Mechanical Systems (MEMS) or Nano Electro mechanical Systems (NEMS). Two basic methods are used in MEMS technology. For sensing applications, a DC polarization is applied to the system [2]. When the goal is driving periodically or stochastically the mechanical arm (which is a moving electrode of a parallel plate capacitor), an electrical signal composed of a DC polarization and an AC component is used to excite harmonic motion or complex dynamics [3-11]. DC current is used to achieve permanent displacement of the beam. Applications of MEMS devices are found in defense, medical surgery, automotive industry and biology.

Despite of the success, many fundamental problems still hold the attention. Some current questions are, the pull in phenomenon, the squeezing damping and the non polynomial form of the electrostatic force. In most of the existing MEMS, as the voltage is directly applied across the capacitor (mechanical part), the resulting electrostatic force is a hyperbolic function of the mechanical displacement. For analytical investigation, one often approximates the electrostatic force by a polynomial function truncated at the first, second or third order [3-8, 10,11]. For optimization purpose, the gap between the electrodes is minimized and the size of the beams (beam making electrodes)are maximized. This fact makes the effect of squeezing film damping more pronounced. Consequently, the modelling equation of the device is more complex. Younis and Nayfeh [6,12] recently presented a robust method to study the behavior of such MEMS. The method is based on finite element method and perturbation technique. Pull-in phenomenon defines the operating condition that leads to the destabilization of the device. There are many contributions concerning this problem, the recurrent solution is based on parametric excitation as proposed by Rhoads et al [11] and Krylov et al [2,10]. However, feedback control algorithm is also used in this target [8].

When MEMS and NEMS are submitted to AC signal, they show very complex behavior[3,4,7-11]. The complexity is a consequence of nonlinearities in the mechanical and transducing parts. One way to have such complexities is to introduce or take into account nonlinear characteristics of the electrical components of the devices. Recently, Taffoti Yollong and Woafor [13] proposed a model made up of an implementation of a Duffing electrical oscillator coupled to a clamped-clamped flexible beam seen as the moving electrode of a linear capacitor. They presented the richness of the behavior of the system and the condition for a complete synchronization in a shift-invariant network of such MEMS.
The goal of the present paper is to model and study the behavior of a self-sustained MEMS where the self-sustained oscillations originate from the electrical component. In fact, results form macro systems, show the possibility of integrating self-sustained electronic circuits in electromechanical devices to perform their functioning; the suitable circuits used are the van der Pol and Rayleigh oscillators [14-16]. Guided by these initial studies, we are considering in this paper self-excited MEMS with capacitive coupling. The modelling of the device dynamics shows that it is described by a partial differential equation (the flexible beam or arm) coupled to a nonlinear differential equation (the electrical circuit). By using the Galerkin modal approach, a set of two coupled differential equations is obtained and constitutes the basis for the analytical and part of the numerical investigations. This is complemented by a direct numerical simulation of the partial differential equations discretized by the finite difference scheme. Focus is made on oscillations at resonance and bifurcation diagrams showing regions of periodic dynamics and chaotic states.

The outline of the paper is the following. Section 2 deals with the modelling of the device using Newton law of dynamics and Kirchhoff’s laws. In section 3, the modal approximation is carried out and the finite difference numerical scheme is presented. The dynamical behavior of the device is analyzed in section 4 using the multiple time scales method and the numerical simulation. We conclude in section 5, giving some hints of application of the device and its dynamical behavior in engineering.

2 The self-sustained electrostatic MEMS

2.1 The self-sustained electrostatic circuit

The device is presented in Figure 1. It is constituted of a nonlinear electrical circuit and a plate capacitor with one beam fixed and the other flexible. The electrical part consists of a nonlinear resistor (NLR), a nonlinear capacitor (NLC) C and an inductor L, all connected in series. The voltage across the capacitor is a nonlinear function of the instantaneous electrical charge $\tilde{q}$ and is expressed as

$$V_C = \frac{1}{C_0} \tilde{q} + a_3 \tilde{q}^3,$$

where $C_0$ is the capacitance of the linear part of the capacitor and $a_3$ is a nonlinear coefficient depending on the type of the capacitor [17]. The current-voltage characteristics of the resistor is given as

