EXOTIC BIALGEBRAS FROM $9 \times 9$ UNITARY BRAID MATRICES

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Abstract

We present the exotic bialgebras that arise from a $9 \times 9$ unitary braid matrix.
1 Introduction

For several years [1–4] our initial collaboration studied the algebraic structures coming from
4x4 R-matrices (solutions of the Yang–Baxter equation) that are not deformations of classical
ones (i.e., the identity up to signs). In parallel, higher dimensional (N^2 × N^2 matrices for all
N > 2) exotic braid matrices have been presented and studied in [5, 6]. More recently, our
follow-up collaboration (with Boucif Abdesselam replacing our deceased friend and co-author
Daniel Arnaudon) constructed N^2 × N^2 unitary braid matrices \( \hat{R} \) for N > 2 generalizing the class
known for N = 2 [7, 8]. Some of these results were applied to model-building [9, 10].

In the present paper we start the systematic study of the bialgebras that arise from these
higher dimensional unitary braid matrices. We start with the simplest possible case N = 3 in
order to get the necessary expertise. However, even this case is complicated enough.

The paper is organized as follows. In Section 2 we give a the overall general setting. Section
3 is devoted to the explicit presentation of the exotic bialgebras.

2 Preliminaries

Our starting point is the following 9×9 braid matrix from [7]:

\[
\hat{R}(\theta) = \begin{pmatrix}
  a_+ & 0 & 0 & 0 & 0 & 0 & 0 & a_- \\
  0 & b_+ & 0 & 0 & 0 & 0 & b_- & 0 \\
  0 & 0 & a_+ & 0 & 0 & a_- & 0 & 0 \\
  0 & 0 & 0 & c_+ & 0 & c_- & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & c_- & 0 & c_+ & 0 & 0 \\
  0 & 0 & a_- & 0 & 0 & a_+ & 0 & 0 \\
  0 & b_- & 0 & 0 & 0 & 0 & b_+ & 0 \\
  a_- & 0 & 0 & 0 & 0 & 0 & 0 & a_+ \\
\end{pmatrix}
\]

(2.1)

where

\[
a_\pm = \frac{1}{2}(e^{m_{11}\theta} \pm e^{m_{11}'\theta}), \quad b_\pm = \frac{1}{2}(e^{m_{22}\theta} \pm e^{m_{22}'\theta}), \quad c_\pm = \frac{1}{2}(e^{m_{22}\theta} \pm e^{m_{22}'\theta}),
\]

(2.2)

and \( m_{ij}^\pm \) are parameters. The above braid matrix satisfies the baxterized braid equation:

\[
\hat{R}_{12}(\theta)\hat{R}_{23}(\theta + \theta')\hat{R}_{12}(\theta') = \hat{R}_{23}(\theta')\hat{R}_{12}(\theta + \theta')\hat{R}_{23}(\theta).
\]

(2.3)

For the RTT relations of Faddeev-Reshetikhin-Takhtajan [11] we need the corresponding R-
matrix, \( R = P\hat{R} \), (P is the permutation matrix):

\[
R(\theta) = \begin{pmatrix}
  a_+ & 0 & 0 & 0 & 0 & 0 & a_- \\
  0 & 0 & c_+ & 0 & c_- & 0 & 0 \\
  0 & 0 & a_- & 0 & 0 & a_+ & 0 \\
  0 & b_+ & 0 & 0 & 0 & 0 & b_- \\
  0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & b_- & 0 & 0 & 0 & b_+ & 0 \\
  0 & 0 & a_+ & 0 & 0 & a_- & 0 \\
  0 & 0 & c_- & 0 & c_+ & 0 & 0 \\
  a_- & 0 & 0 & 0 & 0 & 0 & a_+ \\
\end{pmatrix}
\]

(2.4)
which satisfies the baxterized Yang-Baxter equation:

$$R_{12}(\theta)R_{13}(\theta + \theta')R_{23}(\theta') = R_{23}(\theta')R_{13}(\theta + \theta')R_{12}(\theta)$$  \hspace{1cm} (2.5)

In fact, we need the solutions of the constant YBE which are as follows:

$$a_+ = b_+ = c_+ = 1/2, \quad a_\pm = \pm a_-, \quad b_\pm = \pm b_-, \quad c_\pm = \pm c_- \quad (2.6)$$

In view of (2.2) we see that for \(a_+ = a_-\) the proper limit is obtained, for example, by taking the following limits: first \(m_{11}^- \to -\infty\), and then \(\theta = 0\), while for \(a_+ = -a_-\) the limit may be obtained for \(m_{11}^+ = -\infty\) first, and then \(\theta = 0\). Similarly are obtained the limits for \(b_\pm\) and \(c_\pm\).

