A model operator $H_\mu$, $\mu > 0$ associated to a system of three particles on the three-dimensional lattice $\mathbb{Z}^3$. We study the case where the parameter function $w$ has a special form with the non-degenerate minimum at the $n, n > 1$ points. If the associated Friedrichs model has a zero energy resonance, then we prove that the operator $H_\mu$ has infinitely many negative eigenvalues accumulating at zero. Moreover, we obtain an asymptotic value for the number of negative eigenvalues of $H_\mu$ lying below $z < 0$ with respect to the spectral parameter $z \to -0$. 

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1 INTRODUCTION

We are going to discuss the following remarkable phenomenon of the spectral theory of the three-body Schrödinger operators, known as the Efimov effect: if a system of three particles interacting through pair short-range potentials is such that none of the three two-particle subsystems has bound states with negative energy, but at least two of them have a zero energy resonance, then this three-particle system has an infinite number of three-particle bound states with negative energy accumulating at zero.

For the first time the Efimov effect has been discussed in [7]. An independent proof on a physical level of rigor has also been given in [5] and then many works devoted to this subject, see for example, [6, 11, 12, 13, 14]. A rigorous mathematical proof of the existence of Efimov’s effect was originally carried out in [16].

Denote by \( N(z) \) the number of eigenvalues of the Hamiltonian lying below \( z, z < 0 \). The growth of \( N(z) \) has been studied in [2] for the symmetric case. Namely, the authors of [2] first found (without proofs) the exponential asymptotics of eigenvalues corresponding to spherically symmetric bound states. This result is consistent with the lower bound

\[
\lim_{z \to -0} \inf N(z) |\log|z||^{-1} > 0
\]


In [12] the asymptotics of the form \( N(z) \sim U_0 |\log|z|| \) as \( z \to -0 \) for the number \( N(z) \) of bound states of a three-particle Schrödinger operator below \( z, z < 0 \) was obtained, where the coefficient \( U_0 \) depends only on the ratio of the masses of the particles.

Recently in [15] the existence of the Efimov effect for \( N \)-body quantum systems with \( N \geq 4 \) has been proved and a lower bound on the number of eigenvalues was given.

In [1, 3, 8, 9, 10] the presence of Efimov’s effect for the three-particle discrete Schrödinger operators has been proved and in [1, 3] an asymptotics for the number of eigenvalues similarly to [12, 14] was obtained.

In the present paper, we study the model operator \( H_\mu, \mu > 0 \) associated to a system of three-particles on \( \mathbb{Z}^3 \). Here we are interested in discussing the case where the parameter function \( w \) has a special form with the non degenerate minimum at the \( n, n > 1 \) points of the six-dimensional torus \( \mathbb{T}^6 \). If the associated Friedrichs model has a zero energy resonance, then we prove that the operator \( H_\mu \) has infinitely many negative eigenvalues accumulating at zero (in the considering case zero is the bottom of the essential spectrum of \( H_\mu \)). Moreover, we establish the asymptotic formula

\[
\lim_{z \to -0} \frac{N_\mu(z)}{|\log|z||} = \frac{n \gamma_0}{4\pi}
\]

for the number \( N_\mu(z) \) of eigenvalues of \( H_\mu \) lying below \( z, z < 0 \). Here the number \( n \equiv n(n) \), \( n > 1 \) is defined in Remark 2.3 (see below) and the number \( \gamma_0 \) is a unique positive solution of the
equation
\[ \gamma \sqrt{3} \cos \frac{\pi \gamma}{2} = 8 \sin \frac{\pi \gamma}{6}. \]  \hspace{1cm} (1.1)

The asymptotics obtained in this paper can be considered as a generalization of the asymptotics, which was obtained in [1, 3, 4, 12, 14]. In [4] the non symmetric version of the operator \( H_\mu \) was considered and the spectrum of this operator was analyzed for an arbitrary function \( w \) with \( n = 1 \).

The organization of the paper is as follows. In Section 2 the model operator \( H_\mu \) is introduced as a bounded self-adjoint operator and the main result of the paper is formulated. In Section 3 some spectral properties of the associated Friedrichs model \( h_\mu(p), p \in (-\pi, \pi]^3 \) are studied. In Section 4, we reduce the eigenvalue problem by the principle of Birman-Schwinger. Section 5 is devoted to the proof of the main result of the paper.