$$V_R = -R_0 i_0 \left( \frac{i}{i_0} - \frac{1}{3} \left( \frac{i}{i_0} \right)^3 \right).$$

where $R_0$ and $i_0$ are, respectively, the characteristic resistance and current; $i = \frac{dq}{dt}$ is the current through the resistor. This nonlinear resistor can be realized using a block consisting of two transistors [18], a series of diodes [19] or operational amplifier and diodes [20]. With this resistor
the system has the property to exhibit self excited oscillations. The current-voltage characteristics of the inductor is

$$V_L = L \frac{di}{d\tau},$$

(3)

where $\tau$ is the time.

Besides the above electrical components, there is also the one coming from the mechanical part. Indeed, as the electrical potential across the beam of the capacitor varies, the flexible plate vibrates. Thus the gap between the beam varies, inducing the variation of the electrostatic potential \[13\] given by

$$V_{\text{beam}} = \frac{Q_{\text{total}}}{C} = \frac{Q_{\text{total}}\tilde{g}}{\varepsilon S_0},$$

(4)

where $Q_{\text{total}}$ is the total electrical charge between the beams, $C = \frac{\varepsilon S_0}{g}$ is the capacitance of the plate capacitor, $\tilde{g}$ is the deformed gap between the electrodes, $\varepsilon$ is the permittivity of the air and $S_0$ is the area of each beam.

For the device, $V_{\text{beam}}$ can be rewritten as

$$V_{\text{beam}} = \frac{Q_0 + \tilde{q}}{C_1} \left[ 1 - \frac{W(X, \tau)}{g} \right],$$

(5)

where $W$ is the transverse deflection of the flexible beam, $C_1 = \frac{\varepsilon S_0}{g}$ is the value of the capacitance of the plate capacitor at rest ($W=0$), $g = 0.1\mu m$ is the undeformed gap and $Q_0$ is the charge due to polarization voltage $U_0$.

The equation of the electrical part of the device is thus (using the Kirchoff’s laws)

$$L \frac{d^2\tilde{q}}{d\tau^2} - b \left( \frac{a}{b} - \left( \frac{d\tilde{q}}{d\tau} \right)^2 \right) \frac{d\tilde{q}}{d\tau} + \tilde{q} \left( \frac{1}{C_0} + \frac{1}{C_1} \right) + a_3\tilde{q}^3 = \frac{Q_0 + \tilde{q}}{C_1 g} W(X, \tau),$$

(6)

where $a = 3i_0^2$ and $b = \frac{R_0}{3i_0^2}$.

### 2.2 The mechanical arm

The mechanical arm is a $0.2\mu m \times 0.1\mu m \times 0.5\mu m$ micro clamped-clamped beam of Young modulus $E = 158 \times 10^9 N m^{-2}$ and mass density $\rho = 2330 kg m^{-3}$ [5]. Its dynamics is described by the following equation [21]

$$\rho S \frac{\partial^2 W}{\partial \tau^2} + EI \frac{\partial^4 W}{\partial X^4} + \lambda \frac{\partial W}{\partial \tau} = F(X, \tau),$$

(7)

where $I$ is the moment of inertia, $-\lambda \frac{\partial W}{\partial \tau}$ is the load arising from the squeezed air film between the electrodes (see Ref[2] for derivation) and $F(X, \tau)$ represents the electrostatic force per unit length of the beam given as

$$F(X, \tau) = \frac{(Q_0 + \tilde{q})^2}{2C_1 gl_1},$$

where $l_1$ is the length of the beam.
With the expression of potential in equation (5), the electrostatic load, which is assumed small is this paper, does not vary with the beam deflection. The boundaries conditions are given as [22]

\[ W(X, \tau) = 0 \quad \text{and} \quad \frac{\partial W(X, \tau)}{\partial X} = 0 \quad \text{at both ends}, \]  

which express that the displacement and the rotation are constrained to zero at the ends.