So we have eight \(R\) matrices satisfying the constant YBE:

$$\begin{align*}
(\pm, \pm, \pm), &(-, \pm, +), (+, \pm, -), (+, -), (-, -), (-, +) \quad (2.7)
\end{align*}$$

where the \(\pm\) signs denote respectively the signs of \(a_+ = \pm a_-, \quad b_\pm = \pm b_-, \quad c_\pm = \pm c_-\).

For the elements of the \(3 \times 3\) \(T\) matrix we introduce the notation:

$$T = \begin{pmatrix} k & p & l \\ q & r & s \\ m & t & n \end{pmatrix} \quad (2.8)$$

3 Solutions of the RTT equations and exotic bialgebras

We consider matrix bialgebras which are unital associative algebras generated by the nine elements from (2.8) \(k, l, m, n, p, q, r, s, t\). The co-product and co-unit relations are the classical ones:

$$\begin{align*}
\delta (T) &= T \otimes T \quad (3.9a) \\
\varepsilon (T) &= 1_3 \quad (3.9b)
\end{align*}$$

We expect the bialgebras under consideration not to be Hopf algebras which, as in the S03 case [2], would be easier to check after we find the dual bialgebras.

In the next subsections we obtain the desired bialgebras by applying the RTT relations of [11]:

$$RT_1 T_2 = T_2 T_1 R \quad , \quad (3.10)$$

where \(T_1 = T \otimes 1_2\), \(T_2 = 1_2 \otimes T\), for \(R = R(\theta)\), (2.4), and the parameters are the constants in (2.6) following the eight cases of (2.7).

3.1 Algebraic relations

I) Relations, which do not depend on the parameters \(a_\pm, b_\pm, c_\pm\). We have the set of relations

$$\begin{align*}
(N) &= \{ k^2 = n^2, \quad kn = nk, \quad l^2 = m^2, \quad lm = ml \\
&\quad km = nl, \quad kl = nm, \quad lk = mn, \quad mk = ln \\
r(k-n) = (k-n)r = 0, \quad r(l-m) = (l-m)r = 0 \} \quad (3.11)
\end{align*}$$
The last two relations suggest to introduce the variables:

\[ k = \tilde{k} + \tilde{n}, \quad n = \tilde{k} - \tilde{n}; \quad l = \tilde{l} + \tilde{m}, \quad m = \tilde{l} - \tilde{m}, \]
\[ p = \tilde{p} + \tilde{t}, \quad t = \tilde{p} - \tilde{t}; \quad q = \tilde{q} + \tilde{s}, \quad s = \tilde{q} - \tilde{s}, \]  

(3.12)

In terms of these variables we have:

\[ (N) = \{ \tilde{k} \tilde{n} = \tilde{n} \tilde{k} = 0; \quad \tilde{l} \tilde{m} = \tilde{m} \tilde{l} = 0; \quad \tilde{k} \tilde{m} = \tilde{n} \tilde{l} = 0; \quad \tilde{l} \tilde{n} = \tilde{m} \tilde{k} = 0; \quad \tilde{r} \tilde{m} = \tilde{r} \tilde{n} = 0, \quad \tilde{m} \tilde{r} = \tilde{n} \tilde{r} = 0 \} \]  

(3.13)

II) Relations that do not depend on the relative signs of \((a_-, b_-)\), \((a_-, c_-)\) and \((b_-, c_-)\). In that case we have

IIa) \(a_+ = \pm a_-\)

If \(a_+ = a_-\) we have the set of relations

\[ A_+ = \{ p^2 = t^2, \quad pt = tp; \quad q^2 = s^2, \quad qs = sq \} \]  

(3.14)

Or in terms of the alternative variables we have:

\[ A_+ = \{ \tilde{p} \tilde{t} = \tilde{t} \tilde{p} = 0; \quad \tilde{q} \tilde{s} = \tilde{s} \tilde{q} = 0 \} \]  

(3.15)

If \(a_+ = -a_-\) the set of relations is

\[ A_- = \{ p^2 = -t^2, \quad pt = -tp; \quad q^2 = -s^2, \quad qs = -sq \} \]  

(3.16)

Or alternatively

\[ A_- = \{ p^2 = \tilde{t}^2 = 0; \quad \tilde{q}^2 = \tilde{s}^2 = 0 \} \]  

(3.17)

IIb) \(b_+ = \pm b_-\)

