Abstract

This work is devoted to the study of the vacuum structure, special relativity, electrodynamics of interacting charged point particles and quantum mechanics, and is a continuation of [6, 7]. Based on the vacuum field theory no-geometry approach, the Lagrangian and Hamiltonian reformulation of some alternative classical electrodynamics models is devised. The Dirac type quantization procedure, based on the canonical Hamiltonian formulation, is developed for some alternative electrodynamics models. By means of the developed approach a combined description, both of electrodynamics and gravity, is analyzed.
1. Introduction

The classical relativistic electrodynamics of a freely moving charged point particle in the Minkovski space-time $\mathbb{M}^4 := \mathbb{E}^3 \times \mathbb{R}$ is, as well-known, based [12, 5, 35, 10] on the Lagrangian formalism assigning to it the following Lagrangian function

\begin{equation}
\mathcal{L} := -m_0(1 - u^2)^{1/2},
\end{equation}

where $m_0 \in \mathbb{R}_+$ is the so-called particle rest mass, located at a spatial point $r \in \mathbb{E}^3$, and $u := dr/dt \in \mathbb{E}^3$ is its spatial velocity in the Euclidean space $\mathbb{E}^3$, expressed here and throughout the paper in the light speed units (that is the light speed $c$ units). The least action Fermat principle in the form

\begin{equation}
\delta S = 0, \quad S := -\int_{t_1}^{t_2} m_0(1 - u^2)^{1/2} dt
\end{equation}

for any fixed temporal interval $[t_1, t_2] \subset \mathbb{R}$ gives rise to the well-known relativistic relationships for the mass of the particle

\begin{equation}
m = m_0(1 - u^2)^{-1/2},
\end{equation}

the momentum of the particle

\begin{equation}
p := mu = m_0u(1 - u^2)^{-1/2}
\end{equation}

and the energy of the particle

\begin{equation}
E_0 = m = m_0(1 - u^2)^{-1/2}.
\end{equation}

The origin of Lagrangian (1.1), owing to the reasonings from [12, 35], can be extracted from the action expression

\begin{equation}
S := -\int_{t_1}^{t_2} m_0(1 - u^2)^{1/2} dt = -\int_{\tau_1}^{\tau_2} m_0 d\tau,
\end{equation}

on the suitable temporal interval $[\tau_1, \tau_2] \subset \mathbb{R}$, where, by definition,

\begin{equation}
d\tau := dt(1 - u^2)^{1/2}
\end{equation}

and $\tau \in \mathbb{R}$ is the so-called proper temporal parameter assigned to a freely moving particle with respect to the “rest” reference system $K_r$. The action (1.6), if considered without any \textit{a priori} chosen constraint looks, from the dynamical point of view, slightly controversial, since it is physically defined with respect to the rest reference system $K_r$, giving rise to the constant action $S = -m_0(\tau_2 - \tau_1)$, as limits of integrations $\tau_1 < \tau_2 \in \mathbb{R}$ were taken to be fixed from the very beginning. We here mention, for completeness, that this problem can be in part remedied by imposing on (1.6) the relativistic constraint \[ (dt/d\tau)^2 - |dr/d\tau|^2 \right)^{1/2} = 1, \] equivalent to relationship (1.7), as it was recently shown in [7, 28]. Moreover, considering this particle as charged with a charge $q \in \mathbb{R}$ and moving in the Minkovski space-time $\mathbb{M}^4$ under action of an electromagnetic
field \((\varphi, A) \in \mathbb{R} \times \mathbb{E}^3\), the corresponding classical (relativistic) action functional is chosen (see [12, 5, 35, 10]) as follows:

\[
S := \int_{\tau_1}^{\tau_2} \left[ -m_0 \, dr + q < A, \dot{r} > d\tau - q \varphi (1 - u^2)^{-1/2} d\tau \right],
\]

with respect to the so-called “rest” reference system, parameterized by the Euclidean space-time variables \((r, \tau) \in \mathbb{E}^4\), where as before, \(< \cdot, \cdot >\) is the standard scalar product in the related Euclidean subspace \(\mathbb{E}^3\) and there is denoted \(\dot{r} := dr/d\tau\) in contrast to the definition \(u := dr/dt\).

The action (1.8), with respect to the moving with velocity vector \(u \in \mathbb{E}^3\) reference system, can be rewritten as

\[
S = \int_{t_1}^{t_2} \mathcal{L} dt, \quad \mathcal{L} := -m_0(1 - u^2)^{1/2} + q < A, u > -q \varphi,
\]

on the suitable temporal interval \([t_1, t_2] \subset \mathbb{R}\), giving rise to the following [12, 5, 35, 10] dynamical expressions

\[
P = p + qA, \quad p = mu,
\]

for the particle momentum and

\[
\mathcal{E}_0 = [m_0^2 + (P - qA)^2]^{1/2} + q \varphi
\]

for the particle energy, where, by definition, \(P \in \mathbb{E}^3\) means the common momentum of the particle and the ambient electromagnetic field at a space-time point \((r, t) \in \mathbb{M}^4\).

The obtained expression (1.11) for the particle energy \(\mathcal{E}_0\) also looks slightly controversial, since the potential energy \(q \varphi\), entering additively, has no impact into the particle mass \(m = m_0(1 - u^2)^{-1/2}\). As it was already mentioned [15] by L. Brillouin, the fact that the potential energy has no impact on the particle mass says us that “... any possibility of existing the particle mass related with an external potential energy, is completely excluded”. This and some other special relativity theory and electrodynamics problems, as is well-known, stimulated many other prominent physicists of the past [15, 4, 36, 35, 19] and the present [16, 38, 37, 23, 20, 21, 22] to make significant efforts aiming to develop alternative relativity theories [34, 33, 32, 31, 29, 18, 42, 41, 24, 25, 26, 27, 28] based on completely different space-time and matter structure principles.

