SHAPE CHANGING COLLISIONS OF BRIGHT SOLITONS
IN NONLINEAR ELECTRICAL TRANSMISSION LATTICES

Camus G. Latchio Tiofack
Laboratory of Mechanics, Department of Physics, Faculty of Science,
University of Yaounde I, P.O. Box 812, Yaounde, Cameroon,

Alidou Mohamadou
Laboratory of Mechanics, Department of Physics, Faculty of Science,
University of Yaounde I, P.O. Box 812, Yaounde, Cameroon,
Condensed Matter Laboratory, Department of Physics,
Faculty of Science, University of Douala, P.O. Box 24157, Douala, Cameroon
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy,

Timoléon C. Kofane
Laboratory of Mechanics, Department of Physics, Faculty of Science,
University of Yaounde I, P.O. Box 812, Yaounde, Cameroon
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy

and

Fabien II Ndzana
Laboratory of Mechanics, Department of Physics, Faculty of Science,
University of Yaounde I, P.O. Box 812, Yaounde, Cameroon.

MIRAMARE – TRIESTE
August 2008

1glatchio@yahoo.fr
2mohdoufr@yahoo.fr
3Senior Associate of ICTP, tckofane@yahoo.com
4ndzanafabienii@yahoo.com
Abstract

In the continuous limit, and in suitable scaled coordinates, the telegraphers equations of the electrical transmission line can be described by a set of two coupled nonlinear Schrödinger equations. The corresponding soliton solutions are obtained by using the Hirota’s bilinearization method. These solutions include the bright one- and two-soliton including one-peak solitons. Through the figures for several sample solutions, the stable propagation and elastic collision for these kinds of bright solitons are presented through the line with the use of experimental values of parameters. As in the case of nonlinear optical fibers, we point out the fact that soliton collision in an electrical lattice shows shape changing leading to intensity redistribution.
1 Introduction

Ever since their discovery [1], solitons have fascinated scientists in many widely different fields, such as plasma waves [2], optical pulse in fibers [3], Bose-Einstein condensate (BEC) [4], electrical lattices [5] among others. Solitons arising as a result of competing focusing nonlinearity and anomalous dispersion (diffraction) of pulse (beam) are called bright solitons [6] as they are well localized structures of light. Since 1970s, various investigators have discovered the existence of solitons in nonlinear transmission line (NLTL), through both mathematical models and physical experiments [5,7,8]. It has been shown that the system of equations governing the physics of this network can be reduced to the nonlinear Schrödinger (NLS) equations, the KdV equation or the complex Ginzburg-Landau equation. Marquie et al. [9] have successfully modelled the NLTL by a set of coupled NLS (CNLS) equations. The CNLS equations are often used to describe the interaction among the modes in NLTL. The study of physical and mathematical aspects of CNLS equations is of considerable current interest as these equations arise in diverse areas of science like nonlinear optics, BEC, biophysics, NLTL and plasma physics [9-14].

One of the most exciting phenomena associated with solitons is their collisions. In the case of NLS equation, when two scalar soliton collisions occur sequentially, the outcome of the first collision does not affect the second collision. It then came as a surprise that collision interactions between vector solitons can be very strong [15]. Such strong interactions, besides being fundamentally interesting, have also opened the possibility to use soliton in the signal amplification process [16]. It is a physical mechanism for transferring energy from the pump to the signal. The major virtue of this type of collision-based amplification process is that it does not induce any noise, as it does not make use of any external amplification medium. Many works have been done in this sense in the context of nonlinear optic [15-17]. Radhakrishnan et al. [15] have given an explicit bright two-soliton solution for the (1+1)-dimensional (1D) CNLS equations and derived explicit asymptotic results for collisions in anomalous dispersion regime. They have revealed the fact that the two-soliton solutions of the Manakov model undergo a fascinating shape-changing collision, resulting in an redistribution between the two modes.