If \(b_+ = b_-\) we have the set of relations:

\[ B_+ = \{ rp = rt; \quad rq = rs \} \]  

(3.18)

Alternatively

\[ B_+ = \{ r \tilde{t} = 0; \quad r \tilde{s} = 0 \} \]  

(3.19)

If \(b_+ = -b_-\)
\[ B_- = \{ rp = -rt; \; rq = -rs \} \] (3.20)

Alternatively

\[ B_- = \{ r\tilde{p} = 0; \; r\tilde{q} = 0 \} \] (3.21)

IIc) \( c_+ = c_\pm \)

If \( c_+ = c_- \) we have

\[ C_+ = \{ pr = tr; \; qr = sr \} \] (3.22)

Alternatively

\[ C_+ = \{ \tilde{t}r = 0; \; \tilde{s}r = 0 \} \] (3.23)

If \( c_+ = -c_- \) we have

\[ C_- = \{ pr = -tr; \; qr = -sr \} \] (3.24)

Alternatively

\[ C_- = \{ p\tilde{r} = 0; \; q\tilde{r} = 0 \} \] (3.25)

III) Relations depending on the relative signs of \((a_-, b_-), (a_-, b_0)\) and \((b_-, c_-)\)

IIIa) \( a_- = \pm b_- \)

If \( a_- = b_- \) we have the set of relations

\[
(AB)_+ = \{ pk = tn, \; tk = pn; \; pl = tm, \; tl = pm; \\
qk = sn, \; qn = sk; \; ql = sm, \; sl = qm \}
\] (3.26)

Alternatively

\[
(AB)_+ = \{ p\tilde{m} = \tilde{p}n = \tilde{t}k = \tilde{t}l = 0; \; q\tilde{m} = \tilde{q}n = \tilde{s}k = \tilde{s}l = 0 \}
\] (3.27)

If \( a_- = -b_- \) we have

\[
(AB)_- = \{ pk = -tn, \; tk = -pn; \; pl = -tm, \; tl = -pm; \\
qk = -sn, \; qn = -sk; \; ql = -sm, \; sl = -qm \}
\] (3.28)

Alternatively
\[(AB)_- = \{ \tilde{p}k = \tilde{p}l = \tilde{m}m = \tilde{n}n = 0; \quad \tilde{q}k = \tilde{q}l = \tilde{s}m = \tilde{s}n = 0 \} \quad (3.29)\]

IIIb) \(a_\pm = \pm c_\pm\)

If \(a_- = c_-\) we have

\[(AC)_+ = \{ kp = nt, \quad kt = np; \quad lp = mt, \quad lt = mp; \quad kq = ns, \quad nq = ks; \quad lq = ms, \quad ls = mq \} \quad (3.30)\]

Alternatively

\[(AC)_+ = \{ \tilde{k} \tilde{l} = \tilde{m} \tilde{p} = \tilde{n} \tilde{p} = 0; \quad \tilde{k} \tilde{s} = \tilde{l} \tilde{s} = \tilde{m} \tilde{q} = \tilde{n} \tilde{q} = 0 \} \quad (3.31)\]

If \(a_- = -c_-\) we have

\[(AC)_- = \{ kp = -nt, \quad kt = -np; \quad lp = -mt, \quad lt = -mp; \quad kq = -ns, \quad nq = -ks; \quad lq = -ms, \quad ls = -mq \} \quad (3.32)\]

Alternatively

\[(AC)_- = \{ \tilde{k} \tilde{p} = \tilde{l} \tilde{p} = \tilde{m} \tilde{l} = \tilde{n} \tilde{l} = 0; \quad \tilde{k} \tilde{q} = \tilde{l} \tilde{q} = \tilde{m} \tilde{s} = \tilde{n} \tilde{s} = 0 \} \quad (3.33)\]

IIIc) \(b_\pm = \pm c_\pm\)

If \(b_- = c_-\) we have

\[(BC)_+ = \{ pq = ts, \quad tq = ps; \quad qp = st, \quad qt = sp \} \quad (3.34)\]

Alternatively

\[(BC)_+ = \{ \tilde{p} \tilde{s} = \tilde{i} \tilde{q} = 0; \quad \tilde{s} \tilde{p} = \tilde{q} \tilde{i} = 0 \} \quad (3.35)\]

If \(b_- = -c_-\) we have

\[(BC)_- = \{ pq = -ts, \quad tq = -ps; \quad qp = -st, \quad qt = -sp \} \quad (3.36)\]