There is also another controversial inference from the action expression (1.9). As one can easily show [12, 35, 5, 10], the corresponding dynamical equation for the Lorentz force is given as follows:

\[
dp{p}{t} = F := qE + qu \times B,
\]

where the operation \(\times\) denotes, as before, the standard vector product and we put, by definition,

\[
E := -\partial A/\partial t - \nabla \varphi
\]
for the related electric field and

\[ B := \nabla \times A \]

for the related magnetic field, acting on the charged point particle \( q \); the operation "\( \nabla \)" is here, as before, the standard gradient. The obtained expression (1.12) means, in particular, that the Lorentz force \( F \) depends linearly on the particle velocity vector \( u \in \mathbb{E}^3 \), giving rise to its strong dependence on the reference system with respect to which the charged particle \( q \) moves. Namely, the attempts to reconcile this and some related controversies \([15, 4, 18, 30]\) forced A. Einstein to devise his special relativity theory and proceed further to create his general relativity theory trying to explain the gravity by means of a geometrization of space-time and matter in the Universe. Here we must mention that the classical Lagrangian function \( \mathcal{L} \) in (1.9) is written by means of the mixed combinations of terms expressed by means of both the Euclidean rest reference system variables \((r, \tau) \in \mathbb{E}^4\) and an arbitrarily chosen reference system variables \((r, t) \in \mathbb{M}^4\).

These problems were recently analyzed from another completely "no-geometry" point of view in \([6, 7, 18]\), where new dynamical equations were derived, being free of controversy, as mentioned above. Moreover, the devised approach allowed to avoid the introduction of the well-known Lorentz transformations of the space-time reference systems with respect to which the action functional (1.9) is invariant. From this point of view there are very interesting reasonings of work \([22]\), in which Galilean invariant Lagrangians, possessing the intrinsic Poincare-Lorentz group symmetry, are reanalyzed. Below we will reanalyze the results obtained in \([6, 7]\) from the classical Lagrangian and Hamiltonian formalisms, which will shed a new light on the related physical backgrounds of the vacuum field theory approach to common studying electromagnetic and gravitational effects.

2. The vacuum field theory electrodynamics equations: Lagrangian analysis

2.1. A freely moving point particle - an alternative electrodynamical model. Within the vacuum field theory approach to joint describing the electromagnetism and the gravity, devised in \([6, 7]\), the main vacuum potential field function \( \bar{W} : \mathbb{M}^4 \to \mathbb{R} \), related to a charged point particle \( q \), satisfies in the case of the rested external charged point objects the following \([6]\) dynamical equation

\[ \frac{d}{dt}(-\bar{W}u) = -\nabla \bar{W}, \]

where, as above, \( u := dr/dt \) is the particle velocity with respect to some reference system.

To analyze the dynamical equation (2.1) from the Lagrangian point of view we will write the corresponding action functional as

\[ S := -\int_{t_1}^{t_2} \bar{W} dt = -\int_{\tau_1}^{\tau_2} \bar{W}(1 + r^2)^{1/2} d\tau, \]
expressed with respect to the rest reference system $K_r$. Having fixed proper temporal parameters $\tau_1 < \tau_2 \in \mathbb{R}$, from the least action condition $\delta S = 0$ one easily finds that

$$p : = \partial L / \partial \dot{r} = -\bar{W} \dot{r}(1 + \dot{r}^2)^{-1/2} = -\bar{W} u,$$

$$\dot{p} : = dp / d\tau = \partial L / \partial r = -\nabla W(1 + \dot{r}^2)^{1/2},$$

where, owing to (2.2), the corresponding Lagrangian function

$$\mathcal{L} : = -\bar{W}(1 + \dot{r}^2)^{1/2}.$$  

Recalling now the definition of the particle mass

$$m : = -\bar{W}$$

and the relationships

$$d\tau = dt(1 - u^2)^{1/2}, \quad \dot{r} d\tau = u dt,$$

from (2.3) we easily obtain exactly the dynamical equation (2.1). Moreover, one easily obtains that the dynamical mass, defined by means of expression (2.5), is given as

$$m = m_0(1 - u^2)^{-1/2},$$

coinciding with the result (1.3) of the preceding Section. Thereby, based on the obtained above results, one can formulate the following proposition.

**Proposition 2.1.** The alternative freely moving point particle electrodynamical model (2.1) allows the least action formulation (2.2) with respect to the "rest" reference system variables, where the Lagrangian function is given by expression (2.4). Its electrodynamics is completely equivalent to that of a classical relativistic freely moving point particle, described in Section 1.

2.2. A moving charged point particle - an alternative electrodynamical model. We proceed now to the case when our charged point particle $q$ moves in the space-time with velocity vector $u \in \mathbb{E}^3$ and interacts with another external charged point particle, moving with velocity vector $u_f \in \mathbb{E}^3$ subject to some common reference system $K$. As was shown in [6, 7], the corresponding dynamical equation on the vacuum potential field function $\bar{W} : \mathbb{M}^4 \rightarrow \mathbb{R}$ is given as

$$\frac{d}{dt}[-\bar{W}(u - u_f)] = -\nabla \bar{W}.$$

As the external charged particle moves in the space-time, it generates the related magnetic field $B := \nabla \times A$, whose magnetic vector potential $A : \mathbb{M}^4 \rightarrow \mathbb{E}^3$ is defined, owing to the results of [6, 7, 18], as

$$qA : = \bar{W} u_f.$$

Since, owing to (2.3), the particle momentum $p = -\bar{W} u$, equation (2.7) can be equivalently rewritten as

