THREE MEROMORPHIC MAPPINGS
SHARING SOME COMMON HYPERPLANES

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Abstract

In this paper, using techniques of Nevanlinna theory, we give a degeneracy theorem of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ with few hyperplanes and truncated multiplicities.

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1 Introduction

Let \( f \) be a linearly nondegenerate meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{CP}^n \) with reduced representation \( (f_0 : \cdots : f_n) \). For each hyperplane \( H : a_0w_0 + \cdots + a_nw_n = 0 \) in \( \mathbb{CP}^n \), we put \( (f, H) = a_0f_0 + \cdots + a_nf_n \) and denote by \( \nu(f, H) \) the map of \( \mathbb{C}^m \) into \( \mathbb{N}_0 \) such that \( \nu(f, H)(a) \) \((a \in \mathbb{C}^m)\) is the intersection multiplicity of the image of \( f \) and \( H \) at \( f(a) \).

Take \( q \) hyperplanes \( H_1, \ldots, H_q \) in \( \mathbb{CP}^n \) in general position, a linearly nondegenerate meromorphic mapping \( f \) of \( \mathbb{C} \) into \( \mathbb{CP}^n \) such that

\[
\dim (f^{-1}(H_i) \cap f^{-1}(H_j)) \leq m - 2, \quad \text{for all } 1 \leq i < j \leq q.
\]

Let \( p \) be a positive integer. We consider the family \( \mathcal{F}(\{H_j\}_{j=1}^{q} : f, p) \) of all linearly nondegenerate meromorphic mappings \( g : \mathbb{C}^m \rightarrow \mathbb{CP}^n \) satisfying the conditions:

(a) \( \min \{\nu(g, H_j), p\} = \min \{\nu(f, H_j), p\} \) for all \( j \in \{1, \ldots, q\} \),

(b) \( g = f \) on \( \bigcup_{j=1}^{q} f^{-1}(H_j) \).

The uniqueness problem of meromorphic mappings of \( \mathbb{C}^m \) into \( \mathbb{CP}^n \) means that we want to find conditions for \( q \) (the number of hyperplanes) and \( p \) (the value at which multiplicities are truncated) such that the set \( \mathcal{F}(\{H_j\}_{j=1}^{q} : f, p) \) contains only one mapping (Uniqueness Theorem) or, more generally, we want to study the cardinality of the set \( \mathcal{F}(\{H_j\}_{j=1}^{q} : f, p) \) and find the relations among the mappings of this set.

In 1988, S. Ji [10] showed that

**Theorem 1.1.** Assume that \( q = 3n+1 \) and \( p = 1 \). Then for three maps \( g_1, g_2, g_3 \in \mathcal{F}(\{H_j\}_{j=1}^{q} : f, p) \), the map \( g_1 \times g_2 \times g_3 : \mathbb{C}^m \rightarrow \mathbb{CP}^n \times \mathbb{CP}^n \times \mathbb{CP}^n \) is algebraically degenerate, namely, \( \{(g_1(z), g_2(z), g_3(z)) \mid z \in \mathbb{C}^m\} \) is included in a proper algebraic subset of \( \mathbb{CP}^n \times \mathbb{CP}^n \times \mathbb{CP}^n \).

In 2006, G. Dethloff and T. V. Tan [4] showed that the above result of S. Ji remains valid if \( q \geq \frac{5(n+1)}{2} \).

In 1998, H. Fujimoto [8] obtained the following theorem.

**Theorem 1.2.** Suppose that \( q \geq 2n + 2, p = \frac{n(n+1)}{2} + n \) and take arbitrary \( n + 2 \) mappings \( f_1, \ldots, f_{n+2} \) in \( \mathcal{F}(\{H_j\}_{j=1}^{q} : f, p) \). Then, there are \( n + 1 \) hyperplanes \( H_{j_0}, \ldots, H_{j_n} \) among \( H_j \)'s such that for each pair \((i, k)\) with \( 0 \leq i < k \leq n \), we have that

\[
\frac{(f_2, H_{j_k})}{(f_2, H_{j_i})} = \frac{(f_1, H_{j_k})}{(f_1, H_{j_i})} \frac{(f_3, H_{j_k})}{(f_3, H_{j_i})} \frac{(f_4, H_{j_k})}{(f_4, H_{j_i})} \cdots \frac{(f_{n+2}, H_{j_k})}{(f_{n+2}, H_{j_i})} \frac{(f_1, H_{j_k})}{(f_1, H_{j_i})}
\]

are linearly dependent.

We would like to emphasize here that the Cartan auxiliary function (cf. [1],[8]) and the Second Main Theorem play essential roles in the uniqueness problem. They are used to estimate the counting functions. In this paper, we obtain an improvement concerning the Cartan auxiliary
function. So, the estimate of counting functions which we obtain here is better than the estimates of previous authors. After that, instead of estimating the defect relation as previous authors, we try to replace the value at which multiplicities are truncated by a bigger one. Because of that, our main result (Theorem 1.3) improves considerably the above mentioned results. Furthermore, it also contains many degeneracy theorems taking into account (truncated) orders of the inverse images of the hyperplanes.

