LINEAR SYSTEMS AND MULTIPLICITY OF IDEALS

(in memory of my friend Sevin Recillas)

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Introduction

A result of P. Samuel ([16] p. 186, Chap. II, Théorème 5) says that in a local noetherian ring \((O, \mathfrak{M})\) of Krull dimension \(d\) in which the residual field \(k\) is infinite, the multiplicity of a \(\mathfrak{M}\)-primary ideal \(I\) is equal to the multiplicity of an ideal \((x_1, \ldots, x_d)\) generated by some parameter sequence \(x_1, \ldots, x_d\) contained in \(I\). By a theorem of Rees ([15] p.142 Theorem 9.44), this implies that the ideals \(I\) and \((x_1, \ldots, x_d)\) have the same integral closure in the ring \(O\).

In fact Samuel’s proof shows that the elements of the parameter sequence can be chosen to be general elements of \(I\), namely superficial elements of \(I\).

An interesting consequence of Samuel’s result is that, in the case the local ring \(O\) is a Cohen-Macaulay ring, e.g. a regular or a local intersection ring, the multiplicity of the ideal \(I\) in \(O\) is equal to the length of the \(O\)-module \(O/(x_1, \ldots, x_d)\).

Using a geometric interpretation of the multiplicity by C. P. Ramanujam ([14]), we shall give a geometric way to calculate the multiplicity. We shall consider the particular case of a non-singular complex surface and give an example with a geometric proof of a result of Mumford, as it was suggested to the author by M.S. Narasimhan.

Most of this note is written in the language of complex analytic spaces (see [2] and [1]), but the results can be stated and proved in the case of schemes of finite type (see definition in [3] Chap. IV 1.6.1) over an infinite field with equicharacteristic local rings.

1 Integral closures and blowing-ups

Let \((O, \mathfrak{M})\) be a reduced complex analytic local ring and let \(J\) be an ideal of \(O\). We say that an element \(x\) of \(O\) is integral over the ideal \(J\) if there is a relation

\[
x^n + \sum_{i=1}^{n} a_i x^{n-i} = 0
\]

where \(a_i \in J^i\).

Elements of \(O\) which are integral over \(J\) form an ideal \(\overline{J}\) in \(O\) which contains \(J\). This ideal is called the integral closure of \(J\) in \(O\).

We know that an ideal \(J\) of \(O\) defines an order function \(\nu_J\) defined by

\[
\nu_J(x) := \sup\{k \mid x \in J^k\} \in \mathbb{N} \cup +\infty
\]

for any \(x \in O\).

We can define

\[
\nu_J(x) := \liminf \frac{\nu_J(x_k)}{k} \in \mathbb{N} \cup +\infty
\]
Then, we have the important following theorem (see [11] Théorème 2.1 or [7]):

1.1 Theorem. Let \((\mathcal{O}, \mathfrak{M})\) be a reduced analytic local ring, \(J\) be an ideal of \(\mathcal{O}\) and \(x \in \mathcal{O}\). Denote by \((Z, z)\) a germ of complex analytic space such that \(\mathcal{O}_{Z,z} = \mathcal{O}\). The following conditions are equivalent:

i) The element \(x\) is integral over the ideal \(J\);

ii) We have \(\nu_J(x) \geq 1\);

iii) There is a modification \(\pi : \tilde{Z} \to (Z, z)\) such that the space \(\tilde{Z}\) is normal and \(J\mathcal{O}_{\tilde{Z}}\) is principal and \(x \circ \pi\) is a section of \(J\mathcal{O}_{\tilde{Z}}\).

iv) Let \(\pi : \tilde{Z} \to (Z, z)\) be the normalized blowing-up of \(J\), then \(x \circ \pi\) is a section of \(J\mathcal{O}_{\tilde{Z}}\).

On the other hand we have the following consequence of a theorem of D. Rees ([15] p.142 Theorem 9.44):

1.2 Theorem. Let \((\mathcal{O}, \mathfrak{M})\) be an analytic local ring which is an integral domain. Let \(I \subset J\) be \(\mathfrak{M}\)-primary ideals of \(\mathcal{O}\). Then, these ideals have the same multiplicity if and only if they have the same integral closure in \(\mathcal{O}\).

The preceding theorems give us the important corollary:

1.3 Corollary. Let \(I\) be a \(\mathfrak{M}\)-primary ideal of an analytic local ring \((\mathcal{O}, \mathfrak{M})\) and let \(x_1, \ldots, x_d\) a sequence of parameters in \(I\) which generates an ideal \((x_1, \ldots, x_d)\) having the same multiplicity as the one of \(I\). The normalized blowing-up of \(I\) equals the normalized blowing-up of \((x_1, \ldots, x_d)\).

Proof: Let \((Z, z)\) be a germ of complex analytic space such that \(\mathcal{O}_{Z,z} = \mathcal{O}\). From the theorem of Rees, it is enough to prove that the normalized blowing-up \(\pi : \tilde{Z} \to (Z, z)\) of \(I\mathcal{O}_{\tilde{Z}}\) is also the normalized blowing-up of the integral closure \(\overline{T}\) of \(I\) in \(\mathcal{O}\). We have \(\overline{T} \supset I\), so \(\overline{T}\mathcal{O}_{\tilde{Z}} \supset I\mathcal{O}_{\tilde{Z}}\). Theorem 1.1 implies that \(\overline{T}\mathcal{O}_{\tilde{Z}} \subset I\mathcal{O}_{\tilde{Z}}\), so

\[ \overline{T}\mathcal{O}_{\tilde{Z}} = I\mathcal{O}_{\tilde{Z}}. \]

Therefore \(\overline{T}\mathcal{O}_{\tilde{Z}}\) is invertible and \(\pi\) factorizes uniquely by \(\sigma\) through the normalized blowing-up \(\overline{T} : \tilde{Z}' \to (Z, z)\) of \(T\):

\[
\begin{array}{c}
\tilde{Z} \\
\pi \\
\downarrow \\
(Z, z)
\end{array} \quad \sigma \\
\overline{T} \quad \pi \\
\downarrow \\
\tilde{Z}'
\]

