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VARIATIONAL APPROACH TO THE MODULATIONAL INSTABILITY
OF BOSE-EINSTEIN CONDENSATE IN A PARABOLIC TRAP

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Abstract

By means of the time-dependent variational approach, we study the modulational instability of Bose-Einstein condensates, with both two- and three-body interatomic interactions, trapped in an external parabolic potential. Within this framework, we derive and analyze ordinary differential equations for the explicit time evolution of the amplitude and phase of modulational perturbation. The effects of the trapping coefficients as well as the quintic nonlinear interaction on the dynamics of the BEC are examined. Numerical simulations are carried out in order to support our theoretical findings.
1 Introduction

Research on nonlinear properties of matter wave has received much attention due to the experimental realization of Bose-Einstein condensate (BEC) in weakly interacting atomic gases [1]. Amongst the experimental achievements is the demonstration of atomic four-wave mixing [2], the observation of solitons [3] and matter-wave amplification [4] in BEC. Conceptually, as a kind of macroscopic quasiparticle, solitons provide a link of BEC physics with fluid mechanics, nonlinear optics, and fundamental particle physics. For a condensate, the sign and the magnitude of the nonlinearity are determined by the scattering length $a$. Two types of solitons have been created, namely, dark solitons [3] for condensates with repulsive interactions and bright ones [4] for condensates with attractive interactions. The interactions are repulsive for $a > 0$ and attractive for $a < 0$. If the interactions are repulsive, the condensate is stable and its size and number have no fundamental limit. However, if the interactions are attractive, only a limited number of atoms can form a condensate [5], where in this case, the condensate is stabilized against collapse [6-8]. If the confining potential is made asymmetric, such that the atoms can only undergo one-dimensional (1D) motion, they are predicted to form a stable, self-focusing BEC or matter-wave soliton [9-11].

Dark solitons are density dips characterized by a phase jump of the wave function at the position of the dip, and, thus, can be generated by means of phase-engineering techniques. The properties of dark solitons have been studied theoretically [12] and they have been created experimentally [3] in elongated BECs. Bright solitons, which were only recently created in BECs of $^7$Li, are characterized by a localized maximum in the density profile without any phase jump across it. In the relevant experiment, this type of soliton was formed upon utilizing a Feshbach resonance to change the sign of the scattering length from positive to negative [3]. Bright solitons have been created in two experiments [3] with BECs of $^7$Li atoms. In these experiments, a tunable magnetic field has been used to adjust the inter atomic interactions from repulsive to weakly attractive. For example, the harmonic potential that we have used in this work is relevant to experimental setups in which the (magnetic) trap is strongly confined in the two directions, while it is much shallower in the third one [13]. It is now well established that the nonlinear term is cubic in the atomic field, in analogy with the well-known Kerr nonlinearity in optics. A dark soliton is a localized absence of atoms or depression of the atomic field. Dark solitons are constrained to propagate in the nonlinear medium, which in this case is the condensate itself. Bright solitons, on the other hand, are themselves condensates and have no such constraint on the propagation medium. Furthermore, and since the generation of the above-mentioned coherent nonlinear structures is typically induced by instabilities [14], these developments have placed a renewed emphasis in the study of relevant instabilities. Of particular interest, among others, is the modulational instability (MI) in 1D that gives rise to solitons, and which has been studied both experimentally [15,16] and theoretically [14, 17-20]. Another focal point is the transverse MI in 2D that gives rise to vortices [14, 21-23].
Our aim in this paper is to revisit, using the time-dependent variational approach (TDVA) and numerical calculations, the MI of a trapped BEC immersed in a highly elongated harmonic trap, considering both two- and three-body interactions, and study the results in comparison with the standard linear stability calculations. Today, such a study is relevant, as modulational instability (MI) is a fundamental mechanism that leads to the formation of localized solitary-wave structures in a variety of settings. It is clear that in low temperature and atom density, where interatomic distance is much greater than the distance scale of atom-atom interactions, two-body s-wave scattering should be important and three-body interactions can be neglected. On the other hand, if the atom density is high, for example, in the case of the miniaturization of devices in the integrated atom optics, three-body interactions can start to play an important role [24-26]. As reported in Ref. [27], even for a small strength of the three-body force, the region of stability for the condensate can be extended considerably. It was also reported in Ref. [28] that a sufficiently dilute and cold Bose gas exhibits similar three-body dynamics for both signs of the s-wave atom-atom scattering length. These authors concluded that the long-range three-body interaction between neural atoms is effectively repulsive for either sign of the scattering length. Experimentally, by detuning the two-body interactions via the Feshbach resonance techniques, we can also come to the BEC with dominating three-body interactions. In the tight-binding approximation the Gross-Pitaevskii (GP) equation with two- and three-body interactions and a periodic potential can be reduced to the cubic-quintic discrete nonlinear Schrödinger (CQ-DNLS) equation. The GP equation with cubic and quintic nonlinearities also appears in the description of the evolution of a broad gap soliton under the joint action of linear and nonlinear optical lattices [29]. If the nonlinear optical lattice rapidly varies in space in comparison with the periodical potential one, averaging the GP equation over this modulation leads to the appearance of the effective quintic nonlinearity [30,31].