2 MODEL OPERATOR AND STATEMENT OF THE MAIN RESULT

Let us introduce some notations used in this work. Denote by \( \mathbb{T}^3 \) the three-dimensional torus, the cube \((-\pi, \pi]^3\) with appropriately identified sides. The torus \( \mathbb{T}^3 \) will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on the three-dimensional space \( \mathbb{R}^3 \) modulo \((2\pi \mathbb{Z})^3\). Let \((\mathbb{T}^3)^2 = \mathbb{T}^3 \times \mathbb{T}^3\) be a Cartesian product, \( L_2(\mathbb{T}^3) \) be the Hilbert space of square-integrable (complex) functions defined on \( \mathbb{T}^3 \) and \( L_2^s((\mathbb{T}^3)^2) \) be the Hilbert space of square-integrable symmetric (complex) functions defined on \((\mathbb{T}^3)^2\).

Let us consider a model operator \( H_\mu \) acting on the Hilbert space \( L_2^s((\mathbb{T}^3)^2) \) as

\[ H_\mu = H_0 - \mu V_1 - \mu V_2, \]

where

\[ (H_0 f)(p, q) = w(p, q) f(p, q), \]
\[ (V_1 f)(p, q) = \varphi(p) \int_{\mathbb{T}^3} \varphi(s) f(s, q) ds, \]
\[ (V_2 f)(p, q) = \varphi(q) \int_{\mathbb{T}^3} \varphi(s) f(p, s) ds. \]

Here \( \mu \) is a positive real number, the function \( \varphi(\cdot) \) is a real-valued analytic even function on \( \mathbb{T}^3 \) and the function \( w \) has form

\[ w(p, q) = \varepsilon(p) + \varepsilon(p + q) + \varepsilon(q) \]

with

\[ \varepsilon(p) = \sum_{j=1}^{3} (1 - \cos mp^{(j)}), p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{T}^3, \]
where $m$ is the positive integer number.

Under these assumptions the operator $H_\mu$ is bounded and self-adjoint in $L^2_s(\mathbb{T}^3)$. To formulate the main result of the paper we introduce the Friedrichs model $h_\mu(p), p \in \mathbb{T}^3$, which acts in $L^2(\mathbb{T}^3)$ as

$$h_\mu(p) = h_0(p) - \mu v,$$

where

$$(h_0(p)f_1)(q) = w(p, q)f(q),$$

$$(vf)(q) = \varphi(q)\int_{\mathbb{T}^3} \varphi(s)f(s)ds.$$ 

The perturbation $\mu v$ of the operator $h_0(p), p \in \mathbb{T}^3$ is a self-adjoint operator of rank one. Therefore, in accordance with the invariance of the essential spectrum under finite rank perturbations the essential spectrum $\sigma_{\text{ess}}(h_\mu(p))$ of $h_\mu(p), p \in \mathbb{T}^3$ fills the following interval on the real axis:

$$\sigma_{\text{ess}}(h_\mu(p)) = [m(p); M(p)],$$

where the numbers $m(p)$ and $M(p)$ are defined by

$$m(p) = \varepsilon(p) + 2\sum_{j=1}^{3}(1 - \cos\frac{mp^{(j)}_p}{2}), p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{T}^3,$$

$$M(p) = \varepsilon(p) + 2\sum_{j=1}^{3}(1 + \cos\frac{mp^{(j)}_p}{2}), p = (p^{(1)}, p^{(2)}, p^{(3)}) \in \mathbb{T}^3.$$ 

The following Theorem [4] describes the location of the essential spectrum of $H_\mu$.

**Theorem 2.1** For the essential spectrum $\sigma_{\text{ess}}(H_\mu)$ of the operator $H_\mu$ the equality

$$\sigma_{\text{ess}}(H_\mu) = \bigcup_{p \in \mathbb{T}^3} \sigma_{\text{disc}}(h_\mu(p)) \cup [0; \frac{27}{2}]$$

holds, where $\sigma_{\text{disc}}(h_\mu(p))$ is the discrete spectrum of $h_\mu(p), p \in \mathbb{T}^3$.