Now we show that \(I\mathcal{O}_{\tilde{Z}'} = \overline{T}\mathcal{O}_{\tilde{Z}'}\). First, notice that \(I\mathcal{O}_{\tilde{Z}'} \subset \overline{T}\mathcal{O}_{\tilde{Z}'}\), and, for \(k \geq 0\) \(I^k\mathcal{O}_{\tilde{Z}'} \subset \overline{T}^k\mathcal{O}_{\tilde{Z}'}\). By definition we have that \(\overline{T}\mathcal{O}_{\tilde{Z}'}\) is locally principal. Since \(\mathcal{O}_{Z,z}\) is noetherian, the ideal \(\overline{T}\) is
finitely generated. Let $f_1, \ldots, f_k$ be generators of $T$. Let $y \in \pi^{-1}(z)$. Since $\mathcal{T}O_{\tilde{Z}, y}$ is principal, one of the $f_1 \circ \pi$'s, say $f_1 \circ \pi$, generates $\mathcal{T}O_{\tilde{Z}, y}$. On the other hand $f_1$ is integral over $I$, there is a relation:

$$f_1^N + \sum_{1}^{N} a_k f_1^{N-k} = 0$$

where $a_k \in I^k$. Therefore in $\mathcal{T}O_{\tilde{Z}, y}$, we have:

$$(f_1 \circ \pi)^N + \sum_{1}^{N} (a_k \circ \pi)p(f_1 \circ \pi)^{N-k} = 0$$

and by dividing by $(f_1 \circ \pi)^N$:

$$1 + \sum_{1}^{N} \frac{(a_k \circ \pi)}{(f_1 \circ \pi)^k} = 0,$$

which yields

$$f_1 \circ \pi = - \sum_{1}^{N} \frac{(a_k \circ \pi)}{(f_1 \circ \pi)^{k-1}}.$$ 

Since $a_k \circ \pi$ belongs to $I^k$, $a_k \circ \pi \in \mathcal{T}^{k-1}I$ and we have, for $1 \leq k \leq N$,

$$\frac{(a_k \circ \pi)}{(f_1 \circ \pi)^{k-1}} \in \mathcal{T}O_{\tilde{Z}, y},$$

so $f_1 \circ \pi \in \mathcal{T}O_{\tilde{Z}, y}$ and at $y$:

$$\mathcal{T}O_{\tilde{Z}, y} = \mathcal{T}O_{\tilde{Z}, y} = (f_1 \circ \pi)O_{\tilde{Z}, y}.$$ 

Therefore the sheaf $\mathcal{T}O_{\tilde{Z}}$ is invertible. It follows that $\pi$ factorizes uniquely by $\tau: \tilde{Z} \to \tilde{Z}$ through the morphism $\pi$:

$$\xymatrix{ \tilde{Z} \ar[r]^\tau \ar[d]_\pi & \tilde{Z} \ar[d]^{\pi} \\
(Z, z) \ar[ru]^\sigma }$$

The uniqueness of the morphism implies that necessarily $\sigma$ is the inverse morphism of $\tau$, which shows that the normalized blowing-ups of $I$ and its integral closure $T$ in $O$ are the same.

2 Geometry of Multiplicities

In [14], C. P. Ramanujam gave an interesting geometrical interpretation of the multiplicity. First recall that for an invertible sheaf $\mathcal{L}$ on a proper scheme $X$ (resp. on a compact analytic space), the Euler characteristic $\chi(\mathcal{L}^n)$ of the cohomology on $X$ of the $n$-th power $\mathcal{L}^n$ of $\mathcal{L}$ is a function of $n$ which coincides with a polynomial $P_\mathcal{L}(n)$ of degree $m \leq d := \dim X$ in $n$. The coefficient of $n^d$ in this polynomial is

$$\frac{1}{d!}d(\mathcal{L})$$
and $d(\mathcal{L})$ is called the degree of $\mathcal{L}$.

In the case of local analytic rings the result of C.P. Ramanujam (see [14] Theorem p. 64 and Remark (1) p. 66) can be stated in the following way:

2.1 Theorem. Let $(\mathcal{O}, \mathfrak{M})$ be a reduced local analytic local ring and $I$ a $\mathfrak{M}$-primary ideal of $\mathcal{O}$. Let $(Z, z)$ be a germ of analytic space such that $\mathcal{O}_{Z,z} = \mathcal{O}$. Let $\pi : Z' \to (Z, z)$ be a bimeromorphic map such that $\pi^* I$ is an invertible sheaf on $Z'$. The degree of the restriction of $\pi^* I$ to the space defined by $\pi^* I$ is equal to the multiplicity of the ideal $I$.

Considering the space defined by the coherent ideal sheaf $\pi^* I = I\mathcal{O}_{Z'}$, we have the exact sequence

$$0 \to I^{n+1}\mathcal{O}_{Z'} \to I^n\mathcal{O}_{Z'} \to I^n\mathcal{O}_{Z'} \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_{Z'}/I\mathcal{O}_{Z'} \to 0$$

which yields that the degree of the restriction of $\pi^* I$ to the space $< \pi^* I >$ defined by $\pi^* I$ itself equals the degree of $I\mathcal{O}_{Z'}$ because

$$\chi(I^n\mathcal{O}_{Z'}) - \chi(I^{n+1}\mathcal{O}_{Z'}) = \chi(I^n\mathcal{O}_{Z'} \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_{Z'}/I\mathcal{O}_{Z'})$$

and $\chi(I^n\mathcal{O}_{Z'}) - \chi(I^{n+1}\mathcal{O}_{Z'}) = P_{I\mathcal{O}_{Z'}}(n) - P_{I\mathcal{O}_{Z'}}(n + 1)$ is a polynomial of degree $d - 1$ with a term of degree $d - 1$ equal to

$$-\frac{1}{(d - 1)!} d(I\mathcal{O}_{Z'}) n^{d-1}.$$

Since $\chi(I^n\mathcal{O}_{Z'} \otimes_{\mathcal{O}_{Z'}} \mathcal{O}_{Z'}/I\mathcal{O}_{Z'})$ has a term of degree $d - 1$ equal to

$$\frac{1}{(d - 1)!} d(I\mathcal{O}_{Z'}|_{<\pi^* I>}) n^{d-1},$$

Ramanujam’s theorem implies

2.2 Corollary. The multiplicity of $I$ equals

$$d(I\mathcal{O}_{Z'}|_{<\pi^* I>}) = -d(I\mathcal{O}_{Z'}).$$

Ramanujam’s theorem in particular applies to the cases when $\pi$ is the normalized blowing-up of $I$ or a resolution of $(Z, z)$ in which $\pi^* I$ is an invertible sheaf.