As we said above, we would consider the classical explicit time-dependent criteria for MI for such a system by means of the TDVA. For this aim, we perform a modified lens-type transformation which converts our initial GP equation, with real position-dependent parabolic potential and constant coefficients of nonlinearity, into a GP equation without potential and with a constant coefficient of quintic nonlinearity and a time-dependent coefficient of cubic nonlinearity. The resultant GP equation is then treated by the TDVA to propose the MI conditions known from the classical analysis. The paper is structured as follows: in section 2, we present the mathematical framework in which we derive the MI conditions of the CQ-NLS equation by both linear stability analysis and TDVA. Then in section 3, we perform numerical integrations to check the validity of the MI conditions stated in section 2. The obtained results are compared to those of the analytical methods and excellent agreements are found. Section 4 concludes our findings.
2 Analytical results

The experimental observations of solitons and vortices in BEC have stimulated intensive studies of the nonlinear excitations of BEC matter waves, including such aspects as the soliton propagation, vortex dynamics, interference patterns, domain walls in binary BEC and the MI. One of the most important aspects of the BEC solitary wave is its dynamics, since it is believed that the generation, dynamics and management of a BEC solitary wave is important for a number of BEC applications, like atomic interferometry, and different kinds of quantum phase transitions, as well as in the context of the nonlinear physics, including nonlinear optics and hydrodynamics. We focus our attention on a BEC with two- and three-body interatomic interactions. It is well known that at zero temperature the dynamics of weakly interacting bosonic gases trapped in a potential \( V \) is well described by the GP equation, which has the form of a CQ-NLS equation, written on the normalized form:\[32,33]\:

\[
\frac{\partial \psi}{\partial T} = -\frac{\partial^2 \psi}{\partial X^2} + V(X)\psi + g_0|\psi|^2\psi + \chi|\psi|^4\psi
\]  

where \( \psi \) is the normalized macroscopic wavefunction depending on space \( X \) and time \( T \). The expression \( V(X) = \alpha X^2 \) is the external harmonic (magnetic) potential that will be taken as attractive \[34\], i.e. the trapping coefficient \( \alpha \) (a real constant here) is positive. The real constants \( g_0 \) and \( \chi \) are, respectively, the two- and three-body interatomic interactions coefficients.

Although the GP equation is widely accepted as a valid model for the dynamics of the BEC at \( T \approx 0 \) K, the knowledge of the dynamics of the condensates is scarce since the GP equation is nonintegrable. However, various methods have been proposed in order to find explicit solutions to this nonintegrable GP equation:

- One approach to the dynamics of the condensate is the time-dependent variational technique \[35\], which assumes a fixed profile and computes the evolution of some parameters such as the width by variational techniques.

- Numerical simulations of condensate breathing and scissor modes trapped in harmonic potentials have been studied by making use of a spectral-Galerkin method \[36\], using a basis set of harmonic-oscillator functions, and the Gauss-Hermite quadrature, on one hand, while another numerical procedure, which is based on the steepest descent method for functional minimization has been carried out, on the other \[37\].

- Exact solutions of the dynamics of Bose-Einstein rotating condensates have been obtained by spatial translations and Galilean-like symmetry in inhomogeneous nonlinear Schrödinger equations (NLSEs) \[38\], that lacks translational invariance. The symmetry implies a decoupling of the dynamics of the center of mass with respect to all other properties of the wave packet.

- Thomas-Fermi theory for Bose-Einstein condensates in inhomogeneous traps has been reported \[39\]. The Thomas-Fermi kinetic energy has been calculated for any number of particles. Good agreement between the Gross-Pitaevskii and Thomas-Fermi kinetic energies has been found even for low and intermediate particle numbers \( N \) \[39\].
Recently, following the inverse scattering transformation technique the exact bright matter-wave solutions to the GP equation with the expulsive harmonics potential and uniform feeding term have been proposed [40].

Despite important efforts made to find solutions to the nonintegrable GP equation, we have used, in the present paper, the transformation (2) which is a modified lens-type transformation [41]. The virtue of the modified lens-type transformation (2) is two-fold, namely, (i) to show that a NLS equation for inhomogeneous medium can be reduce to a NLS equation for a homogeneous medium with time dependent coefficient by means of an ingenious transformation without explicit spatial dependence, and to show that (ii) the lens-transformation property of the nonintegrable NLS equation allows us to map a given solution of NLS equation to a solution of NLS equation with the same beam which passes through a thin lens with a given focal length [42]. In particular, if the original beam collapses, then the lens transformation can be used to compute the effect of the lens on the collapse distance.

Let us now investigate the MI in such a BEC. In order to get the exact analytical condition to describe the MI, we first need to find an expression with space-independent coefficients for equation (1). We begin with a modified lens-type transformation. For this, we set [34]

\[
\psi(X,T) = \frac{1}{\ell(T)} \phi(x,t) \exp(if(T)X^2),
\]

where

\[\ell(T) = |\cos(2\sqrt{\alpha} T)|, \quad x = \frac{X}{\ell(T)}, \quad t(T) = \frac{1}{2\sqrt{\alpha}} \tan(2\sqrt{\alpha} T) \quad \text{and} \quad f(T) = \frac{-\sqrt{\alpha}}{\ell(T)} \tan(2\sqrt{\alpha} T).\]

The rescaling signals the existence of negative \( t \) and is valid for any \( T \neq \frac{(2n+1)\pi}{4\sqrt{\alpha}} \) (where \( n \) is a positive integer) in the \((T,t)\) plane. We consider the case where \( T \) goes from zero to \( \frac{\pi}{4\sqrt{\alpha}} \) to ensure a variation of \( t \) from zero to infinity. Then Eq. (1), in terms of rescaled variables \( x \) and \( t \), is reduced to:

\[
i\frac{\partial \phi}{\partial t} = -\frac{\partial^2 \phi}{\partial x^2} + g(t) |\phi|^2 \phi + \chi |\phi|^4 \phi,
\]

where

\[g(t) = g_0 |\cos(2\sqrt{\alpha} T)| = g_0 / \sqrt{1 + 4\alpha t^2}.\]

This rescaled equation has the form of a CQ-NLS equation with a time-dependent coefficient and, interestingly, without potential. Thus, the importance of the lens-type transformation is that, it converts the GP equation into a modified CQ-NLS equation without explicit spatial dependence.