Alternatively

\[(BC)_- = \{ \tilde{p} \tilde{q} = \tilde{i} \tilde{s} = 0; \quad \tilde{q} \tilde{p} = \tilde{s} \tilde{i} = 0 \} \quad (3.37)\]
3.2 Classification of bialgebras

Thus we have the following solutions

- for \( a_+ = b_+ = c_- = c_- \) we have the set of relations

\[
(+, +, +) = \{ N \cup A_+ \cup B_+ \cup C_+ \cup (AB)_+ \cup (AC)_+ \cup (BC)_+ \}
\] (3.38)

Explicitly we have:

\[
\begin{align*}
\tilde{k} \tilde{m} &= \tilde{n} \tilde{k} = 0; \quad \tilde{k} \tilde{n} = \tilde{n} \tilde{k} = 0; \quad \tilde{k} \tilde{t} = \tilde{t} \tilde{k} = 0; \quad \tilde{k} \tilde{s} = \tilde{s} \tilde{k} = 0; \\
\tilde{l} \tilde{m} &= \tilde{n} \tilde{l} = 0; \quad \tilde{l} \tilde{n} = \tilde{n} \tilde{l} = 0; \quad \tilde{l} \tilde{t} = \tilde{t} \tilde{l} = 0; \quad \tilde{l} \tilde{s} = \tilde{s} \tilde{l} = 0; \\
\tilde{p} \tilde{m} &= \tilde{n} \tilde{p} = 0; \quad \tilde{p} \tilde{n} = \tilde{n} \tilde{p} = 0; \quad \tilde{p} \tilde{t} = \tilde{t} \tilde{p} = 0; \quad \tilde{p} \tilde{s} = \tilde{s} \tilde{p} = 0; \\
\tilde{q} \tilde{m} &= \tilde{n} \tilde{q} = 0; \quad \tilde{q} \tilde{n} = \tilde{n} \tilde{q} = 0; \quad \tilde{q} \tilde{t} = \tilde{t} \tilde{q} = 0; \quad \tilde{q} \tilde{s} = \tilde{s} \tilde{q} = 0; \\
\tilde{r} \tilde{m} &= \tilde{m} \tilde{r} = 0; \quad \tilde{r} \tilde{n} = \tilde{n} \tilde{r} = 0; \quad \tilde{r} \tilde{t} = \tilde{t} \tilde{r} = 0; \quad \tilde{r} \tilde{s} = \tilde{s} \tilde{r} = 0.
\end{align*}
\] (3.39)

From (3.39) we see that the algebra \( A_{++} \) is a direct sum of two subalgebras: \( A^1_{++} \) with generators \( \tilde{k}, \tilde{l}, \tilde{p}, \tilde{q}, \tilde{r} \), and \( A^2_{++} \) with generators \( \tilde{m}, \tilde{n}, \tilde{s}, \tilde{t} \). Both subalgebras are free, with no relations, and thus, no PBW bases.

- for \( a_+ = a_- = b_+ = c_- \) we have the set of relations

\[
(-, +, +) = \{ N \cup A_- \cup B_- \cup C_- \cup (AB)_- \cup (AC)_- \cup (BC)_- \}
\] (3.40)

Explicitly we have:

\[
\begin{align*}
\tilde{k} \tilde{m} &= \tilde{n} \tilde{k} = 0; \quad \tilde{k} \tilde{n} = \tilde{n} \tilde{k} = 0; \quad \tilde{k} \tilde{p} = \tilde{p} \tilde{k} = 0; \quad \tilde{k} \tilde{q} = \tilde{q} \tilde{k} = 0; \\
\tilde{l} \tilde{m} &= \tilde{n} \tilde{l} = 0; \quad \tilde{l} \tilde{n} = \tilde{n} \tilde{l} = 0; \quad \tilde{l} \tilde{p} = \tilde{p} \tilde{l} = 0; \quad \tilde{l} \tilde{q} = \tilde{q} \tilde{l} = 0; \\
\tilde{r} \tilde{m} &= \tilde{m} \tilde{r} = 0; \quad \tilde{r} \tilde{n} = \tilde{n} \tilde{r} = 0; \quad \tilde{r} \tilde{t} = \tilde{t} \tilde{r} = 0; \quad \tilde{r} \tilde{s} = \tilde{s} \tilde{r} = 0; \\
\tilde{s} \tilde{m} &= \tilde{m} \tilde{s} = 0; \quad \tilde{s} \tilde{n} = \tilde{n} \tilde{s} = 0; \quad \tilde{s} \tilde{p} = \tilde{p} \tilde{s} = 0; \quad \tilde{s}^2 = \tilde{r}^2 = \tilde{l}^2 = 0.
\end{align*}
\] (3.41)

