GEOMETRY OF EXPANDING ABSOLUTELY CONTINUOUS INVARIANT MEASURES AND THE LIFTABILITY PROBLEM

José F. Alves¹, Carla L. Dias²
Departamento de Matemática Pura, Faculdade de Ciências do Porto,
Rua do Campo Alegre 687, 4169-007 Porto, Portugal

and

Stefano Luzzatto³
Mathematics Department, Imperial College
180 Queen’s Gate, London SW7, United Kingdom

and

The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy⁴.

Abstract

We consider a quite broad class of maps on compact manifolds of arbitrary dimension possibly admitting critical points, discontinuities and singularities. Under some mild nondegeneracy assumptions we show that $f$ admits an induced Gibbs-Markov map with integrable inducing times if and only if it has an ergodic invariant probability measure which is absolutely continuous with respect to the Riemannian volume and has all Lyapunov exponents positive.

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¹jalves@fc.up.pt
²carlald@fc.up.pt
³luzzatto@ictp.it
⁴Current address.
1. Introduction and statement of results

Let $M$ be a compact Riemannian manifold and $f : M \to M$ a measurable map preserving an ergodic probability measure $\mu$ which is absolutely continuous with respect to the Riemannian volume (Lebesgue measure) $\text{Leb}$ on $M$. Supposing that $f$ is differentiable along the orbit of some point $x \in M$ we define the (forward) Lyapunov exponent of $x$ along the direction $v \in T_xM$ as

$$\lambda(x, v) = \liminf_{n \to \infty} \frac{1}{n} \log \|Df^n(x)v\|.$$ 

If $f$ is differentiable $\mu$ almost everywhere in $M$ and $\log |Df| \in L^1(\mu)$ (for example if $f$ is a $C^1$ diffeomorphism or a local diffeomorphism), then the limit above actually exists for $\mu$ almost every point $x$ and any vector $v \in T_xM$, then we say that the measure $\mu$ is hyperbolic. If $\lambda(x, v) > 0$ for $\mu$ almost every $x \in M$ and every $v \in T_xM$, then we say that the measure $\mu$ is expanding.

The local geometry (local stable and unstable manifolds, absolute continuity of stable and unstable foliations, etc.) of hyperbolic measures has been extensively studied in the pioneering and fundamental papers of Pesin [18] for absolutely continuous invariant measures for $C^{1+\epsilon}$ diffeomorphisms; Ruelle [21] for arbitrary invariant measures for $C^{1+\epsilon}$ diffeomorphisms and [22] for not necessarily invertible $C^{1+\epsilon}$ endomorphisms; Katok and Strelcyn [16] for the case of $C^{1+\epsilon}$ diffeomorphisms with discontinuities. More recently Barreira, Pesin and Schmeling [7] have resolved a long standing conjecture about the pointwise dimension of a hyperbolic measure.

In this paper we are interested in the global geometry of absolutely continuous expanding measures. We shall show that such measures imply a remarkable Markov structure. We recall that if $\Delta \subseteq M$, then an induced map $F : \Delta \to \Delta$ is a map such that $F(x) = f^{R(x)}(x)$ where $R : \Delta \to \mathbb{N}$ is an inducing time function with the property that $f^{R(x)}(x) \in \Delta$ whenever $x \in \Delta$.

**Definition 1.1.** We say that an induced map $F : \Delta \to \Delta$ is Gibbs-Markov if $\Delta \subseteq M$ is an open set and there exists a (Leb mod 0) partition $\mathcal{P}$ of $\Delta$ into open subsets such that $R$ is constant on each element $U \in \mathcal{P}$ and $f^{R(U)} : U \to \Delta$ is a uniformly expanding diffeomorphism with uniformly bounded volume distortion: there are $0 < \kappa < 1$ and $K > 0$ such that for all $U \in \mathcal{P}$ and all $x \in U$

\[i) \quad \|DF(x)^{-1}\| < \kappa;\]
\[ii) \quad \log \left| \frac{\det DF(x)}{\det DF(y)} \right| \leq K \text{dist}(F(x), F(y)).\]

Moreover, if the inducing time function $R$ is integrable with respect to Leb, then we say that the induced map has integrable return times.

We remark that the Gibbs-Markov structure of an induced map is quite a strong and non-trivial condition which cannot be expected to hold in general. It is for example generally not true that the first return time induced map to some arbitrary open set will have a Gibbs-Markov structure. Supposing that one can find or construct an induced Gibbs-Markov map it is also non-trivial, and not true in general, that the return time function will be integrable.
1.1. **Global geometry of expanding measures.** The main purpose of our paper is to show that in fact, in great generality, the abstract measure-theoretic conditions on the existence of an absolutely continuous expanding invariant measure are sufficient to imply the much more concrete geometrical structure of an induced Gibbs-Markov map with integrable return times. More specifically, we shall define below three increasingly general classes of maps:

1. $C^2$ local diffeomorphisms;
2. $C^2$ maps with critical points;
3. $C^2$ maps with critical points, discontinuities and singularities (infinite derivative).

In the local diffeomorphism case we shall show without any additional assumptions that the existence of an expanding absolutely continuous invariant measure implies the existence of a Gibbs-Markov map with integrable return times. In the other two cases we shall obtain the same conclusion by adding some mild nondegeneracy conditions on the set of critical points, discontinuities and singularities. We note that the result is non-trivial even in the simplest setting of a local diffeomorphism even though it is of course most remarkable in the setting with critical points, discontinuities and singularities since these are all significant potential obstructions to the existence of regular geometric structures. We mention that a version of this statement in the one-dimensional setting was proved in [11] by quite different arguments.

We remark also that these results should have applications beyond the purely geometrical structure of such systems. Indeed, the existence of induced Gibbs-Markov maps in various specific systems has already been used extensively to study various properties such as decay of correlations and statistical stability, see for example [25, 8, 4, 12]. In particular we give a complete solution to the so-called “liftability problem” in the case of absolutely continuous invariant measures as described below.

1.2. **Solution of the liftability problem.** In the last few years there has been a significant amount of work in which the construction of induced Gibbs-Markov maps has been used to prove the existence of ergodic absolutely continuous invariant probability measures, see for example [25, 8, 4, 13]. Indeed, given an induced map $F$ as above with integrable return times, it is a classical result that $F$ admits an ergodic absolutely continuous invariant probability measure (acip) $\nu$ with bounded density. It is then possible to define a measure

$$
\mu = \frac{\sum_{j=0}^{\infty} f^j \nu(\{R > j\})}{\sum_{j=0}^{\infty} \nu(\{R > j\})}.
$$

The integrability condition with respect to the Riemannian volume and the bounded density of $\nu$ imply the integrability of $R$ with respect to $\nu$ and thus guarantees that the denominator is finite. Therefore, by well known standard calculations it follows from fact that $\nu$ is an acip for $F$, that $\mu$ is an expanding acip for $f$.

**Definition 1.2.** We say that $\nu$ is the lift of the measure $\mu$ (to the induced map).
In the light of this definition we have the following natural problem which is often referred to as the liftability problem.

**Question 1.** Does every ergodic acip admit a lift (to an induced Gibbs-Markov map)?

The answer to this question if negative in full generality as even such well known systems as area preserving Anosov diffeomorphisms [6] cannot admit induced Gibbs-Markov maps since they have a contracting direction. However, as a corollary of our results, the lack of an expanding direction is essentially the only obstruction to liftability. Thus, for the classes of maps we consider and which we will define precisely below we obtain the following

**Theorem.** An ergodic acip $\mu$ is liftable if and only if $\mu$ is expanding.

This gives a rigorous conceptual justification of the importance and scope of induced Gibbs-Markov maps in the theory, and shows that this condition can essentially be used as a definition of nonuniformly expanding.

1.3. **Innovations in the construction.** Finally, before giving the formal statements of our results, we say a few words on the construction of the induced Gibbs-Markov maps in this paper and in comparison to analogous constructions carried out in previous papers.

One key difference between the construction carried out here and that given in other papers such as [25, 8, 14, 9, 10] is that we use as a fundamental technical tool in the construction the notion of a hyperbolic time. This idea which was introduced in [2] and has since then been widely applied in a variety of settings including the construction of induced Gibbs-Markov maps in several situations such as those considered in [4, 13], where the aim was to prove the existence of an acip. In all these papers (irrespective of the use of hyperbolic times or not) the construction of induced Gibbs-Markov map is quite involved and technical, using a mixture of relatively sophisticated combinatorial, analytic and probabilistic arguments. In this paper we introduce a different strategy which is based on hyperbolic times but on a significantly simplified combinatorial argument. This allows us give the complete inductive definition of the construction in just a few paragraphs and with a minimal amount of additional notation and indexing.