On the other hand, NLTL are very convenient tools to study the wave propagation in one-dimensional nonlinear dispersive media. In particular, they provide a useful way to check how the nonlinear excitations behave inside the nonlinear medium. So, it will also be interesting to extend the study of solitons collision in such systems. The purpose of this work is then to investigate the interactions of the bright two-soliton solution of two counterpropagating waves in an experimental transmission line. The paper is organized as follows. In section 2, we derive a set of two CNLS equations from the NLTL. In section 3, the details of Hirota bilinearization procedure for the CNLS equations is given and the bright one and two-solitons are obtained. Section 4 is devoted to a detailed analysis of shape changing (intensity redistribution) collision exhibited by these solutions. Section 5 concludes the paper.
2 The model

We consider a nonlinear network with $N$ cells (see Fig. 1). Each cell contains two linear inductances, $L_1$ in series and $L_2$ in parallel, and a variable-capacitance diode (BB112), biased by a constant voltage $V_0$. Its capacitance $C(V_n) = C_0(1 - \alpha V_n + \beta V_n^2)$ depends nonlinearly on the voltage $V_n$ of the $n$th cell, with positive parameters $C_0$, $\alpha$, and $\beta$. During computations, the following experimental values of parameters of the network are used: $L_1 = 220 \mu H$, $L_2 = 470 \mu H$, $C_0 = 320 p F$, $\alpha = 0.21 V^{-1}$, $\beta = 0.02 V^{-2}$ [5,9,18]. Neglecting in a first approximation the losses of the components, we easily derive from the Kirchhoff’s laws the system of nonlinear discrete equations (for $n = 1, 2, ..., N$)

\[
\frac{d^2 V_n}{dt^2} + \omega_0^2 V_n + u_0^2 \frac{d(V_{n-1} - 2V_n + V_{n+1})}{dt} = \alpha \frac{d^2 V_n^2}{dt^2} - \beta \frac{d^2 (V_n)^3}{dt^2},
\]

with $u_0^2 = \frac{1}{L_1 C_0}$ and $\omega_0^2 = \frac{1}{L_2 C_0}$. From Eq. (1), one gets the linear dispersion relation

\[
\omega^2 = \omega_0^2 + 4u_0^2 \sin^2 \left( \frac{k}{2} \right),
\]

which corresponds to a bandpass filter with a lower cutoff frequency $f_0 = \frac{\omega_0^2}{2\pi}$ and an upper one, $f_{\text{max}} = \sqrt{\frac{(\omega_0^2 + 4u_0^2)}{2\pi}}$, introduced by the lattice effects (see Fig. 2). In order to use the reductive perturbation method [19], we introduce the independent multiple-scale variables $X_i = \varepsilon^i n$ and $T_i = \varepsilon^i t$, where $\varepsilon \ll 1$. Moreover, the solution of Eq. (1) is assumed to have the following form:

\[
V_n(t) = \varepsilon [V_{1+}(X_1, X_2, ..., T_1, T_2, ...)] \exp (i\theta_+) + \varepsilon^2 [V_{2+}(X_1, X_2, ..., T_1, T_2, ...)] \exp (2i\theta_+) + \varepsilon [V_{1-}(X_1, X_2, ..., T_1, T_2, ...)] \exp (i\theta_-) + \varepsilon^2 [V_{2-}(X_1, X_2, ..., T_1, T_2, ...)] \exp (2i\theta_-) + c.c.,
\]

where the carrier phases are $\theta_+ = kn - \omega t$ and $\theta_- = -kn - \omega t$, and c.c. represents the complex conjugate of the preceding terms. The second-harmonic terms $V_{2\pm}$ are added to the fundamental ones $V_{1\pm}$ in order to take the asymmetry of the variable-capacitance charge into account.