The structure of this algebra, denoted \( A_{++} \), is more complicated. There are two quasi-free subalgebras: \( A^1_{++} \) with generators \( \tilde{k}, \tilde{l}, \tilde{p}, \tilde{q} \), and \( A^2_{++} \) with generators \( \tilde{m}, \tilde{n}, \tilde{p}, \tilde{q} \). They are quasi-free due to the last line of (3.41). They do not form a direct sum due to the existence of the following 12 two-letter building blocks of the basis \( A_{++} : \ r \tilde{k}, r \tilde{l}, r \tilde{p}, r \tilde{q}, \tilde{p} \tilde{l}, \tilde{q} \tilde{s} \) plus the reverse order.

- for \( a_+ = a_- = b_- = c_- \) we have
Explicitly we have:

\[ \tilde{k}\tilde{m} = \tilde{m}\tilde{k} = 0; \quad \tilde{k}\tilde{n} = \tilde{n}\tilde{k} = 0; \quad \tilde{k}\tilde{t} = \tilde{t}\tilde{k} = 0; \quad \tilde{p}\tilde{k} = \tilde{q}\tilde{k} = 0; \quad (3.43) \]

The structure of this algebra, denoted \( A_{+-+-} \), is also complicated. There are four free subalgebras: \( A^1_{+-+-} \) with generators \( \tilde{k}, \tilde{l}, r \), \( A^2_{+-+-} \) with generators \( \tilde{m}, \tilde{n}, \tilde{a} \), \( A^3_{+-+-} \) with generators \( \tilde{p}, \tilde{s}, \tilde{q}, \tilde{t} \). Only the first two are in direct sum, otherwise all are related by the following 20 two-letter building blocks: \( \tilde{k}\tilde{p}, \tilde{k}\tilde{q}, \tilde{s}\tilde{k}, \tilde{i}\tilde{k}, \tilde{i}\tilde{p}, \tilde{i}\tilde{q}, \tilde{s}\tilde{l}, \tilde{i}\tilde{l}, \tilde{r}\tilde{s}, \tilde{r}\tilde{i}, \tilde{p}\tilde{r}, \tilde{q}\tilde{r}, \tilde{m}\tilde{s}, \tilde{m}\tilde{l}, \tilde{p}\tilde{m}, \tilde{q}\tilde{m}, \tilde{n}\tilde{s}, \tilde{n}\tilde{l}, \tilde{p}\tilde{n}, \tilde{q}\tilde{n} \).

There is no overall ordering. There is some partial order if we consider the subalgebra formed by the generators of the latter three subalgebras: \( \tilde{m}, \tilde{n}, \tilde{p}, \tilde{s}, \tilde{q}, \tilde{t} \), namely, we have:

\[ \tilde{p}, \tilde{q} > \tilde{m}, \tilde{n} > \tilde{s}, \tilde{t} \quad (3.45) \]

But for the natural subalgebra formed by generators \( \tilde{k}, \tilde{l}, r, \tilde{p}, \tilde{s}, \tilde{q}, \tilde{t} \), we have cyclic ordering:

\[ \tilde{p}, \tilde{q} > r > \tilde{s}, \tilde{i} > \tilde{k}, \tilde{l} > \tilde{p}, \tilde{q} \quad (3.46) \]

i.e., no ordering. We have seen this phenomenon in the simpler exotic bialgebra \( S03, [2] \).

- for \( a_+ = a_- = b_- = -c_- \) we have the set of relations

\[ (+, +, -) = \{ N \cup A_+ \cup B_- \cup C_+ \cup (AB)_- \cup (AC)_+ \cup (BC)_- \} \quad (3.47) \]

Explicitly we have:

\[ \tilde{k}\tilde{m} = \tilde{m}\tilde{k} = 0; \quad \tilde{k}\tilde{n} = \tilde{n}\tilde{k} = 0; \quad \tilde{k}\tilde{t} = \tilde{t}\tilde{k} = 0; \quad \tilde{p}\tilde{k} = \tilde{q}\tilde{k} = 0; \quad \tilde{p}\tilde{r} = \tilde{q}\tilde{r} = 0; \quad \tilde{r}\tilde{s} = \tilde{s}\tilde{r} = 0; \quad \tilde{m}\tilde{n} = \tilde{n}\tilde{m} = 0; \quad \tilde{p}\tilde{i} = \tilde{i}\tilde{p} = 0; \quad \tilde{s}\tilde{l} = \tilde{l}\tilde{s} = 0; \quad \tilde{m}\tilde{l} = \tilde{l}\tilde{m} = 0; \quad \tilde{q}\tilde{p} = \tilde{p}\tilde{q} = 0; \quad \tilde{s}\tilde{q} = \tilde{q}\tilde{s} = 0; \quad \tilde{q}\tilde{n} = \tilde{n}\tilde{q} = 0; \quad \tilde{m}\tilde{n} = \tilde{n}\tilde{m} = 0; \quad \tilde{m}\tilde{s} = \tilde{s}\tilde{m} = 0; \quad \tilde{m}\tilde{l} = \tilde{l}\tilde{m} = 0; \quad \tilde{p}\tilde{n} = \tilde{n}\tilde{p} = 0; \quad \tilde{n}\tilde{s} = \tilde{s}\tilde{n} = 0. \quad (3.49) \]
This algebra, denoted $A_{++-}$, is a conjugate of the previous one. It has the same four free algebras, and the only difference is that the subalgebras are related by 20 two-letter building blocks which are in reverse order w.r.t. the previous case: $\tilde{k}s, \tilde{k}t, \tilde{p}k, \tilde{q}k, \tilde{I}s, \tilde{I}t, \tilde{p}l, \tilde{q}l, r\tilde{p}, r\tilde{q}, s\tilde{r}, s\tilde{r}, \tilde{m}\tilde{p}, \tilde{m}\tilde{q}, \tilde{s}\tilde{m}, \tilde{t}\tilde{n}, \tilde{n}\tilde{p}, \tilde{n}\tilde{q}, \tilde{s}\tilde{n}, \tilde{t}\tilde{n}.$

- for $a_+ = a_- = b_- = c_-$ we have the set of relations

\[(+, -, -) = \{N \cup A_+ \cup B_- \cup C_- \cup (AB)_- \cup (AC)_- \cup (BC)_+\} \tag{3.50}\]

Explicitly we have:

\[
\begin{align*}
\tilde{k}m = \tilde{n}\tilde{k} = 0; & \quad \tilde{k}n = \tilde{n}\tilde{k} = 0; \quad \tilde{k}\tilde{p} = \tilde{p}\tilde{k} = 0; \quad \tilde{k}\tilde{q} = \tilde{q}\tilde{k} = 0; \\
\tilde{l}m = \tilde{n}\tilde{l} = 0; & \quad \tilde{l}n = \tilde{n}\tilde{l} = 0; \quad \tilde{l}\tilde{p} = \tilde{p}\tilde{l} = 0; \quad \tilde{l}\tilde{q} = \tilde{q}\tilde{l} = 0; \\
rm = \tilde{n}r = 0; & \quad \tilde{n}r = \tilde{n}r = 0; \quad \tilde{r}\tilde{p} = \tilde{p}\tilde{r} = 0; \quad \tilde{r}\tilde{q} = \tilde{q}\tilde{r} = 0; \\
sm = \tilde{s}\tilde{s} = 0; & \quad \tilde{s}\tilde{n} = \tilde{n}\tilde{s} = 0; \quad \tilde{s}\tilde{p} = \tilde{p}\tilde{s} = 0; \quad \tilde{s}\tilde{q} = \tilde{q}\tilde{s} = 0; \\
\tilde{t}m = \tilde{m}\tilde{l} = 0; & \quad \tilde{t}n = \tilde{n}\tilde{l} = 0; \quad \tilde{t}\tilde{p} = \tilde{p}\tilde{l} = 0; \quad \tilde{t}\tilde{q} = \tilde{q}\tilde{l} = 0. \tag{3.52}
\end{align*}
\]

This algebra, denoted $A_{+-+}$, is a conjugate of the algebra $A_{+++}$, obtained by the exchange of the pairs of generators $(\tilde{p}, \tilde{q})$ and $(\tilde{s}, \tilde{t})$.

- for $a_+ = a_- = b_- = c_-$ we have the set of relations

\[(-, +, -) = \{N \cup A_- \cup B_+ \cup C_- \cup (AB)_- \cup (AC)_+ \cup (BC)_-\} \tag{3.53}\]