3 Modal equations and the numerical scheme

To facilitate the analysis, we use the dimensionless variables \( Y = \frac{W}{g}, \ x = \frac{X}{l}, \ q = \frac{\tilde{Q}}{Q} \left( \frac{\tilde{Q}}{Q} = \frac{1}{\omega \sqrt{\frac{E}{I}}}, \right) \), \( t = \tau \omega \left( \omega = \frac{k_m^2}{m_l^2} \right) \), \( \eta_0 = \frac{1}{LC_1 \omega^2}, \ \eta_1 = \frac{\tilde{Q}^2}{2g^2 l_1 C_1 \rho S \omega^2}, \ \omega_0^2 = \frac{1}{\omega^2 L} \left( \frac{1}{C_0} + \frac{1}{C_1} \right), \ a_1 = \frac{1}{k_m^2}, \ Q = \frac{Q_0}{Q}. \)

The boundaries conditions are also transformed into

\[ Y(x, t) = 0 \quad \text{and} \quad \frac{\partial Y(x, t)}{\partial x} = 0 \quad \text{at both ends}. \]  

3.1 Formulation of the modal approximation

To have some mathematical analysis of the system, the modal approach is necessary. The transversal deflection of the beam is decomposed in the following form

\[ Y(x, t) = \sum_{m=1}^{\infty} y_m(t) \Phi_m(x), \]  

where (13) \( y_m(t) \) is the time dependent function of each mode and \( \Phi_m(x) \) is the shape function obtained from the undamped equation

\[ \frac{\partial^2 Y}{\partial t^2} + a_1^2 \frac{\partial^4 Y}{\partial x^4} = 0, \]  

with boundaries conditions given in equations (12). Substituting the resulting mode decomposition in equations (10) and (11) and projecting back on the \( i^{th} \) mode [15,16] yields the following set of equations

\[ \frac{d^2 q}{dt^2} - \varepsilon_1 \left( 1 - \left( \frac{dq}{dt} \right)^2 \right) \frac{dq}{dt} + \omega_0^2 q + \beta q^3 = \eta_0 m \ y_m(q + Q), \]  

\[ \frac{d^2 y_m}{dt^2} + \varepsilon_2 \frac{dy_m}{dt} + y_m = \eta m(q + Q)^2, \]  

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with \( \eta_{im} = \int_0^1 \Phi_m(x) \, dx, \quad i = 0, 1 \) and
\[
\Phi_m(X) = -\frac{\sin k_m - \sinh k_m}{\cos k_m - \cosh k_m} \left[ \cos k_m X - \cosh k_m X \right] + \sin k_m X - \sinh k_m X. \quad (17)
\]
The constant \( k_m \) is the solution of the transcendental equation
\[
\cos k_m \cosh k_m - 1 = 0. \quad (18)
\]

### 3.2 The Finite difference algorithm

For obtaining a numerical solution of equations (10) and (11), we use the finite difference scheme as defined by Kitio Kwuimy and Woafó [15, 16]. In this respect, we divide the non-dimensional beam length in \( n \) intervals of length \( h_x \), e.g. \( h_x = \frac{1}{n} \). Also the time is discretized in units of length \( h_t \). Therefore one can write \( x_i = (i - 1)h_x \) and \( t_j = jh_t \) where \( i \) and \( j \) are integer variables. Consequently, equations (10) and (11) become

\[
\frac{d^2 q}{dt^2} - \varepsilon_1 \frac{dq}{dt} \left( 1 - \left( \frac{dq}{dt} \right)^2 \right) + w_0^2 q + bq^3 = \eta_0 Y_{i,j}(q + Q), \quad (19)
\]

\[
A_1 Y_{i,j+1} + A_2 Y_{i,j} + A_3 Y_{i,j-1} + A_4 (Y_{i+2,j} + Y_{i-2,j}) + A_5 (Y_{i+1,j} + Y_{i-1,j}) = \eta_1 (q + Q)^2, \quad (20)
\]

for \( i = 2, \ldots, n \) and \( \forall j \in \mathbb{N} \),

with
\[
A_1 = \frac{1}{h_t^2} + \frac{\varepsilon_2}{2h_t}, \quad A_2 = \frac{-2}{h_t^2} + \frac{6a}{h_x^2}, \quad A_3 = \frac{1}{h_t^2} - \frac{\varepsilon_2}{2h_t}, \quad A_4 = \frac{a^2}{h_x^4}, \quad A_5 = 4A_4.
\]