A second key issue in which the argument here differs from previously applied arguments is in the strategy for controlling the tail of the inducing time. Previous papers consider situation with various rates of decay (e.g. exponential, polynomial) of the inducing time function, depending on various additional assumptions on the map. However none of them include the case in which the return times are simply integrable as the constructions there are not efficient enough, so starting from assumptions that certain quantities decay exponentially or polynomially it is possible to obtain similar rates with possibly some loss in the exponents but starting from only integrability assumptions, the lack of an efficient construction algorithm does not make it possible to obtain integrable inducing times. The new strategy we apply in this paper gives a much more efficient construction which in particular yields the integrability of the inducing time function without
in fact having to make any assumptions at all on the integrability of any other quantity. We remark that a different and also very efficient construction has been also implemented in the recent preprint [20] to obtain some related results.

1.4. Local diffeomorphisms. We are now ready to give the formal statements of our results. As mentioned above we shall consider three increasingly general classes of functions. First of all let \( f : M \to M \) be a \( C^2 \) local diffeomorphism.

**Theorem 1.** Let \( \mu \) be an ergodic \( f \)-invariant absolutely continuous probability measure. There exists an induced Gibbs-Markov map \( F \) with an invariant probability measure \( \nu \) such that \( \mu \) and \( \nu \) are related by (\*\*) if and only if \( \lambda(x,v) > 0 \) for \( \mu \) almost every \( x \) and all \( v \in T_x M \).

The “only if” part is a relatively standard easy calculation because the Lyapunov exponents of \( f, \mu \) are the simply the Lyapunov exponents of \( (F, \hat{\mu}) \) divided by the average value of the inducing times, see e.g. [5, Lemma 4.1]. In this paper we will prove the “if” part that any measure with positive Lyapunov exponents admits an induced Markov map.

1.5. Smooth maps with critical points.

**Definition 1.3.** We say that \( x \) is a critical point if \( Df(x) \) is not invertible. We say that a set of critical points \( C \) is non-degenerate if \( \text{Leb}(C) = 0 \) and, letting \( \text{dist}(x,C) \) denote the distance between the point \( x \) and the set \( C \), there are constants \( B > 1 \) and \( \beta, \beta' > 0 \) such that for every \( x \in M \setminus C \)

\[
(C1) \quad \frac{1}{B} \text{dist}(x,C)^\beta \leq \min_{\|v\|=1} \|Df(x)v\| \leq B \text{dist}(x,C)^{\beta'}. 
\]

Moreover, the functions \( \log \det Df \) and \( \log \|Df^{-1}\| \) are locally Lipschitz at points \( x \in M \setminus C \): for every \( x, y \in M \setminus C \) with \( \text{dist}(x,y) < \text{dist}(x,C)/2 \) we have

\[
(C2) \quad |\log \|Df(x)^{-1}\| - \log \|Df(y)^{-1}\| | \leq B \frac{\text{dist}(x,y)}{\text{dist}(x,C)^{\beta}}, \\
(C3) \quad |\log |\det Df(x)| - \log |\det Df(y)| | \leq B \frac{\text{dist}(x,y)}{\text{dist}(x,C)^{\beta}}. 
\]

**Remark 1.4.** Notice that the formulation of (C1) differs slightly from analogous definitions used in other papers in that it includes an upper bound. This guarantees a similar upper bound for \( |\det Df(x)| \) which, together with the fact that \( \log |\det Df| \in L^1(\mu) \) for general \( C^2 \) maps [17, Remark 1.2], implies that \( \log \text{dist}(\cdot, C) \in L^1(\mu) \).

**Theorem 2.** Let \( f : M \to M \) be \( C^2 \) and such that for each \( n \geq 1 \), \( f^n \) is a local diffeomorphism outside a non-degenerate critical set \( C_n \), and let \( \mu \) be an ergodic \( f \)-invariant absolutely continuous probability measure. There exists an induced Gibbs-Markov map \( F \) with an invariant probability measure \( \nu \) such that \( \mu \) and \( \nu \) are related by (\*\*) if and only if \( \lambda(x,v) > 0 \) for \( \mu \) almost every \( x \) and all \( v \in T_x M \).
1.6. Maps with discontinuities.

**Definition 1.5.** We say that \(x\) is a **singular point** if \(Df(x)\) is not invertible or simply does not exist (including the case in which \(f\) is discontinuous at \(x\)). We say that a set of singular points \(C\) is **non-degenerate** if \(\text{Leb}(C) = 0\) and, letting \(\text{dist}(x, C)\) denote the distance between the point \(x\) and the set \(C\), there are constants \(B > 1\) and \(\beta > 0\) such that for every \(x \in M \setminus C\), conditions (C2), (C3) of the previous definition are satisfied, and condition (C1) is replaced by

\[
(C1') \quad \frac{1}{B} \text{dist}(x, C)^\beta \leq \frac{\|Df(x)v\|}{\|v\|} \leq B \text{dist}(x, C)^{-\beta}, \quad \forall v \in T_x M.
\]

**Theorem 3.** Let \(f : M \to M\) be such that for each \(n \geq 1\), \(f^n\) is a \(C^2\) local diffeomorphism outside a non-degenerate singular set \(C_n\), and let \(\mu\) be an ergodic \(f\)-invariant absolutely continuous probability measure such that \(\log \text{dist}(\cdot, C_n) \in L^1(\mu)\). There exists an induced Gibbs-Markov map \(F\) with an invariant probability measure \(\nu\) such that \(\mu\) and \(\nu\) are related by \((\ast)\) if and only if \(\lambda(x, v) > 0\) for \(\mu\) almost every \(x\) and all \(v \in T_x M\).

1.7. **Overview of the paper.** We remark first of all that the statement of Theorem 3 includes Theorems 1 and 2 as special cases. Indeed, in the setting of Theorem 1, the singular set is empty, and in the setting of Theorem 2, for \(C^2\) maps, (C1) is clearly more restrictive than (C1') and therefore the critical set of Theorem 2 satisfies the nondegeneracy conditions of the more general singular set of Theorem 3. Moreover, the integrability assumption \(\log \text{dist}(\cdot, C_n)\) of Theorem 3 is automatically satisfied in the settings of Theorem 1 and Theorem 2. In Theorem 1 this is immediate since the derivative is bounded above and below and the singular set is empty. In Theorem 2 this follows by known results as discussed in Remark 1.4. We shall therefore concentrate on the proof of the most general setting as formulated in Theorem 3.

In Section 2 we show that some power of \(f\) satisfies some stronger expansion condition and also some slow recurrence to the singular set. These are the standard conditions which are usually assumed in the setting of so-called **nonuniformly expanding maps**. In Section 3 we recall some known properties of nonuniformly expanding maps including the crucial notion of **hyperbolic time**. We also prove the important fact that the support of an invariant measure for a nonuniformly expanding map contains a ball. This is important in our setting because, unlike the situation in other papers such as [4, 13, 19], we are not assuming that the map is nonuniformly expanding on the whole manifold.

In Section 4 we give the complete construction of the induced Gibbs-Markov map. We mention here some key differences between our construction and that of [4, 13, 19]. One of the shortcomings of [4] was a relatively inefficient construction which led to significantly larger inducing times than necessary, thus allowing only polynomial estimates to be obtained. This aspect of the construction was improved in [13, 19] where a global partition of the manifold was introduced, leading to significantly more efficient construction where the inducing times are essentially optimal. This strategy cannot be used here since our assumptions do not necessarily imply the map to be nonuniformly expanding on the whole manifold. We therefore return to a more local construction.
but develop a new strategy to improve the effectiveness of the inducing time estimates. Finally, in Subsection 4.4 we prove the integrability of the inducing times for the constructed Gibbs-Markov map.

2. Non-uniform expansion and slow recurrence

Definition 2.1. Let $f : M \to M$ be a $C^2$ local diffeomorphism outside a non-degenerate singular set $C$. We say that $f$ is non-uniformly expanding (NUE) on a set $A \subset M$ if there is $\lambda > 0$ such that for every $x \in A$ one has

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|Df(f^j(x))^{-1}\| < -\lambda.$$ 

We say that $f$ has slow recurrence (SR) if given any $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in A$ we have

$$\limsup_{n \to +\infty} \frac{1}{n} \sum_{j=1}^{n} -\log \text{dist}_\delta(f^j(x), C) \leq \epsilon,$$

where

$$\text{dist}_\delta(x, C) = \begin{cases} 1, & \text{if } \text{dist}(x, C) \geq \delta; \\ \text{dist}(x, C), & \text{otherwise}. \end{cases}$$

In Sections 3 and 4 we shall prove the following

Theorem 2.2. Let $f : M \to M$ be a $C^2$ local diffeomorphism outside a non-degenerate singular set $C$. Assume that $f$ satisfies NUE and SR on a forward invariant set $A$ having a positive Lebesgue measure subset of points whose orbit is dense in $A$. Then there is a ball $\Delta \subset M$ with $\text{Leb}(\Delta \setminus A) = 0$ and an induced Gibbs-Markov map $F : \Delta \to \Delta$ with Lebesgue integrable return times.