Inserting Eq. (3) in Eq. (1) yields, after some standard calculations, to a set of two CNLS equations:

\[
i \frac{\partial V_{1+}}{\partial T_2} + iV_g \frac{\partial V_{1+}}{\partial X_2} + P \frac{\partial^2 V_{1+}}{\partial X_1^2} + [Q_1 |V_{1+}|^2 + Q_2 |V_{1-}|^2] V_{1+} = 0,
\]

\[
i \frac{\partial V_{1-}}{\partial T_2} - iV_g \frac{\partial V_{1-}}{\partial X_2} + P \frac{\partial^2 V_{1-}}{\partial X_1^2} + [Q_1 |V_{1-}|^2 + Q_2 |V_{1+}|^2] V_{1-} = 0,
\]

where $V_g = \frac{d\omega}{dk} = \frac{u_0^2 \sin(k)}{\omega}$ is the group velocity and the group velocity dispersion $P = \frac{d^2 \omega}{dk^2} = \frac{u_0^2 \cos(k) - V_g^2}{2\omega}$ is calculated from Eq. (2). The nonlinear coefficients $Q_1$ and $Q_2$ are, respectively,

\[
Q_1 = Q_1' = \omega \left[ \frac{3\beta}{2} - \frac{4\alpha^2 \omega^2}{3\omega_0^2} + 16u_0^2 \sin^4 \left( \frac{k}{2} \right) \right], \quad Q_2 = 3\beta \omega f \text{ for } f < \frac{f_{\text{max}}}{2},
\]

(5)
\[ Q_1 = Q_1'' = \frac{3\beta}{2} \omega, Q_2 = 3\beta \omega \text{ for } f > \frac{f_{\text{max}}}{2}. \]  

(6)

Physically, the set of two variables \((T = T_2 \text{ and } X = X_1 = X_2)\) is sufficient to describe the evolution of the envelopes, which correspond to letting \(\varepsilon\) tend to unity.

### 3 Bilinearization and bright soliton solutions

Using the transformation

\[
V_{1+} = V_1 \exp[-i \frac{V_g}{2F}(X - \frac{V_g}{2}T)], V_{1-} = V_1 \exp[i \frac{V_g}{2F}(X + \frac{V_g}{2}T)],
\]

(7)

and by rescaling \(X\) and \(T\) by

\[
X = x, T = Pt,
\]

(8)

we get

\[
\begin{align*}
\frac{\partial V_1}{\partial t} + \frac{\partial^2 V_1}{\partial x^2} + \frac{1}{F}[Q_1 |V_1|^2 + Q_2 |V_2|^2]V_1 &= 0, \\
\frac{i}{F} \frac{\partial V_2}{\partial t} + \frac{\partial^2 V_2}{\partial x^2} + \frac{1}{F}[Q_1 |V_2|^2 + Q_2 |V_1|^2]V_2 &= 0.
\end{align*}
\]

(9)

Equations (9) are a special case of the fairly general and frequently studied system of CNLS equations (Manakov model) known to be integrable for \(Q_1 = Q_2\). Then by solving Eq. (5) for \(Q_1 = Q_2\) with \(\omega\) given by Eq. (2), we obtain the relation

\[
48\beta u_0^2 \sin^4 \left(\frac{k}{2}\right) + 32\alpha^2 u_0^2 \sin^2 \left(\frac{k}{2}\right) + (9\beta + 8\alpha^2) \omega_0^2 = 0,
\]

(10)

the solutions of this equation are

\[
k = 2 \arcsin \sqrt{\frac{-32\alpha^2 u_0^2 \pm \sqrt{(32\alpha^2 u_0^2)^2 - 192\beta u_0^2 \omega_0^2 (9\beta + 8\alpha^2)}}{96\beta u_0^2}}.
\]

(11)

In this section, by applying Hirota’s technique we point out that the most general one-soliton and two-soliton solutions for the system Eq. (9) can be obtained.