The boundary conditions are (\( \forall j \in \mathbb{N} \))
\[
Y_{1,j} = Y_{n+1,j} = 0, \quad Y_{0,j} = Y_{2,j} \quad \text{and} \quad Y_{n+2,j} = Y_{n,j}. \quad (21)
\]

One can show that the discretization scheme is stable if
\[
\frac{8}{h_x^2} \leq \frac{1}{h_t^2} \left[ 1 + \sqrt{1 - \frac{(\varepsilon_2 h_t)^2}{4}} \right], \quad (22)
\]

with \( \varepsilon_2 h_t \leq 2 \).

The mathematical and a part of the numerical analysis of the system are based on equations (15) and (16). We restrict the analysis to the first mode \( m = 1 \). The results obtained from the modal analysis are complemented by the numerical simulation of equations (19) and (20) with \( h_x = 0.1 \) and \( h_t = 0.001 \) to ensure the stability of the discretization scheme according to equation (22).

### 4 Dynamical behavior of the device

#### 4.1 The resonant states

In MEMS applications such as actuation, the resonant motion is useful as the resonant energy compared to non-resonant motion predominates. Therefore, in this section, the resonant conditions of the MEMS in equations (11) and (10) will be determined through the multiple time scales
method [21]. We use following decomposition

\[ q_1 = q_0 (T_0, T_1) + \varepsilon_0 q_0 (T_0, T_1) + O (\varepsilon_0^2), \]

\[ y_m = y_{0m} (T_0, T_1) + \varepsilon_0 y_{1m} (T_0, T_1) + O (\varepsilon_0^2), \]

where \( T_0 = t \) and \( T_1 = \varepsilon_0 t \) are respectively the fast scale (associated to the unperturbed system) and the slow scale (associated to the amplitude and phase modulation induced by the global first order perturbation). \( \varepsilon_0 \) is a small dimensionless parameter. Using the formalism described and intensively used in Nayfeh and Mook [21] one can show that at the zero order, \( q_0 \) and \( y_{0m} \) can be given as

\[ q_0 (T_0, \varepsilon_0) = \frac{1}{2} A \text{exp} (\omega_0 T_0) + C.C, \]

\[ y_{0m} (T_0, \varepsilon_0) = \frac{1}{2} B \text{exp} (J T_0) + C.C, \]

where \( C.C \) stands for the complex conjugate of each preceding term, \( J^2 = -1 \), \( A \) and \( B \) are complex functions who are determined latter. At the first order, one can obtains

\[ D_0^2 q_1 + \omega_0^2 q_1 = \left[ J \varepsilon_1 \omega_0 A \left( 1 - 3 \omega_0^2 A \bar{A} \right) - 2 J \omega_0 D_1 A - 3 \beta A^2 \bar{A} \right] \text{exp} (\omega_0 T_0) + \]

\[ + \eta_{0m} \left[ - J Q B \text{exp} (J T_0) + A B \text{exp} (1 + \omega_0 T_0) + \bar{A} B \text{exp} (1 - \omega_0 T_0) \right] \]

\[ (J \varepsilon_1 \omega_0^3 - \beta) A^3 \text{exp} (3 \omega_0 T_0) + C.C, \]

\[ D_0^2 y_{1m} + y_{1m} = \left[ - J \varepsilon_2 B - 2 J D_1 B \right] \text{exp} (J T_0) + \]

\[ \eta_{1m} \left[ A^2 \text{exp} (2J \omega_0 T) + 2 A \bar{A} + 2 AQ \text{exp} (J \omega_0 T) + Q^2 \right] + C.C, \]

where \( D_0 = \frac{d}{dT_0} \) and \( D_1 = \frac{d}{dT_1} \). An analysis of these equations shows two interesting resonant states in which the flexible beam vibrates.