In the rest of this section we show that we can reduce Theorem 3 to Theorem 2.2.

Proposition 2.3. Let $\mu$ be an ergodic $f$-invariant absolutely continuous probability measure with all Lyapunov exponents positive such that $\log \text{dist}(\cdot, C_n) \in L^1(\mu)$ for all $n \geq 1$. Then, for all $N$ large enough, $f^N$ satisfies NUE and SR on a forward $f^N$-invariant set $A$ having a positive Lebesgue measure subset of points whose $f^N$-orbit is dense in $A$.

To complete the proof of Theorem 3 using Theorem 2.2 and Proposition 2.3 we just need to discuss the relationship between the original measure $\mu$ and the lift of the measure $\nu$ to the induced Gibbs-Markov map. Let $\nu$ be the absolutely continuous ergodic $F$-invariant measure for the Gibbs-Markov map of Theorem 2.2 with integrable return time function $R$. Using the well-known fact that $\nu$ has density with respect to Lebesgue measure on $\Delta$ bounded from above and below by positive constants, we easily get that $R$ is also Lebesgue integrable. Keeping in mind that this return time is defined in terms of $f^N$ we define $\tilde{R} = N \cdot R$ and the corresponding $f$-invariant probability measure $\tilde{\mu}$ by

$$\tilde{\mu} = \frac{\sum_{j=0}^{\infty} f^j(\nu(\{\tilde{R} > j\}))}{\sum_{j=0}^{\infty} \nu(\{\tilde{R} > j\})} \quad (1)$$
It just remains to show that $\hat{\mu} = \mu$. This follows from the standard fact that two distinct ergodic absolutely continuous invariant measures cannot both give positive mass to the same positive Lebesgue measure set. In this case we have that $\hat{\mu}$ and $\mu$ are both ergodic absolutely continuous $f$-invariant measures which contain $\Delta$ in their support and therefore they must be equal. This completes the proof of Theorem 3 (and therefore also of Theorems 1 and 2) modulo Theorem 2.2 and Proposition 2.3.

In the remaining part of this section we prove Proposition 2.3.

**Lemma 2.4.** Let $\varphi \in L^1(\mu)$ and $(B_n)_n$ a sequence of sets with $\mu(B_n) \to 0$ as $n \to \infty$. Then

$$\frac{1}{n} \sum_{j=0}^{n-1} \int_{B_n} \varphi \circ f^j \, d\mu \to 0, \text{ as } n \to \infty.$$  

**Proof.** From the $L^1$ Ergodic Theorem (see e.g. [23, Corollary 1.14.1]) we have

$$\frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j \xrightarrow{L^1(\mu)} \varphi^*, \text{ as } n \to \infty. \quad (2)$$

Observe that

$$\left| \int \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j \, d\mu - \int B_n \varphi^* \, d\mu \right| = \int \left( \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j \, d\mu - \varphi^* \right) \chi_{B_n} \, d\mu \leq \int \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j \, d\mu - \varphi^* \, d\mu.$$  

It follows from (2) that this last quantity converges to 0 when $n \to \infty$. Since we also have

$$\int B_n \varphi^* \, d\mu \to 0, \text{ when } n \to \infty,$$  

the conclusion then holds. \qed

Now we give a result on the existence of finitely many ergodic attractors for powers of $f$.

**Lemma 2.5.** Given $N \geq 1$, there are $1 \leq \ell \leq N$ and $f^N$-invariant Borel sets $C_1, \ldots, C_\ell$ such that:

1. $\{C_1, \ldots, C_\ell\}$ is a partition $\mu$-mod 0 of $M$ with $\mu(C_j) \geq 1/N$ for each $1 \leq j \leq \ell$;
2. $(f^N, \mu|C_j)$ is ergodic for each $1 \leq j \leq \ell$.

**Proof.** We start by proving that if $C$ is an $f^N$-invariant subset with positive measure, then $\mu(C) \geq 1/N$. Indeed, assume by contradiction that $\mu(C) < 1/N$. Consider the $f$-invariant set $\bigcup_{j=0}^{N-1} f^{-j}(C)$.

We have that

$$0 < \mu \left( \bigcup_{j=0}^{N-1} f^{-j}(C) \right) \leq \sum_{j=0}^{N-1} \mu(f^{-j}(C)) < 1.$$  

This gives a contradiction, because the set is $f$-invariant and $\mu$ is ergodic.
Now, if \((f^N, \mu)\) is not ergodic, then we may decompose \(M\) into a union of two \(f^N\)-invariant disjoint sets with positive measure. If the restriction of \(\mu\) to some of these sets is not ergodic, then we iterate this process. Note that this must stop after a finite number of steps with at most \(N\) disjoint subsets, since \(f^N\)-invariant sets with positive measure have its measure bounded from below by \(1/N\).

For a given \(N \geq 1\) we shall refer to the sets \(A_i = \text{supp}(\mu|C_i)\), with \(1 \leq i \leq \ell\) and \(A_1, \ldots, A_\ell\) given by the previous lemma, as the \textit{ergodic attractors} of \((f^N, \mu)\). Observe that if \(\mu\) is ergodic with respect to \(f^N\), then it has exactly one ergodic attractor.

\textbf{Proof of Proposition 2.3.} We start by proving that for all sufficiently large \(N\), \(f^N\) satisfies NUE on at least one of the ergodic attractors \(A = A_i\) defined above. Using that \(\log \text{dist}(\cdot, C_N) \in L^1(\mu)\) and \((C1')\) applied to \(f^N\) and \(C_N\), it easily follows that \(\log \|Df^N\|^{-1} \in L^1(\mu)\). We will show that for all large enough \(N\)

\[\int \log \|Df^N\|^{-1} d\mu < 0. \tag{3}\]

From the assumptions that \(\mu\) has all Lyapunov exponents positive and the fact that \(\mu\) is ergodic, it follows by standard ergodic theory that there are at most a finite number (bounded above by the dimension of the manifold \(M\)) of possible values that the Lyapunov exponents can attain for \(\mu\) almost every \(x\). Therefore we have a uniform lower bound and for \(\mu\) almost every \(x\)

\[\liminf_{N \to \infty} \frac{1}{N} \log \|Df^N(x)\| \geq \lambda > 0, \quad \forall v \in T_x M. \tag{4}\]

Thus, there exists a sequence of sets \((B_N)_N\) such that \(\mu(B_N) \to 0\) when \(N \to \infty\) and \(\|Df^N(x)v\| \geq \lambda N\) for every \(x \in M \setminus B_N\) and every \(v \in T_x M\). This is equivalent to saying that \(\log \|Df^N(x)\|^{-1} \leq -\lambda N\) for every \(x \in M \setminus B_N\) and therefore

\[\int_{M \setminus B_N} \log \|Df^N(x)^{-1}\| d\mu < -\lambda N(1 - \mu(B_N)). \tag{5}\]

By the chain rule we have

\[\int_{B_N} \log \|Df^N(x)^{-1}\| d\mu \leq \sum_{j=0}^{N-1} \int_{B_N} \log \|Df(f^j(x))^{-1}\| d\mu =: N b_N. \tag{6}\]

Applying Lemma 2.4 with \(\varphi = \log \|(Df)^{-1}\|\) we get that \(b_N \to 0\) when \(N \to \infty\). Therefore, combining (5) and (6) we get

\[\int \log \|Df^N(x)^{-1}\| d\mu \leq -\lambda N(1 - \mu(B_N)) + N b_N = N(-\lambda(1 - \mu(B_N)) + b_N).\]

This last quantity is obviously negative for big \(N\) and therefore we have proved (3). This immediately implies that there is some ergodic attractor \(A = A_i\) of \((f^N, \mu)\) such that

\[\int_{A_i} \log \|Df^N(x)^{-1}\| d\mu < 0.\]

Thus, by Birkhoff’s Ergodic Theorem for \(\mu\) almost every \(x \in A\) one has

\[\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{Nj}(x)^{-1}\| = \int_{A} \log \|Df^N(x)^{-1}\| d\mu < 0.\]
This proves NUE for $f^N$ on the set $A$.