**Theorem 1.3.** Take arbitrary three mappings \( f_1, f_2, f_3 \) in \( F(\{H_j\}^q_{j=1}, f, p) \). Assume that one of the following conditions satisfies

i) \( p = n \) and \( q > \frac{n+4+\sqrt{7n^2+2n+1}}{2} \), or

ii) \( 1 \leq p < n \) and there exists a positive integer \( t \) in \( \{p, \ldots, n-1\} \) such that

\[
3q + \frac{18t}{n-t} \left( \frac{3qn}{2q+3p-6} - q + n + 1 \right) < \frac{(q-n-1)(2q+3t-3)}{n}.
\]

Then there exist constants \( \alpha, \beta \in \mathbb{C} \) and a pair \((i_0, j_0)\) with \( 1 \leq i_0 \neq j_0 \leq q \), such that \( \alpha \left( \frac{f_2^* H_{h_0}}{(f_2^*, H_{h_0})} \right) + \beta \left( \frac{f_3^* H_{h_0}}{(f_3^*, H_{h_0})} - \frac{f_1^* H_{h_0}}{(f_1^*, H_{h_0})} \right) \equiv 0. \)

**Corollary 1.1.** Under the same assumption as in Theorem 1.3, we have that the mapping \( f_1 \times f_2 \times f_3 \) is linearly degenerate (with the algebraic structure in \( \mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n \) given by the Segre imbedding into \( \mathbb{C}P^{(n+1)^3-1} \)).

The most interesting special cases of Theorem 1.3 are the cases \( p = n \) and \( p = 1 \).

The case \( p = n \) is the one which gives the degeneracy theorem with the fewest number of hyperplanes.

The case \( p = 1 \) is the one where multiplicities of the inverse images of the hyperplanes are not taken into account. In this case, the inequality in the assertion ii) of Theorem 1.3 will become the following

\[
(\ast) \quad 3q + \frac{18t}{n-t} \left( \frac{3qn}{2q-3} - q + n + 1 \right) < \frac{(q-n-1)(2q+3t-3)}{n}.
\]

For each positive integer \( c \), take \( q = \lfloor 2.4n \rfloor - c \), \( t = \lfloor \frac{n}{3} \rfloor \), where we denote \( \lfloor x \rfloor := \max \{ k \in \mathbb{Z} : k \leq x \} \) for a constant \( x \). Then the right side of (\ast) \( \geq 8.12n - O(1) \), and the left side of (\ast) \( \leq 8.1n + O(1) \).

So, in this case there exists a positive integer \( N(c) \) depending only on \( c \) such that (\ast) is satisfied for all \( n > N(c) \).

## 2 Preliminaries

We set \( \|z\| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2} \) for \( z = (z_1, \ldots, z_m) \in \mathbb{C}^m \) and define

\[
B(r) := \{ z \in \mathbb{C}^m : \|z\| < r \}, \quad S(r) := \{ z \in \mathbb{C}^m : \|z\| = r \} \text{ for all } 0 < r < \infty.
\]

Define \( d^c := \sqrt{-1} \frac{1}{4\pi} (\overline{\partial} - \partial) \), \( v := (dd^c\|z\|^2)^{m-1} \) and \( \sigma := d^c\log\|z\|^2 \wedge (dd^c\log\|z\|^2)^{m-1} \).
Let $F$ be a nonzero holomorphic function on $\mathbb{C}^m$. For each $a \in \mathbb{C}^m$, expanding $F$ as $F = \sum P_i(z - a)$ with homogeneous polynomials $P_i$ of degree $i$ around $a$, we define 

$$\nu_F(a) := \min\{i : P_i \not= 0\}.$$ 

Let $\varphi$ be a nonzero meromorphic function on $\mathbb{C}^m$. We define the divisor $\nu_\varphi$ as follows: For each $z \in \mathbb{C}^m$, we choose nonzero holomorphic functions $F$ and $G$ on a neighborhood $U$ of $z$ such that $\varphi = \frac{F}{G}$ on $U$ and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leq m - 2$ and then we put $\nu_\varphi(z) := \nu_F(z)$.

Let $\nu$ be a divisor in $\mathbb{C}^m$ and let $k$ be a positive integer or $+\infty$. Set $|\nu| := \{z : \nu(z) \not= 0\}$ and $\nu^{[k]}(z) = \min\{\nu(z), k\}$.

The counting function is defined by 

$$N^{[k]}(r, \nu) := \int_1^r \frac{n^{[k]}(t)}{t^{2m-1}} dt, \quad r > 1$$ 

where 

$$n^{[k]}(t) := \int_{|\nu| \cap B(r)} \nu^{[k]}.v$$ for $m \geq 2$ and $n^{[k]}(t) := \sum \nu^{[k]}(z)$ for $m = 1$.

For a nonzero meromorphic function $\varphi$ on $\mathbb{C}^m$, we set $N^{[k]}_{\varphi}(r) := N^{[k]}(r, \nu_\varphi)$.

For brevity we will omit the character $^{[k]}$ in the counting function and in the divisor if $k = +\infty$.

We have the following Jensen’s formula:

$$N_\varphi(r) - N_{\varphi^2}(r) = \int_{S(r)} \log|\varphi|\sigma - \int_{S(1)} \log|\varphi|\sigma.$$

For a closed subset $A$ of a purely $(m - 1)$-dimensional analytic subset of $\mathbb{C}^m$, we define

$$N^{[1]}(r, A) := \int_1^r \frac{n^{[1]}_A(t)}{t^{2m-1}} dt, \quad r > 1$$

where 

$$n^{[1]}_A(t) := \begin{cases} \int_{A \cap B(t)} v & \text{for } m \geq 2 \\ \#(A \cap B(t)) & \text{for } m = 1. \end{cases}$$

Let $f : \mathbb{C}^m \rightarrow \mathbb{C}P^n$ be a meromorphic mapping. For an arbitrary fixed homogeneous coordinate system $(w_0 : \cdots : w_n)$ in $\mathbb{C}P^n$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each $f_i$ is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension $\geq 2$. Set $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$. The characteristic function $T_f(r)$ of $f$ is defined by

$$T_f(r) := \int_{S(r)} \log\|f\|\sigma - \int_{S(1)} \log\|f\|\sigma, \quad r > 1.$$
For a meromorphic function \( \varphi \) on \( \mathbb{C}^m \), the characteristic function \( T_\varphi(r) \) of \( \varphi \) is defined by considering \( \varphi \) as a meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{C}P^1 \).