### 4.1.1 First resonant state

The first resonant state appears at \( \omega_0 = 1 + \sigma \), where \( \sigma \) is the detuning parameter defining the nearness of \( \omega_0 \) and the natural frequency of the mechanical microbeam. The secularity condition in this state corresponds to the following coupled differential equation satisfied by \( A \) and \( B \)

\[ -2 j \omega_0 D_1 A + j \varepsilon_1 \omega_0 A \left( 1 - 3 \omega_0^2 A \bar{A} \right) - 3 \beta A^2 \bar{A} - J \eta_{0m} Q B = 0, \]

\[ -2 J D_1 B - J \varepsilon_2 B + 2 \eta_{1m} A Q = 0. \]

Expressing \( A \) and \( B \) in polar form

\[ A = \frac{1}{2} a \text{exp} (j \phi_1), \]

\[ B = \frac{1}{2} b \text{exp} (j \phi_2), \]

\[ C.C, \]

\[ D. \]
\[ B = \frac{1}{2} b \exp(j \phi_2), \] (32)

where \( a \) and \( \phi_1 \) respectively \( b \) and \( \phi_2 \) are the amplitude and the phase of fundamental solution, we obtain the following set of first order differential equation for the amplitudes and phases,

\[
\begin{align*}
\frac{da}{dt} &= \frac{\epsilon_1 a}{2} \left( 1 - \frac{3}{4} a^2 \omega_0^2 \right) + \frac{\eta_{0m} Q b}{2 \omega_0} \sin \varphi, \\
\frac{db}{dt} &= -\frac{\epsilon_2 b}{2} - \eta_{1m} a Q \sin \varphi, \\
\frac{d\varphi}{dt} &= \sigma - \frac{3 \beta a^2}{8 \omega_0} - \left( \frac{Q \eta_{1m} a}{b} - \frac{Q \eta_{0m} b}{2 a \omega_0} \right) \cos \varphi,
\end{align*}
\] (33-35)

where \( \varphi = \phi_1 - \phi_2 \) is the phase difference between \( q_0 \) and \( y_{0m} \). The stationary values of \( a \) and \( b \) are solutions of the following algebraic equations

\[
\begin{align*}
c_6 a^6 + c_4 a^4 + c_2 a^2 + c_0 &= 0, \\
b &= M (4 - 3a^2 \omega_0^2),
\end{align*}
\] (36)

with

\[
\begin{align*}
c_6 &= \frac{9 \omega_0^6 \xi_2}{64}, \\
c_4 &= -\frac{9 \omega_0^6 \xi_2 \eta_{1n}}{Q^2 \eta_{0m}} (\xi_1 - 1) + \frac{9}{16} \eta_{1n} \xi_2 \omega_0^2 (\xi_2 - 1), \\
c_2 &= \frac{3}{2} \omega_0^2 \xi_2 \eta_{1n} \xi_2 \left( 1 - \frac{\xi_1 \eta_{1n}}{2} (\xi_1 - 1)^2 \right) - \eta_{1n} \xi_2 \omega_0^2 (\xi_1 - 1) + \frac{4 \omega_0^2}{Q \eta_{0m}} \left( \sigma - \frac{\beta}{2 \omega_0^2} \right), \\
c_0 &= \eta_{1n} \xi_2 (\xi_1 - 1)^2 (\xi_2 - 1) + \frac{4 \omega_0^2}{Q \eta_{0m}} \xi_2 \eta_{1n} \sigma, \\
M &= \frac{\xi_1 \omega_0 \eta_{0m}}{8 \xi_2 \eta_{0m}}, \quad \xi_1 = \frac{2 \xi_2}{\xi_1}, \quad \xi_2 = \frac{\xi_2 \xi_1 \omega_0^2}{4 \eta_{1n} \eta_{0m} Q^2}.
\end{align*}
\]