Explicitly we have:

\[
\begin{align*}
\tilde{k}m = \tilde{n}\tilde{k} = 0; & \quad \tilde{k}n = \tilde{n}\tilde{k} = 0; \quad \tilde{k}\tilde{t} = \tilde{k}\tilde{s} = 0; \quad \tilde{p}k = \tilde{q}\tilde{k} = 0; \\
\tilde{l}m = \tilde{n}\tilde{l} = 0; & \quad \tilde{l}n = \tilde{n}\tilde{l} = 0; \quad \tilde{l}\tilde{t} = \tilde{l}\tilde{s} = 0; \quad \tilde{p}\tilde{l} = \tilde{q}\tilde{l} = 0; \\
rm = \tilde{n}r = 0; & \quad \tilde{n}r = \tilde{n}r = 0; \quad \tilde{r}\tilde{t} = \tilde{r}\tilde{s} = 0; \quad \tilde{p}\tilde{r} = \tilde{q}\tilde{r} = 0; \\
\tilde{s}\tilde{t} = \tilde{l}\tilde{s} = 0; & \quad \tilde{s}\tilde{n} = \tilde{s}\tilde{n} = 0; \quad \tilde{l}\tilde{m} = \tilde{l}\tilde{n} = 0; \\
\tilde{p}\tilde{q} = \tilde{q}\tilde{p} = 0; & \quad \tilde{m}\tilde{p} = \tilde{n}\tilde{p} = 0; \quad \tilde{m}\tilde{q} = \tilde{n}\tilde{q} = 0; \\
\tilde{p}^2 = \tilde{q}^2 = \tilde{s}^2 = \tilde{t}^2 = 0.
\tag{3.54}
\end{align*}
\]

The structure of this algebra, denoted $A_{+-+}$, is very complicated. There are two free subalgebras: $A^{1}_{+++}$ with generators $\tilde{k}, \tilde{l}, \tilde{r}$, $A^{2}_{++-}$ with generators $\tilde{m}, \tilde{n}, \tilde{l}$, and four quasi-free subalgebras: $A^{4}_{++-}$ with generators $\tilde{p}, \tilde{s}$, $A^{5}_{+++}$ with generators $\tilde{q}, \tilde{t}$, $A^{6}_{+-+}$ with generators $\tilde{p}, \tilde{t}$, $A^{7}_{--+}$ with generators $\tilde{q}, \tilde{s}$. The first four subalgebras have generators as in the $A_{++-}$ case (but taking into account the last line of (3.54)). Only the first two subalgebras are in direct sum,
and there are intersections between the last four. Furthermore, all are related by the following 20 two-letter building blocks: \( \tilde{k}\tilde{p}, \tilde{k}\tilde{q}, \tilde{s}\tilde{k}, \tilde{t}\tilde{k}, \tilde{l}\tilde{q}, \tilde{s}\tilde{l}, \tilde{l}\tilde{l}, \tilde{r}\tilde{p}, \tilde{r}\tilde{q}, \tilde{s}\tilde{r}, \tilde{t}\tilde{r}, \tilde{n}\tilde{s}, \tilde{n}\tilde{l}, \tilde{p}\tilde{m}, \tilde{q}\tilde{m}, \tilde{m}\tilde{s}, \tilde{m}\tilde{l}, \tilde{p}\tilde{n}, \tilde{q}\tilde{n} \), which are the same as in the \( A_{++} \) case, except those involving \( r \).

The last difference makes things better. Indeed, there is no overall ordering, more precisely we have:

\[
p,q > m,n > \tilde{s},\tilde{t} > \tilde{k},\tilde{l},r > \tilde{p},\tilde{q}
\]

(3.55)
i.e., we have some cyclic order.

Thus, the bialgebra \( A_{++} \) may turn out to be the easiest to handle, as the exotic bialgebra \( S03, [2] \).

- for \( a_+ = -a_- = -b_- = c_- \) we have the set of relations

\[
(-,-,) = \{ N \cup A_- \cup B_- \cup C_+ \cup (AB)_+ \cup (AC)_- \cup (BC)_- \}
\]

(3.56)
Explicitly we have:

\[
\tilde{k}\tilde{n} = \tilde{n}\tilde{k} = 0; \quad \tilde{k}\tilde{n} = \tilde{n}\tilde{k} = 0; \quad \tilde{k}\tilde{p} = \tilde{k}\tilde{q} = 0; \quad \tilde{s}\tilde{k} = \tilde{t}\tilde{k} = 0;
\]

