SINGULAR AND MARCINKIEWICZ INTEGRALS
WITH $H^1$ KERNELS ON PRODUCT SPACES

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Abstract

In this paper we shall prove that for $\Omega \in H^1(S^{n-1} \times S^{m-1})$, which satisfies the cancellation condition $\int_{S^{n-1}} \Omega(x', y')dx' = \int_{S^{m-1}} \Omega(x', y')dy' = 0 \ (\forall (x', y') \in S^{n-1} \times S^{m-1})$, the Calderón-Zygmund singular integral operator $T_\Omega$, its maximal operator $T^*_\Omega$ and the Marcinkiewicz integral operator $\mu_\Omega$ are bounded on $L^p(R^n \times R^m)$ for $1 < p < \infty$. 

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Let $R^l$ ($l \geq 2$) be the $l$-dimensional Euclidean space and $S^{l-1}$ be the unit sphere in $R^l$. For nonzero point $z \in R^l$, we denote $z' = z/|z|$. The Calderón-Zygmund singular integral operator and its maximal operator, with a homogeneous kernel on the product space $R^n \times R^m$, are defined by

$$
T_\Omega(f)(x,y) = p.v. \int_{R^n \times R^m} K(u,v) f(x-u,y-v) dudv
$$

$$
T^*_\Omega(f)(x,y) = \sup_{\epsilon' > 0, \epsilon'' > 0} \left| \int_{|u| > \epsilon', |v| > \epsilon''} K(u,v) f(x-u,y-v) dudv \right|
$$

(1)

where $\Omega \in L^1(S^{n-1} \times S^{m-1})$ and satisfies the following cancellation condition

$$
\int_{S^{n-1}} \Omega(x', y') dx' = \int_{S^{m-1}} \Omega(x', y') dy' = 0 (\forall (x', y') \in S^{n-1} \times S^{m-1}).
$$

(2)

The Marcinkiewicicz integral operator on the product space $R^n \times R^m$ is defined by

$$
\mu_\Omega(f)(x,y) = \left( \int_{R^n \times R^m} |\psi_{t,s} * f(x,y)|^p \frac{dt ds}{ts} \right)^{1/p}
$$

(3)

where $\Omega \in L^1(S^{n-1} \times S^{m-1})$ and satisfies (2), $\psi_{t,s}(x,y) = t^{-n} s^{-m} \psi(\frac{x}{t}, \frac{y}{s})$ and

$$
\psi(x,y) = \Omega(x', y') |x|^{-1-n} |y|^{1-m} \chi_{|z| \leq 1}(x) \chi_{|y| \leq 1}(y).
$$

(4)

The $L^p$-boundedness ($1 < p < \infty$) of $T_\Omega$, $T^*_\Omega$ and $\mu_\Omega$ has been studied extensively in both product and non-product cases. One important issue, among many others, is to find some size conditions on $\Omega$ as weak as possible to ensure the $L^p$-boundedness of the above mentioned operators. The readers can see [2], [4], [5], [7]-[9], [12]-[15], [17], [18], [20], [21], [23], [24], [26]-[30], [32], [33], etc. for various conditions on $\Omega$ and some interesting developments.

More specifically, the largest function class, so far, containing $\Omega$ to ensure the $L^p$-boundedness (for all $p \in (1, \infty)$) of $T_\Omega$ and $T^*_\Omega$ in the non-product case is the Hardy space $H^1(S^{n-1})$. This fact was discovered in 1978 by Connett [12] and by Ricci and Weiss [24]. See also [17] for the similar theorem on a more general operator. On the product case, however, the largest function class (that was found so far) containing $\Omega$ to ensure the $L^p$-boundedness (for all $p \in (1, \infty)$) is the $L((\ln + L)^2(S^{n-1} \times S^{m-1})$ space. As $L((\ln + L)^2(S^{n-1} \times S^{m-1})$ is a proper subspace of $H^1(S^{n-1} \times S^{m-1})$, comparing the result in the non-product case, one naturally asks the following question:

Are operators $T_\Omega, T^*_\Omega$ bounded on $L^p(R^n \times R^m)$ if $\Omega \in H^1(S^{n-1} \times S^{m-1})$ ?

This problem is an extension from non-product to the product case. But it is not a trivial one. Unlike the non-product case, a famous counterexample of Carleson tells us that there is no standard atomic decomposition on the product space. While the proofs for theorems on non-product spaces [12], [14], [17] heavily depend on the atomic characterization of $H^1(S^{n-1})$. Thus, to solve the above question, it is expected to find a non-atomic characterization of the Hardy space $H^1$ on the product space $S^{n-1} \times S^{m-1}$ which may allow us to execute proofs throughout the product space. We note that Ricci and Weiss did use the non-atomic characterization of $H^1$
decomposition in their proof on non-product case. In [24], Ricci and Weiss used the “Riesz system” and characterized the $H^1(S^{n-1})$ space by the Riesz transforms. Their method can be partially adopted in our product case (when we employ the rotation method), but not all. Alternately, we will use another characterization of $H^1(S^{n-1} \times S^{m-1})$ which actually is a “restriction” of $H^1(R^n \times R^m)$ on $S^{n-1} \times S^{m-1}$ (see [25] for the non-product case). The following is one of the two main theorems in this paper.

**Theorem 1** Suppose $\Omega \in H^1(S^{n-1} \times S^{m-1})$ and satisfies (2). Then $T_\Omega$ and $T_\Omega^*$ are bounded on $L^p$ for $1 < p < \infty$.

Similarly, for Marcinkiewicz integral operator $\mu_\Omega$, the largest function class containing $\Omega$ to ensure the $L^p$-boundedness (for all $p \in (1, \infty)$) of $\mu_\Omega$ in the non-product case is $H^1(S^{n-1}) \cup L(\ln + L)^{1/2}(S^{n-1})$, see [14], [26], also see [32] for a different proof for $\Omega \in H^1(S^{n-1})$. On the product case, however, the largest function class containing $\Omega$ to ensure the $L^p$-boundedness (for all $p \in (1, \infty)$) is the $L \ln + L(S^{n-1} \times S^{m-1})$ space. As $L \ln + L(S^{n-1} \times S^{m-1})$ does not contain $H^1(S^{n-1} \times S^{m-1})$, one naturally asks the following question:

Is operator $\mu_\Omega$ bounded on $L^p(R^n \times R^m)$ if $\Omega \in H^1(S^{n-1} \times S^{m-1})$?

The following is the other main theorem in this paper.

**Theorem 2** Suppose $\Omega \in H^1(S^{n-1} \times S^{m-1})$ and satisfies (2). Then $\mu_\Omega$ is bounded on $L^p$ for $1 < p < \infty$.

As we mentioned before, our proofs of the theorems will be based on a “restriction” of $H^1(R^n \times R^m)$ on $S^{n-1} \times S^{m-1}$ and based on the rotation method of Calderón and Zygmund. In the section one, we will review the definition of the Hardy space $H^1$ on the product space $S^{n-1} \times S^{m-1}$ and discuss its properties if we view it as a “restriction” of the Hardy space on $R^n \times R^m$. For this purpose, we will give a unified definition of $H^1$ on the Euclidean space and the unit sphere, since they are both special cases of a previous results in [8], [10], [31] on the Hardy space $H^1$ on a Riemannian manifold which can be employed in this special case. We will prove Theorem 1 in section 2. In which the proof for the operator $T_\Omega$ can be followed along some ideas in [4], but the proof for $T_\Omega^*$ is quite different and even more difficult. Thus, in section 2 we will mainly address the proof for the operator $T_\Omega^*$. We will use the rotation method of Calderón and Zygmund to reduce the function $\Omega(x, y)$ in the kernel of $T_\Omega^*$ to a new function $\tilde{\Omega}(x, y)$ such that it is odd in both $x$ and $y$ variables. Thus the $L^p$-boundedness of $T_\Omega^*$ is reduced to the $L^p$-boundedness of certain maximal directional Hilbert transforms on the product space $R \times R$. Though this method is well-known and expected, there exist quite a few technical difficulties when we work on the product case. In particular, since we use a new characterization of $H^1$ that was not used in papers mentioned above. In section 3, we will prove Theorem 2. In this case we deal with the Marcinkiewicz integral. Again, we will use the rotation method, but it must be modified. As we know that a singular
integral composed with a Riesz transform is again a singular integral. But an easy computation, even in the non-product case, shows that the Marcinkiewicz integral has no such a useful property. Thus, we introduce some auxiliary functions and prove that the $L^1(S^{n-1} \times S^{m-1})$-norm of these functions can be controlled by the $H^1(S^{n-1} \times S^{m-1})$-norm of $\Omega$ (see Lemma 10). Finally, we decompose the Marcinkiewicz integral $\mu_\Omega$ as a sum of several sub-operators and show that the $L^p$-norm of each sub-operator can be dominated by the $L^1(S^{n-1} \times S^{m-1})$-norm of these auxiliary functions. In section 4, we will give some further discussions on various definitions of the Hardy space on the product space $S^{n-1} \times S^{m-1}$.