In MEMS technology, fringe effect, squeeze-film and air flow damping affect the mechanical dissipative coefficient. The amplitude curves are plotted in Figures 2 as function of dissipative coefficient \( \epsilon_2 \) for \( C_0 = 1.77 \times 10^{-18} F, \quad C_1 = 5.31 \times 10^{-18} F, \quad L = 5.41 H, \quad a = 1.211 \times 10^8 A^2, \quad b = 1.137 \times 10^{23} \Omega A^{-2}, \quad and \quad a_3 = 9.84 \times 10^{48} V C^{-3} (\epsilon_1 = 0.06, \quad \beta = 0.1, \quad \eta_0 = 0.25, \quad \eta_1 = 0.75, \omega_0 = 1). \)

Newton-Raphson algorithm is used to plot the results from analytical investigation (curve with points) while for the results of the semi analytical or modal approach (curve with stars), the fourth-order Runge Kutta procedure is used to solve the differential equations (15) and (16). These amplitude curves present for \( \epsilon_2 < 0.05 \) two possible amplitudes for analytical results. Further investigations show that only one of them is stable (the higher one) as shown by numerical results. Consequently, in practice the MEMS can not vibrate with the lower amplitude.

Increasing \( \epsilon_2 \) leads to the non complete quenching phenomenon for the mechanical arm vibrations. This results is understandable since of the DC polarization induces a permanent static deflection. Note that, the corresponding electrical variable is not null but very weak. The domain of no oscillation (also obtained by setting \( c_0 = 0 \)) corresponds to the domain of stability of the fixed point \( (q = 0, \frac{dq}{dt} = 0, y_m = 0, \frac{dy_m}{dt} = 0) \) and, there is a little difference between the results
from analytical treatment and those from semi-analytical approach. After the region of quenching, the amplitude of the electrical oscillator is an increasing function of $\varepsilon_2$ while for the mechanical arm the amplitude increases and decreases with a maximum at $\varepsilon_2 = 0.146$. The finite difference simulation algorithm gives the results of Figures 2 in cross. The results qualitatively agree with those of modal and analytical approaches, but in place of quenching phenomenon, we have a minimum amplitude. In fact, for some value of the system modal approximation and finite difference simulation results do not converge. This result was also obtained by Kitio Kwuimy and Woafò [16] when studied a self-sustained electromechanical device with nonlinear coupling.

### 4.1.2 Second resonant state

The second resonant state is obtained from equations (27) and (28) by setting $\omega_0 = 2 + \sigma$. In this case the complex amplitudes $A$ and $B$ satisfy the secular equations

\[
-2j\omega_0 D_1 A + j\varepsilon_1 \omega_0 A \left(1 - 3\omega_0^2 AA\right) - 3\beta A^2 \bar{A} - J\eta_1 m AB = 0,
\]

\[
-2JD_1 B - J\varepsilon_2 B + 2\eta_1 m A \bar{A} = 0.
\]

(37)

(38)

Express $A$ and $B$ in polar form as above, one obtains the following set of differential for the amplitudes and phases.

\[
\frac{da}{dt} = \frac{\varepsilon_1 a}{2} \left(1 - \frac{3}{4} a^2 \omega_0^2\right) + \frac{\eta_1 m ab}{4\omega_0} \sin \varphi,
\]

\[
\frac{db}{dt} = -\frac{\varepsilon_2 b}{2} + \frac{\eta_1 m a}{4} \sin \varphi,
\]

\[
\frac{d\varphi}{dt} = \sigma - \frac{3\beta a^2}{4\omega_0} + \left(\frac{\eta_1 m a}{4B} + \frac{\eta_1 m b}{2\omega_0}\right) \cos \varphi.
\]

(39)

(40)

(41)

The stationary solutions satisfy the following set of algebraic equations

\[
c_6 a^6 + c_4 a^4 + c_2 a^2 + c_0 = 0,
\]

\[b = M (4 - 3a^2 \omega_0^2),\]

(42)