**Definition 2.2** The set $\bigcup_{p \in \mathbb{T}^3} \sigma_{\text{disc}}(h_\mu(p))$ resp. $[0; \frac{27}{2}]$ is called two- resp. three-particle branch of the essential spectrum $\sigma_{\text{ess}}(H_\mu)$ of the operator $H_\mu$, which will be denoted by $\sigma_{\text{two}}(H_\mu)$ resp. $\sigma_{\text{three}}(H_\mu)$.

Denote by $n \equiv n(m)$ the number of all the points of the form $(p_i, q_j) \in (\mathbb{T}^3)^2$ with $p_i = (p^{(1)}_i, p^{(2)}_i, p^{(3)}_i)$ and $q_j = (q^{(1)}_j, q^{(2)}_j, q^{(3)}_j)$ such that $p^{(k)}_i, q^{(k)}_j \in \{0, \pm \frac{2\pi}{m}; \pm \frac{4\pi}{m}; \cdots; \pm \frac{m'}{m}\pi\}$, $k = 1, 2, 3$ and $p_s \neq p_l, q_s \neq q_l$ for $s \neq l$, where

$$m' = \begin{cases} m - 2, & \text{if the number } m \text{ is even} \\ m - 1, & \text{if the number } m \text{ is odd.} \end{cases}$$
It is easy to check that the function \( w(\cdot, \cdot) \) has the non-degenerate minimum at that points \((p_i, q_j) \in (T^3)^2 \) and \( n = (m' + 1)^6 \).

Now we additionally assume that \( m \geq 3 \). Because, it is easy to show that, if \( m = 1, 2 \), then \( n = 1 \). In this paper we are interested in studying the case where \( n > 1 \).

We denote that \( 1, n = \{1, 2, \cdots, n\} \).

**Remark 2.3** In our analysis of the discrete spectrum of \( H_\mu \), a crucial role is played by the zeroes of the function \( \varphi(\cdot) \) at the points \( q_j \in T^3, j = 1, \sqrt{n} \) (see, for example [4]). Suppose that at only \( n_1 < n \leq n \) points of the set \( \{q_j\}_{j=1}^{\sqrt{n}} \) the value of the function \( \varphi(\cdot) \) is nonzero. We consider the set \( \{(p_i, q_j) \in (T^3)^2 : i = 1, n, \text{ such that } \varphi(q_j) \neq 0, i = 1, n \text{ and } \varphi(q_j) = 0, i = n + 1, n \} \). Throughout this paper we shall use this notation without further comments.

**Remark 2.4** Note that the equality \( h_\mu(p_{s_i}) \equiv h_\mu(p_{s_i}), i = 2, n \) holds.

Let \( C(T^3) \) (resp. \( L_1(T^3) \)) be the Banach space of continuous (resp. integrable) functions on \( T^3 \).

**Definition 2.5** The operator \( h_\mu(p_{s_i}) \) is said to have a zero energy resonance if the number 1 is an eigenvalue of the integral operator

\[
(G\psi_\alpha)(q) = \frac{\mu\varphi(q)}{2} \int_{T^3} \frac{\varphi(s)\psi(s)ds}{\varepsilon(s)}, \quad \psi \in C(T^3)
\]

and at least one (up to normalization constant) of the associated eigenfunctions \( \psi \) satisfies the condition \( \psi(q_{s_i}) \neq 0, i = 1, n \).

**Remark 2.6** We notice that if the operator \( h_\mu(p_{s_i}) \) has a zero energy resonance, then the function

\[
f(q) = \frac{\mu\varphi(q)}{2\varepsilon(q)} \in L_1(T^3) \setminus L_2(T^3),
\]

obeys the equation \( h_\mu(p_{s_i})f = 0 \) (see Lemma 3.7).

Set

\[
\mu_0 = 2 \left( \int_{T^3} \varphi^2(s)ds \varepsilon(s) \right)^{-1}.
\]

**Remark 2.7** We remark that the operator \( h_\mu(p_{s_i}) \) has a zero energy resonance if and only if \( \mu = \mu_0 \) (see Lemma 3.2).