1 Some basic facts on Hardy spaces $H^1$

We first give a unified definition of $H^1(R^l)$ and $H^1(S^{l-1})$. Let $\mathcal{H}$ be a Hilbert space. Let $D$ be a complete Riemannian manifold of dimension $l_D$ with non-negative Ricci curvatures, $\Delta_D$ its Laplace-Beltrami operator, $\nabla_D$ its gradient operator, $P_D = \{P^D_t\}_{t>0}$ its Poisson semi-group, $\{P^1_t\}_{t>0}$ the Poisson kernels, $d_Dx$ the volume element. For $f \in L^1(D \rightarrow \mathcal{H})$, define its radial maximal function $P^+_D(f)$, non-tangential maximal function $P^\ast_D(f)$, area integral function $A_D(f)$ and Riesz transform $R_D(f)$ as follows

$$P^+_D(f)(x) = \sup_{t>0} \|P^D_t(f)(x)\|$$

$$P^\ast_D(f)(x) = \sup_{(y,t) \in \Gamma_x} \|P^D_t(f)(y)\|$$

$$A_D(f)(x) = \left( \int_{\Gamma_x} \|\nabla_D^\perp P^D_t(f)(y)\|^2 \frac{tdtdu}{|B_D(x,t)|} \right)^{1/2}$$

$$R_D(f)(x) = \nabla_D(-\Delta_D)^{-1/2}(f)(x)$$

where $\nabla_D^\perp = (\nabla_D, \partial_t)$, $B_D(x,t)$ is the geodesic ball in $D$ with center $x$ and radius $t$, $\Gamma^D_x = \{(y,t) \in D \times R^l_+ : \rho_D(x,y) < t\}$ in which $\rho_D$ is the geodesic distance on $D$. We call $f \in L^1(D \rightarrow \mathcal{H})$ an $H^1(D \rightarrow \mathcal{H})$ function if $\|f\|_{H^1} := \|P^+_D(f)\|_1 < \infty$. We have

Proposition 3 [8, Theorem 3] For $f \in L^1(D \rightarrow \mathcal{H})$ which satisfies the following cancellation condition

$$\int_D f(x)d_Dx = 0, \quad (5)$$

we have $\|f\|_{H^1} \cong \|P^+_D(f)\|_1 \cong \|A_D(f)\|_1 \cong \|R^\ast_D(f)\|_1$, where $R^\ast_D = (R_D, I_d_D)$.

For $D = R^l$ with the Euclidean metric, the definition of $H^1(D \rightarrow \mathcal{H})$ is just the definition in [19]. For $D = S^{l-1}$ with the standard metric induced from $R^l$, the definition is equivalent to the definition in [11] by the atomic decomposition theorem (see [8, Theorem 1]). The following proposition gives a relation between $H^1(S^{l-1} \rightarrow \mathcal{H})$ and $H^1(R^l \rightarrow \mathcal{H})$.

Proposition 4 [25, Lemma 2.5] If $\Omega \in H^1(S^{l-1} \rightarrow \mathcal{H})$ and satisfies (5), $\varphi$ is a measurable function on $R^l_+$ and satisfies $\Phi(r) := \sup_{t \geq r} \left|\varphi(t)^{l-1}\right| \in L^1(R^l_+, dr)$, then $f(x) = \Omega(x/|x|)\varphi(|x|) \in H^1(R^l \rightarrow \mathcal{H})$. Furthermore, we have $\|f\|_{H^1} \leq C_1 \|\Phi\|_1 \|\Omega\|_{H^1_1}$.
Let $N$ and $M$ are two complete Riemannian manifolds with non-negative Ricci curvatures. For $f \in L^1(N \times M)$, define its radial maximal function $P^+(f)$, non-tangential maximal function $P^*(f)$ and area integral function $A(f)$ as follows

$$P^+(f)(x,y) = \sup_{t,s>0} \left| (P_t^N \otimes P_s^M)(f)(x,y) \right|,$$

$$P^*(f)(x,y) = \sup_{(u,v)\in \Gamma_x^N \times \Gamma_y^M} \left| (P_t^N \otimes P_s^M)(f)(u,v) \right|,$$

$$A(f)(x,y) = \left( \int_{\Gamma_x^N \times \Gamma_y^M} \left| (\nabla_t^N \otimes \nabla_y^M)(P_t^N \otimes P_s^M)(f)(u,v) \right|^2 \frac{tdudv}{|B_S(x,t)| |B_M(v,s)|} \right)^{1/2}.$$

We call $f \in L^1(N \times M)$ an $H^1(N \times M)$-function if $\|f\|_{H^1} =: \|P^+(f)\|_1 < \infty$. We have

**Proposition 5** For $f \in L^1(N \times M)$, we have $\|A(f)\|_1 \leq C_{n,m} \|f\|_{H^1}$. 

**Proof.** By [10, Theorem 1], we have $\|A(f)\|_1 \leq C_{n,m} \|P^*(f)\|_{H^1}$. By sub-mean inequality of harmonic functions, it’s easy to prove that

$$\|P^*(f)\|_1 \leq C_{n,m} \|P^+(f)\|_1.$$

So, the proposition holds.

For $N \times M = S^{n-1} \times S^{m-1}$, we have

**Lemma 6** If $\Omega \in H^1(S^{n-1} \times S^{m-1})$ and satisfies (2), $\varphi^{(k)}$ is a measurable function on $R^1_+$ and satisfies $\Phi^{(k)}(r) =: \sup_{t \geq r} \varphi^{(k)}(t)t^{l_k-1}$ $\in L^1(R^1_+, dr)$ for $l_1 = n$, $l_2 = m$, $k = 1, 2$, then

$$\|\langle \mathcal{R}_R \otimes \mathcal{R}_M \rangle (f)\|_{L^1(R^1_+ \times R^m)} \leq C_{n,m} \|\Phi^{(1)}\|_1 \|\Phi^{(2)}\|_1 \|\Omega\|_{H^1(S^{n-1} \times S^{m-1})},$$

$$\|\langle \mathcal{R}_R \otimes I \mathcal{R}_M \rangle (f)\|_{L^1(R^1_+ \times R^m)} \leq C_{n,m} \|\Phi^{(1)}\|_1 \|\Phi^{(2)}\|_1 \|\Omega\|_{H^1(S^{n-1} \times S^{m-1})},$$

$$\|\langle I \mathcal{R}_R \otimes \mathcal{R}_M \rangle (f)\|_{L^1(R^1_+ \times R^m)} \leq C_{n,m} \|\Phi^{(1)}\|_1 \|\Phi^{(2)}\|_1 \|\Omega\|_{H^1(S^{n-1} \times S^{m-1})},$$

where $f(x,y) = \Omega(x', y')\varphi^{(1)}(|x|)\varphi^{(2)}(|y|)$.

**Proof** We only consider $\langle \mathcal{R}_R \otimes \mathcal{R}_M \rangle (f)$. Set $\Omega_x(y) = \mathcal{R}_R(f_y)(x)$ where $f_y(x) = \Omega(x', y')\varphi^{(1)}(|x|)$, we have

$$\|\langle \mathcal{R}_R \otimes \mathcal{R}_M \rangle (f)\|_{L^1(R^1_+ \times R^m)} = \int_{R^1_+} \left( \int_{R^m} \left| \mathcal{R}_R \left( \Omega_x \varphi^{(2)} \right)(y) \right| dy \right) dx \leq C_{n} \int_{R^1_+} \left( \|\Omega_x\|_{H^1(S^{n-1})} \right) \|\Phi^{(2)}\|_1 dx$$

(by Propositions 4 and 3)

$$\leq C_{n} \int_{R^1_+} \Phi^{(2)}(y) \left( \int_{R^m} \left| \mathcal{R}_M \left( \varphi^{(1)} \right)(y') \right| dy' \right) dx \leq C_{n} \int_{S^{n-1}} \left( \int_{R^1_+} \left| \mathcal{R}_M \left( \varphi^{(1)} \right)(y') \right| dy' \right) \omega_{y'}(x) dx \leq \omega_{y'}(x) \left( \int_{R^m} \left| \mathcal{R}_M \left( \varphi^{(1)} \right)(y') \right| dy' \right) \omega_{y'}(x) dx,$$

where $H = H_{y'} = L^2(\Gamma_{y'}^s, |B_{S^{m-1}}(y', s)|^{-1} dsdv')$ and $\omega_{y'}(x) = \nabla_{S^{n-1}}^\perp P_{S^{m-1}}^S(\Omega(x', \cdot) |y') (w', s) \in \Gamma_{y'}^{s-1}$, and take $\omega_{y'}$ as an $H_{y'}$-valued function, then we have

$$\int_{R^1_+} \left( \int_{S^{n-1}} \mathcal{R}_M (\omega_{y'}) \varphi^{(1)}(y) \right) dx \leq C_{n} \left\| \Phi^{(1)} \right\|_1 \left\| \omega_{y'} \right\|_{H^1(S^{n-1} \rightarrow H_{y'})}$$

(by Propositions 4 and 3)
by Proposition 3. So, we have
\[ \| (\mathcal{R}_{R^n} \otimes \mathcal{R}_{R^m})(f) \|_{L^1(R^n \times R^m)} \leq C_{n,m} \| \Phi^{(1)} \|_1 \| \Phi^{(2)} \|_1 \| A(\Omega) \|_{L^1(S^{n-1} \times S^{m-1})} \]
\[ \leq C_{n,m} \| \Phi^{(1)} \|_1 \| \Phi^{(2)} \|_1 \| \Omega \|_{H^1(S^{n-1} \times S^{m-1})} \]
by Proposition 5.

2 Singular integrals on product spaces

For \( \Omega \in L^1(S^{n-1} \times S^{m-1}) \) which satisfies (2), decompose
\[ \Omega = \Omega_{\alpha,o} + \Omega_{\alpha,e} + \Omega_{e,o} + \Omega_{e,e} \tag{6} \]
where
\[ \Omega_{\alpha,o}(x', y') = \frac{1}{2} (\Omega(x', y') - \Omega(-x', y') - \Omega(x', -y') + \Omega(-x', -y')) \]
\[ \Omega_{\alpha,e}(x', y') = \frac{1}{2} (\Omega(x', y') - \Omega(-x', y') + \Omega(x', -y') - \Omega(-x', -y')) \]
\[ \Omega_{e,o}(x', y') = \frac{1}{2} (\Omega(x', y') + \Omega(-x', y') - \Omega(x', -y') - \Omega(-x', -y')) \]
\[ \Omega_{e,e}(x', y') = \frac{1}{2} (\Omega(x', y') + \Omega(-x', y') + \Omega(x', -y') + \Omega(-x', -y')) \]
which means that there are \( \omega_{\alpha,o} \) on \( S^{n-1} \times S^{m-1} \) such that
\[ K_{\alpha,o}(x, y) = \omega_{\alpha,o}(x', y') |x|^{-n} |y|^{-m} \tag{8} \]
for \( \alpha = \alpha, e \). Furthermore, all above \( \omega \)'s are odd both in \( x \) and \( y \). Similarly, there are \( \omega, \Omega' \) and \( \Omega'' \) such that
\[ (\mathcal{R}_{R^n} \otimes \mathcal{R}_{R^m})(p.v. \Omega(x', y'))(x, y) = \frac{\omega(x', y')}{|x|^n |y|^m} \]
for \( \alpha = o \) and \( \beta \in \{o, e\} \). Furthermore, all above \( \omega \)'s are odd both in \( x \) and \( y \). Similarly, there are \( \omega, \Omega' \) and \( \Omega'' \) such that
\[ (\mathcal{R}_{R^n} \otimes \mathcal{R}_{R^m})(p.v. \Omega(x', y'))(x, y) = \frac{\omega(x', y')}{|x|^n |y|^m} \tag{9} \]
We have

**Lemma 7** For \( \Omega \in (2) \cap H^1(S^{n-1} \times S^{m-1}) \), \( \omega, \Omega' \) and \( \Omega'' \in L^1(S^{n-1} \times S^{m-1}) \), and their \( L^1 \)-norms \( \leq C_{n,m} \| \Omega \|_{H^1(S^{n-1} \times S^{m-1})} \). Especially, we have \( \omega_{\alpha,o} \in L^1(S^{n-1} \times S^{m-1}) \) for all \( \alpha \) and \( \beta \in \{o, e\} \).
Proof It can be proved by Lemma 6 and the ideas in [4]. We omit details here.