For instance, when the bimeromorphic map $\pi$ of the preceding theorem is a resolution of singularities $\pi$ of $(Z, z)$ for which $\pi^* I$ is invertible, we have:

2.3 Corollary. Assume that the map $\pi$ of the preceding theorem is a resolution of singularities for which $\pi^* I$ is invertible and $(\mathcal{O}, \mathfrak{M})$ is an integral domain, then the multiplicity of $I$ equals $(-1)^{d-1}(D)^d$, where $d$ is the Krull dimension of $\mathcal{O}$, $D$ is the divisor defined by $\pi^* I$ on $Z'$ and $(D)^d$ the $d$-th self-intersection of $D$.

Proof: According to Ramanujam’s theorem the multiplicity of $I$ equals

$$d(I\mathcal{O}_{Z'}|_{<\pi^* I>}).$$
The preceding corollary gives
\[ d(I_{OZ'})_{\pi^*I} = -d(I_{OZ'}). \]
Let \( D = \pi^*I \) be the divisor of \( Z' \) defined by the invertible sheaf \( I_{OZ'} \). Hirzebruch-Riemann-Roch theorem (see [H] Theorem 4.1 Appendix A) gives that the degree \( d(I_{OZ'}) \) of \( I_{OZ'} \) equals \((-1)^d(D)^d\). Precisely,
\[ \chi(I^n_{OZ'}) = ch(I^n_{OZ'}) Todd(T_{Z'}) \cap [Z'] \]
where \([Z']\) is the fundamental class of \( Z' \) and \( ch(I^n_{OZ'}) \) is the Chern character and \( Todd(T_{Z'}) \) is the Todd class of the tangent bundle of \( Z' \):
\[ Todd(T_{Z'}) = 1 + \frac{1}{2} c_1(T_{Z'}) + \ldots, \]
and, since \( I^n_{OZ'} \) is invertible, we have:
\[ ch(I^n_{OZ'}) = 1 + n c_1(I_{OZ'}) + \ldots + \frac{1}{d!} n^d c_d(I_{OZ'}). \]
By comparing the terms of degree \( d \) in \( n \), for \( n \gg 0 \), on each side of the equality of Hirzebruch-Riemann-Roch theorem, we have:
\[ d(I_{OZ'}) = c_1^d(I_{OZ'}) \cap [Z']. \]
Since \( I_{OZ'} \) is \( O(-D) \) we have:
\[ c_1^d(I_{OZ'}) \cap [Z'] = (-D)^d \]
and the multiplicity of \( I \) is \(-d(I_{OZ'}) = -(D)^d = (-1)^{d-1}(D)^d\).

3 Linear Systems

Let \((y_1, \ldots, y_k)\) be generators of an ideal \( J \) of \( O_{Z,z} \). We can construct the blowing-up of \( J \) in the following way.

Let \( Z \) be a representative of the germ \((Z, z)\) such that the germs \( y_i \) \((1 \leq i \leq k)\) are defined by holomorphic functions defined on \( Z \) also denoted by \( y_k \) and let \( Y \) be a representative of the support of \( J \) in \( Z \). Then on \( Z \setminus Y \) we define the map \( \lambda \) into the complex projective space \( \mathbb{P}^{k-1} \) by:
\[ \lambda(z') = (y_1(z') : \ldots : y_k(z')) \]
for any \( z' \in Z \setminus Y \).

The graph \( G \) of \( \lambda \) is an analytic subspace of \( Z \times \mathbb{P}^{k-1} \). The topological closure \( \overline{G} \) of \( G \) is naturally an analytic subspace of \( Z \times \mathbb{P}^{k-1} \), because \( G \) is the difference of the analytic set defined by
\[ (y_1 : \ldots : y_k) = (u_1 : \ldots : u_k) \]
in \( Z \times \mathbb{P}^{k-1} \), the \( u_i \)'s are the homogeneous coordinates of \( \mathbb{P}^{d-1} \), and \( Y \times \mathbb{P}^{k-1} \) (use e.g. Lemma 3.9 of [18]). One can show that the restriction to \( \overline{G} \) of the first projection onto \( Z \) is a representative of
the blowing-up $p : Z_J \to (Z, z)$ of the ideal $J = (y_1, \ldots, y_k)$ in $(Z, z)$. Let $n$ be the normalization of $\mathcal{O}_J$, then by corollary 1.3 $p \circ n$ is also the normalized blowing-up $\pi : \tilde{Z} \to (Z, z)$ of $I$ in $(Z, z)$.

Consider the special case $J$ is generated by $d \geq 2$ generators where $d$ is the Krull dimension of $\mathcal{O}_{Z, z}$ and $Y = \{z\}$. The blowing-up $Z_J$ of $J$ is given in $Z \times \mathbb{P}^{d-1}$ by the equations

$$u_{i+1}y_i - u_iy_{i+1} = 0$$

where $1 \leq i \leq d - 1$. Therefore the second projection induces a map

$$\lambda_J : Z_J \to \mathbb{P}^{d-1}$$

which can be viewed as the family of curves defined by the linear system generated by $y_1, \ldots, y_d$.

On the other hand the underlying set $|p^{-1}(z)|$ of the exceptional divisor of the blowing-up $p : Z_J \to (Z, z)$ is contained in $\{z\} \times \mathbb{P}^{d-1}$, so

$$|p^{-1}(z)| = \{z\} \times \mathbb{P}^{d-1}.$$ 

Let $a = (a_1, \ldots, a_d)$ be a general point of $\mathbb{P}^{d-1}$. The fiber $\lambda_J^{-1}(a)$ is a general curve in the linear system of curves generated by $y_1, \ldots, y_d$. Therefore, after normalization, the inverse image $n^{-1}(\lambda_J^{-1}(a))$ is a non-singular curve transverse to the exceptional divisor of the normalized blowing-up $p \circ n$. Since all the components of the exceptional divisor of $p \circ n$ project onto $\mathbb{P}^{d-1}$, the curve $n^{-1}(\lambda_J^{-1}(a))$ intersects all these components.