### 2.1 Linear stability analysis

In order to examine the MI of the BEC briefly through the linear stability framework, we may use the ansatz:

\[
\phi = (A_0 + \delta \phi) \exp(-i \int_0^t \omega(s) ds).
\]

In Eq. (4), \( A_0 \) is a real constant representing the amplitude of the wave while \( \omega(t) \) is a real time-dependent function representing the nonlinear frequency shift. Substituting Eq. (4) into Eq. (3),
neglecting second order terms in $\delta \phi$, and taking $\omega(t) = A_0^2(g(t) + \chi A_0^2)$, we obtain the following equation describing the dynamics of the perturbation $\delta \phi$:

$$i \frac{\partial (\delta \phi)}{\partial t} = -\frac{\partial^2 (\delta \phi)}{\partial x^2} + \Delta(t)(\delta \phi + \delta \phi^*) .$$

(5)

In Eq. (5), $\Delta(t) = A_0^2(g(t) + 2\chi A_0^2)$ and the asterisk stands for the complex conjugation. Letting $\delta \phi = u_1 + iu_2$ transforms Eq. (5) into the following two coupled equations:

$$\frac{\partial u_1}{\partial t} + \frac{\partial^2 u_2}{\partial x^2} = 0,$$

(6)

$$-\frac{\partial u_2}{\partial t} + \frac{\partial^2 u_1}{\partial x^2} - 2\Delta(t)u_1 = 0.$$

(7)

Now, we consider that the variation of the perturbation follows the expressions:

$$u_1 = \text{Re}[U_1 \exp(i(qx - \int_0^t \Omega(s)ds))]$$

and

$$u_2 = \text{Im}[U_2 \exp(i(qx - \int_0^t \Omega(s)ds))],$$

(8)

where $qx - \int_0^t \Omega(s)ds$ is the modulation phase in which $q$ and $\Omega$ are, respectively, the wave number and frequency of the modulation. After some manipulations, we obtain the following explicit time-dependent dispersion relation:

$$\Omega^2 = q^4[1 + 2(A_0^2)^2\left(\frac{g_0}{\sqrt{1 + 4\alpha t^2}} + 2\chi A_0^2\right)].$$

(9)

The development of MI requires (as sufficient condition) that the frequency of modulation must be imaginary. Such a condition is simply realized if:

$$q^2 < -2A_0^2(\frac{g_0}{\sqrt{1 + 4\alpha t^2}} + 2\chi A_0^2).$$

(10)

For attractive (to which we restrict this study) two-body interatomic interaction, i.e. $g_0$ is negative, relation (10) can be rewritten in the form:

$$2\left(\frac{A_0}{q}\right)^2 \frac{|g_0|}{\sqrt{1 + 4\alpha t^2}} > 4\chi A_0^2(\frac{A_0}{q})^2 + 1.$$

(11)

Relation (10) is thus the MI condition, for plane waves with wavenumbers $q$, given by the linear stability analysis.

### 2.2 Time-dependent variational approach

We examine the MI using a TDVA and study the results in comparison with the linear stability calculation. The use of the TDVA for the study of solitons, at the classical [43] and even at the quantum [44] level, is not novel. What distinguishes the present study and those of [34] from these earlier ones is the use of the MI-motivated ansatz in the TDVA. One can also note that in
passing the MI and solitons from a quantum mechanical point of view have been considered in a number of references including (but not limited to) [43]. Furthermore, similar in spirit, three-mode approximation was systematically developed in the work of Ref. [44]. Let us now attempt to identify the intervals of unstable wave numbers by means of TDVA. The process should start by finding the lagrangian density generating Eq. (3), which is:

\[ \mathcal{L}(\phi) = \frac{i}{2}(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}) - \frac{1}{2} |\frac{\partial \phi}{\partial x}|^2 - \frac{1}{2} g(t)|\phi|^4 - \frac{1}{3} \chi |\phi|^6. \]  

(12)

An exact solution of the cubic-quintic NLS equation (3) is a wave of the form

\[ \phi(x, t) = A_0 \exp[i(kx - \int_0^t \omega(s) ds)], \]

(13)

to which the time-dependent dispersion relation is associated

\[ \omega(t) = k^2 + g(t)A_0^2 + \chi A_0^4. \]

(14)

We may use, as a variational ansatz for the wave function of the condensate, a modulation of the solution (13) defined by:

\[ \phi(x, t) = \{A_0 + a_1(t) \exp[i(qx + b_1(t))] + a_2(t) \exp[i(-qx + b_2(t))]\} \exp[i(kx - \int_0^t \omega(s) ds)]. \]

(15)

Following the standard procedure (ref.[32]), we should insert the variational ansatz into the density (12) and calculate the effective lagrangian,

\[ L = \int_{-\infty}^{+\infty} \mathcal{L}(\phi) dx. \]

(16)

But here, we consider an annular (one-dimensional) geometry, which imposes periodic boundary conditions on the wave function \( \phi(x, t) \) and integration limits \( 0 \leq x < 2\pi \) in Eq. (16). This leads to the quantization of the wave numbers i.e. \( k, q = 0, \pm 1, \pm 2, \pm 3, \ldots \). However, these results can be generalized to the case of an infinite open system and to higher dimensions. Let’s substitute the ansatz (15) into Eqs. (12) and (16) considering the new geometry. We obtain the effective lagrangian:

\[ L = -\pi \left\{ 2(a_1^2 \frac{\partial b_1}{\partial t} + a_2^2 \frac{\partial b_2}{\partial t}) + 2[q^2 + g(t)A_0^2 + 2\chi A_0^4](a_1^2 + a_2^2) + 4qk(a_1^2 - a_2^2) + [g(t) + 6\chi A_0^2](a_1^2 + a_2^2 + 4a_1^2a_2^2) - 4[A_0^2g(t) + \frac{4}{3} \chi A_0^2] + \frac{2}{3} \chi (a_1^2 + a_2^2 + 9a_1^4a_2^2 + 9a_1^2a_2^4) + 4A_0^2g(t) + 2A_0^4(a_1a_2 \cos(b_1 + b_2) + 12\chi A_0^2(a_1^2 + a_2^2)a_1a_2 \cos(b_1 + b_2) \right\}. \]

(17)
From this lagrangian, we may interpret the pair $b_1(t), b_2(t)$ as the generalized coordinates of the system, while the pair $A_1(t), A_2(t)$, with $A_1(t) = 2a_1(t)$ and $A_2(t) = 2a_2(t)$, is the corresponding momenta. The expression of the hamiltonian of the system is

$$H_s = -L + \int_{-\infty}^{+\infty} i \left( \phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t} \right) dx,$$