(43)

with

\[
c_6 = \frac{9}{16} \left(\frac{3\xi_1 \omega_0^2}{4} - 1\right), \quad c_4 = \frac{3}{4} \left[\frac{9\omega_0^4 \xi_1}{4} + \frac{3\beta \xi_2}{4\omega_0} + 2\omega_0^2\right], \quad c_2 = \frac{3}{2} \left[\frac{3\omega_0^2 \xi_1}{2} - \frac{\beta \xi_2}{\omega_0}\right],
\]

\[c_0 = \xi_2 \sigma^2 - \xi_1, \quad M = \frac{\varepsilon_1 \omega_0 \eta_1 n_1}{8 \varepsilon_2 \omega_0 \eta_0 m \eta_1 n}, \quad \xi_1 = \frac{4\varepsilon_1 \varepsilon_2 \omega_0 \omega_0 n}{\eta_0 n_1 \eta_1 n}, \quad \xi_2 = \frac{16 \omega_0 \varepsilon_1}{\varepsilon_2 \omega_0 \eta_1 n \eta_0 m}.
\]

This resonant case can be obtained with the values of Figures 2 where $C_0 = 2.45 \times 10^{-16} F$. The amplitude curves of the system are plotted in Figures 3 for $\varepsilon_2 \in [0, 0.5]$. The figures show that the amplitude of the beam deflection is a decreasing function of the dissipative coefficient while the amplitude of the electrical part first decreases and increases with $\varepsilon_2$ the minimum appears at $\varepsilon_2 = 0.1$. Results from finite difference simulation are with crosses, those from modal approach are
in point and those from analytical treatment are in lines. In this case, the first mode approximation
gives good satisfaction comparing to the results of finite difference simulation. However, there
is a difference between the two numerical approaches and the analytical investigation. This is
a consequence of approximation made in analytical treatment for this type of resonance and
coupling.

The difference is more pronounced for small value of $\varepsilon_2$ (mechanical arm) or when $\varepsilon_2$ is near $\varepsilon_1$ (electrical part).
The MEMS thus present two resonant states which can be used for positioning or others MEMS
applications. Attention should be made on the fact that depending on the value of the mechanical
dissipation (fringe effect, squeeze-film or air flow damping affect) the system can be at rest (first
resonant state) or has a minimal displacement (second resonant state).

4.2 Bifurcation and chaos

The presence of nonlinear terms in a physical system can have positive or negative interest ac-
cording to the utility of the system. It is thus important to find how a particular nonlinear system
will exhibit the complexity of chaos or the simplicity of order. Chaotic motion due to various
mechanisms has been reported for MEMS with separated drive actuators for signal encryption
applications [3] and MEMS based on variable gap capacitors [4,7]. Recently, DeMartini et al [23]
used the Melnikov’s method to define the regions of parameter space where homoclinic chaos
can occur for a MEMS governed by the nonlinear Mathieu equation. To determine chaos, one
generally evaluates the divergence rate of initially nearby trajectories in the state space. This
can be done through the Lyapunov exponent coupled to the bifurcation diagram. The Lyapunov
exponent expresses the convergence (when negative) or the divergence (when positive) of nearby
trajectories. Therefore, a system is said to be chaotic if the exponent is positive which corresponds
in the bifurcation diagram to a cloud of points. For a multi periodic dynamics, the exponent is
asymptotically null. Krylov [24] recently used with good results the Lyapunov exponent to study
pull-in instability of electrostatically actuated microstructure.

For the modal approximation, we define the exponent as
\[
ly = \lim_{t \to \infty} \frac{\ln(d_0(t))}{t} \quad \text{with} \quad d_0 = \sqrt{\left(\frac{dq}{dt}\right)^2 + \left(\frac{dy_m}{dt}\right)^2},
\]
while for the partial differential equation, we define the Lyapunov exponent by
\[
lyn = \lim_{t \to \infty} \frac{\ln(d_1(t))}{t} \quad \text{with} \quad d_1 = \sqrt{\left(\frac{dq}{dt}\right)^2 + \sum_{i=1}^{n} \left(\frac{\partial dY_i}{\partial t}\right)^2 + \sum_{i=1}^{n} \left(\frac{dY_i}{dt}\right)^2},
\]
where $dq$, $dy_m$ and $dY_i$ are respectively the variations of $q$, $y_m$ and $Y_i$.