Apply these results to the case of a $\mathcal{M}_{Z, z}$-primary ideal $I$ of the reduced analytic local ring $\mathcal{O}_{Z, z}$. The result of P. Samuel tells us that the ideal $I$ is integral over a ideal $J$ generated by $d$ general elements $x_1, \ldots, x_d$ of $I$, where $d$ is the Krull dimension of $\mathcal{O}_{Z, z}$. We have seen that the normalized blowing-up $\pi : \tilde{Z} \to (Z, z)$ of $I$ coincides with the normalized blowing-up of the ideal generated by $x_1, \ldots, x_d$. Let $\Gamma$ be a general curve in the linear system of curves generated by $x_1, \ldots, x_d$. From what precedes we observe that the strict transform $\tilde{\Gamma}$ of $\Gamma$ by $\pi$ is a non-singular curve which intersects transversally all the components $D_\alpha$, $\alpha \in A$ of $|\pi^{-1}(z)|$. This strict transform of a general curve in the linear system of curves generated by $x_1, \ldots, x_d$ can be obtained in the following way:

- Let $\pi_J : Z_J \to (Z, z)$ be the blowing-up of the ideal $J$. We have a map $\lambda_J : Z_J \to \mathbb{P}^{d-1}$ defined by the generators $x_1, \ldots, x_d$ of $J$.

- Consider a general point $m$ of $\mathbb{P}^{d-1}$, it is defined by $d - 1$ linear equations $\sum_{i=1}^d \alpha_i^j \xi_i = 0$, $1 \leq j \leq d - 1$, where $\xi_1, \ldots, \xi_d$ are the homogeneous coordinates of $\mathbb{P}^{d-1}$.

- The fiber $\lambda_J^{-1}(m)$ of $\lambda_J$ over $m$ is the strict transform by the blowing-up $\pi_J$ of the curve $\Gamma$ on $(Z, z)$ defined by $\sum_{i=1}^d \alpha_i^j x_i = 0$, $1 \leq j \leq d - 1$. Since $m$ is a general point of $\mathbb{P}^{d-1}$, the germ of curve $(\Gamma, z)$ is a general curve in the linear system of curves generated by $x_1, \ldots, x_d$. The strict transform of $\Gamma$ by the normalized blowing-up $\pi$ is $n^{-1}(\lambda_J^{-1}(m))$. 

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Let $d_\alpha$ be the number of components of the strict transform $\tilde{\Gamma}$ which have a non-empty intersection with $D_\alpha$. Let $e_\alpha$ be the multiplicity of $D_\alpha$ in the divisor defined by $IO_{\tilde{Z}}$. Then, we have:

**3.1 Theorem.** The multiplicity of the ideal $I$ equals $\sum_{\alpha \in A} e_\alpha d_\alpha$.

Proof: Let $\varphi : Z \to \tilde{Z}$ be a resolution of singularities of $\tilde{Z}$. The sheaf $(\overline{\varphi} \circ \varphi)^* IO_Z = IO_Z$ generated by $I$ on $Z$ is invertible. Let $D$ be the divisor of $Z$ defined by $IO_Z$. According to corollary 2.3, the multiplicity of $I$ equals $(-1)^{d-1}(D)^d$. We shall prove:

**3.2 Lemma.**

$$(-1)^{d-1}(D)^d = \sum_{\alpha \in A} e_\alpha d_\alpha.$$  

Using Ramanujam’s result, this lemma obviously implies our theorem.

Proof of the lemma: First we observe that, since the image of $D$ by the map $\overline{\varphi} \circ \varphi$ is a point $\{z\}$, we have

$$D.\text{div}(f \circ \overline{\varphi} \circ \varphi) = 0,$$

for any germ of functions $f \in M_{Z,z} \subset O_{Z,z}$. In particular, if $f$ is a general element of the ideal $I$, we have:

$$\text{div}(f \circ \overline{\varphi} \circ \varphi) = D + H(f)$$

where $H(f)$ is the strict transform of $\{f = 0\}$.

Now let us choose $\alpha_i^j \in \mathbb{C}$, such that the $d-1$ linear equations $\sum_{i=1}^{d} \alpha_i^j \xi_i = 0$, $1 \leq j \leq d-1$, are general and define a general point of $\mathbb{P}^{d-1}$. Let $f_j := \sum_{i=1}^{d} \alpha_i^j x_i = 0$, $1 \leq j \leq d-1$. The functions $f_j$, $1 \leq j \leq d-1$, are general elements of the ideal $I$. The curve $\Gamma$ on $Z$ defined by $\{f_1 = \ldots = f_{d-1}\}$ is a general curve in the linear system of curves generated by $x_1,\ldots,x_d$. The strict transform of $\Gamma$ by $\overline{\varphi} \circ \varphi$ is the curve $H(f_1) \cap \ldots \cap H(f_{d-1})$.

The lemma will be consequence of the equality

$$(-1)^{d-1}(D)^d = (D.H(f_1)\ldots.H(f_{d-1})).$$

In fact, since $(D.D + H(f_i)) = 0$, for $1 \leq i \leq d-1$, we have

$$(D.H(f_1)\ldots.H(f_{d-1})) = -(D.H(f_1)\ldots.H(f_{d-2}).D)$$

Therefore, by induction we can prove

$$(D.H(f_1)\ldots.H(f_{d-1})) = (-1)^{d-2}(D.H(f_1).D\ldots.D) = (-1)^{d-1}(D^d).$$