(18)

that is, in the new geometry,

$$H_s = \pi \left\{ \begin{array}{l} A_0^2[2k^2 + g(t)A_0^2 + \frac{2}{3} \chi A_0^4] + [q^2 + k^2 + 2g(t)A_0^2 + 3\chi A_0^4] (A_1 + A_2) \\
+2gk(A_1 - A_2) + \frac{1}{4}[g(t) + 6\chi A_0^2] (A_1^2 + A_2^2 + 4A_1 A_2) \\
+ \frac{1}{12} \chi (A_1^3 + A_2^3 + 9A_1 A_2^2 + 9A_1^2 A_2) \\
+ 2A_0^2 g(t) + 2\chi A_0^2 \sqrt{A_1 \sqrt{A_2}} \cos(b_1 + b_2) \\
+ 3\chi A_0^2 (A_1 + A_2) \sqrt{A_1 \sqrt{A_2}} \cos(b_1 + b_2) \end{array} \right\} \quad (19)$$

But, in particular, the pairs $A_1(t), b_1(t)$ and $A_2(t), b_2(t)$ are canonically conjugate with respect to the effective hamiltonian $H$ (i.e $\frac{2A_0}{\partial} = -\frac{\partial H}{\partial b}$ and $\frac{\partial q}{\partial b} = \frac{\partial H}{\partial A}$) given by:

$$H = \left\{ \begin{array}{l} [q^2 + g(t)A_0^2 + 2\chi A_0^4] (A_1 + A_2) \\
+ 2gk(A_1 - A_2) + \frac{1}{4}[g(t) + 6\chi A_0^2] (A_1^2 + A_2^2 + 4A_1 A_2) \\
+ \frac{1}{12} \chi (A_1^3 + A_2^3 + 9A_1 A_2^2 + 9A_1^2 A_2) \\
+ 2A_0^2 g(t) + 2\chi A_0^2 \sqrt{A_1 \sqrt{A_2}} \cos(b_1 + b_2) \\
+ 3\chi A_0^2 (A_1 + A_2) \sqrt{A_1 \sqrt{A_2}} \cos(b_1 + b_2) \end{array} \right\} \quad (20)$$

This effective hamiltonian is an exact integral of motion on the subspace spanned by the ansatz (15). To derive the evolution equations for the time-dependent parameters of Eq. (15), we use the corresponding Euler-Lagrange equations based on the variational lagrangian $L$. In the generalized form, these equations are:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\xi}_i} - \frac{\partial L}{\partial \xi_i} = 0,$$

(21)

where $\xi_i$ is the generalized coordinate and $\dot{\xi}_i$ is the corresponding generalized momentum.

Hence, the evolution equation corresponding to the coordinate $b_1(t)$ is

$$\frac{\partial b_1}{\partial t} = C_1 + C_2 \frac{a_2}{a_1} \cos(b_1 + b_2)$$

$$+ \left\{ \frac{1}{3} C_3 [1 + 2(\frac{a_2}{a_1})^2] + \frac{1}{2} C_4 \frac{a_2}{a_1} [3 + (\frac{a_2}{a_1})^3] \cos(b_1 + b_2) \right\} a_1^2$$

$$+ \frac{1}{10} C_5 [1 + 6(\frac{a_2}{a_1})^2 + 3(\frac{a_2}{a_1})^4] a_1^4 \quad (22)$$

The parameters $C_i$, $i = 1, ..., 5$ are given by: $C_1 = -g(t)A_0^2 - 2\chi A_0^4 - 2gk$, $C_2 = -g(t)A_0^2 - 2\chi A_0^4$, $C_3 = -3g(t) - 18\chi A_0^2$, $C_4 = -6\chi A_0^2$, $C_5 = -10\chi$ and we let $C_6 = -g(t)A_0^2 - 2\chi A_0^4 - q^2 + 2gk$.

For the coordinate $a_1(t)$, we have:

$$\frac{\partial a_1}{\partial t} = \left\{ C_2 a_2 + \frac{1}{2} C_4 [1 + (\frac{a_1}{a_2})^2] a_2^2 \right\} \sin(b_1 + b_2).$$

(23)
The coordinate $a_2(t)$ yields the evolution equation:

\[ \frac{\partial b_2}{\partial t} = C_6 + C_2 \frac{a_1}{a_2} \cos(b_1 + b_2) \]
\[ + \left\{ \frac{1}{3} C_3 [1 + 2 \left( \frac{a_1}{a_2} \right)^2] + \frac{1}{2} C_4 \frac{a_1}{a_2} [3 + (\frac{a_1}{a_2})^3 \cos(b_1 + b_2)] \right\} a_2^2 \]
\[ + \frac{1}{10} C_5 [1 + 6 (\frac{a_1}{a_2})^2 + 3 (\frac{a_1}{a_2})^4] a_2^4. \] (24)

The evolution of the variational parameter $a_2(t)$ is provided by the coordinate $b_2(t)$:

\[ \frac{\partial a_2}{\partial t} = \{ C_2 a_1 + \frac{1}{2} C_4 [1 + (\frac{a_2}{a_1})^2] \} \sin(b_1 + b_2). \] (25)

For simplicity, we may use a variant of the ansatz (15) for which

\[ a_2 = a_1 \equiv a. \] (26)

Then adding together Eqs. (22) and (24), we obtain with Eq. (23) (or Eq. (25)), under the condition (26), the system of equations:

\[ \frac{\partial a}{\partial t} = (C_2 a + C_4 a^3) \sin(b(t)) \]
\[ \frac{\partial b(t)}{\partial t} = (C_1 + C_6) + 2[C_2 \cos(b(t)) + (C_3 + 2C_4 \cos(b(t))) a^2 + C_5 a^4], \] (27)

where $b(t) = b_1 + b_2$.