$$\frac{d}{dt}(p + qA) = -\nabla \bar{W}.$$
To represent the dynamical equation (2.9) within the classical Lagrangian formalism, we start from the following action functional naturally generalizing functional (2.2):

\[
(2.10) \quad S := - \int_{\tau_1}^{\tau_2} W(1 + |\dot{r} - \dot{\xi}|^2)^{1/2} \, d\tau,
\]

where we denoted by \( \dot{\xi} = u_f dt/d\tau \), \( d\tau = dt(1 - |u - u_f|^2)^{1/2} \), which takes into account the relative velocity of our charged point particle \( q \) with respect to the reference system \( K_f \), moving with velocity vector \( u_f \in \mathbb{E}^3 \) subject to the reference system \( K \). In this case, evidently, our charged point particle \( q \) moves with the velocity vector \( u - u_f \in \mathbb{E}^3 \) subject to the reference system \( K_f \), and with respect to which the external charged particle \( q_f \) is, respectively, in rest.

Compute now the least action variational condition \( \delta S = 0 \), taking into account that, owing to (2.10), the corresponding Lagrangian function is given as

\[
(2.11) \quad \mathcal{L} := -W(1 + |\dot{r} - \dot{\xi}|^2)^{1/2}.
\]

Thereby, the generalized particle momentum

\[
(2.12) \quad P := \frac{\partial \mathcal{L}}{\partial \dot{r}} = -\dot{W}(\dot{r} - \dot{\xi})(1 + |\dot{r} - \dot{\xi}|^2)^{-1/2} = -\dot{W}(1 + |\dot{r} - \dot{\xi}|^2)^{-1/2} + \dot{W}\dot{\xi}(1 + |\dot{r} - \dot{\xi}|^2)^{-1/2} = mu + qA := p + qA,
\]

and the corresponding dynamical equation is given as

\[
(2.13) \quad \frac{d}{d\tau}(p + qA) = -\nabla \dot{W}(1 + |\dot{r} - \dot{\xi}|^2)^{1/2}.
\]

Taking into account that \( d\tau = dt(1 - |u - u_f|^2)^{1/2} \) and \((1 + |\dot{r} - \dot{\xi}|^2)^{1/2} = (1 - |u - u_f|^2)^{-1/2} \), we obtain finally from (2.13) exactly the dynamical equation (2.9). Thus, we can formulate our result as the next proposition.

**Proposition 2.2.** The alternative classical relativistic electrodynamical model (2.7) allows the least action formulation (2.10) with respect to the "rest" reference system variables, where the Lagrangian function is given by expression (2.11).

### 2.3. A moving charged point particle - a dual to the classical alternative electrodynamical model.

It is easy to observe that the action functional (2.10) is written taking into account the classical Galilean transformations of reference systems. If we now consider the action functional (2.2) for a charged point particle, moving with respect the reference system \( K_r \), and take into account its interaction with an external magnetic field, generated by the vector potential \( A : \mathbb{M}^4 \to \mathbb{E}^3 \), it can be naturally generalized as

\[
(2.14) \quad S := \int_{t_1}^{t_2} (\dot{W}dt + q < A, \dot{r} >) = \int_{t_1}^{t_2} [-\dot{W}(1 + \dot{r}^2)^{1/2} + q < A, \dot{r} >] d\tau,
\]

where we accepted that \( d\tau = dt(1 - u^2)^{1/2} \).
Thus, the corresponding common particle-field momentum looks as follows:

\begin{equation}
P = \partial L / \partial \dot{r} = -\dot{W}r(1 + \dot{r}^2)^{-1/2} + qA = \mu u + qA := p + qA,
\end{equation}

satisfying the equation

\begin{equation}
\dot{P} = dP/d\tau = \partial L / \partial r = -\nabla \dot{W}(1 + \dot{r}^2)^{-1/2} + q\nabla < A, \dot{r} > = -\nabla \dot{W}(1 - u^2)^{-1/2} + q\nabla < A, u > (1 - u^2)^{-1/2},
\end{equation}

where

\begin{equation}
L := -\dot{W}(1 + \dot{r}^2)^{1/2} + q < A, \dot{r} >
\end{equation}
is the corresponding Lagrangian function. Taking now into account that $d\tau = dt(1 - u^2)^{1/2}$, one easily finds from (2.16) that

\begin{equation}
dP/dt = -\nabla \dot{W} + q\nabla < A, u >.
\end{equation}

Upon substituting (2.15) into (2.18) and making use of the well-known \[12\] identity

\begin{equation}
\nabla < a, b > = < a, \nabla > b + < b, \nabla > a + b \times (\nabla \times a) + a \times (\nabla \times b),
\end{equation}

where $a, b \in \mathbb{R}^3$ are arbitrary vector functions, we obtain finally the classical expression for the Lorentz force $F$, acting on the moving charged point particle $q$:

\begin{equation}
dp/dt := F = qE + qu \times B,
\end{equation}

where, by definition

\begin{equation}
E := -\nabla \dot{W}q^{-1} - \partial A/\partial t
\end{equation}
is the corresponding electric field and

\begin{equation}
B := \nabla \times A
\end{equation}
is the corresponding magnetic field.

With the result obtained we can formulate the next proposition.

**Proposition 2.3.** The classical relativistic Lorentz force (2.20) allows the least action formulation (2.14) with respect to the rest reference system variables, where Lagrangian function is given by expression (2.17). Its electrodynamics, described by the Lorentz force (2.20), is completely equivalent to the classical relativistic moving point particle electrodynamics, described by means of the Lorentz force (1.12) in Section 2.