It remains to prove that $(D.H(f_1)\ldots.H(f_{d-1})) = \sum_{\alpha \in A} e_\alpha d_\alpha$. The curve $\Gamma$ being a general curve in the linear system of curves generated by $x_1,\ldots,x_d$ the strict transform $\tilde{\Gamma}$ of $\Gamma$ by $\overline{\varphi}$ is non-singular and transverse to the components of $|\overline{\varphi}^{-1}(z)|$. Since $\varphi$ is a resolution of singularities $\tilde{Z}$ and

$$\varphi^{-1}(\tilde{\Gamma}) = H(f_1) \cap \ldots \cap H(f_{d-1})$$
the intersection points of $H(f_1) \cap \ldots \cap H(f_{d-1})$ and $D$ are the inverse images by $\varphi$ of the intersection points of $\bar{\Gamma}$ and $|\pi^{-1}(z)|$ and the multiplicity $e_\alpha$ of $D_\alpha$ in $\bar{Z}$ equals the multiplicity of the corresponding component in $Z$. Since the intersection of $\bar{\Gamma}$ with the divisor of $\bar{Z}$ defined by $IO_{\bar{Z}}$ is $\sum_{\alpha \in A} e_\alpha d_\alpha$, we have

$$D.H(f_1)\ldots.H(f_{d-1}) = \sum_{\alpha \in A} e_\alpha d_\alpha.$$ 

4 An example

Let us consider the simple case when $O_{Z,z}$ is a regular local ring of Krull dimension 2. The multiplicity $e(I)$ of a $M_{Z,z}$-primary ideal $I$ is the multiplicity of an ideal generated $(f, g)$ by two general elements of $I$. Since $O_{Z,z}$ is regular, it is Cohen-Macaulay, so:

$$e(I) = \dim_C O_{Z,z}(f, g).$$

Therefore, the multiplicity of $I$ is the intersection number of $f = 0$ and $g = 0$ at $z$.

The blowing-up $\pi_J$ of the ideal $J := (f, g)$ gives the surface $Z_J$ defined by $\beta f - \alpha g = 0$ in $Z \times \mathbb{P}^1$. The projection onto $Z$ restricted to $Z_J$ is the blowing-up $\pi_J$ and the projection onto $\mathbb{P}^1$ restricted to $Z_J$ extends to $Z_J$ the map $\lambda$ from $Z \setminus \{z\}$ into $\mathbb{P}^1$ defined by $\lambda(z') = (f(z') : g(z'))$, for $z' \in Z \setminus \{z\}$.

In [17] M. Spivakovsky shows that the singularities of the normalization $\tilde{Z}$ of $Z_J$ are rational. He calls these singularities Sandwich singularities (see also [8]).

Let $\varphi : Z \to \tilde{Z}$ be the minimal resolution of $\tilde{Z}$. The map $\pi_J \circ n \circ \varphi$, where $n$ is the normalization of $Z_J$, is a bimeromorphic map from a non-singular surface $\bar{Z}$ onto $Z$:

$$\bar{Z} \xrightarrow{\varphi} \tilde{Z} \xrightarrow{n} Z_J \xrightarrow{\pi_J} (Z, z).$$

Therefore, it is the composition of a sequence of point blowing-up. In fact, since the strict transforms $H(f)$ and $H(g)$ of $f = 0$ and $g = 0$ by $\pi_J \circ n \circ \varphi$ are non-singular, distinct and transverse to $|((\pi_J \circ n)^{-1}(z))|$, the map $\pi_J \circ n \circ \varphi$ is an embedded resolution of the plane curve $fg = 0$.

Conversely let $\sigma : Z' \to (Z, z)$ be the minimal embedded resolution of the germ of curve $\{fg = 0\}$ in $(Z, z)$. Let $D_\alpha, \alpha \in A$, be the components of the exceptional divisor $\mathcal{E}$ of $\sigma$ which intersect the strict transforms of $\{fg = 0\}$. Consider the connected components of the closure of $\mathcal{E} \setminus \cup_{\alpha \in A} D_\alpha$ and the singular surface $\tilde{Z}'$ obtained from $Z'$ by contracting these components:

$$\varphi' : Z' \to \tilde{Z}'.$$

Since $\sigma$ is the minimal embedded resolution of the germ of curve $\{fg = 0\}$ in $(Z, z)$, the only components of $\mathcal{E}$ which might be of self-intersection $-1$ are among the components $D_\alpha, \alpha \in A$. Therefore, the contraction $\varphi'$ is the minimal resolution of $\tilde{Z}'$. 

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The contraction of the components $D_\alpha$, $\alpha \in A$, defines a holomorphic map:

$$\pi' : \tilde{Z}' \rightarrow (Z, z)$$

We have:

4.1 Lemma. The ideal sheaf $(f, g)\mathcal{O}_{\tilde{Z}'} = \pi'\ast (f, g)\mathcal{O}_Z$ is invertible.

Proof: Let $\tilde{H}(f)$ and $\tilde{H}(g)$ be the strict transforms of $f = 0$ and $g = 0$ by $\pi'$, then the valuation along $\varphi'(D_\alpha)$ of any function $h = uf + vg \in \mathcal{O}_{Z, z}$ being more that the one of $f$ or $g$, at any non-singular point $y$ of $\cup_{\alpha \in A}\varphi'(D_\alpha)$ which is neither a singular point of $\tilde{Z}'$ nor a point of $\tilde{H}(f)$ (resp. a point of $\tilde{H}(g)$), $f \circ \pi'$ (resp. $g \circ \pi'$) is a generator of $(f, g)\mathcal{O}_{\tilde{Z}', y}$.

On the other hand $f \circ \pi'$ (resp. $g \circ \pi'$) does not vanish on

$$\tilde{Z}' \setminus (\tilde{H}(f) \cup_{\alpha \in A} \varphi'(D_\alpha)) \quad (\text{resp. } \tilde{Z}' \setminus (\tilde{H}(g) \cup_{\alpha \in A} \varphi'(D_\alpha))).$$

Therefore, for any function $h$ in $(f, g)$, the meromorphic function $(h/f) \circ \pi'$ is bounded on

$$\tilde{Z}' \setminus (\tilde{H}(f) \cup \Sigma \cup \Sigma_{\tilde{Z}'})$$

where $\Sigma$ is the finite set of singular points of $\cup_{\alpha \in A}\varphi'(D_\alpha)$ and $\Sigma_{\tilde{Z}'}$ is the finite set of singular points of $\tilde{Z}'$. Since $\tilde{Z}'$ is normal, this implies that $(h/f) \circ \pi'$ is holomorphic on $\tilde{Z}' \setminus \tilde{H}(f)$. Similarly $(h/g) \circ \pi'$ is holomorphic on $\tilde{Z}' \setminus \tilde{H}(g)$. It shows that the ideal sheaf $(f, g)\mathcal{O}_{\tilde{Z}'} = \pi'\ast (f, g)\mathcal{O}_Z$ is invertible.