The proximity function \( m(r, \varphi) \) is defined by considering \( \varphi \) as a meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{C}P^1 \).

\[
m(r, \varphi) = \int_{S(r)} \log^+|\varphi| \, \sigma,
\]

where \( \log^+ x = \max\{\log x, 0\} \) for \( x \geq 0 \).

We state the First and Second Main Theorems in Value Distribution Theory:

**First Main Theorem** ([8], Theorem 2.1).

1) For a nonzero meromorphic function \( \varphi \), on \( \mathbb{C}^m \) we have

\[
T_\varphi(r) = N_\varphi(r) + m(r, \varphi) + O(1).
\]

2) Let \( f \) be a meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{C}P^n \), and \( H \) be a hyperplane in \( \mathbb{C}P^n \) such that \((f, H) \neq 0\). Then

\[
N(f, H)(r) \leq T_f(r) + O(1) \quad \text{for all } r > 1.
\]

As usual, by the notation "\( P \)" we mean the assertion \( P \) holds for all \( r \in [1, \infty) \) excluding a Borel subset \( E \) of the interval \([1, \infty)\) with \( \int_E dr < \infty \).

**Second Main Theorem** ([7], Proposition 6.2). Let \( f \) be a linearly nondegenerate meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{C}P^n \) and \( H_1, \ldots, H_q \) \((q \geq n + 1)\) hyperplanes in \( \mathbb{C}P^n \) in general position. Then

\[
\|(q - n - 1)T_f(r) \leq \sum_{j=1}^{q} N_{(f, H_j)}(r) + o(T_f(r)).
\]

The following so-called logarithmic derivative lemma plays an essential role in Value Distribution Theory.

**Lemma 2.1** ([10], Lemma 3.1). Let \( \varphi \) be a non-constant meromorphic function on \( \mathbb{C}^m \). Then for any \( i, \ 1 \leq i \leq m \) we have

\[
\| m(r, \frac{\partial}{\partial z^i} \varphi) = o(T_\varphi(r)).
\]

**3 Cartan auxiliary function**

Let \( F, G, H \) be nonzero meromorphic functions on \( \mathbb{C}^m \). For each \( s, \ 1 \leq s \leq m \), we define the Cartan auxiliary function of \( F, G, H \) by

\[
\Phi^s(F, G, H) := F \cdot G \cdot H \cdot \left| \begin{array}{ccc}
1 & 1 & 1 \\
\frac{\partial}{\partial z^s}(\frac{1}{F}) & \frac{\partial}{\partial z^s}(\frac{1}{G}) & \frac{\partial}{\partial z^s}(\frac{1}{H})
\end{array} \right|.
\]
Lemma 3.1 ([8], Proposition 3.3). Let $F, G, H$ be nonzero meromorphic functions on $\mathbb{C}^m$. Assume that $\Phi^s(F, G, H) \equiv 0$ for all $s \in \{1, \ldots, m\}$. Then there exist constants $\alpha, \beta \in \mathbb{C}$ such that

$$\alpha \left( \frac{1}{G} - \frac{1}{F} \right) + \beta \left( \frac{1}{H} - \frac{1}{F} \right) \equiv 0.$$

Let $f$ be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and let $\{H_j\}_{j=1}^q$ be $q$ ($q \geq n + 1$) hyperplanes in $\mathbb{C}P^n$ in general position. Let $p$ be a positive integer and let $f_1, f_2, f_3$ be three mappings in $\mathcal{F}(\{H_j\}_{j=1}^q, f, p)$. Set $\gamma^{ij}_k := \frac{(f_k H_j)}{(f_k H_j)}$ ($k \in \{1, 2, 3\}$, $i \neq j \in \{1, \ldots, q\}$) and $T(r) = T_{f_1}(r) + T_{f_2}(r) + T_{f_3}(r)$.

Lemma 3.2. Assume that there exist $i_0, j_0 \in \{1, \ldots, q\}$, $s \in \{1, \ldots, m\}$ and a closed subset $A$ of a purely $(m - 1)$-dimensional analytic subset of $\mathbb{C}^m$ such that:

1. $\Phi^s_{i_0 j_0} := \Phi^s(\gamma_{i_0 j_0}, \gamma_{j_0}, \gamma_{1 j_0}) \neq 0$, and
2. $\min \{\nu(f_{i_1} H_k), \ell + 1\} = \min \{\nu(f_{j_2} H_k), \ell + 1\}$ on $\mathbb{C}^m \setminus A$ for $k \in \{i_0, j_0\}$, where $\ell$ is a nonnegative integer.