Concerning the previously obtained dynamical equation (2.13) we can easily observe that it can be equivalently rewritten as follows:

\begin{equation}
dp/dt = (-\nabla \dot{W} - qdA/dt + q\nabla < A, u >) - q\nabla < A, u >.
\end{equation}
The latter, owing to (2.18) and (2.20), takes finally the following Lorentz type force in the form

\begin{equation}
dp/dt = qE + qu \times B - q\nabla < A, u >,
\end{equation}
previously found in [6, 7, 18].

Expressions (2.20) and (2.24) are equal to each other up to the gradient term \( F_c := -q\nabla < A, u > \), which allows to reconcile the Lorentz forces acting on a charged moving particle \( q \) with respect to different reference systems. This fact is important for our vacuum field theory approach since no special geometry needs to use, making it possible to analyze both electromagnetic and gravitational fields simultaneously, based on a new definition of the dynamical mass by means of expression (2.5).

3. The vacuum field theory electrodynamics equations: Hamiltonian analysis

It is well-known [1, 10, 2, 17, 9] that any Lagrangian theory allows the equivalent canonical Hamiltonian representation via the classical Legendrian transformation. As we have already formulated above our vacuum field theory of a moving charged particle \( q \) in the Lagrangian form, we proceed now to its Hamiltonian analysis making use of the action functionals (2.2), (2.11) and (2.14).

Take, first, the Lagrangian function (2.4) and the momentum expression (2.3) for defining the corresponding Hamiltonian function

\[
H := <p, \dot{r}> - \mathcal{L} = -<p, p> + \mathcal{L}^{-1} - p^2\mathcal{L}^{-1} + \mathcal{L}^{-1} = -(\mathcal{L} - p^2)^{1/2}.
\]

As a result, we easily obtain [2, 1, 10, 9] that the Hamiltonian function (3.1) is a conservation law of the dynamical field equation (2.1), that is for all \( \tau, t \in \mathbb{R} \)

\[
dH/dt = 0 = dH/d\tau,
\]

which naturally allows to interpret it as the energy expression. Thus, we can write the particle energy

\[
E = (\mathcal{L} - p^2)^{1/2}.
\]

The suitable Hamiltonian equations, equivalent to the vacuum field equation (2.1), look as follows:

\[
\dot{r} := dr/d\tau = \partial H/\partial p = p(\mathcal{L} - p^2)^{1/2}
\]

\[
\dot{p} := dp/d\tau = -\partial H/\partial r = W\nabla \mathcal{L}(\mathcal{L} - p^2)^{1/2}.
\]

Thereby, based on the results obtained above, one can formulate the following proposition.

**Proposition 3.1.** The alternative freely moving point particle electrodynamical model (2.1) allows the canonical Hamiltonian formulation (3.4) with respect to the rest reference system variables, where the Hamiltonian function is given by expression (3.1). Its electrodynamics is formally completely equivalent to the classical relativistic freely moving point particle electrodynamics, described in Section 2.
Now based on the Lagrangian expression (2.11) one can construct in the same way as above, the Hamiltonian function for the dynamical field equation (2.9), describing the motion of charged particle \( q \) in external electromagnetic field in the canonical Hamiltonian form:

\[
\begin{align*}
\dot{r} := \frac{dr}{d\tau} &= \frac{\partial H}{\partial P}, \\
\dot{P} := \frac{dP}{d\tau} &= -\frac{\partial H}{\partial r},
\end{align*}
\]

where

\[
H := < P, \dot{r} > - L = < P, \dot{\xi} > - P \bar{W}^{-1}(1 - P^2/\bar{W}^2)^{-1/2} + \bar{W} \bar{W}^{-1}(\bar{W}^2 - P^2)^{-1/2} = < P, \dot{\xi} > + P^2(\bar{W}^2 - P^2)^{-1/2} - \bar{W}^2(\bar{W}^2 - P^2)^{-1/2} = -(\bar{W}^2 - P^2)^{1/2} - q < A, P > (\bar{W}^2 - P^2)^{-1/2}.
\]

Here we took into account that, owing to definitions (2.8) and (2.12),

\[
qA := \bar{W}u = \bar{W}d\xi/dt = \bar{W}d\dot{\xi}/d\tau = \bar{W}\dot{\xi}(1 - |u - v|^2)^{1/2} = -\bar{W}\dot{\xi}(\bar{W}^2 - P^2)^{1/2} - \bar{W}^{-1} = -qA(\bar{W}^2 - P^2)^{-1/2},
\]

or

\[
\dot{\xi} = -qA(\bar{W}^2 - P^2)^{-1/2},
\]

where \( A : \mathbb{M}^4 \rightarrow \mathbb{R}^3 \) is the related magnetic vector potential, generated by the moving external charged particle.

Thereby we can state that the Hamiltonian function (3.6) satisfies the energy conservation conditions

\[
dH/dt = 0 = dH/d\tau,
\]

for all \( \tau, t \in \mathbb{R} \), that is, the suitable energy expression

\[
\mathcal{E} = (\bar{W}^2 - P^2)^{1/2} + q < A, P > (\bar{W}^2 - P^2)^{-1/2}
\]

holds. The result (3.10) essentially differs from that obtained in [12], which makes use of the well-known Einsteinian Lagrangian for a moving charged point particle \( q \) in external electromagnetic field. Thereby, our result can be formulated as follows.

**Proposition 3.2.** The alternative classical relativistic electrodynamical model (2.7) allows the Hamiltonian formulation (3.5) with respect to the "rest" reference system variables, where the Hamiltonian function is given by expression (3.6).