Let us denote by \( \tau_{ess}(H_\mu) \) the bottom of the essential spectrum \( \sigma_{ess}(H_\mu) \) of \( H_\mu \) and by \( N_\mu(z) \) the number of eigenvalues of \( H_\mu \) lying below \( z \), \( z < \tau_{ess}(H_\mu) \).

**Remark 2.8** We note that \( \tau_{ess}(H_{\mu_0}) = 0 \) (see Lemma 3.6).
The main result of this paper is the following

**Theorem 2.9** The operator $H_{\mu_0}$ has infinitely many negative eigenvalues accumulating at zero and for the function $N_{\mu_0}(\cdot)$ the relation

$$
\lim_{z \to -0} \frac{N_{\mu_0}(z)}{|\log|z||} = \frac{n\gamma_0}{4\pi}
$$

holds, where the number $n$ is defined in Remark 2.3 and the number $\gamma_0$ is a positive solution of equation (1.1).

**Remark 2.10** Clearly, the infinite cardinality of the negative discrete spectrum of $H_{\mu_0}$ follows automatically from the positivity of the number $\gamma_0$.

**Remark 2.11** We point out that the asymptotics (2.2) is new and similar asymptotics have not yet been obtained for the three-particle Schrödinger operators on $\mathbb{R}^3$ and $\mathbb{Z}^3$.

## 3 SPECTRAL PROPERTIES OF THE OPERATOR $h_{\mu}(p)$

In this section we study some spectral properties of the Friedrichs model $h_{\mu}(p), p \in \mathbb{T}^3$, which plays a crucial role in our analysis of the discrete spectrum of the operator $H_{\mu}$.

Let $\mathbb{C}$ be the field of complex numbers. For any $p \in \mathbb{T}^3$ we define an analytic function $\Delta_{\mu}(p; \cdot)$ (the Fredholm determinant associated with the operator $h_{\mu}(p), p \in \mathbb{T}^3$) in $\mathbb{C} \setminus \sigma_{\text{ess}}(h_{\mu}(p))$ by

$$
\Delta_{\mu}(p; z) = 1 - \mu \int_{\mathbb{T}^3} \frac{\varphi^2(q) dq}{w(p, q) - z}.
$$

The following statement (see [4]) establishes a connection between eigenvalues of $h_{\mu}(p), p \in \mathbb{T}^3$ and zeroes of the function $\Delta_{\mu}(p; \cdot), p \in \mathbb{T}^3$.

**Lemma 3.1** For any $p \in \mathbb{T}^3$ the operator $h_{\mu}(p)$ has an eigenvalue $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(h_{\mu}(p))$ if and only if $\Delta_{\mu}(p; z) = 0$.

Since the function $w(\cdot, \cdot)$ has the non-degenerate minimum at the points $(p_{s_i}, q_{s_i}) \in (\mathbb{T}^3)^2$, $i = 1, n$ and the function $\varphi(\cdot)$ is an analytic function on $\mathbb{T}^3$, the integral

$$
\int_{\mathbb{T}^3} \frac{\varphi^2(q) dq}{w(p, q)}, \quad p \in \mathbb{T}^3
$$

is finite.

By Lebesgue’s dominated convergence theorem and the equality $\Delta_{\mu}(p_{s_i}; 0) = \Delta_{\mu}(p_{s_1}; 0)$, $i = 2, n$ it follows that

$$
\Delta_{\mu}(p_{s_1}; 0) = \lim_{p \to p_{s_i}} \Delta_{\mu}(p; 0), \quad i = 1, n.
$$

We remark that the following three statements, which are useful for the proof of main result can be proven similarly to corresponding statements of [1, 4] and hence, for completeness, we only reproduce here these statements without proofs.
Lemma 3.2 The operator $h_\mu(p_{s_1})$ has a zero energy resonance if and only if $\mu = \mu_0$.

Lemma 3.3 The following decomposition holds

$$\Delta_{\mu_0}(p; z) = 2\pi^2 \mu_0 \sum_{j=1}^n \phi^2(q_{s_j}) \sqrt{\frac{3}{4}|p - p_{s_j}|^2 - z + O(|p - p_{s_j}|^2) + O(|z|)}$$

as $|p - p_{s_j}| \to 0$, $i = 1, n$ and $z \to 0$.