(3.57)
\[
\tilde{l}\tilde{m} = \tilde{m}\tilde{l} = 0; \quad \tilde{l}\tilde{n} = \tilde{n}\tilde{l} = 0; \quad \tilde{l}\tilde{p} = \tilde{l}\tilde{q} = 0; \quad \tilde{s}\tilde{l} = \tilde{l}\tilde{l} = 0;
\]

\[
r\tilde{m} = \tilde{m}\tilde{r} = 0; \quad r\tilde{n} = \tilde{n}\tilde{r} = 0; \quad r\tilde{p} = \tilde{r}\tilde{q} = 0; \quad r\tilde{r} = \tilde{s}\tilde{r} = 0;
\]

\[
\tilde{s}\tilde{l} = \tilde{l}\tilde{s} = 0; \quad \tilde{m}\tilde{s} = \tilde{n}\tilde{s} = 0; \quad \tilde{m}\tilde{l} = \tilde{n}\tilde{l} = 0;
\]

\[
\tilde{p}\tilde{q} = \tilde{q}\tilde{p} = 0; \quad \tilde{p}\tilde{m} = \tilde{m}\tilde{p} = 0; \quad \tilde{q}\tilde{m} = \tilde{m}\tilde{q} = 0;
\]

\[
\tilde{p}^2 = \tilde{q}^2 = \tilde{s}^2 = \tilde{l}^2 = 0.
\]

(3.58)
This algebra, denoted \( A_{--} \), is a conjugate of the previous algebra obtained by the exchange of the pairs of generators \( (\tilde{p},\tilde{q}) \) and \( (\tilde{s},\tilde{t}) \).

- for \( a_+ = -a_- = -b_- = -c_- \) we have the set of relations

\[
(-,-,-) = \{ N \cup A_- \cup B_- \cup C_- \cup (AB)_+ \cup (AC)_- \cup (BC)_+ \}
\]

(3.59)
Explicitly we have:

\[
\tilde{k}\tilde{n} = \tilde{n}\tilde{k} = 0; \quad \tilde{k}\tilde{n} = \tilde{n}\tilde{k} = 0; \quad \tilde{k}\tilde{p} = \tilde{k}\tilde{q} = 0; \quad \tilde{k}\tilde{t} = \tilde{k}\tilde{k} = 0;
\]

(3.60)
\[
\tilde{l}\tilde{m} = \tilde{m}\tilde{l} = 0; \quad \tilde{l}\tilde{n} = \tilde{n}\tilde{l} = 0; \quad \tilde{l}\tilde{s} = \tilde{s}\tilde{l} = 0; \quad \tilde{l}\tilde{l} = \tilde{l}\tilde{l} = 0;
\]

\[
r\tilde{m} = \tilde{m}\tilde{r} = 0; \quad r\tilde{n} = \tilde{n}\tilde{r} = 0; \quad r\tilde{p} = \tilde{r}\tilde{p} = 0; \quad r\tilde{q} = \tilde{q}\tilde{r} = 0;
\]

\[
\tilde{p}\tilde{m} = \tilde{m}\tilde{p} = 0; \quad \tilde{p}\tilde{m} = \tilde{n}\tilde{p} = 0; \quad \tilde{p}\tilde{s} = \tilde{s}\tilde{p} = 0;
\]

\[
\tilde{q}\tilde{m} = \tilde{m}\tilde{q} = 0; \quad \tilde{q}\tilde{n} = \tilde{n}\tilde{q} = 0; \quad \tilde{q}\tilde{t} = \tilde{l}\tilde{q} = 0;
\]

\[
\tilde{p}^2 = \tilde{q}^2 = \tilde{s}^2 = \tilde{l}^2 = 0.
\]

(3.61)
This algebra, denoted $\mathcal{A}_{-\cdots}$, is a conjugate of the algebra $\mathcal{A}_{++}$, obtained by the exchanges of generators: $\tilde{p} \leftrightarrow \tilde{s}$ and $\tilde{q} \leftrightarrow \tilde{t}$.

4 Summary and Outlook

Thus, taking into account conjugation, we have found four different bialgebras:

$$
\mathcal{A}_{++} \cong \mathcal{A}_{-\cdots}, \quad \mathcal{A}_{--} \cong \mathcal{A}_{+-}, \quad \mathcal{A}_{+-} \cong \mathcal{A}_{++}, \quad \mathcal{A}_{-+} \cong \mathcal{A}_{--}
$$

Thus, for future use we shall use shorter notation:

$$
\mathcal{A}_{++} \equiv \mathcal{A}_{++}, \quad \mathcal{A}_{--} \equiv \mathcal{A}_{--}, \quad \mathcal{A}_{+-} \equiv \mathcal{A}_{+-}, \quad \mathcal{A}_{-+} \equiv \mathcal{A}_{-+}
$$

The first two bialgebras have no ordering. The first one is simpler, since it is split in two subalgebras with five and four generators. The third subalgebra has partial ordering in one subalgebra. The last one, is the most promising since it has partial cyclic ordering.

The next task in this line of research is to find the dual bialgebras, analogously, as done for the four-element exotic bialgebras in [1, 2], cf. [12].

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