Thus, the contraction $\pi'$ factorizes uniquely through the normalized blowing-up

$$\bar{\pi} := \pi \circ n : \tilde{Z} \rightarrow (Z, z)$$

of the ideal $(f, g)$:

$$\bar{\pi}' = \bar{\pi} \circ \tilde{\theta}$$

Since $\sigma : Z' \rightarrow (Z, z)$ is the minimal embedded resolution of $\{fg = 0\}$ in $Z$, the map $\bar{\pi} \circ \varphi$ factorizes uniquely through $\sigma$:

$$\bar{\pi} \circ \varphi = \sigma \circ \eta.$$
The map $\varphi' \circ \eta$ is constant on the exceptional fibers of $\varphi$ and the space $\tilde{Z}$ is normal, so it gives a unique holomorphic map

$$\pi': \tilde{Z} \to \tilde{Z}'$$

such that $\pi \circ \varphi = \varphi' \circ \eta$.

Necessarily, because of the uniqueness of the factorizations, $\bar{\eta}$ is the inverse of $\bar{\pi}$.

So, we have proved:

**4.2 Lemma.** The map $\pi': \tilde{Z}' \to (Z, z)$ obtained by contraction is the normalized blowing-up of $I$.

The preceding results yields:

**4.3 Theorem.** Let $(\mathcal{O}_{Z,z}, \mathfrak{M}_{Z,z})$ be the analytic local ring of the germ of a non-singular complex surface $(Z, z)$. Let $I$ be a $\mathfrak{M}_{Z,z}$-primary ideal of $\mathcal{O}_{Z,z}$. Consider $f$ and $g$, such that, the Milnor number of $f$ and $g$ at $z$ is minimum among the Milnor numbers at $z$ of the elements of $I$ and assume that the ideal $I$ is integral over the ideal $(f,g)$ generated by $f$ and $g$. The normalized blowing-up of $I$ in $(Z, z)$ is obtained from the minimal embedded resolution of the curve $fg = 0$ by contracting the exceptional components which do not intersect the strict transform of $fg = 0$.

Recall that the Milnor number of $f$ at $z$ (introduced in [12]) is a topological invariant of the germ of $f = 0$ at $z$ (see [6] p. 261). It is the number of vanishing cycles of $f = 0$ at $z$ and equals the first Betti number of $\{f = t\}B_\varepsilon(z)$, where $B_\varepsilon(z)$ is a sufficiently small ball centered at $z$ and $\varepsilon \gg |t| > 0$.

Elements $f$ with the minimum Milnor number in $I$ have the same topology by using the results of [9]. Moreover, since $f$ and $g$ belongs the linear system $\lambda f + \nu g$ and have the minimum Milnor number in this linear system, because $\lambda f + \nu g \in I$, one can show that they have the same embedded resolution (see e.g. [10] §2).

Theorem 4.3 indicates that superficial elements $(f, g)$ of $I$ have to be elements of $I$ with minimum Milnor number at $z$ and such that $I$ and the ideal $(f, g)$ have the same multiplicity.

In this context Theorem 3.1 tells us that the multiplicity of $I$ equals the intersection number of the strict transform $\tilde{H}(f)$ of $f$ by the normalized blowing-up of $I$ and the exceptional divisor of this normalized blowing-up.
The preceding discussion also gives the following result:

Let \( f \) be an element of \( I \) having the smallest Milnor number at \( z \). Let \( \tau : Z \to (Z, z) \) be the minimal embedded resolution of \( f = 0 \).

4.4 Corollary. The ideal sheaf is \( \tau^* I \) is invertible except possibly at the points where the strict transform of \( f = 0 \) intersects the exceptional divisor of \( \tau \). It becomes invertible after a sequence of blowing-ups which separates non-singular branches at these points.

Proof: As we indicate above, elements of \( I \) with the minimum Milnor number have the same embedded resolution. So, the minimal embedded resolution of \( f \) is also the minimal embedded resolution of \( g \). However the minimal embedded resolution of \( f \) might not be the minimal embedded resolution of \( fg = 0 \), if the strict transforms of \( f = 0 \) and \( g = 0 \) in the minimal embedded resolution of \( f = 0 \) have common points on the exceptional divisor, in which case one has to separate the strict transforms of \( f = 0 \) and \( g = 0 \), by a sequence of point blowing-ups to separate tangent non-singular branches.

We have already seen above that, in the embedded resolution of \( fg = 0 \), the pull-back of \( I \) is invertible. In fact, one can check that the points of the embedded minimal resolution of \( f = 0 \), where the strict transforms of \( f = 0 \) and \( g = 0 \) have common points, are precisely the points where the pull-back of \( I \) is not invertible.

In summary, the ideal \( \tau^*(f, g) \) is invertible on the minimal embedded resolution of \( f = 0 \) or on the modification of this minimal embedded resolution obtained by a sequence of point blowing-ups to separate the branches of \( f = 0 \) and \( g = 0 \) passing through common points on the exceptional divisor. Since this embedded resolution of \( fg = 0 \) is non-singular, it is normal. So, it dominates the normalized blowing-up of \( (f, g) \) which is also the normalized blowing-up of \( I \) by corollary 1.3. This implies that on a resolution of \( fg = 0 \), the inverse image of \( I \) is invertible as stated.

M.S. Narasimhan showed me the following result of D. Mumford (see [13] Lemma p. 91-92) which can be obtained by using this viewpoint.

Consider \((Z, z) = (C^2, O)\) and the ideal \( I \) generated by the monomials \( x^{r_0}y^{s_0}, \ldots, x^{r_n}y^{s_n} \). Let \( \alpha = p/q \) (\( p \) and \( q \) being relatively prime). Denote by \( \nu_\alpha \) the discrete valuation of rank 1 on \( O_{C^2, O} \) centered at \( O \) such that

\[
\nu_\alpha(\sum_{i,j} a_{i,j} x^i y^j) = \min_{a_{i,j} \neq 0} (ip + jq).
\]

4.5 Proposition. The exceptional divisors of the normalized blowing-up of \( I \) are those prime divisors of the field of fractions of \( O_{C^2, O} \) corresponding to valuations \( \nu_\alpha \) where the least integer in the sequence of integers \( r_i p + s_i q \) (\( 0 \leq i \leq n \)) occurs at least twice.