Set

$$U_\delta(p_0) = \{p \in \mathbb{T}^3 : |p - p_0| < \delta\}, \quad p_0 \in \mathbb{T}^3, \quad \delta > 0.$$

Lemma 3.4 There exist positive numbers $C_1, C_2, C_3$ and $\delta$ such that

$$C_1|p - p_{s_j}|^2 \leq |\Delta_{\mu_0}(p; 0)| \leq C_2|p - p_{s_j}|^2, \quad p \in U_\delta(p_{s_j}), \quad i = n + 1, n;$$

$$|\Delta_{\mu_0}(p; 0)| \geq C_3, \quad p \in \mathbb{T}^3 \setminus \bigcup_{i=1}^n U_\delta(p_{s_j}).$$

From the representation

$$w(p, q) = |p - p_{s_j}|^2 + (p - p_{s_j}, q - q_{s_j}) + |q - q_{s_j}|^2 + O(|p - p_{s_j}|^4) + O(|q - q_{s_j}|^4)$$

as $|p - p_{s_j}|, |q - q_{s_j}| \to 0$, $i = 1, n$ it follows

Lemma 3.5 There exist the numbers $C_1, C_2, C_3 > 0$ and $\delta > 0$ such that

1) $C_1(|p - p_{s_j}|^2 + |q - q_{s_j}|^2) \leq w(p, q) \leq C_2(|p - p_{s_j}|^2 + |q - q_{s_j}|^2)$ for $(p, q) \in U_\delta(p_{s_j}) \times U_\delta(q_{s_j}), \quad i = 1, n;\quad$

2) $w(p, q) \geq C_3$ for all $p, q$, which at least one of the conditions $p \notin \bigcup_{i=1}^n U_\delta(p_{s_j})$ and $q \notin \bigcup_{i=1}^n U_\delta(q_{s_j})$ is fulfilled.

Lemma 3.6 The operator $h_{\mu_0}(p), p \in \mathbb{T}^3$ has no negative eigenvalues.

Proof. First we show that for any $p \in \mathbb{T}^3 \setminus \{p_{s_1}, p_{s_2}, \ldots, p_{s_n}\}$ the inequality $\Delta_{\mu_0}(p; 0) > \Delta_{\mu_0}(p_{s_1}; 0)$ holds. Denote

$$\Lambda(p) = \int_{\mathbb{T}^3} \frac{\phi^2(q) dq}{w(p, q)}.$$

Since the functions $\phi(\cdot)$ and $w(\cdot, \cdot)$ are even, the function $\Lambda(\cdot)$ is also even. Then

$$\Lambda(p) - \Lambda(p_{s_1}) = \frac{1}{4} \int_{\mathbb{T}^3} \frac{2w(p_{s_1}, q) - (w(p, q) + w(-p, q))}{w(p, q)w(-p, q)w(p_{s_1}, q)}[w(p, q) + w(-p, q)]^2 \phi^2(q) dq -$$

$$- \frac{1}{4} \int_{\mathbb{T}^3} \frac{[w(p, q) + w(-p, q)]^2}{w(p, q)w(-p, q)w(p_{s_1}, q)} \phi^2(q) dq. \quad (3.1)$$
By the equalities
\[ w(p_{s_1}, q) - \frac{w(p, q) + w(-p, q)}{2} = \sum_{j=1}^{3} (\cos m p(j) - 1)(1 + \cos m q(j)) \]
and (3.1) we have the inequality \( \Lambda(p) - \Lambda(p_{s_1}) < 0 \) for any \( p \in T^3 \setminus \{p_{s_1}, p_{s_2}, \ldots, p_{s_n}\} \).