Keeping only terms to $O(a)$ (linear approximation) in this system yields the equations:

\[ \frac{\partial a}{\partial t} = C_2 a \sin(b(t)), \] (28)

and:

\[ \frac{\partial b(t)}{\partial t} = (C_1 + C_6) + 2C_2 \cos(b(t)). \] (29)

To obtain the MIF criterion, we rewrite the effective Hamiltonian (20) considering Eq. (26). Then, with $A = 2a^2$, we have

\[ H(A) = \left\{ \begin{array}{c} 2[q^2 + g(t)A_0^2 + 2\chi A_0^4]A + \frac{3}{2} \frac{g(t)}{A_0^2} + 6\chi A_0^2]A^2 \\ + \frac{5}{3} \chi A^3 + 2A_0^2 [g(t) + 2\chi A_0^2]A \cos(b(t)) \\ + 6\chi A_0^2 A^2 \cos(b(t)) \end{array} \right\}. \] (30)

Using $A(t = 0) = 0$ (without loss of generality) in Eq. (30) yields $H_0 = H(A(t = 0)) = 0$. Since the Hamiltonian is conserved, $H_0 = H(A)$ and thus, we obtain

\[ 2[q^2 + g(t)A_0^2 + 2\chi A_0^4 + q^2]A + \frac{3}{2} \frac{g(t)}{A_0^2} + 6\chi A_0^2]A^2 + \frac{5}{3} \chi A^3 \\ + 2A_0^2 [g(t) + 2\chi A_0^2 + 3\chi A]A \cos(b(t)) = 0. \] (31)
Rewriting the first equation of (27) in terms of $A$ and eliminating $b(t)$ between the resultant equation, Eq. (31) leads us to the following “energy equation” for $A$:

$$
\frac{1}{2} \dot{A}^2 + V = 0,
$$

(32)

where the effective potential

$$
V = \begin{cases} 
2A^2 q^2 [q^2 + 2g(t)A_0^2 + 4\chi A_0^4] + 3A^4 [q^2 (g(t) + 6\chi A_0^2)] \\
+ 4g(t)A_0^2 \chi + g(t)^2 A_0^4 + 4\chi^2 A_0^6 + A_0^4 \left[ \frac{101}{6} g(t)A_0^2 \chi + \frac{9}{8} g(t)^2 \right] \\
+ \frac{175}{6} \chi^2 A_0^4 + \frac{10}{3} q^2 \chi] + \frac{5}{2} A^5 \chi [g(t) + 6\chi A_0^2] + \frac{25}{18} A^6 \chi^2
\end{cases}
$$

(33)

The evaluation of the curvature of the potential at $A = 0$ may determine whether the dynamics is stable or not. Indeed, the potential will be convex (corresponding to a stable dynamics) for $\frac{\partial^2 V}{\partial A^2}|_{A=0} > 0$, while for $\frac{\partial^2 V}{\partial A^2}|_{A=0} < 0$ it will be concave (unstable dynamics). Thus, as we have $\frac{\partial^2 V}{\partial A^2}|_{A=0} = 4q^2[q^2 + 2g(t)A_0^2 + 4\chi A_0^4]$, the instability condition for Eq. (3) appears for perturbation wave numbers verifying the time-dependent condition:

$$q^2 < -2g(t)A_0^2 - 4\chi A_0^4.
$$

(34)

For attractive two-body interaction (negative $g_0$), the above condition of instability may take the form:

$$Q^2 < -\chi + \chi_c |\cos(2\sqrt{\alpha}T)|,
$$

(35)

where $\chi_c = \frac{1}{2A_0} |g_0|$ and $Q = \frac{1}{2A_0} q$. This relation is nothing but the modulational instability condition (11) obtained in the classical study of MI of the CQ-NLS equation (3) within the linear stability framework. The different modulational stability/instability regions in the plane $(\chi, Q)$ provided by that condition are plotted in Fig. 1, where they are demarcated by the curves $Q = \sqrt{-\chi + \chi_c}$ and $Q = \sqrt{-\chi}$. The modes in region (I) are stable while the ones in regions (II)+(III) are unstable, at the initial time (just when the modulational perturbation is introduced). The parameters $g_0 = -1$, $A_0 = 1.0$ are used in all numerical calculations related to both analytical and numerical investigations. In Fig. 2, the effective potential is plotted as a function of $A$, for the modulational stable, unstable and marginal cases. These cases are easily distinguishable through the concavity of the curve $V_{(eff)}(A)$. We have chosen the values of $A$ in the interval $[0.0, 1.4, 10^{-3}]$ to ensure that we are observing the curvature sufficiently close to the origin. From the figure, we can realize that the presence of the quintic nonlinearity induces a shift (either positive or negative) in the threshold wave number $q_c = \sqrt{2C_2(t)}$ separating the stable from the unstable wave number ranges. Indeed, when $\chi$ increases, a mode (e.g. $q_2$) can cross from the unstable range (Fig. 2b) to the threshold (Fig. 2c) and finally to the stable range (Fig. 2d). Thus, due to the shift, some wave numbers are even converted from unstable to stable or vice-versa. Moreover, when this threshold wave number becomes imaginary (\(\chi\) crosses the critical value $\chi_c$), the range of unstable wave numbers disappears. Figure 2 also justifies the
stability/instability of the region in Fig. 1 from where the couples \((\chi, Q)\) have been taken. The analysis of the MI criterion (35) also shows that a wavenumber stable (region III) at \(t = 0\) remains stable when the time evolves. A wavenumber unstable at \(t = 0\) can subsequently remain unstable (region I) or, on account of the trapping, become stable (region II) after a threshold time

\[
t_c = \frac{1}{2\sqrt{\alpha}} \sqrt{(\chi_c - \chi - Q^2)(\chi_c + \chi + Q^2)}.
\]  

Equation (35) gives the possibility to annihilate the appearance of unstable wave numbers (by choosing \(\chi > \chi_c\)), and thus to stabilize the BEC (see Fig. 2d on which all modes are stable). Such a possibility is nonexistent when only the cubic nonlinearity is considered. Consequently, the criterion presents the interest that resides in the simultaneous consideration of cubic and quintic nonlinearities.