Proof: First, notice that the ideal \( I \) might not be primary for the maximal ideal \( M_{C^2, O} \) of \( O_{C^2, O} \). However, there are unique integers \( a \) and \( b \) and a unique \( M_{C^2, O} \)-primary ideal \( I \), such that:

\[
I = (x^a)(y^b)I'.
\]
Now, it is clear that, since the ideal \((x^a)\) and \((y^b)\) are invertible, the normalized blowing-up of \(I'\) is also the normalized blowing-up of \(I\).

Let \(x^{r_0}y^{s_0}, \ldots, x^{r_n}y^{s_n}\) be the generators of \(I'\), so, for \(0 \leq i \leq n\):

\[
r_i' = r_i - a \quad \text{and} \quad s_i' = s_i - b.
\]

As we have seen in our example above, the components of the normalized blowing-up of the \(\mathfrak{m}_{\mathbb{C}^2,O}\)-primary ideal \(I'\) come from components of the minimal embedded resolution:

\[
\pi : Z \to (\mathbb{C}^2, O)
\]

of \(FG\), where \(F\) and \(G\) are two linear combinations:

\[
\sum_{i=0}^{n} \lambda_i x^{r_i'}y^{s_i'}, \quad \text{with } j = 0, 1
\]

having the minimal Milnor number at \(O\) and such that \(I'\) and \((F,G)\) have the same multiplicity.

Precisely, consider the exceptional components \(D_{\alpha}, \alpha \in A\), of this embedded resolution which intersect the strict transforms of \(FG = 0\). Now contract the exceptional components of \(\pi\) which are not among the \(D_{\alpha}\)'s. We obtain:

\[
\gamma : Z \to Z_1
\]

and \(\pi\) defines a unique morphism \(\pi_1 : Z_1 \to (\mathbb{C}^2, O)\), such that \(\pi = \pi_1 \circ \gamma\). We saw above that \(\pi_1\) is the normalized blowing-up of \(I\).

Corollary 4.4 suggests to consider first the minimal embedded resolution

\[
Z' \to (\mathbb{C}^2, O)
\]

of \(F = 0\).

Consider the set \(B\) of exponents \((r_i', s_i')\), for \(0 \leq i \leq n\), in the real plane. The convex hull of \(B\) is called the Newton Polyhedron of the set of exponents \(B\). The Newton Polygon \(\mathcal{N}(B)\) of \(B\) is the set of faces viewed from the origin \((0,0)\). Call \(A_1, \ldots, A_\ell\) the sides of the Newton Polygon \(\mathcal{N}(B)\) with respective slopes

\[
-\frac{p_1}{q_1} \leq -\frac{p_2}{q_2} \leq \ldots \leq -\frac{p_\ell}{q_\ell}.
\]

Notice that the slopes of the edges of \(\mathcal{N}(B)\) are given by the linear forms \(p\alpha + q\beta\) which attain their minimum at two exponents of \(B\) at least.

Denote \(F_{A_j} := \sum_{(r_i', s_i') \in A_j} \lambda_i x^{r_i'}y^{s_i'}\) where the \(\lambda_i\) are general complex numbers.

4.6 Lemma. In the linear family of polynomials

\[
F = \sum_{i=0}^{n} \lambda_i x^{r_i'}y^{s_i'}
\]
where $\lambda_i \neq 0$, for $0 \leq i \leq n$, are general complex numbers, the plane curve singularity $F = 0$ at $O$ is isolated and has the topological type at $O$ of

$$F_1 = F_{A_1} \ldots F_{A_\ell}.$$  

Proof: Consider a polynomial $F_0(x, y) = \sum_{i=0}^{n} \beta_i x^{r_i'} y^{s_i'} = G_{A_1} + \ldots + G_{A_\ell}$ of this linear family, where:

$$G_{A_j} = \sum_{(r_i',s_i') \in A_j} \beta_i x^{r_i'} y^{s_i'}.$$

The proof of Puiseux Theorem (see [18] Chap. IV §3) shows that the series:

$$y - cx^{\gamma_i} - \ldots$$

where $\gamma_i = q_i/p_i$ and $c$ is a root of $P_i(t) := \sum_{(r_j',s_j') \in A_i} \beta_j t^{s_j'}$ divide $F_0$ in the ring $\mathbb{C}[y][[x^{1/n}]]$.

For a general choice of the coefficients $\beta_j$, the polynomial $P_i$ has $b_i$ distinct solutions, where:

$$b_i := \sup_{(r_j',s_j') \in A_i} s_j' - \inf_{(r_j',s_j') \in A_i} s_j'.$$

Therefore, for a general choice of the coefficients $\beta_j$, the Puiseux series are all distinct which implies that the product:

$$\prod_{i=1}^{\ell} \prod_{P_i(c)=0} (y - cx^{\gamma_i} - \ldots)$$

divides $F_0$:

$$F_0 = u \prod_{i=1}^{\ell} \prod_{P_i(c)=0} (y - cx^{\gamma_i} - \ldots),$$

where $u$ is a unit in $\mathbb{C}[[x,y]]$, because $\sum_{i=1}^{\ell} b_i = \sup_{(r_j',s_j') \in N(B)} s_j'$. This shows that $F_0 = 0$ has an isolated singularity at $O$ and:

$$G_{A_i} = u_i \prod_{P_i(c)=0} (y - cx^{\gamma_i} - \ldots),$$

where $u_i$ is a unit in $\mathbb{C}[[x,y]]$.

In particular, for a general choice of the $\beta_i$, each plane curve $G_{A_i} = 0$ has an isolated singularity at $O$. Moreover, since $G_{A_i}$ is a weighted homogeneous polynomial, each branch of $G_{A_i} = 0$ is also defined by a weighted homogeneous polynomial with the same weights. This implies that in each Puiseux series above has the simple form $y - cx^{\gamma_i}$. The Milnor number of these branches is $(p_i - 1)(q_i - 1)$ and their pairwise intersection numbers are $p_i q_i$.