Proof of the $L^p(R^n \times R^m)$-boundedness of $T_\Omega$. Note that $R_{R'} \cdot R_{R'} = -Id_{R'}, (R_{R^n} \otimes R_{R^m}) \cdot (R_{R^n} \otimes R_{R^m}) = Id_{R^n+m}$, we have

$$T_\Omega(f) = T_{\alpha,\beta}(f) - R_{R^n} \cdot T_{\alpha,\beta}(f) - R_{R^m} \cdot T_{\alpha,\beta}(f) + (R_{R^n} \otimes R_{R^m}) \cdot T_{\alpha,\beta}(f)$$

by (1), (6) and (7), where

$$T_{\alpha,\beta}(f)(x,y) = K_{\alpha,\beta} * f(x,y) = p.v. \int_{R^n \times R^m} K_{\alpha,\beta}(u,v)f(x - u, y - v)du dv$$

for $\alpha$ and $\beta \in \{0, e\}$. By the boundedness of Riesz transforms, it is enough to prove the $L^p(R^n \times R^m)$-boundedness of all $T_{\alpha,\beta}$. However, $K_{\alpha,\beta}$ is odd and homogeneous of order $-n$ in $x$ and $-m$ in $y$, so

$$T_{\alpha,\beta}(f)(x,y) = \frac{1}{4} \int_{S_{n-1} \times S_{m-1}} \omega_{\alpha,\beta}(u',v')(H_{u'}^{R^n} \otimes H_{v'}^{R^m})(f)(x,y)du' dv'$$

by rotation method, where

$$H_{u'}^{R^n}(f)(x,y) = p.v. \int_{R^n} f(x - tu', y) \frac{du}{t}$$

$$H_{v'}^{R^m}(f)(x,y) = p.v. \int_{R^m} f(x, y - sv') \frac{dv}{s}$$

are directional Hilbert transforms. Therefore, the boundedness of directional Hilbert transforms implies the boundedness of $T_{\alpha,\beta}$.

Proof of the $L^p(R^n \times R^m)$-boundedness of $T_\Omega$. It is easy to check that

$$T_\Omega(f)(x,y) \leq \sum_{\alpha,\beta \in \{0, e\}} \left( T_{\alpha,\beta}^{\alpha,\beta}(f)(x,y) + T_{\alpha,\beta}^{\alpha,\beta,1}(f)(x,y) + T_{\alpha,\beta}^{\alpha,\beta,2}(f)(x,y) + 16 \int_{S_{n-1} \times S_{m-1}} |\Omega_{\alpha,\beta}(u',v')| M_{u',v'}(f)(x,y)du' dv' \right)$$

(12)

where (we give more notations here for further use below)

$$M_{u',v'}(f)(x,y) = \sup_{t,s>0} \left| \int_{(0,t) \times (0,s)} |f(x - \delta u', y - \eta v')| d\delta d\eta \right|$$

$$M_{u'}^{R^n}(f)(x,y) = \sup_{t>0} \left| \int_{(0,t)} |f(x - \delta u', y)| d\delta \right|$$

$$M_{v'}^{R^m}(f)(x,y) = \sup_{s>0} \left| \int_{(0,s)} |f(x, y - \eta v')| d\eta \right|$$

(13)

and

$$T_{\alpha,\beta}^{\alpha,\beta}(f)(x,y) = \sup_{t,s>0} \left| \int_{R^n \times R^m} L_{t,s}^{\alpha,\beta}(u,v)f(x - u, y - v)du dv \right|$$

$$T_{\alpha,\beta}^{\alpha,\beta,1}(f)(x,y) = \sup_{t,s>0} \left| \int_{\{|u|<t\} \times R^m} L_{t,s}^{\alpha,\beta}(u,v)f(x - u, y - v)du dv \right|$$

$$T_{\alpha,\beta}^{\alpha,\beta,2}(f)(x,y) = \sup_{t,s>0} \left| \int_{R^n \times \{|v|<s\}} L_{t,s}^{\alpha,\beta}(u,v)f(x - u, y - v)du dv \right|$$

in which

$$L_{t,s}^{\alpha,\beta}(x,y) = t^{-m-s} L^{\alpha,\beta}(x/t, y/s), L^{\alpha,\beta}(x,y) = K_{\alpha,\beta}(x,y) \tilde{\chi}(|x|) \tilde{\chi}(|y|)$$

$$\tilde{\chi} \in C^\infty((0,\infty)), \tilde{\chi}(r) = 0 \text{ for } r \leq \frac{1}{2}, \tilde{\chi}(r) = 1 \text{ for } r \geq 1, 0 \leq \tilde{\chi} \leq 1.$$  

(14)

We shall only deal with $T_{\alpha,\beta}^{\alpha,\beta}(f)$ below because $T_{\alpha,\beta}^{\alpha,\beta,j}(f)$ ($j = 1$ or 2) are much easier to be treated.

Now, let

$$T_{\alpha,\beta}^{\alpha,\beta}(f)(x,y) = \sup_{t,s>0} \left| L_{t,s}^{\alpha,\beta} * f(x,y) \right|$$

$$L_{t,s}^{\alpha,\beta}(x,y) = t^{-m-s} L^{\alpha,\beta}(x/t, y/s)$$

$$L^{\alpha,\beta}(x,y) = (R_{R^n} \otimes R_{R^m})(L^{\alpha,\beta})(x,y)$$

(15)
in which $\mathfrak{R}_m^\alpha$ is defined in (7). Note that $\mathcal{L}_{t,s}^{\alpha,\beta}(x,y) = t^{-\eta}s^{-\eta} \mathcal{L}^{\alpha,\beta}(x/t, y/s) = (\mathfrak{R}_m^\alpha \otimes \mathfrak{R}_m^\beta)(\mathcal{L}_{t,s}^{\alpha,\beta})(x,y)$ (see (14) for $L_{t,s}^{\alpha,\beta}$), we have

$$T_{s,t}^{\alpha,\beta}(f)(x,y) = \sup_{s,t>0} \left| L_{t,s}^{\alpha,\beta} * f(x,y) \right| = T_s^{\alpha,\beta}(f^\alpha \cdot f^\beta)(x,y)$$

(16)

where $f^\alpha = (\mathfrak{R}_m^\alpha \otimes \mathfrak{R}_m^\beta)(f)$.

Now, we estimate $T_s^{\alpha,\beta}(f)$.

**Case I:** $(\alpha, \beta) = (\alpha, o)$. We have

$$T_{s,t}^{0,o}(f)(x,y) = \sup_{s,t>0} \left| \int \int S_{o-1} S_{o-1} \left( \int \int_{(R_n^+)^2} L_{n,n}^{o,\nu} (\delta u', \eta v') \cdot f(x-\delta u', y-\eta v') \frac{du}{\eta} \right) \right|$$

$$\leq \frac{1}{4} \sup_{s,t>0} \left| \int \int S_{o-1} S_{o-1} \left( \int \int_{(R_n^+)^2} H_{n,n}^{o,\nu}(f)(x,y) du dv \right) \right|$$

(17)

by rotation method, where (we give more notations here for further use below)

$$\hat{H}_{u',v'}^{R_n,\nu}(f)(x,y) = \sup_{t,s>0} \left| \left( \hat{H}_{u',v'}^{R_n,\nu} \circ \hat{H}_{v',s}^{R_n,\nu} \right)(f)(x,y) \right|$$

(18)

are the maximal double directional Hilbert transforms on $R^n \times R^n$, in which

$$\hat{H}_{u',v'}^{R_n,\nu}(f)(x,y) = \int f(x,y-\eta v') \chi(\frac{x}{\eta}) \frac{du}{\eta} = \int f(x,y-\nu v') \chi(\frac{x}{\nu}) \frac{du}{\nu}$$

$$\hat{H}_{u',v'}^{R_n,\nu}(f)(x,y) = \int f(x-\delta u', y-\eta v') \chi(\frac{x}{\nu}) \frac{du}{\nu}$$

See (14) for $\hat{\chi}$ and $\chi = \chi_{[1,\infty)}$.