To make this difference more clear, we will analyze below the Lorentz force (2.20) from the Hamiltonian point of view based on the Lagrangian function (2.17). Thus, we obtain that the
corresponding Hamiltonian function

\[
H = \langle P, \dot{r} \rangle - \mathcal{L} = \langle P, \dot{r} \rangle + \bar{W}(1 + \dot{r}^2)^{1/2} - q \langle A, \dot{r} \rangle = \\
= \langle P - qA, \dot{r} \rangle + \bar{W}(1 + \dot{r}^2)^{1/2} = \\
= - \langle p, p \rangle + \bar{W}^{-1}(1 - p^2/\bar{W}^2)^{-1/2} + \bar{W}(1 - p^2/\bar{W}^2)^{-1/2} = \\
= - (\bar{W}^2 - p^2)(\bar{W}^2 - p^2)^{-1/2} = -(\bar{W}^2 - p^2)^{1/2}.
\]

Since \( p = P - qA \), expression (3.11) takes the final “no interaction” [12, 35, 39, 40] form

\[
(3.12) \quad H = -[(\bar{W}^2 - (P - qA)^2)]^{1/2},
\]

being conservative with respect to the evolution equations (2.15) and (2.16), that is,

\[
(3.13) \quad \frac{dH}{dt} = \frac{dH}{d\tau} = 0
\]

for all \( \tau, t \in \mathbb{R} \). The latter are simultaneously equivalent to the following Hamiltonian system:

\[
(3.14) \quad \dot{r} = \frac{\partial H}{\partial P} = (P - qA)[\bar{W}^2 - (P - qA)^2]^{-1/2}, \\
\dot{P} = -\frac{\partial H}{\partial r} = (\bar{W} \nabla \bar{W} - \nabla \langle qA, (P - qA) \rangle)[\bar{W}^2 - (P - qA)^2]^{-1/2},
\]

that can easily be checked by direct calculations. Really, the first equation

\[
(3.15) \quad \dot{r} = (P - qA)[\bar{W}^2 - (P - qA)^2]^{-1/2} = p(\bar{W}^2 - p^2)^{-1/2} = \\
= mu(\bar{W}^2 - p^2)^{-1/2} = -\bar{W}u(\bar{W}^2 - p^2)^{-1/2} = u(1 - u^2)^{-1/2},
\]

holds, owing to the condition \( d\tau = dt(1 - u^2)^{1/2} \) and definitions \( p := mu, m = -\bar{W} \), postulated from the very beginning. Similarly we obtain that

\[
(3.16) \quad \dot{P} = -\nabla \bar{W}(1 - p^2/\bar{W}^2)^{-1/2} + \nabla \langle qA, u \rangle (1 - p^2/\bar{W}^2)^{-1/2} = \\
= -\nabla \bar{W}(1 - u^2)^{-1/2} + \nabla \langle qA, u \rangle (1 - u^2)^{-1/2},
\]

exactly coinciding with equation (2.18) subject to the evolution parameter \( t \in \mathbb{R} \). We now formulate our result as the next proposition.

**Proposition 3.3.** *The dual to the classical relativistic electrodynamical model (2.20) allows the Hamiltonian formulation (3.14) with respect to the “rest” reference system variables, where the Hamiltonian function is given by expression (3.12).*

### 4. The quantization of electrodynamics models within the vacuum field theory no-geometry approach

#### 4.1. The problem setting

In our recent works [6, 7] a new regular no-geometry approach was devised to deriving from the first principles the electrodynamics of a moving charged point particle \( q \) in external electromagnetic field. This approach has, in part, reconciled the existing mass-energy controversy [15] within the classical relativistic electrodynamics. Based on the vacuum field theory approach proposed in [6, 7, 18] we re-analyzed this problem in the sections above both from Lagrangian and Hamiltonian points of view having derived crucial expressions for the
corresponding energy functions and Lorentz type forces, acting on moving charge point particle \( q \).

Since all of our electrodynamics models were represented here in the canonical Hamiltonian form, they are suitable for applying to them the Dirac type quantization procedure [11, 3, 14] and regularly obtaining the related Schrödinger type evolution equations. This Section is devoted to this problem.

4.2. **Free point particle electrodynamics model and its quantization.** Charged point particle electrodynamics models, discussed in detail in Sections 2 and 3, were also considered in [7] from the dynamical point of view, where an attempt of applying the quantization Dirac type procedure to the corresponding conserved energy expressions was done. Nevertheless, within the canonical point of view, the true quantization procedure should be based on the suitable canonical Hamiltonian formulation of the models, which in the case under consideration is given by (3.4), (3.5) and (3.14).

In particular, consider a free charged point particle electrodynamics model, governed by the following Hamiltonian equations:

\[
\frac{dr}{d\tau} = \frac{\partial H}{\partial p} = -p(\bar{W}^2 - p^2)^{-1/2},
\]

\[
\frac{dp}{d\tau} = -\frac{\partial H}{\partial r} = -\bar{W}\nabla\bar{W}(\bar{W}^2 - p^2)^{-1/2},
\]

where we denoted, as before, the corresponding vacuum field potential by \( \bar{W} : \mathbb{M}^4 \to \mathbb{R} \), characterizing vacuum medium field structure, the standard canonical coordinate-momentum variables by \( (r, p) \in \mathbb{E}^3 \times \mathbb{E}^3 \), the proper rest reference system \( \mathcal{K}_r \) time parameter related with our moving particle by \( \tau \in \mathbb{R} \), and by \( H : \mathbb{E}^3 \times \mathbb{E}^3 \to \mathbb{R} \) the Hamiltonian function

\[
H := -(\bar{W}^2 - p^2)^{1/2},
\]

expressed here and throughout the paper in the light speed units. The rest reference system \( \mathcal{K}_r \), parameterized by variables \( (r, \tau) \in \mathbb{E}^4 \), is related to any other reference system \( \mathcal{K} \) subject to which our charged point particle \( q \) moves with velocity vector \( u \in \mathbb{E}^3 \), and which is parameterized by variables \( (r, t) \in \mathbb{M}^4 \), via the following Euclidean infinitesimal relationship:

\[
dt^2 = d\tau^2 + dr^2,
\]

which is equivalent to the Minkovskian infinitesimal relationship

\[
d\tau^2 = dt^2 - dr^2.
\]