One can also prove:

**4.7 Lemma.** The minimum Milnor of a linear combination of elements of $I'$ equals the Kushnirenko number $2S - a - b + 1$, where $S$ is the area below the Newton polygon $N(B)$, $a$ is $\sup_{\alpha_i \neq 0} r_i'$ and $b$ is $\sup_{\alpha_i \neq 0} s_i'$. 

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Proof: The number of branches of \( G_{A_i} = 0 \) at \( O \) equals \( b_i/q_i \) and the Milnor number each branch of \( G_{A_i} = 0 \) at \( O \) is \((p_i - 1)(q_i - 1)\). The pairwise intersection numbers of these branches are equal to \( p_iq_i \).

Defining:

\[
a_i := \sup_{(r'_j, s'_j) \in A_i} r'_j - \inf_{(r'_j, s'_j) \in A_i} r'_j,
\]

the number of branches of \( G_{A_i} = 0 \) at \( O \) also equals \( a_i/p_i \).

The pairwise intersection numbers of branches of \( G_{A_i} = 0 \) and \( G_{A_j} = 0 \), for \( i < j \), are equal to \( p_iq_j \).

So, the Milnor number \( \mu(F_0, O) \) at \( O \) for a general choice of the coefficients \( \beta_i \) equals (see [12] Theorem 10.5 and Remark 10.10):

\[
\sum_{i=1}^{i=\ell} \frac{b_i}{q_i} (p_i - 1)(q_i - 1) + \sum_{i=1}^{i=\ell} 2p_iq_i \frac{b_i}{2q_i} (\frac{b_i}{q_i} - 1) + 2 \sum_{i=1}^{i=\ell} \sum_{i<j} p_iq_j \frac{b_i}{q_i} \frac{b_j}{q_j} - \sum_{i=1}^{i=\ell} \frac{b_i}{q_i} + 1.
\]

On the other hand:

\[
2S = \sum_{i=1}^{i=\ell} \frac{b_i^2}{q_i} p_iq_i + 2 \sum_{i=1}^{i=\ell-1} a_i b_{i+1}
\]

\[
a = \sum_{i=1}^{i=\ell} a_i \quad \text{and} \quad b = \sum_{i=1}^{i=\ell} b_i.
\]

Using the equality:

\[
\frac{a_i}{p_i} = \frac{b_i}{q_i}
\]

we obtain:

\[
\mu(F_0, O) = 2S - a - b + 1.
\]

To finish the proof of Lemma 4.6, it is enough to notice that \( F \) and \( F_1 \) belong to the same linear system and the minimum Milnor number is the minimum Milnor number among the analytic functions having the support of their Newton principal part on \( \mathcal{N}(B) \), i.e. the Kushnirenko number (see [5] 1.10).

Now, the minimal embedded resolution of \( F_0 = 0 \) for a general choice of the coefficients \( \beta_i \), is also an embedded resolution for \( G_{A_i} = 0 \). We have noticed that the branches of \( G_{A_i} = 0 \) are weighted homogeneous curves \( \lambda x^{q_i} + \nu y^{p_i} = 0 \). This implies that the multiplicities of the coordinates \( x \) and \( y \) along the component intersected by the strict transforms of the branches of \( G_{A_i} = 0 \) are respectively equal to \( p_i \) and \( q_i \). Thus, this component defines a divisorial valuation of the field of fractions of \( \mathcal{O}_{\mathbb{C}^2, O} \) given by \( v_i(x) = p_i \) and \( v_i(y) = q_i \). Therefore:

\[
v_i\left( \sum c_{\alpha,\beta} x^\alpha y^\beta \right) = \inf_{c_{\alpha,\beta} \neq 0} (p_i\alpha, q_i\beta).
\]

Each slope \(-p_i/q_i\) of the Newton Polygon of \( B \) defines such a valuation. By definition of the Newton Polygon, these valuations are defined by pairs of integers \((p, q)\), for which the minimum
of the linear form $p\alpha + q\beta$ is obtained for at least two distinct pairs among $\{(r'_i, s'_i)\}$. These valuations correspond to the ones given by Proposition 4.5. To prove that these are the divisorial valuations of the exceptional components of the normalized blowing-up of $I'$, it remains to prove that the minimal embedded resolution of $F_0 = 0$ already gives after contraction the normalized blowing-up of $I'$.

As remarked before, we have to prove that the strict transform of a curve singularity $G = 0$ defined by a general element $G$ of $I'$, such that $I'$ and the ideal $(F_0, G)$ have the same multiplicity, is disjoint from the strict transform of $F_0 = 0$ in the minimal embedded resolution of $F_0 = 0$.

To obtain this last assertion, notice that, in the minimal embedded resolution of $F_0 = 0$, the strict transforms of the branches $\lambda x^{q_i} + \nu y^{p_i} = 0$ given by the edges of the Newton polygon for distinct $(\lambda : \nu)$ are disjoint. This implies that the strict transform of $G_0 = 0$, given by another general choice of the coefficients $\beta_i$, is disjoint from the strict transform of $F_0 = 0$.

As seen in §3 above, a general element $G$ of $I'$ to be considered can be chosen as $G = G_0 + H$, where $H$ is a general linear combination of monomial of $B$ which are not on $\mathcal{N}(B)$. The Puiseux series associated to $G$ are of the type $y - cx^\gamma + \ldots$. This shows that the strict transforms of the branches of $G = 0$ intersect the strict transforms of the branches of $G_0 = 0$ in the minimal embedded resolution of $F_0 = 0$. This yields that the strict transforms of the branches of $G = 0$ are disjoint from the strict transforms of the branches of $F_0 = 0$ in the minimal embedded resolution of $F_0 = 0$.

So, the normalized blowing-up of $I'$ is already obtained from the minimal embedded resolution of $F_0 = 0$.

Therefore, the components of the minimal embedded resolution of $F_0 = 0$ intersected by the strict transforms of the branches of $F_0 = 0$ give the components of the normalized blowing-up of $I'$.

As we proved above, the divisorial valuations of the exceptional components of the normalized blowing-up of $I'$ are effectively the valuations given in the Proposition 4.5.

This ends our proof.

References