**Case II:** $(\alpha, \beta) = (o, e)$. We have

$$\mathcal{L}_{t,s}^{o,e}(x,y) = \hat{\chi}(\frac{|x/t|}{T}) p.v. \int_{R^n} \frac{\Omega_{o,e}(x', w')}{|x'| |w'|} \hat{\chi}(\frac{|w|}{s}) dw$$

$$= \hat{\chi}(\frac{|x|}{T}) \left\{ \chi(|y|) \mathcal{L}_{t,s}^{o,e,1}(x,y) + \chi(|y|) \mathcal{L}_{t,s}^{o,e,2}(x,y) \right\}$$

where

$$\mathcal{L}_{t,s}^{o,e,1}(x,y) = \int_{R^n} \frac{\Omega_{o,e}(x', w')}{|x'| |w'|} (\hat{\chi}(\frac{|w|}{s}) - 1) \frac{\Omega_{o,e}(x', w')}{|x'| |w'|} dw$$

$$\mathcal{L}_{t,s}^{o,e,2}(x,y) = \int_{R^n} \frac{\Omega_{o,e}(x', w')}{|x'| |w'|} \hat{\chi}(\frac{|w|}{s}) \frac{\Omega_{o,e}(x', w')}{|x'| |w'|} dw$$

$$\mathcal{L}_{t,s}^{o,e,3}(x,y) = \int_{R^n} \frac{\Omega_{o,e}(x', w')}{|x'| |w'|} (\hat{\chi}(\frac{|w|}{s}) - \hat{\chi}(\frac{|w|}{s})) dw$$

Note that all these functions are odd both in $x$ and in $y$, by rotation method, we have

$$T_{s,t}^{0,e}(f)(x,y) \leq \sup_{s,t>0} \left| \int \int S_{o-1} S_{o-1} \omega_{o,e}(u', v') \left( \hat{H}_{u',v'}^{R_n,\nu} \circ \hat{H}_{v',s}^{R_n,\nu} \right)(f)(x,y) du dv \right|$$

(19)
Obviously, for $T_{0,0}^{\rho,e}(f)$, we have

$$T_{0,0}^{\rho,e}(f)(x, y) \leq C_{n,m} \int_{S^{-1} \times S^{-1}} |\omega_{o,e}(u', v')| \tilde{H}_{u', v'}^{*, 1}(f)(x, y) du' dv'$$

(20)

see (18) for the definition of $\tilde{H}_{u', v'}^{*, 1}$. For $T_{e,1}^{\rho,e}(f)$, we have

$$T_{e,1}^{\rho,e}(f)(x, y) \leq C_{n,m} \int_{S^{-1} \times S^{-1}} |\Omega_{o,e}(u', v')| (\Phi_{R_m} \circ \tilde{H}_{u'}^{*, R_m})(f)(x, y) du' dv'$$

(21)

where (we give more notations here for further use below)

$$\tilde{H}_{u', v'}^{*, R_m}(f)(x, y) = \sup_{t > 0} \left| \tilde{H}_{u', v'}^{*, R_m}(f)(x, y) \right|$$

$$\Phi_{R_m}(f)(x, y) = \sup_{t > 0} \left| \Phi_{R_m}(f)(x, y) \right|$$

(22)

(23)

because for $|v| \geq 2s$

$$\left| L_{t, s}^{\rho,e, 1}(u', v') \right| = \int_{R^m} \left( \frac{c_m(v-w)}{|v-w|^{m+1}} - \frac{c_m v}{|v|^{m+1}} \right) \left( \tilde{\chi}(\frac{|v|}{s}) \right) \frac{|\Omega_{o,e}(u', w')|}{|w|^{m+1}} du \leq C_m \int_{|w| \leq s} \frac{|\Omega_{o,e}(u', w')|}{|w|^{m+1}} dw \leq C_m \Phi_{R_m}(v) \int_{S^{-1}} |\Omega_{o,e}(u', w')| dw'.$$

For $T_{e, 2}^{\rho,e}(f)$, we have

$$T_{e, 2}^{\rho,e}(f)(x, y) \leq C_{n,m} \int_{S^{-1} \times S^{-1}} |\Omega_{o,e}(u', v')| (\Phi_{R_m} \circ \tilde{H}_{u'}^{*, R_m})(f)(x, y) du' dv'$$

(24)

because for $|v| \leq s/4$, we have

$$\left| L_{t, s}^{\rho,e, 2}(u', v) \right| \leq C_m \int_{|w| \geq s/2} \frac{|\Omega_{o,e}(u', w')|}{|w|^{m+1}} dw \leq C_m s^{-m} \int_{S^{-1}} |\Omega_{o,e}(u', w')| dw'.$$

For $T_{e, 3}^{\rho,e}(f)$, it is obvious that

$$T_{e, 3}^{\rho,e}(f)(x, y) \leq C_{n,m} \int_{S^{-1} \times S^{-1}} |\omega_{o,e}(u', v')| (M_{v'}^{R_m} \circ \tilde{H}_{u'}^{*, R_m})(f)(x, y) du' dv'$$

(25)

where $M_{v'}^{R_m}(f)$ is defined in (13). For $T_{e, 4}^{\rho,e}(f)$, we have

$$T_{e, 4}^{\rho,e}(f)(x, y) \leq C_{n,m} \int_{S^{-1} \times S^{-1}} |\Omega_{o,e}(u', v')| (\Phi_{R_m} \circ \tilde{H}_{u'}^{*, R_m})(f)(x, y) du' dv'$$

(26)

because for $|v| \in (s/4, 2s)$, we have

$$\left| L_{t, s}^{\rho,e, 3}(u', v') \right| = \int_{R^m} \frac{c_m(v-w) \Omega_{o,e}(u', w')}{|v-w|^{m+1}} (\tilde{\chi}(\frac{|v|}{s}) - \tilde{\chi}(\frac{|w|}{s})) dw \leq C_m \left( \int_{|w| > 3s} \frac{|\Omega_{o,e}(u', w')|}{|w|^{m+1}} dw + \int_{|w| \leq s/8} \frac{|\Omega_{o,e}(u', w')|}{s^{m+1}|w|^{m-1}} dw \right.$$ \n
$$\left. + \int_{s/8 < |w| \leq s/4} \frac{|\Omega_{o,e}(u', w')|}{s^{m+1}|w|^{m-1}} dw \right) \leq C_m s^{-m} \int_{S^{-1}} |\Omega_{o,e}(u', w')| dw'.$$
By (19)-(21), (24)-(26), we have

\[ T_s^{\alpha,0}(f)(x,y) \leq C_{n,m} \int_{S^{n-1} \times S^{m-1}} |\omega_{\alpha,e}(u',v')| \tilde{H}_{\alpha}^{s,1}(f)(x,y)du' dv' + C_{n,m} \int_{S^{n-1} \times S^{m-1}} (|\Omega_{\alpha,e}| + |\omega_{\alpha,e}|)(u',v') \cdot (\Phi_{Rm}^{s} + M_{Rm}^{e}) \circ \tilde{H}_{\alpha}^{s,1}(f)(x,y)du' dv' \]  

(27)

Similarly, for \((\alpha, \beta) = (e, o)\), we have

\[ T_s^{e,o}(f)(x,y) \leq C_{n,m} \int_{S^{n-1} \times S^{m-1}} |\omega_{e,o}(u',v')| \tilde{H}_{e}^{s,2}(f)(x,y)du' dv' + C_{n,m} \int_{S^{n-1} \times S^{m-1}} (|\Omega_{e,o}| + |\omega_{e,o}|)(u',v') \cdot (\Phi_{Rm}^{s} + M_{Rm}^{e}) \circ \tilde{H}_{e}^{s,2}(f)(x,y)du' dv' \]  

(28)

where \(\Phi_{Rm}^{s}, \tilde{H}_{\alpha}^{s,1}, \tilde{H}_{e}^{s,2}\) and \(M_{Rm}^{e}\) are defined in (23), (22), (18) and (13) respectively.

**Case III**: \((\alpha, \beta) = (e, e)\). Let

\[ E^{R,0}_t = \{ x \in R^l : |x| \leq t/4 \}, \quad E^{R,1}_t = \{ x \in R^l : |x| \leq 2t \}, \quad E^{R,\infty}_t = \{ x \in R^l : |x| \geq 2t \}, \quad E^{R,i}_t = E^{R,i} \times E^{R,m,j} \]

and

\[ \mathcal{L}_{^{R,e}}(x,y) = \sum_{i,j} \chi_{E^{R,i}_t}(x) \mathcal{L}_{^{R,e}}(x,y) = \sum_{i,j} \mathcal{L}_{^{R,e},t,s}(x,y), \]

we have

\[ T_s^{e,e}(f)(x,y) \leq \sum_{i,j} \sup_{L_{s,t,s}} \mathcal{L}_{^{R,e},t,s}(f)(x,y) =: \sum_{i,j} T_s^{e,e}(f)(x,y). \]  

(29)

For \(\mathcal{L}_{^{R,e},t,s} : |x| \geq 2t, |y| \geq 2s\), we have

\[ \mathcal{L}_{^{R,e}}(x,y) = \mathcal{K}_{e,e}(x,y) - |y|^{-m} \int_{R^n} \left( \frac{c_{n}}{|x-z|^{n+1}} - \frac{c_{n}}{|x|^{n+1}} \right) \frac{\eta_{e}^{n}(z',y')}{|z|} (\tilde{\chi}(|z|)) - 1) dz \]

\[- |x|^{-m} \int_{R^n} \left( \frac{c_{n}}{|y-w|^{m+1}} - \frac{c_{n}}{|y|^{m+1}} \right) \frac{\eta_{e}^{m}(z',w')}{|w|} (\tilde{\chi}(|w|)) - 1) dw \]

\[+ \int_{R^{n} \times R^{m}} \left( \frac{c_{n}}{|x-z|^{n+1}} - \frac{c_{n}}{|x|^{n+1}} \right) \frac{\eta_{e}^{n}(z',w')}{|z|} (\tilde{\chi}(|z|)) - 1) dz dw \]

\[= \mathcal{K}_{e,e}(x,y) - \int_{|z| \leq 1} O_{x,z,t}(1) \frac{\eta_{e}^{n}(z',y')}{|x-z|^{n+1}|y|^{m}} dz \]

\[+ \int_{|w| \leq 1} O_{y,w,s}(1) \frac{\eta_{e}^{m}(z',w')}{|y-w|^{m+1}|x|} dw \]

\[+ \int_{|z| \leq 1, |w| \leq 1} O_{x,z,t,y,w,s}(1) \frac{\eta_{e}^{m}(x',w')}{|w|^{m+1}|x|^{n+1}|y|^{m}} dz \]

where \(O_{a,b,c,...}(1)\) is a function depending only on \(a, b, c, ...\) and satisfying \(|O_{a,b,c,...}(1)| \leq C_{n,m}\) (the same below), and

\[ \eta_{e}^{n}(x',y') = |x'|^{n} \Re_{R^n}(\Omega_{e,e}(x',y') \cdot |^{-n}(x) \]

\[ \eta_{e}^{m}(x',y') = |y'|^{m} \Re_{R^n}(\Omega_{e,e}(x',y') \cdot |^{-m}(y) \]

So, by the rotation method and the oddness of \(\mathcal{K}_{e,e} \) in \(x\) and \(y\), \(\Omega_{e}^{n} \) in \(y\), \(\Omega_{e}^{m} \) in \(x\), we have

\[ T_{^{R,e},t,s}^{R,e}(f)(x,y) \leq \frac{1}{4} \int_{S^{n-1} \times S^{m-1}} |\omega_{e,e}(u',v')| \tilde{H}_{e}^{s,0}(f)(x,y)du' dv' + C_{n,m} \int_{S^{n-1}} \left\| \Omega_{e,e}^{n}(u',v') \right\|_{L^1(S^{n-1})} (\Phi_{Rm}^{s} \circ H_{e}^{s,0})(f)(x,y)du' dv' + C_{n,m} \int_{S^{n-1}} \left\| \Omega_{e,e}^{m}(u',v') \right\|_{L^1(S^{m-1})} (\Phi_{Rm}^{s} \circ H_{e}^{s,0})(f)(x,y)du' dv' \]  

(30)
where $H_{w_0}^{t,R_m}$ and $\Phi_R^t$ are defined in (22) and (23).