Then

$$\| 2 \sum_{j=1, j \neq i_0, j_0}^q N^{[1]}_{(f_k, H_j)}(r) + 2N^{[\ell+1]}_{(f_1, H_{i_0})}(r) + N^{[\ell]}_{(f_2, H_{i_0})} - N^{[\ell]}_{(f_3, H_{i_0})} \leq T(r) + 3\ell N^{[1]}(r, A) + o(T(r))$$

for all $i \in \{1, 2, 3\}$.

Proof. Without loss of generality, we may assume that $s = 1$. For each $j \in \{1, \ldots, q\} \setminus \{i_0, j_0\}$, let $a$ be an arbitrary point in $f_1^{-1}(H_j) (= f_2^{-1}(H_j) = f_3^{-1}(H_j))$ such that $a \notin f_1^{-1}(H_{i_0}) \cup f_2^{-1}(H_{j_0})$ (if there exist any). Then there exists a neighborhood $U$ of $a$ such that $(f_1, H_{i_0}), (f_1, H_{j_0})$ have no zero point on $U$. We have $\frac{1}{\gamma^{i_0 j_0}} = \frac{1}{\gamma^{j_0}} = \frac{1}{\gamma^{1 j_0}}$ on $B := f_1^{-1}(H_j) \cap U$. Choose $a$ such that $a$ is a regular point of $B$. By shrinking $U$, we may assume that there exists a holomorphic function $h$ on $U$ such that $dh$ has no zero point and $U \cap \{h = 0\} = B$. Then $\frac{1}{\gamma^{i_0 j_0}} - \frac{1}{\gamma^{j_0}} = h \varphi_2$ and $\frac{1}{\gamma^{j_0}} - \frac{1}{\gamma^{1 j_0}} = h \varphi_3$ on $U$ where $\varphi_2, \varphi_3$ are holomorphic functions on $U$. Hence, we have

$$\Phi^{1}_{i_0 j_0} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial}{\partial z_1} (\frac{1}{\gamma^{i_0 j_0}}) & \varphi_2 & h \varphi_3 \\ \frac{\partial}{\partial z_1} \varphi_2 & \varphi_3 & h \frac{\partial}{\partial z_1} \varphi_3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ \frac{\partial}{\partial z_1} \varphi_2 & \varphi_3 & h \frac{\partial}{\partial z_1} \varphi_3 \end{vmatrix}.$$ 

So $a$ is a zero point of $\Phi^{1}_{i_0 j_0}$ with multiplicity $\geq 2$. Thus for each $j \in \{1, \ldots, q\} \setminus \{i_0, j_0\}$, there exists an analytic set $M \subset \mathbb{C}^m$ with codimension $\geq 2$ such that

$$\nu^{\Phi^{1}_{i_0 j_0}} \geq 2 \text{ on } f_1^{-1}(H_j) \setminus M.$$ (3.1)

Let $b$ be an arbitrary point in $f_1^{-1}(H_{i_0}) \setminus (A \cup f_1^{-1}(H_{j_0}))$. (6)
We have $\nu(f_1, H_{i_0})(b) \leq \ell$, then $\ell_0 := \nu(f_1, H_{i_0})(b) = \nu(f_2, H_{i_0})(b) = \nu(f_3, H_{i_0})(b) \leq \ell$. Then, it is easy to see that there exists a neighborhood $U$ of $b$ such that $\nu(f_i, H_{i_0}) \leq \ell$ on $U$ for all $i \in \{1, 2, 3\}$. We can choose $U$ such that $U \cap (f_i^{-1}(H_{j_0}) \cup A) = \emptyset$. Then $\nu_{\gamma_{i_0}^{j_0}} = \nu_{\gamma_{i_0}^{j_0}} = \nu_{\gamma_{i_0}^{j_0}} \leq \ell$ on $U$. Choose $b$ such that $b$ is regular point of $U \cap \{\gamma_{i_0}^{j_0} = 0\} = U \cap \{\gamma_{2}^{j_0} = 0\} = U \cap \{\gamma_{3}^{j_0} = 0\}$.

By shrinking $U$ we may assume that there exists a holomorphic function $h$ on $U$ such that $dh$ has no zero point and $\gamma_{i_0}^{j_0} = h^{i_0}u_i$ on $U$, where $u_i (i = 1, 2, 3)$ are nowhere vanishing holomorphic functions on $U$ (note that $\nu_{\gamma_{i_0}^{j_0}}(b) = \ell_0, i \in \{1, 2, 3\}$).

Then, we have

$$\Phi_{i_0,j_0}^1 = u_1 \frac{\partial u_2 - u_2 \partial u_3}{u_2 u_3} + u_2 \frac{\partial u_1 - u_3 \partial u_1}{u_1 u_3}$$

This implies that

$$\nu_{\Phi_{i_0,j_0}^1}(b) \geq \ell_0.$$  \hspace{1cm} (3.2)

**Case 2.** If $\nu(f_1, H_{i_0})(b) \geq \ell + 1$, then $\nu(f_i, H_{i_0})(b) \geq \ell + 1$, $i \in \{1, 2, 3\}$. It means that $b$ is a zero point of $\gamma_{i_0}^{j_0} (i \in \{1, 2, 3\})$ with multiplicity $\geq \ell + 1$.