By the definition of \( \mu_0 \) we have \( \Delta \mu_0(p_{s_1}; 0) = 0 \). Hence the inequality
\[ \Delta \mu_0(p; z) > \Delta \mu_0(p_{s_1}; 0) = 0 \]
holds for any \( p \in T^3 \) and \( z < 0 \). By Lemma 3.1 it means that, the operator \( h_{\mu_0}(p), p \in T^3 \) has no negative eigenvalues. \( \square \)

**Lemma 3.7** The function \( f \), which is defined by (2.1), obeys the equation \( h_{\mu_0}(p_{s_1}) f = 0 \).

**Proof.** First we show that \( f \in L_1(T^3) \setminus L_2(T^3) \), that is,
\[ \int_{T^3} |f(q)| dq < \infty \quad \text{and} \quad \int_{T^3} |f(q)|^2 dq = \infty. \]

From the definition of \( \mu_0 \) it follows that \( \Delta \mu_0(p_{s_1}; 0) = 0 \). By the construction of the set \( \{(p_{s_i}, q_{s_i}) \in (T^3)^2 : i = 1, n\} \) we have that \( \varphi(q_{s_i}) \neq 0, i = 1, n \) and \( \varphi(q_{s_i}) = 0, i = n + 1, n \).

Using these facts and the definition of the function \( \varepsilon(\cdot) \) we obtain that there exist the numbers \( C_1, C_2, C_3 > 0 \) and \( \delta > 0 \) such that
\[ C_1 |q - q_{s_i}|^2 \leq \varepsilon(q) \leq C_2 |q - q_{s_i}|^2, \quad q \in U_\delta(q_{s_i}), \quad i = 1, n, \]
\[ \varepsilon(q) \geq C_3, \quad q \in T^3 \setminus \bigcup_{i=1}^{n} U_\delta(q_{s_i}), \]
\[ |\varphi(q)| \geq C_3, \quad q \in U_\delta(q_{s_i}), \quad i = 1, n \]
and in the case where \( n < n \) we have that
\[ C_1 |q - q_{s_i}|^2 \leq |\varphi(q)| \leq C_2 |q - q_{s_i}|^2, \quad q \in U_\delta(q_{s_i}), \quad i = n + 1, n. \]

Applying the latter inequalities we obtain that
\[ \int_{T^3} |f(q)| dq \leq C_1 \sum_{j=1}^{n} \int_{U_\delta(q_{n_j})} \frac{dq}{|q - q_{n_j}|^2} + C_2 < \infty, \]
\[ \int_{T^3} |f(q)|^2 dq \geq C_1 \sum_{j=1}^{n} \int_{U_\delta(q_{n_j})} \frac{dq}{|q - q_{n_j}|^4} + C_2 = \infty. \]

It is easy to check that the function \( f \) obeys the equation \( h_{\mu_0}(p_{s_1}) f = 0. \) \( \square \)
4 THE BIRMAN-SCHWINGER PRINCIPLE

For a bounded self-adjoint operator $A$, acting in Hilbert space $\mathcal{H}$, we define $d(\lambda, A)$ as

$$d(\lambda, A) = \sup \{ \dim F : (Au, u) > \lambda, u \in F \subset \mathcal{H}, ||u|| = 1 \}.$$ 

$d(\lambda, A)$ is equal to the infinity, if $\lambda$ is in the essential spectrum and if $d(\lambda, A)$ is finite, it is equal to the number of the eigenvalues of $A$ bigger than $\lambda$.

By the definition of $N_\mu(z)$ we have

$$N_\mu(z) = d(-z, -H_\mu), -z > -\tau_{ess}(H_\mu).$$

Since the function $\Delta_\mu(\cdot; \cdot)$ is positive on $T^3 \times (-\infty, \tau_{ess}(H_\mu))$, the positive square root of $\Delta_\mu(p; z)$ exists for any $p \in T^3$ and $z < \tau_{ess}(H_\mu)$.

In our analysis of the spectrum of $H_\mu$ the crucial role is played, the compact integral operator $T_\mu(z), z < \tau_{ess}(H_\mu)$, which acts in $L_2(T^3)$ with the kernel

$$\frac{\mu \varphi(p)\varphi(q)}{\sqrt{\Delta_\mu(p; z)}\sqrt{\Delta_\mu(q; z)}(w(p, q) - z)}.$$ 

The following lemma is a realization of the well-known Birman-Schwinger principle for the operator $H_\mu$ (see [1, 3, 4, 10, 12, 14, 15]).