Let us now prove the slow recurrence condition SR for $f^N$ in the same ergodic component $A$. By assumption we have $\log \text{dist}(\cdot, C_N) \in L^1(\mu)$. Therefore, by the monotone convergence theorem we have

$$\int_A -\log \text{dist}_\delta(\cdot, C_N) d\mu \to 0, \quad \text{when } \delta \to 0.$$ 

So, by Birkhoff’s Ergodic Theorem, given any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \text{dist}_\delta(f^N(x), C_N) = \int_A -\log \text{dist}_\delta(\cdot, C_N) d\mu \leq \epsilon$$

for $\mu$ almost every $x \in A$.

One is left to show that a positive Lebesgue measure subset of points in $A$ has $f^N$-dense orbit in $A$. This follows from the well-known fact that orbits in the support of an ergodic measure have almost all dense orbit in that support, see e.g. [15, Proposition 4.1.18], and the fact that $\mu$ is absolutely continuous with respect to Lebesgue measure.

3. Hyperbolic times

We now begin the proof of Theorem 2.2. Assume that $f$ satisfies NUE and SR on a forward invariant set $A$ with positive Lebesgue measure subset of points whose orbit is dense in $A$. In this section we recall the definition and properties of hyperbolic times and prove the existence of a ball $B$ contained in $A$.

**Definition 3.1.** Given $0 < \sigma < 1$ and $\delta > 0$, we say that $n$ is a $(\sigma, \delta)$-hyperbolic time for $x \in M$ if for all $1 \leq k \leq n$,

$$\prod_{j=n-k+1}^{n} \|Df(f^j(x))^{-1}\| \leq \sigma^k \quad \text{and} \quad \text{dist}_\delta(f^{n-k}(x), C) \geq \sigma^k.$$  

(7)

In the case $C = \emptyset$ the definition of hyperbolic time reduces to the first condition in (7).

We denote

$$H_j(\sigma, \delta) = \{x \in M : j \text{ is a } (\sigma, \delta)\text{-hyperbolic time for } x\}.$$  

A fundamental consequence of properties NUE and SR is the existence of hyperbolic times as in the following result whose proof can be found in [3, Lemma 5.4].

**Proposition 3.2.** There are $\delta > 0$, $0 < \sigma < 1$ and $\theta > 0$ such that

$$\limsup_{n \to \infty} \frac{1}{n} \#\{1 \leq j \leq n : x \in H_j(\sigma, \delta)\} \geq \theta,$$

for every $x \in A$.

From now on we consider $\delta, \sigma, \theta$ fixed as in Proposition 3.2 and let $H_j = H_j(\delta, \sigma)$.

**Remark 3.3.** It easy to see that if $x \in H_j$ for a given $j \in \mathbb{N}$, then $f^i(x) \in H_m$ for any $1 \leq i < j$ and $m = j - i$.  

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The next lemma gives the main properties of the hyperbolic times such as uniform backward contraction and bounded distortion. For the proof see [3, Lemma 5.2, Corollary 5.3].

**Lemma 3.4.** There exists $\delta_1, C_1 > 0$ such that if $n$ is a $(\sigma, \delta)$-hyperbolic time for $x$, then there is neighborhood $V_n$ of $x$ such that:

1. $f^n$ maps $V_n$ diffeomorphically onto a ball of radius $\delta_1$ around $f^n(x)$;
2. for every $1 \leq k \leq n$ and $y, z \in V_n$,
   \[
   \text{dist}(f^{n-k}(y), f^{n-k}(z)) \leq \sigma^{k/2} \text{dist}(f^n(y), f^n(z));
   \]
3. for any $y, z \in V_n$
   \[
   \log \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \leq C_1 \text{dist}(f^n(y), f^n(z)).
   \]

We call the sets $V_n$ hyperbolic pre-balls and their images $f^n(V_n)$ hyperbolic balls. The latter are actually balls of radius $\delta_1 > 0$. Notice that $\delta_1 > 0$ can be taken arbitrarily small for a fixed choice of $\delta > 0$, and this legitimates the assumption in the next result.

**Lemma 3.5.** Assume that $2\delta_1 < \delta < 1$. There is $C_2 > 0$ such that if $n$ is a $(\sigma, \delta)$-hyperbolic time for $x$ and $V_n$ is the corresponding hyperbolic pre-ball, then:

1. for every $1 \leq k \leq n$ and $y \in V_n$,
   \[
   \text{dist}(f^{n-k}(y), C) \geq \frac{1}{2} \min\{\delta, \sigma^{b(n-k)}\};
   \]
2. for any Borel sets $Y, Z \subset V_n$,
   \[
   \frac{1}{C_2 \text{Leb}(Z)} \leq \frac{\text{Leb}(f^n(Y))}{\text{Leb}(f^n(Z))} \leq \frac{C_2 \text{Leb}(Y)}{\text{Leb}(Z)}
   \]

**Proof.** Since $n$ is a hyperbolic time for $x$, then using the second item of Lemma 3.4 we obtain

\[
\text{dist}(f^{n-k}(y), C) \geq \text{dist}(f^{n-k}(x), C) - \text{dist}(f^k(x), f^k(y)) \geq \text{dist}(f^{n-k}(x), C) - \delta_1 \sigma^{(n-k)/2}
\]

(8)

Now, if $\text{dist}(f^{n-k}(x), C) = \text{dist}_0(f^{n-k}(x), C)$, recalling that we have taken $b < 1/2$, then using (8) and the definition of hyperbolic time we get

\[
\text{dist}(f^{n-k}(y), C) \geq \frac{1}{2} \sigma^{b(n-k)}
\]

as long as $\delta_1 < 1/2$. Otherwise, we have $\text{dist}(f^{n-k}(x), C) \geq \delta$, and so

\[
\text{dist}(f^{n-k}(y), C) \geq \frac{1}{2} \delta
\]

as long as $\delta_1 < \delta/2$. This proves the first item. Let us now prove the second one. By a change of variables induced by $f^n$ we may write

\[
\frac{\text{Leb}(f^n(Y))}{\text{Leb}(f^n(Z))} = \frac{\int_Y |\det Df^n(y)| \text{d} \text{Leb}(y)}{\int_Z |\det Df^n(z)| \text{d} \text{Leb}(z)} = \frac{|\det Df^n(y_0)|}{|\det Df^n(z_0)|} \frac{\int_Y \frac{|\det Df^n(y)|}{|\det Df^n(y_0)|} \text{d} \text{Leb}(y)}{\int_Z \frac{|\det Df^n(z)|}{|\det Df^n(z_0)|} \text{d} \text{Leb}(z)}.
\]
where \( y_0 \) and \( z_0 \) are chosen arbitrarily in \( Y \) and \( Z \), respectively. Using the third item of Lemma 3.4 we easily find uniform bounds for this expression.

From here on we fix \( \delta_1 > 0 \) for which the previous lemmas are satisfied. The rest of this section is devoted to the proof of the two following results.

**Proposition 3.6.** There is a ball \( B \) of radius \( \delta_1/4 \) such that \( \text{Leb}(B \setminus A) = 0 \).

**Lemma 3.7.** There are \( p \in B \) and \( N_0 \in \mathbb{N} \) such that \( \bigcup_{j=0}^{N_0} f^{-j}\{p\} \) is \( \delta_1/4 \)-dense in \( A \) and disjoint from the singular set \( C \).

This will be used for the choice of our domain \( \Delta \) of definition for the induced map \( F \), which will be contained in the ball \( B \). A similar statement to that of Proposition 3.6 was proved in [3, Lemma 5.6] under some stronger assumptions in the definition of condition NUE.