We have

$$\Phi_{i_0,j_0}^1 = \gamma_i^{j_0} \gamma_3^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_3^{j_0}} \right) - \gamma_i^{j_0} \gamma_2^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_2^{j_0}} \right)$$

$$+ \gamma_2^{j_0} \gamma_1^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_1^{j_0}} \right) - \gamma_2^{j_0} \gamma_3^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_3^{j_0}} \right)$$

$$+ \gamma_3^{j_0} \gamma_2^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_2^{j_0}} \right) - \gamma_3^{j_0} \gamma_1^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_1^{j_0}} \right).$$

On the other hand $\gamma_i^{j_0} \gamma_3^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_3^{j_0}} \right) = -\gamma_i^{j_0} \gamma_3^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_3^{j_0}} \right)$, so $b$ is a zero point of $\gamma_i^{j_0} \gamma_3^{j_0} \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_3^{j_0}} \right)$ with multiplicity $\geq \ell$. By applying the same argument also to all other combinations of indices, we get

$$\nu_{\Phi_{i_0,j_0}^1}(b) \geq \ell.$$  \hspace{1cm} (3.3)

By (3.2), (3.3) and our choice of $b$, there exists an analytic set $N \subset \mathbb{C}^m$ with codimension $\geq 2$ such that

$$\nu_{\Phi_{i_0,j_0}^1} \geq \min\{\nu(f_1, H_{i_0}), \ell\} \text{ on Zero}(f_1, H_{i_0}) \setminus (N \cup A).$$  \hspace{1cm} (3.4)

By (3.1), (3.4) and since $\dim \left( f^{-1}(H_i) \cap f^{-1}(H_j) \right) \leq m - 2$ for all $1 \leq i \neq j \leq q$, we have

$$2 \sum_{j=1, j \neq i_0, j_0}^{q} N_{f_1, H_j}^1(r) + N_{f_1, H_{i_0}}^1(r) \leq N(r, \nu_{\Phi_{i_0,j_0}^1}) + \ell N_{j_0}^1(r, A).$$  \hspace{1cm} (3.5)
We have
\[ \Phi_{i_0}^1 = (\gamma_2^i \gamma_3^j - \gamma_3^i \gamma_2^j) \gamma_1^i \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_3^i} \right) + (\gamma_3^i \gamma_2^j - \gamma_2^i \gamma_3^j) \gamma_3^i \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_2^i} \right). \]  
(3.6)

Set \( B := \{ \gamma_1^i = 0 \} = \{ \gamma_2^i = 0 \} = \{ \gamma_3^i = 0 \} \) and \( C := \{ \gamma_1^i = \infty \} = \{ \gamma_2^i = \infty \} = \{ \gamma_3^i = \infty \} \). By (3.6) we have
\[ \frac{\nu_{i_0}}{\Phi_{i_0}^1} = 0 \text{ on } B \text{ and } \frac{\nu_{i_0}}{\Phi_{i_0}^1} \leq \max_{i=1,2,3} \frac{\nu_{i_0}}{\gamma_{i_0}^1} + 1 \text{ on } C. \]  
(3.7)

By (3.6) we have that a pole of \( \Phi_{i_0}^1 \) is a zero or a pole of some \( \gamma_{i_0}^1 \). Thus, by (3.7) we have
\[ \frac{\nu_{i_0}}{\Phi_{i_0}^1} \leq \max_{i=1,2,3} \frac{\nu_{i_0}}{\gamma_{i_0}^1} + \nu_{i_0}^{[1]} \leq \max_{i=1,2,3} \nu_{f_i(H_{j_0})} + \nu_{f_0}^{[1]} \]  
(3.8)

Since \( \min \{ \nu_{f_1(H_{j_0})}, \ell + 1 \} = \min \{ \nu_{f_2(H_{j_0})}, \ell + 1 \} = \min \{ \nu_{f_3(H_{j_0})}, \ell + 1 \} \) on \( \mathbb{C}^m \setminus A \) we have
\[ \max_{i=1,2,3} \nu_{f_i(H_{j_0})} \leq \sum_{i=1}^3 \nu_{f_i(H_{j_0})} - 2\nu_{(f_1,H_{j_0})}^{[\ell+1]} \text{ on } \mathbb{C}^m \setminus A. \]  
(3.9)

On the other hand, since \( f_1^{-1}(H_{j_0}) = f_2^{-1}(H_{j_0}) = f_3^{-1}(H_{j_0}) \) we have
\[ \max_{i=1,2,3} \nu_{f_i(H_{j_0})} \leq \sum_{i=1}^3 \nu_{f_i(H_{j_0})} - 2\nu_{f_1(H_{j_0})}^{[\ell+1]} + 2\ell \text{ on } A. \]

Hence, combining with (3.8) and (3.9) we get
\[ \frac{\nu_{i_0}}{\Phi_{i_0}^1} \leq \sum_{i=1}^3 \nu_{f_i(H_{j_0})} - 2\nu_{f_1(H_{j_0})}^{[\ell+1]} + \nu_{f_0}^{[1]} + 2\ell \nu_{A}^{[1]}, \]

where \( \nu_{A}^{[1]} = 1 \) on \( A \) and \( \nu_{A}^{[1]} = 0 \) on \( \mathbb{C}^m \setminus A \).