One finds in this section how chaos arises in the self-sustained MEMS as the DC actuation
changes. But this requires an appropriate choice of the values of the components, we set $a_3 =$
5.04 \times 10^{19} V C^{-3}, \quad a = 2.02 \times 10^6 A^2, \quad b = 4.42 \times 10^{22} \Omega A^{-3}, \quad C_0 = 2.19 \times 10^{-17} F, \quad L = 0.90 H (\omega_0 = 1) and increasing the DC parameter from 0 to 5. The Lyapunov exponent of the system is plotted in Figure 4 from modal approximation (Figure 4a) and from finite difference simulation (Figure 4b). The corresponding bifurcation diagram of the mechanical arm is plotted in Figure 5. The two numerical approaches agree for some value of the DC charge as shown in Figure 4. In addition to the remark about the one mode approximation presented in the above section, the divergence of the methods is also due to the fact that equation (44) defines the exponent for four dimensional system while equation (45) is for a $2n+2$ dimensional system.

The MEMS device displays successively a period-2 orbit, chaotic motion, period-2 and period-1 motion as shown in the bifurcation diagram in Figure 5 (from modal approximation). Chaotic phase portraits of both parts of the system are plotted in Figures 6 and 7 for modal approach and finite difference simulation with $Q = 1.5, \quad \varepsilon_1 = 0.006, \quad \beta = 4.1, \quad \eta_0 = 0.15, \quad \eta_1 = 1$ ($Q_0 = 1.71 \times 10^{-16} C, \quad U = 10.77 V, \quad \text{and} \quad L = 0.90 H$).

The second resonance state where $\omega_0 = 2$ also presents a rich bifurcation diagram as illustrated in Figure 8 for $a_3 = 8.64 \times 10^{48} V C^{-3}, \quad a = 1.21 \times 10^8 A^2, \quad b = 1.43 \times 10^{21} \Omega A^{-2}, \quad C_0 = 4 \times 10^{-16} F, \quad \text{and} \quad L = 0.54 H$. The curves show a period-2 dynamics followed by a chaotic region (in which some windows of period doubling bifurcations are observed) and end by another period-2 motion.

## 5 Conclusion

The aim of this paper was to present a self-sustained electrostatic MEMS in which self-sustained behavior originates from an electrical oscillator. After the modelling of the device, a nonlinear analysis of the system showed two resonant states and possibility of chaotic behaviors. The results from modal approach were complemented by those of finite difference simulation of the partial differential equations.

Results of the work are relevant to a broad variety of applications including actuation, cryptography and micro switches. The device present a rich chaotic behavior, thus it is a good candidate to secure communication by using synchronization chaos [3,25]. The quenching phenomenon can be use in switching [5] conditions to avoid oscillation corresponding to a given value of the squeezing air film force. Another important application is the use of the two oscillating frequency; in fact by adjusting the value of the capacitance $C_0$, one can multiply the frequency by two and thus change the behavior of the system.

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References


Figure 1: The self-sustained MEMS.

Figure 2: Amplitude of the parts of the self-sustained oscillator as function of the beam dissipative coefficient for the first resonance: Results from the analytical (curve with point), semi analytical investigation (curve with stars) and finite difference simulation (curve with cross). (a) Amplitude of the electrical part. (b) Amplitude of the mechanical part.
Figure 3: Amplitude of the parts of the self-sustained oscillator as function of the beam dissipative coefficients for the second resonance. (a) Amplitude of the electrical part. (b) Amplitude of the mechanical part.
Figure 4: Top Lyapunov exponent as function of DC polarization for the first resonance. (a) From modal approximation. (b) From finite difference simulation.
Figure 5: Bifurcation diagram of the mechanical arm as function of DC polarization for the first resonance from modal approximation.
Figure 6: Chaotic phase portrait of the beam deflection for the first resonance. (a) From modal approximation. (b) From finite difference simulation.
Figure 7: Chaotic phase portrait of the electrical variable for the first resonance. (a) From modal approximation. (b) From finite difference simulation.
Figure 8: Bifurcation diagram of the mechanical arm as function of DC polarization for the second state from modal approximation.