**Lemma 4.1** For $z < \tau_{ess}(H_\mu)$ the operator $T_\mu(z)$ is compact and continuous in $z$ and one has

$$N_\mu(z) = d(1, T_\mu(z)).$$

This lemma has been proven in [4] for the non symmetric case.

5 THE PROOF OF THE MAIN RESULT

In this section we shall derive the asymptotics (2.2) for the number $N_{\mu_0}(z)$ of eigenvalues of the operator $H_{\mu_0}$ lying below $z$, $z < 0$, that is, we shall prove Theorem 2.9.

We shall first establish the asymptotics for $d(1, T_{\mu_0}(z))$ as $z \to -0$. Then Theorem 2.9 will be deduced by a perturbation argument based on the following lemma.

**Lemma 5.1** Let $A(z) = A_0(z) + A_1(z)$, where $A_0(z)$ (resp. $A_1(z)$) is compact and continuous in $z < 0$ (resp. $z \leq 0$). Assume that for some function $f(\cdot)$, $f(z) \to 0$, $z \to 0$ one has

$$\lim_{z \to -0} f(z) d(\gamma, A_0(z)) = l(\gamma),$$

and is continuous in $\gamma > 0$. Then the same limit exists for $A(z)$ and

$$\lim_{z \to -0} f(z) d(\gamma, A(z)) = l(\gamma),$$

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For the proof of Lemma 5.1, see Lemma 4.9 of [12].

Let $T(\delta; |z|)$ be the integral operator which acts in $L_2(T^3)$ with the kernel

$$\frac{1}{2\pi^2} \sum_{i=1}^{n} \frac{\chi_\delta(p-p_{s_i})\chi_\delta(q-q_{s_i})(\frac{3}{4}|p-p_{s_i}|^2 + |z|)^{-\frac{1}{2}}(\frac{3}{4}|q-q_{s_i}|^2 + |z|)^{-\frac{1}{2}}}{|p-p_{s_i}|^2 + (p-p_{s_i}, q-q_{s_i}) + |q-q_{s_i}|^2 + |z|}.$$

Here $\chi_\delta(\cdot)$ is the characteristic function of $U_\delta(0)$.

The following lemma can be proven using Lemmas 3.3 – 3.5.

**Lemma 5.2** For any $z \leq 0$ and small $\delta > 0$ the error $T_{\mu_\delta}(z) - T(\delta; |z|)$ is Hilbert-Schmidt operator and is continuous in the uniform operator topology at the point $z = 0$.

The space of the functions $f$ having support in $\bigcup_{i=1}^{n} U_\delta(p_{s_i})$, is an invariant subspace for the operator $T(\delta; |z|)$. Let $T_0(\delta; |z|)$ be the restriction of the operator $T(\delta; |z|)$ to this subspace, that is, the integral operator acting in $L_2(\bigcup_{i=1}^{n} U_\delta(p_{s_i}))$ with the kernel

$$\frac{1}{2\pi^2} \sum_{i=1}^{n} \frac{(\frac{3}{4}|p-p_{s_i}|^2 + |z|)^{-\frac{1}{2}}(\frac{3}{4}|q-q_{s_i}|^2 + |z|)^{-\frac{1}{2}}}{|p-p_{s_i}|^2 + (p-p_{s_i}, q-q_{s_i}) + |q-q_{s_i}|^2 + |z|}.$$

Denote by $\text{diag}\{A_1, A_2, \cdots, A_n\}$ the $n \times n$ diagonal matrix with operators $A_1, A_2, \cdots, A_n$ as diagonal entries.