In solving Eqs. (28) and (29) to obtain the expressions of the modulational parameters \(a(t)\) and \(b(t)\), two different cases arise: when \(q^2 > 2C_2\) (case where the instability criterion is not satisfied), the solutions are:

\[
b(t) = 2\arctan\left[\sqrt{1 - \frac{2C_2(t)}{q^2}} \tan \int_0^t \sqrt{q^2(q^2 - 2C_2(s))} \, ds\right]
\]  

for the phase, and

\[
a(t) = a_0 \exp\left\{ \int_0^t \frac{2\sqrt{1 - \frac{2C_2(u)}{q^2}} \tan \int_0^u \sqrt{q^2(q^2 - 2C_2(s))} \, ds} {1 + \sqrt{1 - \frac{2C_2(u)}{q^2}} \tan \int_0^u \sqrt{q^2(q^2 - 2C_2(s))} \, ds} \, du \right\}
\]  

for the amplitude.

When \(q^2 < 2C_2\) (case where the instability criterion is satisfied), the solutions are:

\[
b(t) = 2\arctan\left[\sqrt{-1 + \frac{2C_2(t)}{q^2}} \tanh \int_0^t \sqrt{q^2(-q^2 - 2C_2(s))} \, ds\right]
\]  

for the phase, and

\[
a(t) = a_0 \exp\left\{ \int_0^t \frac{2\sqrt{-1 + \frac{2C_2(u)}{q^2}} \tanh \int_0^u \sqrt{q^2(-q^2 - 2C_2(s))} \, ds} {1 + \sqrt{-1 + \frac{2C_2(u)}{q^2}} \tanh \int_0^u \sqrt{q^2(-q^2 - 2C_2(s))} \, ds} \, du \right\}
\]  

for the amplitude.

These solutions clearly indicate the occurrence of the MI when crossing the threshold wave numbers defined by \(q_c^2 = 2C_2\) (as aforesaid). The solution (38) is plotted in Fig. 3 when the trapping is switched off \((\alpha = 0.0)\), \(q = 2.5\) and for four values of the quintic nonlinearity parameter \((\chi = -0.25, 0.0, 0.25\) and \(0.75\)). The initial value of the modulational perturbation is \(a_0 = 0.01\) (small compared to \(A_0\), in accordance with the linear stability). The modulational perturbation \(a(t)\) oscillates in the time, close to its initial value (below for \(\chi < \chi_c\) (see Figs. 3a, b, c) and above for \(\chi > \chi_c\) (see Fig. 3d)), showing the stability of BEC. When the quintic parameter
increases, the amplitude decreases with increasing frequency. It should be noted that for \( \chi = \chi_c \), the perturbation annihilates just after its introduction. In Fig. 4, we observe the aspect of \( a(t) \) when the trapping is switched on (\( \alpha = 0.01 \)). The modulational perturbation undergoes stable pseudo-oscillations, close to its initial value and with increasing frequency. When the quintic parameter \( \chi \) is less than or equal to 0.0, the amplitude decreases with convergence in the time. For \( 0.0 < \chi < \chi_c \), the amplitude decreases until a given time before increasing with convergence while for \( \chi \geq \chi_c \) it smoothly increases but always with convergence.

For the unstable case, we display in Fig. 5 the solution (40) when the trapping is first switched off (\( \alpha = 0.0 \)) for \( q = 0.5 \) and for four values of the quintic nonlinearity parameter (\( \chi = -0.25, -0.1, 0.0 \) and 0.25). The maximal time is chosen in such a way that the condition \( q^2 < 2C_2(t) \) remains satisfied for \( \chi = 0.25 \) (in region II). The modulational perturbation \( a(t) \) exponentially grows in the time, from its initial value 0.01, characterizing the instability of BEC. When the quintic parameter increases, the occurrence of the exponential growth is more and more delayed and done with less stronger values. This justifies the stabilizing tendency of the quintic parameter. In Fig. 6, we show the evolution of \( a(t) \) (unstable case) when the trapping is switched on (\( \alpha = 0.01 \)). The modulational perturbation always undergoes the exponential growth but more later and with less stronger values in comparison with Fig. 5 (the lines in \( \chi_1 \) and \( \chi_2 \)). This result is not surprising so far as it is known that a decrease in a focusing cubic nonlinearity has stabilizing effects. We have omitted the values of \( \chi \) in region II because the onset time for the growth would be so long to end after the occurrence of the threshold \( q^2 = 2C_2(t) \).

3 Numerical analysis

The above analytical results determine the instability domains in the parameters space and qualitatively predict how the amplitude and phase of a modulation sideband evolve in those domains. These results are however based on the linearization around the unperturbed carrier wave. The validity of such an analysis is limited to amplitudes of perturbation, small in comparison with that of the carrier wave. In other terms, the linear approximation must fail at large time scales as the amplitude of unstable sideband exponentially grows. Furthermore, the solutions (39) and (40) come up against the time-dependent condition \( q^2 < 2C_2(t) \) for modes \((\chi, Q)\) taken in region (II) of Fig. (1). Linear stability analysis therefore can’t tell us, neither the evolution of strongly perturbed wave nor the long-time evolution of a modulated extended nonlinear wave. In order to check the applicability of our analytical results and carrying to overcome the limits of the linearization, let us perform direct numerical integrations. The system of equations (27) is solved using a fourth-order Runge-Kutta algorithm.

In Fig. (7a) for the same parameters as in Fig. (3a), we show the time evolution of the modulational perturbation launched with different initial values when \( \alpha = 0.0 \). For two values of \( a_0 \) (e.g. 0.001 and 0.01) sufficiently small compared to \( A_0 \), we have stable oscillations with
same oscillation rate and apparently the same frequency. The values of $a_0$ less than $A_0$, but relatively strong, induce an evolution with rapidly decreasing frequency (in time) and with a higher oscillation rate constant in time. When $a_0$ is of the same order as $A_0$, the oscillations can even occur in the up demi-plane relatively to $A_0$ (see Fig. (7b)). It is worth noticing through Figs. (7c) and (7d) that, at sufficiently long times the separation between the (two) oscillations (yet merged at short times in Fig. (7a)) becomes more and more visible, showing the variation of the frequency with time. When the trapping is introduced ($\alpha = 0.01$), for the same parameters as in Fig. (7), we show in Fig. (8), in panels (a), (b) and (c), the time evolution of the corresponding modulational perturbation. Similar behaviors are obtained, but the frequency increases in time, and the amplitude too varies in time with (rapid) convergence and the splitting begins to occur later, justifying again the instability softening effect of the varying focusing cubic nonlinearity. Comparing the solid lines in Fig. (8a) and Fig. (8d), we realize that the quintic nonlinearity reduces the perturbation and then stabilizes the system.