The Hamiltonian function (4.2) satisfies, evidently, the energy conservation conditions

\[
dH/dt = 0 = dH/d\tau
\]

for all \( t, \tau \in \mathbb{R} \). This means that the suitable energy value

\[
\mathcal{E} = (\bar{W}^2 - p^2)^{1/2}
\]
can be treated by means of the Dirac type quantization scheme [11] to obtain, as \( \hbar \rightarrow 0 \), (or the light speed \( c \rightarrow \infty \)) the governing Schrödinger type dynamical equation. To do this, similarly to [6, 7], we need to make canonical operator replacements \( E \rightarrow \hat{E} := -\frac{\hbar}{i} \frac{\partial}{\partial \tau}, \ p \rightarrow \hat{p} := \frac{\hbar}{i} \nabla \), as \( \hbar \rightarrow 0 \), in the following energy determining expression:

\[
(4.7) \quad \mathcal{E}^2 := (\hat{E} \psi, \hat{E} \psi) = (\psi, \hat{E}^2 \psi) = (\psi, \hat{H} \hat{H} \psi),
\]

where, by definition, owing to (4.6),

\[
(4.8) \quad \hat{E}^2 = \bar{W}^2 - \hat{p}^2 = \hat{H} + \hat{H}
\]

is a suitable operator factorization in the Hilbert space \( \mathcal{H} := L_2(\mathbb{R}^3; \mathbb{C}) \) and \( \psi \in \mathcal{H} \) is the corresponding normalized quantum vector state. Since the following elementary identity

\[
(4.9) \quad \bar{W}^2 - \hat{p}^2 = \bar{W}(1 - \bar{W}^{-1} \hat{p}^2 \bar{W}^{-1})^{1/2}(1 - \bar{W}^{-1} \hat{p}^2 \bar{W}^{-1})^{1/2} \hat{W}
\]

holds, we can put, by definition, using (4.8) and (4.9), that the operator

\[
(4.10) \quad \hat{H} := (1 - \bar{W}^{-1} \hat{p}^2 \bar{W}^{-1})^{1/2} \bar{W}.
\]

Having calculated the operator expression (2.10) as \( \hbar \rightarrow 0 \) up to operator accuracy \( O(\hbar^4) \), we can easily obtain that

\[
(4.11) \quad \hat{H} = \frac{\hat{p}^2}{2m(u)} + \bar{W} := -\frac{\hbar^2}{2m(u)} \nabla^2 + \bar{W},
\]

where we took into account the dynamical mass definition \( m(u) := -\bar{W} \) (in the light speed units).

Thereby, based now on (4.7) and (4.11), we obtain up to operator accuracy \( O(\hbar^3) \) the following Schrödinger type evolution equation

\[
(4.12) \quad i\hbar \frac{\partial \psi}{\partial \tau} := \hat{E} \psi = \hat{H} \psi = -\frac{\hbar^2}{2m(u)} \nabla^2 \psi + \bar{W} \psi
\]

with respect to the rest reference system \( K_r \) evolution parameter \( \tau \in \mathbb{R} \). Concerning the related evolution parameter \( t \in \mathbb{R} \), parameterizing a reference system \( K \), equation (4.12) takes the following form:

\[
(4.13) \quad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2 m_0}{2m(u)^2} \nabla^2 \psi - m_0 \psi.
\]

Here we took into account that, owing to (4.6), the classical mass relationship

\[
(4.14) \quad m(u) = m_0 \left(1 - u^2\right)^{-1/2}
\]

holds, where \( m_0 \in \mathbb{R}_+ \) is the corresponding rest mass of our point particle \( q \).

The obtained linear Schrödinger equation (4.13) for the case \( \hbar/c \rightarrow 0 \) really coincides with that well-known [12, 11, 5] from classical quantum mechanics.
4.3. Classical charged point particle electrodynamics model and its quantization. We start here from the first vacuum field theory reformulation of the classical charged point particle electrodynamics, considered in Section 3 and based on the conserved Hamiltonian function (3.12)

\[ H := -[\bar{W}^2 - (P - qA)^2]^{1/2}, \]

where \( q \in \mathbb{R} \) is the particle charge and \((\bar{W}, A) \in \mathbb{R} \times \mathbb{E}^3\) is the corresponding electromagnetic field potential and \( P \in \mathbb{E}^3 \) is the common particle-field momentum, defined as

\[ P := p + qA, \quad p := mu, \]

and satisfying the well-known classical Lorentz force equation. Here \( m := -\bar{W} \) is the observable dynamical mass, related with our charged particle, \( u \in \mathbb{E}^3 \) is its velocity vector with respect to a chosen reference system \( \mathcal{K} \), all being expressed here, as previously, in the light speed units.

As our electrodynamics, based on (4.15), is canonically Hamiltonian, the Dirac type quantization scheme

\[ P \rightarrow \hat{P} := \frac{\hbar}{i} \nabla, \quad \mathcal{E} \rightarrow \hat{\mathcal{E}} := -\frac{\hbar}{i} \frac{\partial}{\partial \tau} \]

should be applied to the suitable energy expression

\[ \mathcal{E} := [\bar{W}^2 - (P - qA)^2]^{1/2}, \]

following from the conservation conditions

\[ \frac{dH}{dt} = 0 = \frac{dH}{d\tau}, \]

satisfied for all \( \tau, t \in \mathbb{R} \).