This implies that
\[ N(r, \frac{\nu_{i_0}}{\Phi_{i_0}^1}) \leq N_{f_1(H_{j_0})}(r) + N_{f_2(H_{j_0})}(r) + N_{f_3(H_{j_0})}(r) - 2N_{(f_1,H_{j_0})}^{[\ell+1]}(r) + N_{(f_1,H_{j_0})}(r) + 2\ell N_{(A)}^{[1]}(r, A). \]  
(3.10)

We have
\[ \Phi_{i_0}^1 = \gamma_1^i \left[ \gamma_3^i \gamma_2^j \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_3^i} \right) - \gamma_2^i \gamma_3^j \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_2^i} \right) \right] + \gamma_2^i \left[ \gamma_3^i \gamma_2^j \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_3^i} \right) - \gamma_2^i \gamma_3^j \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_2^i} \right) \right] + \gamma_3^i \left[ \gamma_2^i \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_2^i} \right) - \gamma_1^i \frac{\partial}{\partial z_1} \left( \frac{1}{\gamma_1^i} \right) \right]. \]
so $m(r, \Phi^{i_{0,j_0}}) \leq \sum_{i=1}^{3} m(r, \gamma^i_{i_{0,j_0}}) + 2 \sum_{i=1}^{3} m(r, \gamma^i_{i_{0,j_0}} \frac{\partial}{\partial z_1} \left( \frac{1}{f^{i_{0,j_0}}} \right)) + O(1)$.

On the other hand by Lemma 2.1, we have

$$\| m(r, \gamma^i_{i_{0,j_0}} \frac{\partial}{\partial z_1} \left( \frac{1}{f^{i_{0,j_0}}} \right)) = o(T, \gamma^i_{i_{0,j_0}}(r)) = o(T, f_i(r)).$$

(note that $T, \gamma^i_{i_{0,j_0}}(r) \leq T, (r) + O(1)$). Thus, we get

$$\| m(r, \Phi^{i_{0,j_0}}) \leq \sum_{i=1}^{3} m(r, \gamma^i_{i_{0,j_0}}) + o(T, r).$$

(3.11)

By (3.10), (3.11) and by the First Main Theorem, we have

$$\| N(r, u_{f_{i_{0,j_0}}}) \leq T, \Phi^{i_{0,j_0}}(r) + O(1) = m(r, \Phi^{i_{0,j_0}}) + N(r, \nu_{i_{0,j_0}}) + O(1)$$

$$\leq \sum_{i=1}^{3} m(r, \gamma^i_{i_{0,j_0}}) + \sum_{i=1}^{3} N(f_i, H_{i_{0,j_0}}) (r) - 2N^{[\ell+1]} (f_{i_{0,j_0}}) (r) + N_{[1]}^{(f_{i_{0,j_0}})} (r)$$

$$+ 2\ell N^{[1]}(r, A) + o(T, r))$$

$$\leq \sum_{i=1}^{3} \left( m(r, \gamma^i_{i_{0,j_0}}) + N_{[i]}^{(f_{i_{0,j_0}})} (r) \right) - 2N^{[\ell+1]} (f_{i_{0,j_0}}) (r)$$

$$+ N_{[1]}^{(f_{i_{0,j_0}})} (r) + 2\ell N^{[1]}(r, A) + o(T, r))$$

$$\leq \sum_{i=1}^{3} T, \gamma^i_{i_{0,j_0}} (r) - 2N^{[\ell+1]} (f_{i_{0,j_0}}) (r) + N_{[1]}^{(f_{i_{0,j_0}})} (r)$$

$$+ 2\ell N^{[1]}(r, A) + o(T, r))$$

$$\leq T, (r) - 2N^{[\ell+1]} (f_{i_{0,j_0}}) (r) + N_{[1]}^{(f_{i_{0,j_0}})} (r) + 2\ell N^{[1]}(r, A) + o(T, r))$$

Combining with (3.5) we get

$$\| 2 \sum_{j=1, j \neq i_{0,j_0}}^{q} N_{[1]}^{(f_{j_{0,j_0}})} (r) + 2N^{[\ell+1]}_{[1]} (f_{j_{0,j_0}}) (r) + N_{[1]}^{(f_{j_{0,j_0}})} (r) - N_{[1]}^{(f_{j_{0,j_0}})} (r)$$

$$\leq T, (r) + 3\ell N^{[1]}(r, A) + o(T, r))$$

By applying the same argument also to $f_2, f_3$, we get Lemma 3.2. □

4 Proof of Theorem 1.3

We distinguish two cases:

**Case 1.** For each pair $(i_0, j_0)$ with $1 \leq i_0 \neq j_0 \leq q$, there exists $s \in \{1, \cdots, m\}$ such that $\Phi^s (\gamma^i_{i_{0,j_0}}, \gamma^j_{i_{0,j_0}}, \gamma^s_{i_{0,j_0}}) \neq 0$. 


By Lemma 3.2 (with ℓ = p − 1, A = ∅) for each pair (i₀, j₀) with 1 ≤ i₀ ≠ j₀ ≤ q, we have

\[ \| \sum_{i=1}^{3} \left( 2 \sum_{j=1, j \neq i_0, j_0}^{q} N_{(f_i, H_j)}^{[1]}(r) + 2N_{(f_i, H_0)}^{[p]}(r) + N_{(f_i, H_0)}^{[p-1]}(r) - N_{(f_i, H_0)}^{[1]}(r) \right) \]

\[ \leq 3T(r) + o(T(r)). \]

Taking the sum of both sides of the above inequality over all pair (i₀, j₀), we get

\[ \| \sum_{i=1}^{3} \sum_{j=1}^{q} \left( (2q - 5)N_{(f_i, H_j)}^{[1]}(r) + 2N_{(f_i, H_j)}^{[p]}(r) + N_{(f_i, H_j)}^{[p-1]}(r) \right) \]

\[ \leq 3qT(r) + o(T(r)). \]