In Fig. (9a) for the same parameters as in Fig. (5) (line in $\chi_1$), we show the time evolution of the modulational perturbation launched with different initial values when $\alpha = 0.0$. For many values of $a_0$ (e.g., 0.00001 and 0.0001) extremely small compared to $A_0$, we have exponential growth of the modulational perturbation, signalling the instability of BEC, as provided by the analytical treatment. The very short-time evolution is less dependent on the small initial value. But curiously, as the initial value increases along the separation occurs, the growth diminishes and even gives way to oscillations (as the initial value reaches $A_0$, see Fig. (9b)). Then a system predicted by linear stability to be unstable can be rather stable under greater perturbations.

The most standard mechanism through which bright solitons appear is through the activation of the modulational instability of plane waves.

The long-time evolution of small initial values of the modulational perturbation (Fig. (10a)) presents a solitonic wave having the form of a pulses train whose spacing depends on the initial value of the perturbation.

In this figure, one can see that due to MI, the initial continuous wave solution has been broken up into a train of ultrashort pulses and hence a train of soliton-like pulses has been generated [[18],[34],[44]].

So for the region predicted to be unstable, the application of linear stability should be limited to very short times and to very small values of $a_0$. As we can see in Fig. (10b), the trapping reduces the intervals between pulses and their height. This result indicates that MI is indeed an underlying physical mechanism explaining the formation of matter-wave soliton trains.

In region (II), modes are predicted to cross, when increases the time, from unstable range to stable range, in the case where the trapping is turned on. Figure (11) shows the comparative time evolution of the modulational perturbation for two values of the trapping parameter. We realize that the trapping stops the exponential growth and leads to oscillations. Moreover, when the trapping increases, the growth softens and the oscillations occur earlier. This figure equally shows
that the elongations considerably decreases while the oscillations arise earlier (when passing from panel (a) to panel (b)). In Fig. (11a), the oscillations in the solid line start at about $t = 33.7$; a time very close to the theoretical threshold time provided by Eq. (36). Thus, as already mentioned, the trapping and the quintic nonlinearity parameter soften the unstable growth and then stabilize the system.

4 Conclusion

To summarize, this paper has both analytically and numerically focused on the modulational instability of a BEC described by a CQ-NLS equation with a parabolic potential through two approaches, namely, the linear stability approach and the time-dependent variational approach. The theoretical analysis invokes a lens-type transformation that converts the GP equation into a modified CQ-NLS equation without explicit spatial dependence. We have given the instability conditions with these two approaches. But, the linear stability analysis cannot tell us the long-time evolution of a modulated extended amplitude or phase of the wavefunction nor can it give us the effective potential. However, the variational method provides a very simple physical picture of the behavior of the condensate: the center of the cloud and its parameters evolve like particles governed by classical potentials, the initial slope and curvature playing, respectively, the role of initial speeds of the particles. The evolution equation of the parameters, which are ordinary differential equations, can be obtained and solved numerically. The TDVA presents some advantages to the linear stability analysis. Instead of considering the modulation directly at the level of the equation, the variation of the present approach is that we use the modulational ansatz in the Lagrangian. This constitutes a basic novel ingredient of the variational type approach to MI. With this method, it is possible to derive the equations for the evolution of the following condensate parameters:

- the time evolution of the amplitude of the wave function (see Eq. (38)) as depicted in figures 3, 4, 5, 6, 7, 8, 9, 10 and 11 (but this time evolution of the amplitude cannot be obtained with the linear stability analysis).
- the time evolution of phase of the wave function (see Eq. (39)).
- the effective potential (see Eq. (33)) as depicted in figure 2 (it is not possible with the classical linear stability analysis to obtain the effective potential). Hence, the TDVA can be applicable to the control of the motion of BECs in a nonuniform confining potential.

We have seen from our numerical study that the quintic term can be used to stabilize the system. We have shown that the solitary wave structures may appear by the activation of the modulational instability of plane waves. We have also shown that the linear stability, for the modulational unstable modes, must not hold at large time scales as the amplitude of unstable sideband exponentially grows.

Furthermore, it is expected that matter-wave solitons may prove useful for eventual technological applications of BECs, such as for gyroscopes for ultra-precise navigation, very accurate
atomic clocks, and other devices that use atomic interferometry.

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Figure 1: Regions of modulational stability/instability in the \((\chi, Q)\) plane: (I) Region of unstable modes; (II) Region of modes changing from unstable to stable; (III) Region of stable modes. The curve \(Q = \sqrt{-\chi}\) (solid line) separates (I) from (II), while the curve \(Q = \sqrt{-\chi + \chi_c}\) (dashed line) separates (II) from (III).
Figure 2: The effective potential as function of $A$ for five different wave numbers, and for four values of the quintic parameter. The parameters are: $q_1 = 0.5$, $q_2 = 1.0$, $q_3 = \sqrt{2}$, $q_4 = 2.0$ and $q_5 = 2.5$; (a) $\chi = 0.0$ (cubic NLS equation case). $q_1$ and $q_2$ are modulationally unstable; $q_4$ and $q_5$ are modulationally stable; while $q_3$ is the threshold wave number. (b) $\chi = -0.5$ (a CQ-NLS equation case with focusing nonlinearities). $q_1$, $q_2$ and $q_3$ are modulationally unstable; $q_5$ is modulationally stable; while $q_4$ is the threshold wave number. (c) $\chi = 0.25$ (a CQ-NLS equation case with defocusing quintic nonlinearity). $q_1$ is modulationally unstable; $q_3$, $q_4$ and $q_5$ are modulationally stable; while $q_2$ is the threshold wave number. (d) $\chi > \chi_c$ (CQ-NLS equation case with particular defocusing quintic nonlinearities): $\chi = 0.6$, $\chi_c = 0.5$. All the modes $q_1$ to $q_5$ (but not limited to) are modulationally stable.
Figure 3: The time evolution of the modulational perturbation \(a(t)\) in the stable oscillatory case provided by Eq. (38) for \(\alpha = 0.0, q = 2.5\) and (a) \(\chi = -0.25\) (oscillations below \(a_0\)); (b) \(\chi = 0.0\) (below \(a_0\)); (c) \(\chi = 0.25\) (below \(a_0\)); (d) \(\chi = 0.75\) (above \(a_0\)).