Let us apply the bilinearizing transformation \([20,21]\)

\[
V_1 = \frac{G}{F}, V_2 = \frac{H}{F},
\]

(12)

where \(G(x, t), H(x, t)\) are complex functions while \(F(x, t)\) is a real function. the following bilinear equations can be obtained:

\[
[iD_t + PD_x^2]GF = 0, D_x^2(FF) = \frac{1}{F}[Q_1 |G|^2 + Q_2 |H|^2],
\]

(13)

where \(D_x\) and \(D_t\) are the Hirota’s operators \([20,21]\) defined as

\[
D_t^n D_x^m(GF) = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n G(t, x)F(t', x').
\]

(14)


3.1 Analytical bright one-soliton solutions

In order to obtain one-soliton solutions, we expand the functions $G$, $H$ and $F$ as follows

$$G = \lambda g_1, H = \lambda h_1, F = (1 + \lambda^2 f_2),$$

where $\lambda$ is the formal expansion parameter. The resulting equations, after collecting the terms with the same power in $\lambda$, are solved to obtain the following forms of $g_1$, $h_1$ and $f_2$

$$g_1 = \alpha_1 \exp(\theta_1), h_1 = \beta_1 \exp(\theta_1),$$

$$f_2 = 0.5\frac{Q_1|\alpha_1|^2 + Q_2|\beta_1|^2}{(k_1 + k_1^*)^2} \exp(\theta_1 + \theta_1^*),$$

where $\theta_1 = k_1 x + ik_1^2 t$ with $\alpha_1$, $\beta_1$, $k_1$ as arbitrary complex parameters. The analytical bright one-soliton solutions are yielded

$$V_1 = \frac{\alpha_1 \exp(\theta_1)}{1 + \exp(\theta_1 + \theta_1^* + \eta)} = \frac{\alpha_1}{2} \exp(-\frac{\eta}{2}) \text{sech}(\theta_1 R + \frac{\eta}{2}) \exp(i\theta_1 t),$$

$$V_2 = \frac{\beta_1 \exp(\theta_1)}{1 + \exp(\theta_1 + \theta_1^* + \eta)} = \frac{\beta_1}{2} \exp(-\frac{\eta}{2}) \text{sech}(\theta_1 R + \frac{\eta}{2}) \exp(i\theta_1 t),$$

where

$$\exp(\eta) = 0.5\frac{Q_1|\alpha_1|^2 + Q_2|\beta_1|^2}{(k_1 + k_1^*)^2}.$$ 

Figure 3 shows the evolution of the bright one-soliton solution.

3.2 Analytical bright two-soliton solutions

To obtain the bright two-soliton solutions, the series of Eq. (15) is now expressed as follow:

$$G = \lambda g_1 + \lambda^3 g_3, H = \lambda h_1 + \lambda^3 h_3, F = (1 + \lambda^2 f_2 + \lambda^4 f_4).$$

After proceeding in a similar fashion, as in the case of one soliton, we obtain

$$g_1 = \alpha_1 \exp(\theta_1) + \alpha_2 \exp(\theta_2),$$

$$h_1 = \beta_1 \exp(\theta_1) + \beta_2 \exp(\theta_2),$$

$$f_2 = \exp(\theta_1 + \theta_1^* + \eta_1) \exp(\theta_1 + \theta_2^* + \eta_2) \exp(\theta_2 + \theta_1^* + \eta_2^*) \exp(\theta_2 + \theta_2^* + \eta_2),$$

$$g_3 = \exp(\theta_1 + \theta_1^* + \theta_2 + \gamma_1) \exp(\theta_1 + \theta_2 + \theta_1^* + \gamma_2),$$

$$f_4 = 0.5\frac{Q_1 \exp(\gamma_1 + \gamma_2^*) + Q_2 \exp(\gamma_3 + \gamma_2^*)}{\exp(\eta_1)(k_2 + k_2^*)^2} \exp(\theta_1 + \theta_1^* + \theta_2 + \theta_2^*),$$

where $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, $k_1$, $k_2$ are all arbitrary complex parameters, $\theta_1 = k_1 x + ik_1^2 t$, $\theta_2 = k_2 + ik_2^2 t$,