Then, by the Second Main Theorem, we have

\[ \| \sum_{i=1}^{3} \sum_{j=1}^{q} \left( (2q - 5)N_{(f_i, H_j)}^{[1]}(r) - \frac{p}{n}N_{(f_i, H_j)}^{[n]}(r) \right) + (N_{(f_i, H_j)}^{[p-1]}(r) - \frac{p-1}{n}N_{(f_i, H_j)}^{[n]}(r)) \]

\[ + \sum_{i=1}^{3} \sum_{j=1}^{q} (2q - 5) \left( N_{(f_i, H_j)}^{[1]}(r) - \frac{1}{n}N_{(f_i, H_j)}^{[n]}(r) \right) \]

\[ \leq 3qT(r) - \frac{2q + 3p - 6}{n} \sum_{i=1}^{3} \sum_{j=1}^{q} N_{(f_i, H_j)}^{[n]}(r) + o(T(r)) \]

\[ \leq (3q - \frac{(q - n - 1)(2q + 3p - 6)}{n})T(r) + o(T(r)). \] (4.1)

i) If p = n and q > \frac{n+4+\sqrt{2n^2+2n+4}}{2} then 2q^2 - 2(n+4)q - 3n^2 + 3n + 6 > 0. On the other hand, by (4.1) we get 3q - \frac{(q - n - 1)(2q + 3p - 6)}{n} ≥ 0. It implies that 2q^2 - 2(n+4)q - 3n^2 + 3n + 6 ≤ 0. This is a contradiction.

ii) If 1 ≤ p < n and there exists a positive integer t in \{p, \ldots, n - 1\} such that

\[ 3q + \frac{18t}{n - t} \left( \frac{3qn}{2q + 3p - 6} - q + n + 1 \right) < \frac{(q - n - 1)(2q + 3t - 3)}{n}. \] (4.2)

For each j ∈ \{1, \ldots, q\}, i ∈ \{1, 2, 3\} and k ∈ \{p, \ldots, t\}, set A_{ij}^k := \{ z : \nu(f_i, H_j)(z) = k \}. Then we have \( \overline{A_{ij}^k \setminus A_{ij}^k} \subseteq \text{sing} f_i^{-1}(H_j) \), where the closure is taken with respect to the usual topology and \( \text{sing} f_i^{-1}(H_j) \) means the singular locus of the (reduction of the) analytic set \( f_i^{-1}(H_j) \) of codimension one. Indeed, otherwise there existed \( a \in \overline{A_{ij}^k \setminus A_{ij}^k} \cap \text{reg} f_i^{-1}(H_j) \). Then \( k_0 := \nu(f_i, H_j)(a) \neq k \). Since a is a regular point of \( f_i^{-1}(H_j) \), by the Rückert Nullstellensatz we can choose nonzero holomorphic functions h, u on a neighborhood U of a such that dh and u have no zero point and \( (f_i, H_j) = h^k u \) on U. Since a ∈ \( \overline{A_{ij}^k} \setminus A_{ij}^k \), there exists \( b \in A_{ij}^k \cap U \). Then \( k = \nu(f_i, H_j)(b) = \nu_{h^k u}(b) = k_0 \). This is a contradiction. Thus, \( \overline{A_{ij}^k \setminus A_{ij}^k} \subseteq \text{sing} f_i^{-1}(H_j) \), for all i ∈ \{1, 2, 3\}, j ∈ \{1, \ldots, q\} and k ∈ \{p, \ldots, t\}. This means that \( \overline{A_{ij}^k \setminus A_{ij}^k} \) is included in an analytic set of codimension ≥ 2. On the other hand \( A_{ij}^k \cap A_{ij}^l = \emptyset \) for all \( p \leq k \neq l \leq t \). Hence, for all j ∈ \{1, \ldots, q\}, v ∈ \{0, 1, \ldots, p\} and i ∈ \{1, 2, 3\} we have

\[ v(n - p)N_{(f_i, H_j)}^{[1]}(r, A_{ij}^k) + \cdots + v(n - t)N_{(f_i, H_j)}^{[1]}(r, A_{ij}^l) \leq nN_{(f_i, H_j)}^{[n]} - vN_{(f_i, H_j)}^{[n]}(r) \]
(note that \( v \leq p \leq t < n \)).

This implies that
\[
\frac{v(n-t)}{n} N^{[1]}(r, \bigcup_{k=p}^{t} A_{ij}^k) \leq \frac{v(n-t)}{n} \sum_{k=p}^{t} N^{[1]}(r, A_{ij}^k)
\]
\[
\leq N^{[v]}_{(f,H_j)}(r) - \frac{v}{n} N^{[n]}_{(f,H_j)}(r)
\]
for all \( i \in \{1,2,3\}, j \in \{1,\ldots,q\}, v \in \{1,\ldots,p\} \).