Figure 4: The time evolution of the modulational perturbation \(a(t)\) in the stable pseudo-oscillatory case provided by Eq.(38) for \(\alpha = 0.01, q = 2.5\) and (a) \(\chi = -0.25\); (b) \(\chi = 0.0\); (c) \(\chi = 0.25\); (d) \(\chi = 0.75\).
Figure 5: The time evolution of the modulational perturbation $a(t)$ in the unstable case provided by Eq. (40) for $\alpha = 0.0, q = 0.5$ and $\chi_1 = -0.25; \chi_2 = -0.1; \chi_3 = 0.0; \chi_4 = 0.25$.

Figure 6: The time evolution of the modulational perturbation $a(t)$ in the unstable case provided by Eq. (40) for $\alpha = 0.01$ (the trapping is turned on), $q = 0.5$ and $\chi = -0.25$ (solid line); $\chi = -0.1$ (dashed line).
Figure 7: The numerical time evolution of the modulational perturbation \( a(t) \) in the case predicted to be stable, for \( \alpha = 0.0 \) (the trapping is turned off), \( g_0 = -1.0 \), \( q = 2.5 \) and \( \chi = -0.25 \). (a) The solid line corresponds to two confused curves: \( 10a(t) \) where \( a(t) \) have been launched with the initial value \( a(t = 0) = 0.001 \), and \( a(t) \) launched with \( a(t = 0) = 0.01 \). The dashed-dotted line corresponds to \( 0.1a(t) \) where \( a(t) \) have been launched with \( a(t = 0) = 0.1 \). And the dashed line corresponds to \( \frac{1}{15}a(t) \) where \( a(t) \) have been launched with \( a(t = 0) = 0.15 \). (b) Line corresponding to \( 0.01a(t) \) where \( a(t) \) have been launched with \( a(t = 0) = 1.0 \). (c) The long-time evolution of the two curves \( 10a(t) \) (solid line) and \( a(t) \) (dashed line) confused in (a); the splitting begins (at about \( t = 300 \)). (d) At very long times (\( t = 1000 \)), the separation of the two curves beforehand confused in (a) becomes very net.
Figure 8: The numerical time evolution of the modulational perturbation $a(t)$ in the cases predicted to be stable, for $\alpha = 0.01$ (the trapping is turned on), $g_0 = -1.0$ and $q = 2.5$. (a) The solid line corresponds to two confused curves: $10a(t)$ where $a(t)$ have been launched with the initial value $a(t = 0) = 0.001$, and $a(t)$ launched with $a(t = 0) = 0.01$. The dashed-dotted line corresponds to $0.1a(t)$ where $a(t)$ have been launched with $a(t = 0) = 0.1$. And the dashed line corresponds to $\frac{1}{10}a(t)$ where $a(t)$ have been launched with $a(t = 0) = 0.15$; $\chi = -0.25$. (b) Line corresponding to $0.01a(t)$ where $a(t)$ have been launched with $a(t = 0) = 1.0$; $\chi = -0.25$. (c) The long-time evolution, for $\chi = -0.25$, of the two curves $10a(t)$ (solid line) and $a(t)$ (dashed line) confused in (a); the splitting begins later (at about $t = 700$), due to the trapping. (d) Two confused curves: $10a(t)$ launched with the initial value $a(t = 0) = 0.001$, and $a(t)$ launched with $a(t = 0) = 0.01$; $\chi = 0.25$. 
Figure 9: The numerical short-time evolution of the modulational perturbation $a(t)$ in the case predicted to be unstable, for $\alpha = 0.0$, $g_0 = -1.0$, $q = 0.5$, $\chi_1 = -0.5$ and $\chi_2 = -0.25$. (a) The solid line in $\chi_1$ corresponds to $1000a(t)$ where $a(t)$ have been launched with the initial value $a(t = 0) = 0.00001$ for $\chi = \chi_1$. The solid line in $\chi_2$ corresponds to two confused curves: $1000a(t)$ launched with the initial value $a(t = 0) = 0.00001$, and $100a(t)$ launched with $a(t = 0) = 0.0001$ for $\chi = \chi_2$. The dashed-dotted line corresponds to $10a(t)$ where $a(t)$ have been launched with $a(t = 0) = 0.001$ for $\chi = \chi_2$. And the dashed line corresponds to $a(t)$ launched with $a(t = 0) = 0.01$ for $\chi = \chi_2$. (b) Oscillations of $a(t)$ launched with $a(t = 0) = 1.0$ for $\chi = \chi_2$.

Figure 10: The numerical long-time evolution of the modulational perturbation $a(t)$ in the case predicted to be unstable, for $g_0 = -1.0$, $q = 0.5$ and $\chi = -0.25$. (a) $\alpha = 0.0$; the separation of the two (initially) confused curves of Fig. 9. The solid line corresponds to $1000a(t)$ launched with $a(t = 0) = 0.00001$. The dashed line corresponds to $100a(t)$ launched with $a(t = 0) = 0.0001$. (b) $\alpha = 0.01$; the line corresponding to $1000a(t)$ launched with $a(t = 0) = 0.00001$. 

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Figure 11: The numerical time evolution of $a(t)$ (solid line) and $25a(t)$ (dashed line) for values of the trapping parameter: $\alpha = 0.001$ and $\alpha = 0.01$ respectively; $g_0 = -1.0$ and $q = 0.5$. The simulations are launched with the same value $a(t=0)=0.0001$ for two points taken in region (II). (a) When $\chi = 0.15$, (b) When $\chi = 0.25$. 