$$\exp(\eta_1) = 0.5\frac{Q_1|\alpha_1|^2 + Q_2|\beta_1|^2}{(k_1 + k_1^*)^2},$$

$$\exp(\eta_0) = 0.5\frac{Q_1|\alpha_1|^2 + Q_2|\beta_1|^2}{(k_1 + k_1^*)^2}.$$
The analytical bright two-soliton solutions can be written as

\[
V_1 = \frac{V_1}{V_1}, \quad V_2 = \frac{V_2}{V_2},
\]

with

\[
V_{10} = \alpha_1 \exp(\theta_1) + \alpha_2 \exp(\theta_2) + \exp(\theta_1 + \theta_1^* + \theta_2 + \gamma_1) + \exp(\theta_1 + \theta_2 + \theta_2^* + \gamma_2),
\]

and

\[
V_{11} = 1 + \exp(\theta_1 + \theta_1^* + \eta_1) + \exp(\theta_1 + \theta_2^* + \eta_2) + \exp(\theta_2 + \theta_1^* + \eta_0) + \exp(\theta_2 + \theta_2^* + \eta_1) + \exp(\theta_2 + \theta_2^* + \eta_2) + \exp(\theta_2 + \theta_2^* + \eta_1) + \exp(\theta_2 + \theta_2^* + \eta_2),
\]

\[
V_{20} = \beta_1 \exp(\theta_1) + \beta_2 \exp(\theta_2) + \exp(\theta_1 + \theta_1^* + \theta_2 + \gamma_1) + \exp(\theta_1 + \theta_2 + \theta_2^* + \gamma_2),
\]

\[
V_{22} = 1 + \exp(\theta_1 + \theta_1^* + \eta_1) + \exp(\theta_1 + \theta_2^* + \eta_0) + \exp(\theta_2 + \theta_1^* + \eta_0) + \exp(\theta_2 + \theta_1^* + \eta_1) + \exp(\theta_2 + \theta_1^* + \eta_2) + \exp(\theta_1 + \theta_1^* + \theta_2 + \gamma_2).
\]

4 **Shape changing collisions of solitons**

In this section, we will investigate elastic and inelastic collision between two bright solitons. In order to study the changes of amplitudes during the collision process, we can easily obtain the asymptotic forms of amplitude solutions before and after collisions.

\[
V_1 : S_1 : \frac{\alpha_1}{2} \exp\left(-\frac{\eta_1}{2}\right) \longrightarrow \frac{1}{2} \exp\left[-\left(-\frac{\eta_2 + \eta_3}{2}\right)\right],
\]

\[
S_2 : \frac{1}{2} \exp(\gamma_1) \exp\left[-\left(-\frac{\eta_1 + \eta_3}{2}\right)\right] \longrightarrow \frac{\alpha_2}{2} \exp\left(-\frac{\eta_2}{2}\right).
\]
\[
V_2 : S_1 : \frac{\beta_1}{2} \exp(-\frac{\eta_1}{2}) \rightarrow \frac{1}{2} \exp[-(\frac{\eta_2 + \eta_3}{2})], \\
S_2 : \frac{1}{2} \exp(\gamma_1) \exp[-(\frac{\eta_1 + \eta_3}{2})] \rightarrow \frac{\beta_2}{2} \exp(-\frac{\eta_2}{2}),
\]