**Proof of Proposition 3.6.** It is enough to prove that there exist disks of radius \( \delta_1/4 \) where the relative measure of \( A \) is arbitrarily close to one. Since the set of points with infinitely many hyperbolic times is positively invariant and \( A \) also is positively invariant, we may assume, without loss generality, that every point in \( A \) has infinitely many hyperbolic times. Let \( \epsilon > 0 \) be some small number. By regularity of \( \text{Leb} \), there is a compact set \( A_c \subset A \) and open set \( A_0 \supset A \) such that

\[
\text{Leb}(A_0 \setminus A_c) < \epsilon \text{Leb}(A). \tag{9}
\]

Assume that \( n_0 \) is large enough so that for every \( x \in A_c \), any hyperbolic preball \( V_n(x) \) with \( n \geq n_0 \) is contained in \( A_0 \). Let \( W_n(x) \) be a part of \( V_n(x) \) that is send diffeomorphically by \( f^n \) onto the ball \( B_{\delta_1/4}(f^n(x)) \). By compactness there are \( x_1, \ldots, x_r \in A_c \) and \( n(x_1), \ldots, n(x_r) \geq n_0 \) such that

\[
A_c \subset W_{n(x_1)}(x_1) \cup \ldots \cup W_{n(x_r)}(x_r). \tag{10}
\]

For the sake of notational simplicity we shall write for each \( 1 \leq i \leq r \)

\[
V_i = V_{n(x_i)}(x_1), \quad W_i = W_{n(x_i)}(x_1) \quad \text{and} \quad n_i = n(x_i).
\]

Let \( n_1^* < n_2^* < \cdots < n_s^* \) be the distinct values taken by the \( n_i \)'s. Let \( I_1 \subset \mathbb{N} \) be a maximal subset of \( \{1, \ldots, r\} \) such that for each \( i \in I_1 \) both \( n_i = n_1^* \), and \( W_i \cap W_j = \emptyset \) for every \( j \in I_1 \) with \( j \neq i \). Inductively, we define \( I_k \) for \( 2 \leq k \leq s \) as follows: supposing that \( I_1, \ldots, I_{k-1} \) have already been defined, let \( I_k \) be a maximal set of \( \{1, \ldots, r\} \) such that for each \( i \in I_k \) both \( n_i = n_k^* \), and \( W_i \cap W_j = \emptyset \) for every \( j \in I_1 \cup \ldots \cup I_k \) with \( i \neq j \).

Define \( I = I_1 \cup \ldots \cup I_s \). By construction we have that \( \{W_i\}_{i \in I} \) is a family of pairwise disjoint sets. We claim that \( \{V_i\}_{i \in I} \) is a covering of \( A_c \). To see this, recall that by construction, give any \( W_j \) with \( 1 \leq j \leq r \), there is some \( i \in I \) with \( n(x_i) \leq n(x_j) \) such that \( W_{x_i} \cap W_{x_j} \neq \emptyset \). Taking images by \( f^{n(x_i)} \) we have

\[
f^{n(x_i)}(W_j) \cap B_{\delta_1/4}(f^{n(x_i)}(x_i)) \neq \emptyset.
\]
It follows from Lemma 3.4, item (2) that
\[ \text{diam}(f^{n(x)}(W_j)) \leq \frac{\delta_1}{2} a^{(n(x_j)-n(x_i))/2} \leq \frac{\delta_1}{2}, \]
and so
\[ f^{n(x)}(W_j) \subset B_{\delta_1}(f^{n(x)}(x_i)). \]
This gives that \( W_j \subset V_i \). We have proved that given any \( W_j \) with \( 1 \leq j \leq r \), there is \( i \in I \) so that \( W_j \subset V_i \). Taking into account (10), this means that \( \{V_i\}_{i \in I} \) is a covering of \( A_c \).

By Lemma 3.4, item (3) one may find \( \tau > 0 \) such that
\[ \text{Leb}(W_i) \geq \tau \text{Leb}(V_i), \quad \text{for all } i \in I. \]
Hence,
\[
\text{Leb}\left( \bigcup_{i \in I} W_i \right) = \sum_{i \in I} \text{Leb}(W_i) \\
\geq \tau \sum_{i \in I} \text{Leb}(V_i) \\
\geq \tau \text{Leb}\left( \bigcup_{i \in I} V_i \right) \\
\geq \tau \text{Leb}(A_c).
\]

From (9) one deduces that \( \text{Leb}(A_c) > (1-\epsilon) \text{Leb}(A) \). Noting that the constant \( \tau \) does not depend on \( \epsilon \), choosing \( \epsilon > 0 \) small enough we may have
\[ \text{Leb}\left( \bigcup_{i \in I} W_i \right) > \frac{\tau}{2} \text{Leb}(A). \quad (11) \]
We are going to prove that
\[ \frac{\text{Leb}(W_i \setminus A)}{\text{Leb}(W_i)} < \frac{2\epsilon}{\tau}, \quad \text{for some } i \in I. \quad (12) \]
This is enough for our purpose, since taking \( B = f^{n(x)}(W_i) \) we have by invariance of \( A \) and Lemma 3.4, item (3)
\[ \frac{\text{Leb}(B \setminus A)}{\text{Leb}(B)} \leq \frac{\text{Leb}(f^{n(x)}(W_i \setminus A))}{\text{Leb}(f^{n(x)}(W_i))} \leq C_0 \frac{\text{Leb}(W_i \setminus A)}{\text{Leb}(W_i)} = \frac{2C_0\epsilon}{\tau}, \]
which can obviously be made arbitrarily small. From this one easily deduces that there are disks of radius \( \delta_1/4 \) where the relative measure of \( A \) is arbitrarily close to one.

Finally, let us prove (12). Assume, by contradiction, than it does not hold. Then, using (9) and (11)
\[
\epsilon \text{Leb}(A) > \text{Leb}(A_0 \setminus A_c) \\
\geq \text{Leb}\left( \left( \bigcup_{i \in I} W_i \right) \setminus A \right) \\
\geq \frac{2\epsilon}{\tau} \text{Leb}\left( \bigcup_{i \in I} W_i \right) \\
\geq \epsilon \text{Leb}(A).
\]
This is gives a contradiction.

Proof of Lemma 3.7. Since we are assuming that \( f \) is a local diffeomorphism up to a set of zero Lebesgue measure, then the set

\[
B = \bigcup_{n \geq 0} f^{-n} \left( \bigcup_{m \geq 0} f^m(C) \right)
\]

has Lebesgue measure equal to zero. On the other hand, there is a positive Lebesgue measure subset of points in \( A \) with dense orbit. Thus there must be some point \( q \in A \setminus B \) with dense orbit in \( A \). Take \( N_0 \in \mathbb{N} \) for which \( q, f(q), \ldots, f^{N_0}(q) \) is \( \delta_1/4 \)-dense in \( A \) and \( f^{N_0}(q) \in B \). The point \( p = f^{N_0}(q) \) satisfies the conclusion of the lemma. \( \square \)

4. Markov structure and return times

In this section we give the complete construction of the induced map \( F : \Delta \to \Delta \). We fix once and for all a point \( p \in A \) and \( N_0 \in \mathbb{N} \) satisfying the conclusions of Lemma 3.7, i.e. such that the set of preimages of \( p \) up to \( N_0 \) is \( \delta_1/4 \)-dense in \( A \). We then fix some

\[
\delta_0 \ll \delta_1
\]

where the conditions on \( \delta_0 \) will be determined below. We define the subsets of \( A \)

\[
\Delta = B(p, \delta_0) \quad \text{and} \quad \Delta' = B(p, 2\delta_0).
\]

The next lemma is an important tool in our construction.

Lemma 4.1. If \( \delta_0 \) is sufficiently small, then there are constants \( D_0, K_0 \) such that for any ball \( B \) of radius \( \delta_1 \) contained in \( A \) there are an open set \( V \subset B \) and an integer \( 0 \leq m \leq N_0 \) for which:

1. \( f^m \) maps \( V \) diffeomorphically onto \( \Delta' \);
2. for each \( x, y \in V \)

\[
\log \left| \frac{\det Df^m(x)}{\det Df^m(y)} \right| \leq D_0 \text{dist}(f^m(x), f^m(y));
\]
3. for each \( 0 \leq j \leq m \) and for all \( x \in f^j(V) \) we have

\[
K_0^{-1} \leq \| Df^j(x) \|, \| (Df^j(x))^{-1} \|, | \det Df^j(x) | \leq K_0.
\]

In particular \( f^j(V) \cap C = \emptyset \).

Proof. The proof is similar to [4, Lemma 2.6], though we repeat it here in order to clarify the fact that also it holds in the situation where \( A \) is not necessarily equal to \( M \). Since \( \bigcup_{j=0}^{N_0} f^{-j} \{ p \} \) is disjoint from \( C \), then choosing \( \delta_0 \) sufficiently small we have that each connected component of the preimages of \( B(p, 2\delta_0) \) up to time \( N_0 \) is bounded away from the singular set \( C \) and is contained in a ball of radius \( \delta_1/4 \). Moreover, \( \bigcup_{j=0}^{N_0} f^{-j} \{ p \} \) is \( \delta_1/4 \)-dense in \( A \) and this immediately implies that any ball \( B \) of radius \( \delta_1 \) contained in \( A \) contains a preimage \( V \) of \( B(p, 2\delta_0) \) which is mapped diffeomorphically onto \( B(p, 2\delta_0) \) in most \( N_0 \) iterates, thus giving (1). Moreover, since the number
of iterations and the distance to the critical region are uniformly bounded, we immediately get (2) and (3).