Proceeding now in the same way as above, we can factorize the operator \( \hat{\mathcal{E}}^2 \) as follows:

\[
\bar{W}^2 - (\hat{P} - qA)^2 = \bar{W}[1 - \bar{W}^{-1}(\hat{P} - qA)^2\bar{W}^{-1}]^{1/2} \times [1 - \bar{W}^{-1}(\hat{P} - qA)^2\bar{W}^{-1}]^{1/2} \bar{W} := \hat{H} \hat{H},
\]

where, by definition, (here as \( \hbar/c \rightarrow 0, \hbar c = \text{const}, c \) is the light velocity)

\[ \hat{H} := \frac{1}{2m(u)} \left( \frac{\hbar}{i} \nabla - qa \right)^2 + \bar{W} \]

up to operator accuracy \( O(\hbar^3) \). Thereby, the related Schrödinger type evolution equation in the Hilbert space \( \mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C}) \) reads as

\[ i\hbar \frac{\partial \psi}{\partial \tau} := \hat{\mathcal{E}} \psi = \hat{H} \psi = \frac{1}{2m(u)} \left( \frac{\hbar}{i} \nabla - qA \right)^2 \psi + \bar{W} \psi \]

with respect to the rest reference system \( \mathcal{K}_r \) evolution parameter \( \tau \in \mathbb{R} \). The corresponding Schrödinger type evolution equation with respect to the evolution parameter \( t \in \mathbb{R} \) reads, respectively, as

\[ i\hbar \frac{\partial \psi}{\partial t} = \frac{m_0}{2m(u)^2} \left( \frac{\hbar}{i} \nabla - qA \right)^2 \psi - m_0 \psi. \]

The Schrödinger type evolution equation (4.21) (as \( \hbar/c \rightarrow 0, \hbar c = \text{const} \)) completely coincides [13, 11] with that well-known from the classical quantum mechanics.
Consider now, in the framework of the canonical point of view, the true quantization procedure of the electrodynamics model (2.13), whose Hamiltonian function reads as

\[ H := -(\bar{W}^2 - P^2)^{1/2} - q <A, P > (\bar{W}^2 - P^2)^{-1/2}. \]

This means that the suitable energy function

\[ \mathcal{E} := (\bar{W}^2 - P^2)^{1/2} + q <A, P > (\bar{W}^2 - P^2)^{-1/2}, \]

where, as before,

\[ P := p + qA, \quad p := mu, \quad m := -\dot{\bar{W}}, \]

is a conserved quantity for (2.13), which we will canonically quantize via the Dirac procedure (4.17). To do this, let us consider the quantum condition

\[ \mathcal{E}^2 := (\hat{\mathcal{E}} \psi, \hat{\mathcal{E}} \psi) = (\psi, \hat{\mathcal{E}}^2 \psi), \quad (\psi, \psi) := 1, \]

where, by definition, \( \hat{\mathcal{E}} := -\hbar \frac{\partial}{\partial \tau} \) and \( \psi \in \mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C}) \) is a suitable normalized quantum state vector. Now making use of the energy function (4.24), one can easily obtain that

\[ \mathcal{E}^2 = \bar{W}^2 - (\bar{P} - qA)^2 + q^2 <A, A > + <A, P > (\bar{W}^2 - P^2)^{-1} < P, A >, \]

which, owing to the canonical Dirac type quantization recipe \( P \rightarrow \hat{P} := \frac{\hbar}{i} \nabla \), transforms into the symmetrized operator expression

\[ \hat{\mathcal{E}}^2 = \bar{W}^2 - (\hat{\bar{P}} - qA)^2 + q^2 <A, A > + q^2 <A, \hat{P} > (\bar{W}^2 - \hat{P}^2)^{-1} < \hat{P}, A >. \]

Having factorized operator (4.28) in the form \( \hat{\mathcal{E}}^2 = \hat{H}^+ \hat{H} \), we obtain that up to operator accuracy \( O(\hbar^4) \) (as \( \hbar/c \rightarrow 0, \hbar c = \text{const} \))

\[ \hat{H} := \frac{1}{2m(u)} (\hbar i \nabla - qA)^2 - \frac{q^2}{2m(u)} < A, A > - \frac{q^2}{2m(u)} < A, \frac{\hbar}{i} \nabla > < \frac{\hbar}{i} \nabla, A >, \]

where we put, as before, \( m(u) = -\bar{W} \) in the light speed units. Thus, owing to (4.26) and (4.29), the resulting Schrödinger evolution equation takes the form

\[ i\hbar \frac{\partial \psi}{\partial \tau} = \hat{H} \psi = \frac{1}{2m(u)} (\hbar i \nabla - qA)^2 \psi - \frac{q^2}{2m(u)} < A, \psi > - \frac{q^2}{2m(u)} < A, \frac{\hbar}{i} \nabla > < \frac{\hbar}{i} \nabla, \psi > \]

with respect to the rest reference system \( \mathcal{K}_f \) evolution parameter \( \tau \in \mathbb{R} \). Similarly one obtains also the related Schrödinger type evolution equation with respect to the time parameter \( t \in \mathbb{R} \), which we will not discuss here. The result (4.30) essentially differs from the corresponding classical Schrödinger evolution equation (4.21), thereby forcing us to reanalyze more thoroughly the main physically motivated principles, put into the backgrounds of classical electrodynamical models, described by the Hamiltonian functions (4.15) and (4.23) and giving rise to different Lorentz type force expressions. This analysis we plan to do in a forthcoming paper.
5. Conclusion

Thereby, we can claim that all of the dynamical field equations discussed above are canonical Hamiltonian systems with respect to the corresponding proper rest reference systems $K_r$, parameterized by suitable time parameters $\tau \in \mathbb{R}$. When passing over to the laboratory reference system $K$, parameterized by the time parameter $t \in \mathbb{R}$, the related Hamiltonian structure is naturally lost. This gives rise to a new interpretation of the real particle motion as such one, which has the absolute sense only with respect to the proper rest reference system $K_r$ and which is completely relative from physical point of view with respect to all other reference systems. Concerning the Hamiltonian expressions (3.1), (3.6) and (3.12) obtained above, one observes that all of them depend strongly on the vacuum potential field function $\bar{W} : \mathbb{M}^4 \to \mathbb{R}$, thereby dissolving the mass problem of the classical energy expression, pointed out by L. Brillouin [15]. Here it is necessary to mention that, subject to the canonical Dirac type quantization procedure, it can be applied only to the corresponding Hamiltonian dynamical field equations, describing charged particles evolution with respect to the proper rest reference systems.