For $L_{0,0,t,s}^{e,e}$, \( |x| \leq t/4, |y| \leq s/4 \), we have

\[
\left| L_{t,s}^{e,e}(x,y) \right| \leq C_{n,m} \int_{|z| \geq t/2, |w| \geq s/2} \left| \frac{\Omega_{e,e}(z',w')}{|z|^n|w|^m} \right| dzdw
\]

\[
\leq C_{n,m} \int_{|z| \geq t/2, |w| \geq s/2} \left| \frac{\Omega_{e,e}(z',w')}{|z|^n|w|^m} \right| dzdw
\]

so

\[
T_{0,0,t,s}^{e,e}(f)(x,y) \leq C_{n,m} \left\| \Omega_{e,e} \right\|_L^1(S^{n-1} \times S^{m-1}) (\Phi_R^t \circ \Phi_R^s)(f)(x,y). \tag{31}
\]

For $L_{\infty,0,t,s}^{e,e}$, \( |x| \geq 2t, |y| < s/4 \), we have

\[
L_{\infty,t,s}^{e,e}(x,y) = p.v. \int_{R^n \times R^m} c_n(x-z) \frac{c_m(y-w) \Omega_{e,e}(z',w')}{|x-z|^{n+1} |y-w|^{m+1} |z|^n |w|^m} \hat{\chi}(\frac{w}{s}) dzdw
\]

\[
+ p.v. \int_{R^n \times R^m} \left( \frac{c_n(x-z)}{|x-z|^{n+1} |z|^n} - \frac{c_nx}{|x|^n} \right) \hat{\chi}(\frac{z}{s}) dzdw
\]

so

\[
T_{\infty,0,t,s}^{e,e}(f)(x,y) \leq C_{n,m} \int_{S^{n-1}} \left( \Omega_{e,e}(z',w') \right) \left( \Phi_R^t \circ H_{w_0}^{t,R_m} \right)(f)(x,y) dz'
\]

\[
+ C_{n,m} \left\| \Omega_{e,e} \right\|_L^1(S^{n-1} \times S^{m-1}) (\Phi_R^t \circ \Phi_R^s)(f)(x,y). \tag{32}
\]

Similarly, we have

\[
T_{0,\infty,t,s}^{e,e}(f)(x,y) \leq C_{n,m} \int_{S^{m-1}} \left( \Omega_{e,e}(z',w') \right) \left( \Phi_R^t \circ H_{w_0}^{t,R_m} \right)(f)(x,y) dy'
\]

\[
+ C_{n,m} \left\| \Omega_{e,e} \right\|_L^1(S^{n-1} \times S^{m-1}) (\Phi_R^t \circ \Phi_R^s)(f)(x,y). \tag{33}
\]

For $L_{1,0,t,s}^{e,e}$, \( t/4 \leq |x| \leq 2t, |y| < s/4 \), we have

\[
L_{1,t,s}^{e,e}(x,y) = p.v. \int_{R^n \times R^m} c_n(x-z) \frac{c_m(y-w) \Omega_{e,e}(z',w')}{|x-z|^{n+1} |y-w|^{m+1} |z|^n |w|^m} \hat{\chi}(\frac{w}{s}) dzdw
\]

\[
+ p.v. \int_{R^n \times R^m} \left( \frac{c_n(x-z)}{|x-z|^{n+1} |z|^n} - \frac{c_nx}{|x|^n} \right) \hat{\chi}(\frac{z}{s}) dzdw
\]

so

\[
T_{1,0,t,s}^{e,e}(f)(x,y) \leq C_{n,m} \int_{S^{n-1}} \left( \Omega_{e,e} \right) \left( \Phi_R^t \circ \tilde{H}_{w_0}^{t,R_m} \right)(f)(x,y) dy'
\]

\[
+ C_{n,m} \left\| \Omega_{e,e} \right\|_L^1(S^{n-1} \times S^{m-1}) (\Phi_R^t \circ \Phi_R^s)(f)(x,y). \tag{34}
\]
where \( \Phi_{u,R_n}(f)(x) = \Phi_{u,R_n}^*(f)(x + u) \). Similarly, we have

\[
\mathcal{T}_{0,1,\nu}^{c,e}(f)(x, y) \leq C_{n,m} \frac{1}{S_{m-1}} \left\| \Omega_{e,e}'(\cdot, \cdot') \right\|_{L^1(S_{m-1})} \left( \Phi_{R_n}^* \circ \tilde{H}_{\nu}^{R_n}(f)(x), y \right) dy'
+ C_{n,m} \frac{1}{3 \geq |v| \geq 1/8} \left\| \Omega_{e,e}(\cdot, \cdot') \right\|_{L^1(S_{m-1})} \left( \Phi_{R_n}^* \circ \Phi_{R_n}^*(f)(x), y \right) dv
+ C_{n,m} \left\| \Omega_{e,e} \right\|_{L^1(S_{m-1} \times S_{m-1})} \left( \Phi_{R_n}^* \circ \Phi_{R_n}^* \right)(f)(x, y).
\]

For \( \mathcal{L}_{1,\infty,t,s}^{c,e}(x, y) \), \( t/4 \leq |x| \leq 2t \), \( |y| \geq 2s \), we have

\[
\mathcal{L}_{1,\infty,t,s}^{c,e}(x, y) = \mathcal{K}_{e,e}(x, y) + \mathcal{K}_{e,e}(x, y)
+ \left\{ \int_{|x| \geq 3t} \frac{O_{e,e}(1) \Omega_{e,e}'(\cdot, \cdot')}{|y|^{m+1} [|z|^m |w|^m]} \right\} \int_{|z| \geq 3t} \frac{O_{e,e}(1) \Omega_{e,e}'(\cdot, \cdot')}{|y|^{m+1} [|z|^m |w|^m]} \right\}
+ \left\{ \int_{|z| \geq 3t} \frac{O_{e,e}(1) \Omega_{e,e}'(\cdot, \cdot')}{|y|^{m+1} [|z|^m |w|^m]} \right\} \int_{|z| \geq 3t} \frac{O_{e,e}(1) \Omega_{e,e}'(\cdot, \cdot')}{|y|^{m+1} [|z|^m |w|^m]} \right\}
\]

so

\[
\mathcal{T}_{1,\infty,t,s}^{c,e}(f)(x, y) \leq C_{n,m} \int_{S_{m-1} \times S_{m-1}} \frac{1}{S_{m-1}} \left\| \omega_{e,e}(u', v') \right\|_{L^1(S_{m-1})} \left( \Phi_{R_n}^* \circ \tilde{H}_{\nu}^{R_n}(f)(x), y \right) dy'
+ C_{n,m} \int_{S_{m-1} \times S_{m-1}} \left\| \Omega_{e,e}'(\cdot, \cdot') \right\|_{L^1(S_{m-1})} \left( \Phi_{R_n}^* \circ \tilde{H}_{\nu}^{R_n}(f)(x), y \right) dy'
+ C_{n,m} \int_{S_{m-1} \times S_{m-1}} \left\| \Omega_{e,e}'(\cdot, \cdot') \right\|_{L^1(S_{m-1})} \left( \Phi_{R_n}^* \circ \tilde{H}_{\nu}^{R_n}(f)(x), y \right) dy'
+ C_{n,m} \int_{S_{m-1} \times S_{m-1}} \left\| \Omega_{e,e}'(\cdot, \cdot') \right\|_{L^1(S_{m-1})} \left( \Phi_{R_n}^* \circ \tilde{H}_{\nu}^{R_n}(f)(x, y) \right) dy'
+ C_{n,m} \int_{S_{m-1} \times S_{m-1}} \left\| \Omega_{e,e} \right\|_{L^1(S_{m-1})} \left( \Phi_{R_n}^* \circ \tilde{H}_{\nu}^{R_n}(f)(x, y) \right) dy'.
\]

(36)
Similarly, we have
\[
T_{\infty,1}^{s,e}(f)(x,y) \leq C_{n,m} \iint_{S^{n-1} \times S^{m-1}} |\omega_{e,e}(u',v')| \tilde{H}_{u',v'}^{s,2}(f)(x,y)du'\,dv' + C_{n,m} \iint_{S^{n-1} \times S^{m-1}} \left| \Omega_{e,e}(z',w') \right| (\Phi_{R_n} \circ \tilde{H}_{w}^{s,R_m})(f)(x,y)dz'\,dw' + C_{n,m} \iint_{S^{n-1} \times S^{m-1}} \left| \Omega_{e,e}(z',v') \right| (\Phi_{R_m}^{*} \circ \tilde{H}_{z}^{s,R_n})(f)(x,y)dz'\,dv' + C_{n,m} \iint_{S^{n-1} \times S^{m-1}} \left| \Omega_{e,e}^{*}(z',w') \right| (\Phi_{R_m}^{*} \circ \tilde{H}_{w}^{s,R_n})(f)(x,y)dz'\,dw' + C_{n,m} ||\Omega_{e,e}||_{L^1(S^{n-1} \times S^{m-1})} (\Phi_{R_m} \circ \tilde{F}_{R_m})(f)(x,y) + C_{n,m} \int_{3\geq|w|\geq1/8} ||\omega_{e,e}(\cdot,w')||_{L^1(S^{n-1})} (\Phi_{R_m}^{*} \circ \tilde{F}_{R_m})(f)(x,y) \, dw
\]  
(37)