Before starting the construction we introduce some important notation. Given a point \( x \in H_n \cap \Delta \), by Lemma 3.4 there exists a hyperbolic pre-ball \( V_n(x) \) such that \( f^n(V_n(x)) = B(f^n(x), \delta_1) \). From Lemma 4.1, there are a set \( V \subset B(f^n(x), \delta_1) \) and an integer \( 0 \leq m \leq N_0 \) such that \( f^m(V) = \Delta' \supset \Delta \). Denote

\[
U_{n+m}^x = (f|_{V_n(x)})^{-1}(\Delta). \tag{14}
\]

Given a set \( U \subset \Delta \), we define

\[
H_n^m(U) = \{ x \in H_n : U_{n+m}^x \cap U \neq \emptyset \}. \tag{15}
\]

We shall use often the following easy consequence of the definition of these sets \( U_{n+m}^x \) together with Lemma 3.4 and Lemma 4.1.

**Lemma 4.2.** There exists \( C_3 > 0 \) such that given \( x \in H_n \), a preball \( V_n(x) \) and \( U_{n+m}^x \) as in (14), then

\[
\text{Leb}(V_n(x)) \leq C_3 \text{Leb}(U_{n+m}^x).
\]

**Proof.** By a change of variables we have

\[
\int_\Delta d\text{Leb}(z) = \int_{f^n(U_{n+m}^x)} |\det Df^n(y)|d\text{Leb}(y).
\]

So, by Lemma 4.1,

\[
\text{Leb}(f^n(U_{n+m}^x)) \geq K_0^{-1} \text{Leb}(\Delta).
\]

We get by bounded distortion in the hyperbolic time \( n \)

\[
\frac{\text{Leb}(V_n(x))}{\text{Leb}(U_{n+m}^x)} \leq C_2 \frac{\text{Leb}(f^n(V_n(x)))}{\text{Leb}(f^n(U_{n+m}^x))} \leq C_2 \frac{\text{Leb}(B(f^n(x), \delta_1))}{K_0^{-1} \text{Leb}(\Delta)}.
\]

It is enough to take \( C_3 = C_2 K_0 \text{Leb}(M)/\text{Leb}(\Delta) \). \( \square \)

These sets \( U_{n+m}^x \) are the candidates for elements of the partition of \( \Delta \) corresponding to the induced map \( F \) since they are mapped onto \( \Delta \) with uniform expansion and bounded distortion. Notice that the sets \( U_{n+m}^x \) and \( U_{n+m'}^x \) for distinct points \( x, x' \) are not necessarily disjoint and this is a major complication in the construction. The strategy for dealing with this is additionally complicated by the fact that \( U_{n+m}^x \) does not necessarily contain the point \( x \).

### 4.1. The partitioning algorithm.

Now we describe an inductive construction of the \( \mathcal{P} \) partition \( (\text{Leb mod 0}) \) of \( \Delta \). The candidates to elements of \( \mathcal{P} \) are the sets of type (14) which will be chosen if they are entirely contained in \( \Delta \) and do not overlap each other. In particular, we introduce sets \( \Delta_n \) and \( S_n \) such that \( \Delta_n \) is the part of \( \Delta \) that has not been partitioned up to time \( n \), and \( S_n \), that we call the **satellite set**, corresponding to the portion of a reference hyperbolic pre-ball that was not used for constructing an element of the partition. It is important to note that to each step of the algorithm is associated a unique hyperbolic time and possibly several distinct return times.
First step of induction. We fix some large \( n_0 \in \mathbb{N} \) and ignore any dynamics occurring up to time \( n_0 \). Define \( \Delta^c = M \setminus \Delta \). Consider a maximal family

\[
\mathcal{U}_{n_0} = \{U_{n_0+m_0}^x, U_{n_0+m_1}^x, \ldots, U_{n_0+m_{k_{n_0}}}^x\}
\]

of pairwise disjoint sets of type (14) contained in \( \Delta \). These are the elements of the partition \( \mathcal{P} \) constructed in the \( n_0 \)-step of the algorithm. Set \( R(x) = n_0 + m_i \) for each \( x \in U_{n_0+m_i}^x \) with \( 0 \leq i \leq k_{n_0} \). We define \( n_0 \)-satellite of an element \( U \in \mathcal{U}_{n_0} \) as

\[
S_{n_0}(U) = \bigcup_{m=0}^{N_0} \bigcup_{x \in H_{n_0}^m(U)} V_{n_0}(x) \cap (\Delta \setminus U).
\]

It will be convenient to consider also the \( n_0 \)-satellite associated to \( \Delta^c \)

\[
S_{n_0}(\Delta^c) = \bigcup_{m=0}^{N_0} \bigcup_{x \in H_{n_0}^m(\Delta^c)} V_{n_0}(x) \cap (\Delta \setminus \Delta^c).
\]

We also define

\[
S_{n_0} = \bigcup_{U \in \mathcal{U}_{n_0}} S_{n_0}(U) \cup S_{n_0}(\Delta^c)
\]

and

\[
\Delta_{n_0} = \Delta \setminus \bigcup_{U \in \mathcal{U}_{n_0}} U.
\]

General step of induction. The general step of the construction follows the ideas above with minor modifications. Assume that the set \( \Delta_s \) is defined for each \( s \leq n - 1 \). Consider a maximal family

\[
\mathcal{U}_n = \{U_{n+m_0}^x, U_{n+m_1}^x, \ldots, U_{n+m_{k_n}}^x\}
\]

of pairwise disjoint sets of type (14) contained in \( \Delta_{n-1} \). These are the elements of the partition \( \mathcal{P} \) constructed in the \( n \)-step of algorithm. Set \( R(x) = n + m_i \) for each \( x \in U_{n+m_i}^x \) with \( 0 \leq i \leq k_n \).

We consider the \( n \)-satellite associated to each element of the partition previously constructed. Given \( U \in \mathcal{U}_{n_0} \cup \cdots \cup \mathcal{U}_n \), we define

\[
S_n(U) = \bigcup_{m=0}^{N_0} \bigcup_{x \in H_n^m(U)} V_n(x) \cap (\Delta \setminus U).
\]

It will be convenient to consider also the \( n_0 \)-satellite associated to \( \Delta^c \)

\[
S_n(\Delta^c) = \bigcup_{m=0}^{N_0} \bigcup_{x \in H_n^m(\Delta^c)} V_n(x) \cap (\Delta \setminus \Delta^c).
\]

We also define

\[
S_n = \bigcup_{U \in \mathcal{U}_{n_0} \cup \cdots \cup \mathcal{U}_n} S_n(U) \cup S_n(\Delta^c)
\]

and

\[
\Delta_n = \Delta \setminus \bigcup_{U \in \mathcal{U}_{n_0} \cup \cdots \cup \mathcal{U}_n} U.
\]
4.2. Expansion and bounded distortion. Recall that, by construction, the return time $R$ for an element $U$ of the partition $P$ of $\Delta$ is made by a certain number $n$ of iterations given by the hyperbolic time of a pre-ball $V_n \subset U$, plus a certain number $m \leq N_0$ of additional iterates which is the time it takes to go from $f^n(V_n)$, which could be anywhere in $M$, to $f^{n+m}(V_n)$, which covers $\Delta$ completely. It follows from Lemmas 3.4 and 4.1 that
\[
\|Df^{n+m}(x)^{-1}\| \leq \|Df^m(f^n(x))^{-1}\| \|Df^n(x)^{-1}\| \leq K_0 \sigma^{n/2} \leq K_0 \sigma^{(n_0-N_0)/2}.
\]
Taking $n_0$ sufficiently large we can make this last expression smaller than one.

We also need to show that there exists a constant $K > 0$ such that for any $x, y$ belonging to an element $U \in P$ with return time $R$, we have
\[
\log \left| \frac{\det Df^R(x)}{\det Df^R(y)} \right| \leq K \text{dist}(f^R(x), f^R(y)).
\]
By Lemmas 3.4 and 4.1, it is enough to take $K = D_0 + C_1 K_0$.