Since the space $L_2(\bigcup_{i=1}^{n} U_\delta(p_{s_i}))$ is isomorphic to $\bigoplus_{i=1}^{n} L_2(U_\delta(p_{s_i}))$, the operator $T_0(\delta; |z|)$ can be written as diagonal operator

$$T_0(\delta; |z|) = \text{diag}\{T_0^{(1)}(\delta; |z|), T_0^{(2)}(\delta; |z|), \cdots, T_0^{(n)}(\delta; |z|)\},$$

where $T_0^{(i)}(\delta; |z|), i = \overline{1, n}$ is the integral operator acting in $\bigoplus_{i=1}^{n} L_2(U_\delta(p_{s_i}))$ with the kernel

$$\frac{1}{2\pi^2} \frac{(\frac{3}{4}|p-p_{s_i}|^2 + |z|)^{-\frac{1}{2}}(\frac{3}{4}|q-q_{s_i}|^2 + |z|)^{-\frac{1}{2}}}{|p-p_{s_i}|^2 + (p-p_{s_i}, q-q_{s_i}) + |q-q_{s_i}|^2 + |z|}.$$

One verifies that the operator $T_0(\delta; |z|)$ is unitary equivalent to the operator $T_1(r)$, $r = |z|^{-\frac{1}{2}}$ acting in $\bigoplus_{i=1}^{n} L_2(U_r(0))$ as

$$T_1(r) = \text{diag}\{T_1^{(1)}(r), T_1^{(2)}(r), \cdots, T_1^{(n)}(r)\},$$

where $T_1^{(i)}(r), i = \overline{1, n}$ is the integral operator acting in the $L_2(U_r(0))$ with the kernel

$$\frac{1}{2\pi^2} \frac{(\frac{3}{4}|p|^2 + 1)^{\frac{1}{2}}(\frac{3}{4}|q|^2 + 1)^{\frac{1}{2}}}{(|p|^2 + (p, q) + |q|^2 + 1)}.$$

We note that the equivalence of these operators is performed by the unitary dilation

$$B_r = \text{diag}\{B_r^{(1)}, B_r^{(2)}, \cdots, B_r^{(n)}\} : \bigoplus_{i=1}^{n} L_2(U_\delta(p_{s_i})) \to \bigoplus_{i=1}^{n} L_2(U_r(0)).$$

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Here the operator $B_r^{(i)}: L_2(U_δ(p_s)) \to L_2(U_r(0))$, $i = \overline{1,n}$ acting by

$$(B_r^{(i)}f)(p) = r^{-\frac{3}{2}}f(\frac{1}{r}(p - p_s)).$$

Since the space $\bigoplus_{i=1}^{n} L_2(U_r(0))$ is isomorphous to $L_2(U_r(0))$, we rewrite the operator $T_1(r)$ as an integral operator acting in $L_2(U_r(0))$ with the kernel

$$K_n(p,q) = \frac{1}{2\pi^2 (\frac{3}{4}|p|^2 + 1)^{\frac{3}{2}} (\frac{3}{4}|q|^2 + 1)^{\frac{3}{2}} (|p|^2 + (p,q) + |q|^2 + 1)}.$$

Further, replacing $(\frac{3}{4}|p|^2 + 1)^{\frac{3}{2}}$, $(\frac{3}{4}|q|^2 + 1)^{\frac{3}{2}}$ and $|p|^2 + (p,q) + |q|^2 + 1$ by $(\frac{3}{4}|p|^2)^{\frac{3}{2}} (1 - \chi_1(p))$, $(\frac{3}{4}|q|^2)^{\frac{3}{2}} (1 - \chi_1(q))$ and $|p|^2 + (p,q) + |q|^2$, respectively, we have the operator $T_2(r)$. The error $T_1(r) - T_2(r)$ will be a Hilbert-Schmidt operator and continuous up to $z = 0$.

The space of functions having support in $L_2(U_r(0) \setminus U_1(0))$ is an invariant subspace for the operator $T_2(r)$. The kernel of this operator has form

$$K_n(p,q) = \frac{n}{\sqrt{3\pi^2}} \frac{1}{|p|^2 |q|^2 (|p|^2 + (p,q) + |q|^2)}.$$

Let $T(r)$ be the integral operator acting on $L_2(U_r(0) \setminus U_1(0))$ with the kernel $K_2(p,q)$.

The following lemma was proven in [1].

**Lemma 5.3** The equality

$$\lim_{z \to -0} \frac{d(1,T(z))}{|\log|z||} = \frac{\gamma_0}{2\pi}$$

is satisfied, where $\gamma_0$ is a positive solution of equation (1.1).

Now Theorem 2.9 follows from Lemmas 4.1, 5.1–5.3.

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References