(43)

where all the parameters have been defined as before. We can find the important role that quantities \(\alpha_1, \alpha_2, \beta_1,\) and \(\beta_2\) play in expressions (42) and (43) during the soliton collision process. Figure 4 presents the head-on elastic collision of two bright solitons \(S_1\) and \(S_2\), when \(\frac{\alpha_1}{\alpha_2} = \frac{\beta_1}{\beta_2}\). Solitons \(S_1\) and \(S_2\) pass through each other unaffectedly. After colliding with each other, the intensity and velocity of solitons \(S_1\) or \(S_2\) are the same as those before collision. Figures 5 and 6 depict the case of inelastic collisions between two bright solitons when \(\frac{\alpha_1}{\alpha_2} \neq \frac{\beta_1}{\beta_2}\). The collision scenario shown in Fig. 5 may be viewed as an amplification process in which one of the soliton \((S_1\) in the \(V_1\) component and \(S_2\) in the \(V_2\) component) represents a signal (or data carrier) while the other soliton \((S_2\) in the \(V_1\) component and \(S_1\) in the \(V_2\) component) represents an energy reservoir (pump). The main virtue of this amplification process is that it does not require any external amplification medium and therefore the amplification of soliton does not induce any noise. On the other hand, the collision scenario shown in Fig. 6 depicts an interesting feature. This figure shows the interaction of two solitons \(S_1\) and \(S_2\) which are well separated before and after collision, in the \(V_1\) and \(V_2\) components. After the collision, the first soliton \(S_1\) in the \(V_1\) component gets enhanced in its amplitude while the soliton \(S_2\) is suppressed.

5 Conclusion

In this paper, we have investigated the propagation and the collision of bright soliton type solutions of CNLS equation in an experimental transmission line. By applying Hirota’s bilinear method, we have derived the bright one- and two-soliton solutions of this nonlinear equation. We have found that even in the NLTL, the solitons undergo fascinating shape changing (intensity redistribution) collision, including maintenance of original shapes (elastic collision), enhancement or suppression of original shapes (inelastic collision). Using this property, in principle it becomes possible to promote the collision process without noise generation in which the gain can be tuned over a relatively large range through a careful choice of precollision parameters.

Acknowledgments

This work was done within the framework of the Associateship Scheme of the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy. The authors acknowledge fruitful discussion with B A Malomed. Mohamadou and Kofane thank the CMSPS of the ICTP for their hospitality where this work has been finalized.
References


Figure 1: One unit cell of the discrete nonlinear electrical transmission line.

Figure 2: Linear dispersion curves as a function of the wave number $k$ for $0 \leq k \leq \pi$.

Figure 3: Intensity profile $|V_1|^2$ of the one soliton solution of CNLS equation in an electrical lattice. Parameters: $k_1 = 0.5 + 0.5i$, $\alpha_1 = 1$, $\beta_1 = 1$.
Figure 4: Intensity profile of the head-on collision of the two soliton solution of CNLS equation showing the standard elastic collision. (a) in the first modes $|V_1|^2$. (b) in the second modes $|V_2|^2$. Parameters: $k_1 = 0.5 + 0.5i$, $k_2 = 0.5 - 0.5i$, $\alpha_1 = 1$, $\alpha_2 = 0.05$, $\beta_1 = 1$, $\beta_2 = 0.05$.

Figure 5: Intensity profile of the head-on collision of the two soliton solution of CNLS equation showing the inelastic collision. (a) the soliton $S_1$ gets enhanced and the soliton $S_2$ gets suppressed. (b) the soliton $S_1$ gets suppressed and the soliton $S_2$ gets enhanced. Parameters: $k_1 = 0.5 + 0.5i$, $k_2 = 0.5 - 0.5i$, $\alpha_1 = 1$, $\alpha_2 = \frac{29+80i}{80}$, $\beta_1 = 1$, $\beta_2 = 1$.

Figure 6: Intensity profile of the head-on collision of the two soliton solution of CNLS equation showing the dramatic ways of inelastic collision. (a) the soliton $S_1$ gets enhanced and the soliton $S_2$ gets totally suppressed. (b) the soliton $S_1$ gets suppressed and the soliton $S_2$ gets enhanced. Parameters: $k_1 = 0.5 + 0.5i$, $k_2 = 0.5 - 0.5i$, $\alpha_1 = 1$, $\alpha_2 = \frac{29+80i}{80}$, $\beta_1 = 1$, $\beta_2 = 0.05$. 