Remark 4.3. Analogously to Lemma 3.5, there exists a constant $C_4 > 0$ such that for any Borel sets $Y, Z \subset (f^{n+m}_{|V_n})^{-1}(\Delta')$ we have
\[
\frac{1}{C_4} \frac{\text{Leb}(Y)}{\text{Leb}(Z)} \leq \frac{\text{Leb}(f^{n+m}(Y))}{\text{Leb}(f^{n+m}(Z))} \leq C_4 \frac{\text{Leb}(Y)}{\text{Leb}(Z)}.
\]

4.3. The measure of satellites. Now we are going to show that the algorithm described in Section 4.1 does indeed produce a partition (Leb mod 0) of $\Delta$. The next lemma shows that, for each $n$ and $m$ fixed, the Lebesgue measure of the union of candidates $U^x_{n+m}$ which intersects an element of partition is proportional to the Lebesgue measure of this element. The proportion constant can actually be made uniformly summable in $n$. Consider $L \in \mathbb{N}$ large so that
\[
2K_0 \sigma^{(L-1)N_0/2} < 1,
\]
where $N_0$ and $K_0$ are given by Lemma 3.7 and Lemma 4.1 respectively.

Lemma 4.4. There exists $C_5 > 0$ such that given $0 \leq m \leq N_0$, $k \geq n_0$ and $U \in \mathcal{U}_k$, then for any $n \geq k$ the following hold:

1. If $k \leq n \leq k + LN_0$, then
\[
\text{Leb} \left( \bigcup_{x \in H^m_n(U)} U^x_{n+m} \right) \leq C_5 \text{Leb}(U). \tag{20}
\]

2. If $n > k + LN_0$, then
\[
\text{Leb} \left( \bigcup_{x \in H^m_n(U)} U^x_{n+m} \right) \leq C_5 \sigma^{\frac{n-k}{2}} \text{Leb}(U). \tag{21}
\]
Proof. Consider an integer $k \geq n_0$ and a set $U \in \mathcal{U}_k$. Recall that by construction we have $R[U] = k + m_0$ for some $0 \leq m_0 \leq N_0$. Moreover, $U$ is part of a hyperbolic preball $V_k$ which is sent diffeomorphically onto $\Delta$ by $f^{k+m_0}$; recall (14). We define $$T = (f_{V_k}^{k+m_0})^{-1}(\Delta' \setminus \Delta).$$

(1) Assume that $k \leq n \leq k + LN_0$. Fix some set $U_{n+m}^x$ with $x \in H^m_n(U)$. Having in mind the conclusion we need, it is no restriction to assume that $U_{n+m}^x$ intersects the complement of $U \cup T$ (observe that $U \cup T$ is obviously a proportion of $U$). Hence, there is a point $u \in U_{n+m}^x \cap T$ for which $v = f^{k+m_0}(u)$ satisfies $\text{dist}(v, p) = 3\delta_0/2$.

**Claim:** There is a uniform constant $\tilde{\rho} > 0$ for which $\text{Leb}(f^{n+m}(U_{n+m}^x \cap T)) \geq \tilde{\rho}$.

Consider first that $n + m > k + m_0$. Considering $\ell = (n + m) - (k + m_0)$ we have $0 \leq \ell \leq LN_0 + N_0$. Just by continuity there is $\rho > 0$ and a neighborhood $V_\rho$ of $u$ such that both $$f^{n+m}(V_\rho) = B(f^{\ell}(v), \rho) \cap \Delta \quad \text{and} \quad f^{k+m_0}(V_\rho) \subset \Delta' \setminus \Delta.$$ Observe that $f$ sends $f^{k+m_0}(V_\rho)$ onto $B(f^{\ell}(v), \rho) \cap \Delta$ in $\ell$ iterates. Moreover, when we look back, we see that the $\ell$ backward iterates comprise a certain number of at most $N_0$ backward iterates plus at most $LN_0$ backward iterates of a hyperbolic ball. Thus, by Lemma 4.1 and Lemma 3.5 we guarantee some uniform bound on the derivative of those backward iterates. This means that it is possible to choose $\rho$ uniformly. Hence, there $\tilde{\rho}$ (depending only on $\rho$) for which $$\text{Leb}(f^{n+m}(U_{n+m}^x \cap T)) \geq \tilde{\rho},$$ which gives the claim in this case.

Consider now $n + m < k + m_0$. Taking in this case $\ell = (k + m_0) - (n + m)$, we have $\ell \leq N_0$. By continuity there is $\rho > 0$ for which $$f^{\ell}(B(f^{n+m}(u), \rho) \cap \Delta) \subset \Delta' \setminus \Delta.$$ By Lemma 4.1 we have some uniform bound on the derivative of the backward iterates of $f^{\ell}(B(f^{n+m}(u), \rho) \cap \Delta)$. This means that it is possible to choose $\rho$ uniformly, and so there exists $\tilde{\rho}$ (depending only on $\rho$) for which the claim again holds.

Let us now use the claim to prove the first part of the lemma. Note that we can find $\xi > 0$ such that $$\frac{\text{Leb}(\Delta' \setminus \Delta)}{\text{Leb}(\Delta)} \leq \xi,$$ and so, by bounded distortion in time $k + m_0$, $$\frac{\text{Leb}(T)}{\text{Leb}(U)} \leq C_4 \frac{\text{Leb}(\Delta' \setminus \Delta)}{\text{Leb}(\Delta)} \leq C_4 \xi. \quad (22)$$

By bounded distortion in the time $n + m$, $$\frac{\text{Leb}(U_{n+m}^x)}{\text{Leb}(U_{n+m}^x \cap T)} \leq C_4 \frac{\text{Leb}(\Delta)}{\tilde{\rho}}. \quad (23)$$
Since $U_{n+m}^x \cap T$ is contained in $T$ and the sets $U_{n+m}^x$ are pairwise disjoint for $m, n$ fixed, it follows from (22) and (23) that

$$\text{Leb} \left( \bigcup_{x \in H^m(U)} U_{n+m}^x \right) \leq C^2_4 \xi \frac{\text{Leb}(\Delta)}{\tilde{\rho}} \text{Leb}(U).$$

(2) Assume now that $n > k + LN_0$. Since for each $U_{n+m}^x$ we have

$$\text{diam}(f^{k+m_0}(U_{n+m}^x)) \leq 2\delta_0 K_0 \sigma^{n-(k+m_0)/2}$$

then, by the choice of $L$, the sets $U_{n+m}^x$ are contained in $T \cup U$ for $n > k + LN_0$. Moreover, defining the annulus inside $\Delta'$ around the boundary of $\Delta$

$$A_{n,k} = \{ x \in \Delta' : \text{dist}(x, \partial\Delta) \leq 2\delta_0 K_0 \sigma^{n-(k+N_0)/2} \}$$

we have

$$f^{k+m_0}(U_{n+m}^x) \subset A_{n,k}.$$

By bounded distortion

$$\frac{\text{Leb} \left( \bigcup_{x \in H^m(U)} U_{n+m}^x \right)}{\text{Leb}(U)} \leq C_4 \frac{\text{Leb}(A_{n,k})}{\text{Leb}(\Delta)}.$$

Since there is a constant $\eta > 0$ for which

$$\frac{\text{Leb}(A_{n,k})}{\text{Leb}(\Delta)} \leq \eta \sigma^{\frac{n-k}{2}},$$

then we have

$$\text{Leb} \left( \bigcup_{x \in H^m(U)} U_{n+m}^x \right) \leq \eta C_4 \sigma^{\frac{n-k}{2}} \text{Leb}(U).$$

Take $C_5 = \max\{C^2_4 \xi \text{Leb}(\Delta)/\tilde{\rho}, \eta C_4\}$. \hfill \Box

The following proposition asserts that the sum of the Lebesgue measures of all satellite sets is finite. This is fundamental to assure that Lebesgue almost every point in $\Delta$ belongs to an element in $U_k$, for some $k \geq n_0$.

**Proposition 4.5.** $\sum_{n=n_0}^{\infty} \text{Leb} S_n < \infty$.

**Proof.** Observe that

$$\sum_{n=n_0}^{\infty} \text{Leb} S_n \leq \sum_{n=n_0}^{\infty} \text{Leb} S_n(\Delta^c) + \sum_{k=n_0}^{\infty} \sum_{U \in U_k} \sum_{n=k}^{\infty} \text{Leb} S_n(U). \quad (24)$$

We start by estimating the sum with respect to the satellites of $\Delta^c$. It follows from the definition of $S_n(\Delta^c)$ and Lemma 3.4 that

$$S_n(\Delta^c) \subset \{ x \in \Delta : \text{dist}(x, \partial\Delta) < 2\delta_1 \sigma^{n/2} \}.$$

Thus, we can find $\zeta > 0$ such that

$$\text{Leb}(S_n(\Delta^c)) \leq \zeta \sigma^{n/2}.$$

This obviously implies that the part of the sum respecting $\Delta^c$ in (24) is finite.
Consider now \( k \geq n_0 \) and \( n \geq k \). Fix \( U \in \mathcal{U}_k \) and consider \( S_n(U) \) the \( n \)-satellite associated to it. By definition of \( S_n(U) \) and Lemma 4.2 we have

\[
\text{Leb}(S_n(U)) \leq \sum_{m=0}^{N_0} \sum_{x \in H^m_n(U)} \text{Leb}(V_n(x) \cap (\Delta \setminus U))
\]

\[
\leq C_3 \sum_{m=0}^{N_0} \text{Leb} \left( \bigcup_{x \in H^m_n(U)} U^{\pi}_{n+m} \right)
\]  \hspace{1cm} (25)

In this last step we have used the obvious fact that for fixed \( n, m \) the sets of the form \( U^{\pi}_{n+m} \) with \( x \in H^m_n(U) \) are pairwise disjoint.