Remark 5.1. Some comments can also be made concerning the classical relativity principle. Namely, we have obtained our results without using the Lorentz transformations of reference systems but only the natural notion of the rest reference system and suitable its parametrization, naturally related with other moving reference systems. It looks reasonable since, in reality, the true state changes of a moving charged particle $q$ are exactly realized only with respect to its proper rest reference system. Therefore, the only question, still left open here, is that about the physical justification of the corresponding relationship between time parameters of moving and rest reference systems.

This relationship, being accepted throughout this work, reads as

\begin{equation}
(5.1) \quad d\tau = dt(1 - u^2)^{1/2},
\end{equation}

where $u := dr/dt \in \mathbb{E}^3$ is the velocity vector with which the rest reference system $K_r$ moves with respect to other arbitrarily chosen reference systems $K$. The expression (5.1) means, in particular, that there holds the equality

\begin{equation}
(5.2) \quad dt^2 - dr^2 = d\tau^2,
\end{equation}

which exactly coincides with the classical infinitesimal Lorentz invariant. Its appearance is, evidently, not casual here, since all our dynamical field equations were successively derived [6, 7] from the governing equations on the vacuum potential field function $W : \mathbb{M}^4 \to \mathbb{R}$ in the form

\begin{equation}
(5.3) \quad \partial^2 W/\partial t^2 - \nabla^2 W = \rho, \quad \partial W/\partial t + \nabla(vW) = 0, \quad \partial \rho/\partial t + \nabla(v\rho) = 0,
\end{equation}

being a priori Lorentz invariant, where we denoted by $\rho \in \mathbb{R}$ the charge density and by $v := dr/dt$ the suitable local velocity of the vacuum field potential changes. Thereby, the dynamical
infinitesimal Lorentz invariant (5.2) reflects this intrinsic physical structure of equations (5.3). If rewritten in the following nonstandard Euclidean form:

\[(5.4)\]
\[dt^2 = d\tau^2 + dr^2\]

it gives rise to a completely different time relationship between reference systems \(K\) and \(K_r\):

\[(5.5)\]
\[dt = d\tau(1 + \dot{r}^2)^{1/2},\]

where, as earlier, we denoted by \(\dot{r} = \frac{dr}{d\tau}\) the related particle velocity with respect to the rest reference system. Thus, we observe that all our Lagrangian analysis completed in Section 2 is based on the corresponding functional expressions written in these “Euclidean” space-time coordinates and with respect to which the least action principle was analyzed. So, we see that there exist two alternatives - the first is to apply the least action principle to the corresponding Lagrangian functions expressed in the Minkovski type space-time variables with respect to an arbitrary chosen laboratory reference system \(K\), and the second is to apply the least action principle to the corresponding Lagrangian functions expressed in the Euclidean space-time variables with respect to the rest reference system \(K_r\). Here we mention very skillful analysis of the Maxwell equations and related electron electrodynamics in [26, 27] with respect to the Euclidean rest reference system \(K_r\) coordinates. Regretfully, the important problem of studying the fundamental relationship between the Lorentz type force expression and the Maxwell electromagnetic equations was in [26, 27] omitted, that entailed the equivalent Maxwell equations reformulations, based on the \textit{a priori} postulated Lorentz transformations of physically observable fields.

An additional remark is needed here concerning the quantization procedure of proposed electrodynamics models. If the dynamical vacuum field equations are expressed in the canonical Hamiltonian form, only technical problems are left to quantize them and obtain the corresponding Schrödinger type evolution equations in suitable Hilbert spaces of quantum states. There still exists another important inference from the approach devised in this work, consisting in complete loss of the physical essence of the well-known Einsteinian equivalence principle [12, 35, 5, 4, 30], becoming superfluous for our vacuum field theory of electromagnetism and gravity.

Based on the canonical Hamiltonian formalism devised in this work, concerning the alternative charged point particle electrodynamics models, we succeeded in treating their Dirac type quantization. The obtained results were compared with classical ones, but the physically motivated choice of a true model is left for future studies. Another important aspect of the developed vacuum field theory no-geometry approach to combining the electrodynamics with the gravity consists in singling out the decisive role of the related rest reference system \(K_r\). Namely, with respect to the rest reference system evolution parameter \(\tau \in \mathbb{R}\) all of our electrodynamics models allow both the Lagrangian and Hamiltonian formulations be suitable for the canonical quantization. The physical essence of this fact, by now, is not enough understood. There is, up to now [35, 12, 30, 34, 33, 26, 27, 28], no physically reasonable explanation of this decisive role of
the rest reference system, except that of the very interesting reasonings by R. Feynman who argued in [5] that the relativistic expression for the classical Lorentz force (1.12) has physical sense only with respect to the rest reference system variables \((r, \tau) \in \mathbb{E}^4\). In the sequel of our work we plan to analyze both the revealed Lagrangian and Hamiltonian properties of the electrodynamics models and the related quantization schemes in more detail and make a step toward the vacuum quantum field theory of infinite many particle systems.

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