By (29)-(37), we get
\[
T_{s,e}^{e}(f)(x,y) \leq C_{n,m} \iint_{S^{n-1} \times S^{m-1}} |\omega_{e,e}(u',v')| \tilde{H}_{u',v'}^{s,2}(f)(x,y)du'\,dv' + C_{n,m} \iint_{\{3 \geq |z| \leq 1/8\} \times \{3 \geq |w| \leq 1/8\}} \left( \Omega_{e,e}'' + \Omega_{e,e}' + |\Omega_{e,e}| \right) (z',w') (\Phi_{R_n}^{*} \circ \tilde{H}_{z}^{s,R_n} + \Phi_{R_n}^{*} \circ \tilde{H}_{w}^{s,R_n} + \Phi_{R_n}^{*} \circ \tilde{H}_{z}^{s,R_n} + \Phi_{R_n}^{*} \circ \tilde{H}_{w}^{s,R_n})(f)(x,y)dz'\,dw' + C_{n,m} \iint_{\{3 \geq |z| \leq 1/8\} \times \{3 \geq |w| \leq 1/8\}} \left( \Omega_{e,e}'' + \Omega_{e,e}' + |\Omega_{e,e}| + |\omega_{e,e}| \right) (u',v') (\Phi_{R_n}^{*} \circ \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n})(f)(x,y)du'\,dv' + C_{n,m} \iint_{\{3 \geq |z| \leq 1/8\} \times \{3 \geq |w| \leq 1/8\}} \left( |\omega_{e,e}| \right) (u',v') (\Phi_{R_n}^{*} \circ \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n})(f)(x,y)du'\,dv' + C_{n,m} \iint_{\{3 \geq |z| \leq 1/8\} \times \{3 \geq |w| \leq 1/8\}} \left( |\omega_{e,e}| \right) (u',v') (\Phi_{R_n}^{*} \circ \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n})(f)(x,y)du'\,dv' + C_{n,m} \iint_{\{3 \geq |z| \leq 1/8\} \times \{3 \geq |w| \leq 1/8\}} \left( |\omega_{e,e}| \right) (u',v') (\Phi_{R_n}^{*} \circ \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n})(f)(x,y)du'\,dv' + C_{n,m} \iint_{\{3 \geq |z| \leq 1/8\} \times \{3 \geq |w| \leq 1/8\}} \left( |\omega_{e,e}| \right) (u',v') (\Phi_{R_n}^{*} \circ \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n} + \tilde{H}_{u}^{s,R_n})(f)(x,y)du'\,dv'
\]  
(38)

Finally, combining the estimates (17), (27), (28), (38), we get
\[
T_{s,b}^{e}(f)(x,y) \leq C_{n,m} \iint_{S^{n-1} \times S^{m-1}} (|\omega_{o,o} + |\omega_{e,o} + |\omega_{e,e}|)(u',v') (\tilde{H}_{u',v'}^{b,0} + \tilde{H}_{u',v'}^{b,1} + \tilde{H}_{u',v'}^{b,2})(f)(x,y)du'\,dv' + C_{n,m} \iint_{\{3 \geq |z| \leq 1/8\} \times \{3 \geq |w| \leq 1/8\}} \left( |\omega_{e,e}| + |\Omega_{e,e}' + |\Omega_{e,e}| + |\omega_{e,e}| \right) (u',v') (\Phi_{R_n}^{*} \circ \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n})(f)(x,y)du'\,dv' + C_{n,m} \iint_{\{3 \geq |z| \leq 1/8\} \times \{3 \geq |w| \leq 1/8\}} \left( |\omega_{e,e}| \right) (u',v') (\Phi_{R_n}^{*} \circ \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n})(f)(x,y)du'\,dv' + C_{n,m} \iint_{\{3 \geq |z| \leq 1/8\} \times \{3 \geq |w| \leq 1/8\}} \left( |\omega_{e,e}| \right) (u',v') (\Phi_{R_n}^{*} \circ \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n})(f)(x,y)du'\,dv' + C_{n,m} \iint_{\{3 \geq |z| \leq 1/8\} \times \{3 \geq |w| \leq 1/8\}} \left( |\omega_{e,e}| \right) (u',v') (\Phi_{R_n}^{*} \circ \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n})(f)(x,y)du'\,dv' + C_{n,m} \iint_{\{3 \geq |z| \leq 1/8\} \times \{3 \geq |w| \leq 1/8\}} \left( |\omega_{e,e}| \right) (u',v') (\Phi_{R_n}^{*} \circ \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n} + \tilde{H}_{u}^{b,R_n})(f)(x,y)du'\,dv'
\]  
(39)

Therefore, by Lemma 6, the boundedness of directional Hilbert transforms and maximal operators, we get $L^p$-boundedness of $T_{s}^{\alpha,\beta}$. Similarly, we have $L^p$-boundedness of $T_{s}^{\alpha,\beta,1}$ and $T_{s}^{\alpha,\beta,2}$. So, $T^*$ is $L^p$-bounded.

### 3 Marcinkiewicz integrals on product spaces

At first, we recall some basic results about Littlewood-Paley-Stein $g$-functions, which is defined by
\[
g_\sigma(f)(x) \overset{\text{def}}{=} |T_\sigma(f)(x)|_H,
T_\sigma : f \mapsto T_\sigma(f)(x) = \sigma_\tau \ast f(x)|_{t \in R_+^1} = K_\sigma \ast f(x),
H = L^2(R_+^1, du), K_\sigma(x) = \sigma_\tau(x)|_{t \in R_+^1} \in H,
\]
where $\sigma$ is an $L^1(R)$ function. We have the following known result.
Lemma 8 If
\[
\|\sigma\|_1 < \infty, \int_R \sigma(t) dt = 0, |K_{\sigma}(x)|_{\mathcal{H}} \leq B |x|^{-1},
\]
then, \( \|g_{\sigma}(f)\|_{W_{1,1}} \leq C(\|\sigma\|_1 + B) \|f\|_1, \|g_{\sigma}(f)\|_p \leq C_p(\|\sigma\|_1 + B) \|f\|_p, \ 1 < p < \infty. \)

In the product space case, the Littlewood-Paley-Stein \( g \)-function of one-dimension is defined by
\[
\begin{align*}
\tilde{g}_{\varsigma}(f)(x, y) & \overset{\text{def}}{=} \left| \mathcal{T}_{\varsigma}(f)(x, y) \right|, \\
\mathcal{T}_{\varsigma} : f & \mapsto \tilde{K}_{\varsigma}(f)(x, y) = \mathcal{K}_{\varsigma} \ast f(x, y), \\
\mathcal{K}_{\varsigma} = L^2((R_+^2)^2, \frac{dt}{\varsigma^{1/2}}, \mathcal{H}), \mathcal{K}_{\varsigma}(x) & = s_{t,s}(x, y)|_{(t,s) \in (R_+^2)^2} \in \mathcal{H},
\end{align*}
\]
where \( \varsigma \) is an \( L^1(R \times R) \) function. We have

Lemma 9 If \( \|\varsigma\|_1 < \infty, \varsigma \) is odd both in \( t \) and in \( s \), and \( \exists \alpha \in (0, 1] \) such that
\[
\begin{align*}
\left| \mathcal{K}_{\varsigma}(x, y) \right|_{\mathcal{H}} & \leq B |x|^{-1} |y|^{-1}, \\
\left| \mathcal{K}_{\varsigma}(x + h, y) - \mathcal{K}_{\varsigma}(x, y) \right|_{\mathcal{H}} + \left| \mathcal{K}_{\varsigma}(y, x + h) - \mathcal{K}_{\varsigma}(y, x) \right|_{\mathcal{H}} & \leq B \frac{|h|^{\alpha}}{|x|^{1+\alpha}} |y|^{-1}, \\
\left| \mathcal{K}_{\varsigma}(x + h, y + k) - \mathcal{K}_{\varsigma}(x, y + k) + \mathcal{K}_{\varsigma}(x + h, y) - \mathcal{K}_{\varsigma}(x, y) \right|_{\mathcal{H}} & \leq B \frac{|h|^{\alpha}|k|^{\alpha}}{|x|^{1+\alpha}|y|^{1+\alpha}}
\end{align*}
\]
for all \( |x| \geq 2|h|, |y| \geq 2|k| \), then \( \|\tilde{g}_{\varsigma}(f)\|_p \leq C_p(\|\varsigma\|_1 + B) \|f\|_p, \ 1 < p < \infty. \)

This Lemma can be viewed as a vector-valued generalization of boundedness of singular integrals on product space \( R \times R \), see [20].