Now we consider the cases \( k \leq n \leq k + LN_0 \) and \( n > k + LN_0 \) separately. Assume first that \( k \leq n \leq k + LN_0 \). It follows from (25) and Lemma 4.4 that

\[
\sum_{n=k}^{k+LN_0} \text{Leb}(S_n(U)) \leq C_3 C_5 (N_0 + 1) \sum_{n=k}^{k+LN_0} \text{Leb}(U)
\]

Hence

\[
\sum_{k=n_0}^{\infty} \sum_{U \in \mathcal{U}_k} \text{Leb}(S_n(U)) \leq C_3 C_5 (N_0 + 1)(LN_0 + 1) \text{Leb} \Delta. \hspace{1cm} (26)
\]

Assume now that \( n > k + LN_0 \). We have by (25) and Lemma 4.4

\[
\text{Leb}(S_n(U)) \leq C_3 C_5 (N_0 + 1) \sigma^{n-k} \text{Leb}(U)
\]

For simplicity of the notation we take \( \lambda = \sigma^{1/2} \) and \( C = C_3 C_5 (N_0 + 1) \). Thus,

\[
\sum_{k=n_0}^{\infty} \sum_{U \in \mathcal{U}_k} \sum_{n=k+LN_0}^{\infty} \lambda^n \text{Leb}(U) = C_3 \frac{\lambda^{LN_0+1}}{1 - \lambda} \sum_{k=n_0}^{\infty} \sum_{U \in \mathcal{U}_k} \text{Leb}(U) \]

This gives the conclusion of the result. \( \square \)

Since \( \Delta \supset \Delta_{n_0} \supset \Delta_{n_0+1} \supset \ldots \), for showing that the algorithm described above does indeed produce a partition (Leb mod 0) of \( \Delta \), we only have to check that \( \text{Leb}(\cap_n \Delta_n) = 0 \). First observe that as the sum of the Lebesgue measures of \( S_n \) is finite, it follows from Borel-Cantelli Theorem that Lebesgue almost every point in \( \Delta \) belongs to finitely many \( S'_n \)'s. Since a generic point \( x \in \Delta \) has infinitely many \( \sigma \)-hyperbolic times, then for almost every \( x \in \Delta \) one can find \( n \) such that \( x \in H_n \) and \( x \notin S_j \) for \( j \geq n \). Thus, \( x \in \{ R = n + m \} \) for some \( 0 \leq m \leq N_0 \). Since this is valid for almost \( x \in \Delta \), then \( \text{Leb}(\cap_n \Delta_n) = 0 \).
4.4. Integrability of the inducing times. In the previous sections we proved the existence of a Lebesgue mod 0 partition \( P \) of \( \Delta \) and an inducing time function \( R : \Delta \to \mathbb{N} \) which is constant in the elements of \( P \). Moreover, the map \( F : \Delta \to \Delta \) defined for \( F(x) = f^{R(x)}(x) \) is a \( C^2 \) piecewise uniformly map with uniform bounded distortion. It is well known that such a map has a unique absolutely continuous (with respect to Lebesgue measure) ergodic invariant probability measure \( \nu \) whose density is bounded away from zero and infinity by some constant.

**Proposition 4.6.** The inducing time function \( R \) is \( \nu \)-integrable.

**Proof.** Suppose by contradiction that \( \sum_{U \in P} R|_U \nu(U) = \infty \). By Birkhoff’s Ergodic Theorem

\[
\frac{1}{i} \sum_{k=0}^{i-1} R(F^k(x)) \to \int R d\nu = \sum_{U \in P} R|_U \nu(U) = \infty,
\]

for \( \nu \)-almost every point \( x \in \Delta \). It follows from Proposition 4.5 that Lebesgue almost every \( x \in \Delta \) belongs just to finitely many \( S_i \)'s. Define

\[
s(x) = \#\{i \in \mathbb{N} : x \in S_i\}
\]

As the density of \( \nu \) is uniformly bounded from above, it follows from Proposition 4.5 that also

\[
\sum_{i=n_0}^{\infty} \nu(S_i) < \infty
\]

Consequently, for \( \nu \)-generic points \( x \in \Delta \) we have

\[
\frac{1}{i} \sum_{k=0}^{i-1} s(F^k(x)) \to \int s d\nu = \sum_{i=n_0}^{\infty} \nu(S_i) < \infty. \tag{27}
\]

Let \( x \in \Delta \) be a \( \nu \)-generic point. Define, for every \( i \in \mathbb{N} \),

\[
j_i = j_i(x) = \sum_{k=0}^{i-1} R(F^k(x)).
\]

This means that \( F^i(x) = f^{j_i}(x) \). We define

\[
I = I(x) = \{j_1, j_2, j_3, \ldots\}.
\]

Given \( j \in \mathbb{N} \), there exists a unique integer \( r = r(j) \geq 0 \) such that \( j_r < j \leq j_{r+1} \). Supposing that \( x \in H_j \), then \( F^r(x) \in H_m \), where \( m = j - j_r \); see Remark 3.3. We must consider the following three cases:

1. \( R(F^r(x)) < m \).

   This is implies that \( j_{r+1} - j_r < m = j - j_r \leq j_{r+1} - j_r \), which is impossible.

2. \( R(F^r(x)) = m \).

   Thus \( j_{r+1} - j_r = m = j - j_r \leq j_{r+1} - j_r \), which then implies that \( j = j_{r+1} \in I \).

3. \( R(F^r(x)) > m \).

   In this case we \( F^r(x) \in S_m \) or \( R(F^r(x)) = m + t \), for some \( 0 < t \leq N_0 \).
According to case (3) above we see that the number of integers \( j \) between \( j_r \) and \( j_{r+1} \) such that \( x \in H_j \) is bounded by the number of integers \( m \) such that \( F^r(x) \in S_m \) or \( F^r(x) \in \{ R = m + t \} \) for some \( 0 < t \leq N_0 \). Hence
\[
\# \{ j \in \{ j_r + 1, \ldots, j_{r+1} - 1 \} : x \in H_j \} \leq 1 + s(F^r(x)).
\]
Thus, for each \( n \in \mathbb{N} \) we may write
\[
\# \{ j \leq n : x \in H_j \} \leq \# \{ j \leq n : j \in \mathcal{I} \} + \sum_{r=1}^{r(n)} [1 + s(F^r(x))]
\]
\[
\leq \# \{ j \leq n : j \in \mathcal{I} \} + r(n) + \sum_{r=1}^{r(n)} s(F^r(x))
\]
\[
\leq r(n) + r(n) + \sum_{r=1}^{r(n)} s(F^r(x)).
\]
Therefore,
\[
\frac{1}{n} \# \{ j \leq n : x \in H_j \} \leq \frac{r(n)}{n} \left( 2 + \frac{1}{r(n)} \sum_{r=1}^{r(n)} s(F^r(x)) \right). \tag{28}
\]
By construction, if \( r(n) = i \), that is, \( j_i < n \leq j_{i+1} \), then
\[
\frac{j_i}{i} < \frac{n}{r(n)} \leq \frac{j_{i+1}}{i+1} \left( 1 + \frac{1}{i} \right).
\]
Since
\[
\frac{j_i}{i} = \frac{1}{i} \sum_{k=0}^{i-1} R(F^k(x)) \to \infty, \quad \text{as } i \to +\infty,
\]
it follows from (27) and (28) that
\[
\lim_{n \to \infty} \frac{1}{n} \# \{ 1 \leq j \leq n : x \in H_j \} = \lim_{n \to \infty} \frac{r(n)}{n} = 0.
\]
This contradicts Proposition 3.2, and so one must have that the inducing time function \( R \) is \( \nu \)-integrable. \qed

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**References**