Taking the sum of both sides of the above inequality over all \( i \in \{1,2,3\}, j \in \{1,\ldots,q\}, v \in \{1,p-1,p\} \), we get
\[
\frac{(n-t)(2q+3p-6)}{n} \sum_{i=1}^{3} \sum_{j=1}^{q} N^{[1]}(r, \bigcup_{k=p}^{t} A_{ij}^k)
\]
\[
\leq \sum_{i=1}^{3} \sum_{j=1}^{q} \left( 2(N^{[p]}_{(f,H_j)}(r) - \frac{p}{n} N^{[n]}_{(f,H_j)}(r)) + (N^{[p-1]}_{(f,H_j)}(r) - \frac{p-1}{n} N^{[n]}_{(f,H_j)}(r)) \right)
\]
\[
+ \sum_{i=1}^{3} \sum_{j=1}^{q} (2q-5) \left( N^{[1]}_{(f,H_j)}(r) - \frac{1}{n} N^{[n]}_{(f,H_j)}(r) \right). \tag{4.3}
\]

By (4.1), (4.3) we have
\[
\| \frac{(n-t)(2q+3p-6)}{n} \sum_{i=1}^{3} \sum_{j=1}^{q} N^{[1]}(r, \bigcup_{k=p}^{t} A_{ij}^k) \n\leq (3q - \frac{(q-n-1)(2q+3p-6)}{n}) T(r) + o(T(r)). \tag{4.4}
\]

Set \( B^k_j := \bigcup_{i=1}^{3} A_{ij}^k \ (j \in \{1,\ldots,q\}, k \in \{p,\ldots,t\}) \). It is clear that \( \min\{\nu(f_1,H_j), t+1\} = \min\{\nu(f_2,H_j), t+1\} \) on \( \mathbb{C}^m \setminus (\bigcup_{k=p}^{t} B^k_j) \cup (\mathbb{C}^m \setminus (\bigcup_{k=p}^{t} B^k_j)) \) (note that \( f_i \in \mathcal{F}(\{H_j\}_{j=1}^{q}, f,p) \)).

For each pair \((i_0,j_0)\) with \( 1 \leq i_0 \neq j_0 \leq q \), by Lemma 3.2 (with \( \ell = t \) and \( A = \bigcup_{k=p}^{t} (B^k_{i_0} \cup B^k_{j_0}) \)) we have
\[
\| \sum_{i=1}^{3} \left( 2 \sum_{j=1,j \neq i_0,j_0}^{q} N^{[1]}_{(f,H_j)}(r) + 2 N^{[t+1]}_{(f,H_{i_0})}(r) + N^{[t]}_{(f,H_{i_0})}(r) - N^{[1]}_{(f,H_{j_0})}(r) \right)
\]
\[
\leq 3T(r) + 9t N^{[1]}(r, \bigcup_{k=p}^{t} (B^k_{i_0} \cup B^k_{j_0})) + o(T(r))
\]
\[
\leq 3T(r) + 9t (N^{[1]}(r, \bigcup_{k=p}^{t} B^k_{i_0}) + N^{[1]}(r, \bigcup_{k=p}^{t} B^k_{j_0})) + o(T(r)).
\]
Taking the sum of both sides of the above inequality over all pair \((i_0, j_0)\), we get
\[
\begin{align*}
\| & \sum_{i=1}^{3} \sum_{j=1}^{q} \left((2q - 5)N_{(f_i, H_j)}^{[1]}(r) + 2N_{(f_i, H_j)}^{[l+1]}(r) + N_{(f_i, H_j)}^{[t]}(r)\right) \\
& \leq 3qT(r) + 18t \sum_{j=1}^{q} N_{(r, \cup_{k=p} B_f^k)}^{[t]} + o(T(r)) \\
& \leq 3qT(r) + 18t \sum_{i=1}^{3} \sum_{j=1}^{q} N_{(r, \cup_{k=p} A_{ij}^k)}^{[t]} + o(T(r)) \\
& \leq \left(3q + 18t \left(\frac{3qn}{n-t}(2q+3p-6) - \frac{q-n-1}{n-t}\right)\right)T(r) + o(T(r)) \quad (4.5)
\end{align*}
\]

On the other hand, by the Second Main Theorem we have
\[
\begin{align*}
\| & \sum_{i=1}^{3} \sum_{j=1}^{q} \left((2q - 5)N_{(f_i, H_j)}^{[1]}(r) + 2N_{(f_i, H_j)}^{[l+1]}(r) + N_{(f_i, H_j)}^{[t]}(r)\right) \\
& \geq \sum_{i=1}^{3} \sum_{j=1}^{q} \frac{(2q-5)n}{n} N_{(f_i, H_j)}^{[n]}(r) + \frac{2(t+1)}{n} N_{(f_i, H_j)}^{[t]}(r) + \frac{t}{n} N_{(f_i, H_j)}^{[t]}(r) \\
& \geq \frac{(q-n-1)(2q+3t-3)}{n}T(r) + o(T(r))
\end{align*}
\]

(note that \(t+1 \leq n\).

Combining with (4.5) we get
\[
3q + \frac{18t}{n-t} \left(\frac{3qn}{2q+3p-6} - q + n + 1\right) \geq \frac{(q-n-1)(2q+3t-3)}{n}.
\]

This contradicts to (4.2).

**Case 2.** There exists a pair \((i_0, j_0)\) with \(1 \leq i_0 \neq j_0 \leq q\), such that \(\Phi^s(\gamma_1^{i_0j_0}, \gamma_2^{i_0j_0}, \gamma_3^{i_0j_0}) \equiv 0\), for all \(s \in \{1, \ldots, m\}\). Then, by Lemma 3.1 there exists \(\alpha, \beta \in \mathbb{C}\) such that 
\[
\alpha \left(\frac{(f_2, H_{i_0})}{(f_2, H_{j_0})} - \frac{(f_1, H_{i_0})}{(f_1, H_{j_0})}\right) + \beta \left(\frac{(f_3, H_{i_0})}{(f_5, H_{o})} - \frac{(f_1, H_{i_0})}{(f_1, H_{o})}\right) \equiv 0.
\]

This completes the proof of Theorem 1.3. \(\Box\)

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**References**