Now, for the Marcinkiewicz integral on \( R^n \times R^n \), we have \( \mu_{\Omega}(f) \equiv \tilde{g}_{\psi,\Omega}(f) \). Without loss of generality, we suppose \( \Omega \) is even both in the first and in the second variables (for the odd case, things are much easier to deal with, we omit details here). Set \( \chi(t) = \chi(0,1)(t) \), take \( \lambda \in C_c^\infty(R) \), \( \text{supp}(\lambda) \subseteq [1, 2] \) and \( \int_R \lambda(t) dt = 1 \), let \( \rho(t) = \chi(|t|) - \lambda(|t|) \), and
\[
\begin{align*}
\sigma^{(1,u',v')}(t) & = |t|^{m-1} R^m(\lambda(|\cdot|); |\cdot|^{1-n} \Omega(\cdot, u'))(tu'), \\
\sigma^{(2,u',v')}(s) & = |s|^{m-1} R^m(\lambda(|\cdot|); |\cdot|^{1-m} \Omega(u', \cdot))(sv'), \\
\varsigma^{(u',v')}(t, s) & = |t|^{m-1} |s|^{m-1} R^m \circ R^m(\lambda(|\cdot|); |\cdot|^{1-m} \Omega(\cdot, \cdot))(tu', sv').
\end{align*}
\]
Then it is easy to see that \( \sigma^{(1,u',v')} \) is odd, and \( \varsigma^{(u',v')}(t, s) \) is odd both in \( t \) and in \( s \). Let
\[
\begin{align*}
\Omega_k^*(x', y') & = \sup_{r \in R, j = 0, 1} (1 + r)^{2+j} \left| \partial_{t} \sigma^{(k,x',y')}(r) \right|, k = 1, 2, \\
\Omega^*(x', y') & = \sup_{t, s \in R, i, and j = 0, 1} (1 + t)^{2+j}(1 + s)^{2+i} \left| \partial_{t} \partial_{s} \varsigma^{(x',y')}(t, s) \right|
\end{align*}
\]
We have

Lemma 10 For \( \Omega \in H^{1}(S^{n-1} \times S^{m-1}) \cap (2) \) which is even both in the first and in the second variables, there holds
\[
\left| \sum_{1}^{2} |\Omega_k^*| + |\Omega^*| \right|_{L^1(S^{n-1} \times S^{m-1})} \leq C_{n,m} \|\Omega\|_{H^1(S^{n-1} \times S^{m-1})}.
\]
Proof These estimates can be obtained by a similar estimate appeared in [32]. Here, we shall only consider $\Omega^*$ because the other two are much easier to be estimated. We have

\[
\frac{\partial \tilde{\partial}_s \zeta(t, s)}{(1+t)^{-s}(1+s)^{-s}} = \frac{(1+t)^3(1+s)^3}{(1+s)^{n+1-m}} \int_{\Omega^*} \int_{\Omega^*} \frac{\Omega(z', w') \lambda(|z|) \lambda(|w|)}{|s|^{n-1}|w|^{m-1}} \frac{c_{i,j,k}^n(tw'-z)}{|tw'-z|^{n+m+1}} + \frac{c_{i,j,k}^n(sw'-z)}{|sw'-z|^{n+m+1}} dwdw
\]

where \( \{c_{i,j,k}^n\}_{i,j,k=1}^4 \) are constants depending only on \( i, j, n \) and \( m, \vartheta_1 \) and \( \vartheta_2 \) are homogeneous \( C^\infty \)-functions of order 0 depending only on the first and the second variable respectively. In the following, we assume \( i \) and \( j = 1 \) for simplicity.

For \( t \geq s \geq 4 \), by (43) and (2), we have

\[
\left| \frac{\partial \tilde{\partial}_s \zeta(t, s)}{(1+t)^{-s}(1+s)^{-s}} \right| \leq 4t^{n+1}s^{m+1} \int_{\Omega^*} \int_{\Omega^*} \frac{\Omega(z', w') \lambda(|z|) \lambda(|w|)}{|z|^{n-1}|w|^{m-1}} \frac{c_{i,j,k}^n(tw'-z)}{|tw'-z|^{n+m+1}} + \frac{c_{i,j,k}^n(sw'-z)}{|sw'-z|^{n+m+1}} dwdw
\]

For \( t \leq s \leq 1 \), by (43), we have

\[
\left| \frac{\partial \tilde{\partial}_s \zeta(t, s)}{(1+t)^{-s}(1+s)^{-s}} \right| \leq C_{n,m}t^{n-2}s^{m-2} \int_{\Omega^*} \int_{\Omega^*} \frac{\Omega(z', w') \lambda(|z|) \lambda(|w|)}{|z|^{n-1}|w|^{m-1}} dwdw
\]

For \( t \geq 4 \) and \( s \leq 1 \), by (43) and (2), we have

\[
\left| \frac{\partial \tilde{\partial}_s \zeta(t, s)}{(1+t)^{-s}(1+s)^{-s}} \right| \leq C_{n,m}t^{n+1}s^{m-2} \int_{\Omega^*} \int_{\Omega^*} \frac{\Omega(z', w') \lambda(|z|) \lambda(|w|)}{|z|^{n-1}|w|^{m-1}} dwdw
\]

Similarly, for \( s \geq 4 \) and \( t \leq 1 \), by (43), we have

\[
\left| \frac{\partial \tilde{\partial}_s \zeta(t, s)}{(1+t)^{-s}(1+s)^{-s}} \right| \leq C_{n,m} \|\Omega\|_{L_1(S^{n-1} \times S^{m-1})}
\]

For \( 1/2 \leq t, s \leq 4 \), note that \( r \partial_r \lambda(|r|) = \lambda'(|r|) |r| \), and let \( \tilde{\lambda}(r) = r^{-1} \lambda(r) - \lambda'(r) \), we have (instead of (43))

\[
\frac{\partial \tilde{\partial}_s \zeta(t, s)}{(1+t)^{-s}(1+s)^{-s}} = \frac{(1+t)^3(1+s)^3}{t^2s^2} \int_{\Omega^*} \int_{\Omega^*} \frac{\Omega(z', w') \tilde{\lambda}(tz') \lambda(sw)}{|z|^{n-1}|w|^{m-1}} \frac{c_{i,j,k}^n(tw'-z)}{|tw'-z|^{n+m+1}} + \frac{c_{i,j,k}^n(sw'-z)}{|sw'-z|^{n+m+1}} dwdw
\]

\[
+ \tilde{\lambda}(tw') \int_{\Omega^*} \int_{\Omega^*} \frac{\Omega(z', w') \lambda(sw')}{|z|^{n-1}|w|^{m-1}} \frac{c_{i,j,k}^n(tw'-z)}{|tw'-z|^{n+m+1}} + \frac{c_{i,j,k}^n(sw'-z)}{|sw'-z|^{n+m+1}} dwdw
\]

\[
+ \tilde{\lambda}(sw') \int_{\Omega^*} \int_{\Omega^*} \frac{\Omega(z', w') \lambda(tw')}{|z|^{n-1}|w|^{m-1}} \frac{c_{i,j,k}^n(tw'-z)}{|tw'-z|^{n+m+1}} + \frac{c_{i,j,k}^n(sw'-z)}{|sw'-z|^{n+m+1}} dwdw
\]

\[
+ \int_{\Omega^*} \int_{\Omega^*} \frac{\Omega(z', w') \tilde{\lambda}(tw')(\tilde{\lambda}(sw') - \tilde{\lambda}(tw'))}{|z|^{n-1}|w|^{m-1}} \frac{c_{i,j,k}^n(tw'-z)}{|tw'-z|^{n+m+1}} + \frac{c_{i,j,k}^n(sw'-z)}{|sw'-z|^{n+m+1}} dwdw
\]

\[
(44)
\]
\[
\frac{\partial_t \varphi_s(u', v')_t(s, t)}{(1+t)^{3(1+s)^3}} = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \left( \lambda(tu') \tilde{\lambda}(sv') \omega(u', v') \right) \left( \frac{c_{1,3,3}(u-w)}{|u-w|^{m+1}} \lambda(tz) + \frac{c_{1,3,4}(u-w)}{|u-w|^{m+1}} \lambda(tz) \right) \, dz \, dw \\
+ \int_{\mathbb{R}^2} \Omega'(u', w') (\tilde{\lambda}(tz) - \tilde{\lambda}(tv')) \, dz' + \int_{\mathbb{R}^2} |\Omega''(z', w')| \, dz' \, dw' \\
+ \int_{\mathbb{R}^2} |\Omega''(z', w')| \, dz' \, dw'.
\]

For \(1/2 \leq t \leq 4, s \geq 4\), by the ideas in (43) and (44), we have

\[
\frac{\partial_t \varphi_s(u', v')_t(s, t)}{(1+t)^{3(1+s)^3}} = \frac{1}{2\pi^2} \int_{\mathbb{R}^2} \Omega'(u', w') \Omega''(z', w') \, dz' + \int_{\mathbb{R}^2} |\Omega''(z', w')| \, dz' \, dw'.
\]

Similarly, for \(1/2 \leq s \leq 4, t \geq 4\), we have

\[
\frac{\partial_t \varphi_s(u', v')_t(s, t)}{(1+t)^{3(1+s)^3}} = \int_{\mathbb{R}^2} |\Omega''(u', w')| \, dz' \, dw' + \int_{\mathbb{R}^2} |\Omega''(z', w')| \, dz' \, dw'.
\]
Now, we shall prove Theorem 2. By (4), (42), the evenness of $\Omega$ and the fact that $\mathcal{R}_{\mathcal{R}_l} \cdot \mathcal{R}_{\mathcal{R}_l} = -\text{Id}_{\mathcal{R}_l}$, we have

$$\psi_{\mathcal{R}_l, s}^\Omega \ast f(x, y) = t^{-1}s^{-1} \int_{\mathcal{R}_l} \int_{\mathcal{R}_l} \frac{\Omega(u', v')}{|u|^2 |v|^2} \chi_{|u| \leq \epsilon(u)} \chi_{|v| \leq \epsilon(v)} f(x - u, y - v) du dv$$

$$= \int_{\mathcal{R}_l} \int_{\mathcal{R}_l} \frac{\Omega(u', v')}{|u|^2 |v|^2} (\rho_t(|u|)\rho_t(|v|) + \lambda_t(|u|)\rho_t(|v|)) f(x - u, y - v) du dv$$

$$+ \frac{1}{|u|^2 |v|^2} \int \int_{\mathcal{R}_l} \Omega(u', v') \int \rho_t(|u|)\rho_t(|v|) f(x - t' u', y - s' v') dt' ds' du' dv'$$

which means that

$$\mu\Omega(f)(x, y) = \left| \psi_{\mathcal{R}_l, s}^\Omega \ast f(x, y) \right|_{\mathcal{H}}$$

$$\leq \frac{1}{|u|^2 |v|^2} \int \int_{\mathcal{R}_l} \Omega(u', v') \left| K'_{\mathcal{R}_l, K'_{\mathcal{R}_l}} \ast f(x', y') \right|_{\mathcal{H}} du' dv'$$

$$+ \frac{1}{|u|^2 |v|^2} \int \int_{\mathcal{R}_l} \left| \left( K'_{\mathcal{R}_l, K'_{\mathcal{R}_l}} \right) \ast f(x', y') \right|_{\mathcal{H}} du' dv'$$

$$+ \frac{1}{|u|^2 |v|^2} \int \int_{\mathcal{R}_l} \left| \left( K'_{\mathcal{R}_l, K'_{\mathcal{R}_l}} \right) \ast f(x', y') \right|_{\mathcal{H}} du' dv'$$

(45)

See (3), (40) and (41) for definitions of the related notations. Therefore, what we need to do is to set up the following lemma.

**Lemma 11** $T'_{\mathcal{R}_l, \mathcal{R}_l}, T''_{\mathcal{R}_l, \mathcal{R}_l}, T''_{\mathcal{R}_l, \mathcal{R}_l}$ are bounded on $l^p(R)$, $T''_{\mathcal{R}_l, \mathcal{R}_l}$ is bounded on $l^p(R \times R)$, furthermore,

$$\left\| T'_{\mathcal{R}_l} \right\|_{p, p} \leq C_p \quad \left\| T''_{\mathcal{R}_l} \right\|_{p, p} \leq C_p \left(1 + \Omega'_2(x', y')\right)$$

$$\left\| T'_{\mathcal{R}_l} \right\|_{p, p} \leq C_p \left(1 + \Omega'_2(x', y')\right) \quad \left\| T''_{\mathcal{R}_l} \right\|_{p, p} \leq C_p \left(1 + \Omega'_2(x', y')\right)$$

where $1 < p < \infty$. See (40)-(42) for the related notations.

**Proof** By Lemmas 8 and 9, it is enough for Lemma 11 to check that $K_{\mathcal{R}_l}, K'_{\mathcal{R}_l, \mathcal{R}_l}(i = 1, 2)$ satisfy the conditions in Lemma 8, and $T''_{\mathcal{R}_l, \mathcal{R}_l}$ satisfies the conditions in Lemma 9. These estimates can be obtained by similar estimate approaches appeared in [32]. We omit details here.
Remark 12 From the above discussions, it is easy to see that $K_\lambda$ actually satisfies the conditions in Lemma 8 except that $\int \lambda(t)dt = 0$. The key purpose to consider $\sigma^{(i,u',v')}$ and $\zeta^{(u',v')}$ is to make it odd to ensure $\int \sigma^{(i,u',v')}(t)dt = 0$. Of course, this leads complicated estimates on $K_{\sigma^{(i,u',v')}}$ and $\tilde{K}_{\zeta^{(u',v')}}$.

4 Further discussions about $H^1(S^{n-1} \times S^{m-1})$

The Hardy space $H^1$ is an important space in the study of harmonic analysis. In this section, we will discuss some more details of the Hardy space on the space $S^{n-1} \times S^{m-1}$. First, we have the following proposition on a general space $N \times M$.

Proposition 13 Let $N$ and $M$ be two complete Riemannian manifolds with non-negative Ricci curvatures, $f \in L^1(N \times M)$ and satisfy
\[ \int_N f(x,y)dx = \int_M f(x,y)dy = 0 \text{ for any } (x,y) \in N \times M. \]
Then we have $\|f\|_{H^1} \approx \|P^*(f)\|_1 \approx \|A(f)\|_1 \approx \| (\mathcal{R}'_N \otimes \mathcal{R}'_M)(f) \|_1$ where $\mathcal{R}'_D$ is defined in Proposition 3.

Proof. By [10, Theorem 1], we have $\|A(f)\|_1 \leq C_{n,m} \|P^*(f)\|_1$. By Proposition 3 and iteration technique, we have $\|(\mathcal{R}'_N \otimes \mathcal{R}'_M)(f)\|_1 \leq C_{n,m} \|A(f)\|_1$. By sub-mean formula of harmonic functions, refers to [8, Lemma 9] for non-product case, we have $\|f\|_{H^1} \leq C_{n,m} \|P^+(f)\|_1$. So, it’s enough to prove
\[ \|P^+(f)\|_1 \leq C_{n,m} \|(\mathcal{R}'_N \otimes \mathcal{R}'_M)(f)\|_1 \]
which can be followed from the sub-harmonicity of $\left| (\nabla_{\mathcal{N}} \otimes \nabla_{\mathcal{M}})(P^N \otimes P^M)(f)(x,y) \right|^\alpha$ for $\alpha > \max \left( \frac{1}{N}, \frac{1}{M} \right)$ (both in $(x,t)$ and in $(y,t)$), see the proof of [31, Theorem 3] for non-product case.

Let $\mathcal{C}^{S^{l-1}}$ be the Poisson operator from $L^1(S^{l-1})$ to harmonic functions on the unit ball $B_{R^l} = \{ |z| \leq 1 : z \in R^l \}$, i.e.
\[ \mathcal{C}^{S^{l-1}}(f)(z) = \int_{S^{l-1}} f(u')p(z,u')du'. \]

Define the classical radial maximal, non-tangential maximal and area integral function operators by
\[ \mathcal{C}^{S^{l-1}}_+(f)(z') = \sup_{r \in (0,1)} \left| \mathcal{C}^{S^{l-1}}_+(f)(rz') \right| \]
\[ \mathcal{C}^{S^{l-1}}_-(f)(z') = \sup_{w \in \Theta_{z'}} \left| \mathcal{C}^{S^{l-1}}_-(f)(w) \right| \]
\[ S^{S^{l-1}}(f)(z') = \left( \int_{\Theta_{z'}^{S^{l-1}}} \left| \nabla_w \mathcal{C}^{S^{l-1}}(f)(w) \right|^2 \frac{dw}{|1-|w||^2} \right)^{1/2} \]
where $\Theta_{z'}^{S^{l-1}}$ is the convex hull of $\frac{1}{2}B_{R^l}$ and $z'$. A function $a \in L^1(S^{l-1})$ is called an $H^1$-atom if $\text{supp}(a) \subset B_{S^{l-1}}(z_0, r_0)$, $\int_{S^{l-1}} a(w')dw' = 0$, and $\|a\|_\infty \leq |B_{S^{l-1}}(z_0, r_0)|^{-1}$. We call
\[ f \in L^1(S^{l-1}) \cap (5) \text{ an } H^1_{a}(S^{l-1})\text{-function if} \]

\[
\|f\|_{H^1_{a}} = \inf \left\{ \sum_{1}^{\infty} |c_j| : f = \sum_{1}^{\infty} c_j a_j \text{ where } \{a_j\} \text{ are } H^1\text{-atoms} \right\} < \infty.
\]

In addition, define the generalized conjugate function operator on \( S^{l-1} \) by

\[
\mathcal{R}_{S^{l-1}}(f)(x') = p.v. \int_{S^{l-1}} H(x', y') f(y') dy',
\]

\[
H(x', y') = (y' - (y' \cdot x')x') \frac{1}{|y' - x'|^2} dr,
\]

see [24]. We have

**Proposition 14** For \( f \in L^1(S^{l-1}) \cap (5) \), \( (1) \|f\|_{H^1} \cong C^S_{l-1}(f) \|_{11} \cong C^S_{l-1}(f) \|_{1}, \quad (2) \|f\|_{H^1} \cong R_{S^{l-1}}'(f) \|_{1} \) (i.e. \( \|f\|_{H^1} \) by our definition in section 1) \( \cong \|R_{S^{l-1}}'(f)\|_{1} \), (3) \( \|f\|_{H^1} \cong \|A_{S^{l-1}}(f)\|_{1} \cong \|R_{S^{l-1}}'(f)\|_{1} \) (see Proposition 3 for \( R_{S^{l-1}}' \)).

It is natural to expect the equivalence \( \|f\|_{H^1} \cong \|S^S_{l-1}(f)\|_{1} \) and to know the relation between \( R_{S^{l-1}} \) and \( R_{S^{l-1}}' \).

Now we return to the space \( S^{n-1} \times S^{m-1} \). For \( f \in L^1(S^{n-1} \times S^{m-1}) \), let

\[
C_{+}(f)(x', y') = \sup_{0 < s < 1} \left| C^{S^{n-1}} \times C^{S^{m-1}}(f)(tx', sy') \right|
\]

\[
C_{s}(f)(x', y') = \sup_{(u, w) \in \Theta_{x', y'}} \left| C^{S^{n-1}} \times C^{S^{m-1}}(f)(u, v) \right|
\]

\[
S(f)(x', y') = \left( \int_{\Theta_{x', y'}} \left| \nabla_{u} \nabla_{v} C^{S^{n-1}} \times C^{S^{m-1}}(f)(u, v) \right|^{2} \frac{du}{(1 - |u|^{2})^{n-2}} \frac{dv}{(1 - |v|^{2})^{m-2}} \right)^{1/2},
\]

where \( \Theta_{x', y'} = \Theta_{x'}^{S^{n-1}} \times \Theta_{y'}^{S^{m-1}} \). By Proposition 13 together with an iteration, we have

**Corollary 15** Suppose that \( f \in L^1(S^{n-1} \times S^{m-1}) \) and satisfies, for any \( (x', y') \in S^{n-1} \times S^{m-1} \),

\[
\int_{S^{n-1}} f(x', y') dx' = \int_{S^{m-1}} f(x', y') dy' = 0.
\]

On \( S^{n-1} \times S^{m-1} \), we have

\[
(1) \quad \|f\|_{H^1} \cong \|A(f)\|_{1} \cong \|R_{S^{n-1}} \times R_{S^{m-1}}'(f)\|_{1} \cong \|R_{S^{n-1}}' \times R_{S^{m-1}}'(f)\|_{1}
\]

\[
(2) \quad \|C_{+}(f)\|_{1} \cong \|C_{s}(f)\|_{1}.
\]

It is also natural to expect that \( \|f\|_{H^1} \cong \|S(f)\|_{1} \cong \|C_{s}(f)\|_{1} \), which is known when \( n = m = 2 \) (see [22]), that is the product torus case \( S^{1} \times S^{1} \).

Finally, it should be possible to establish an atomic decomposition of \( H^1(S^{n-1} \times S^{m-1}) \) along the ideas in [3].

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