NEARING 11d EXTREMAL INTERSECTING GIANTS 
AND NEW DECOUPLED SECTORS IN $D = 3, 6$ SCFT’S

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Abstract

We extend the analysis of arXiv:0801.4457 [hep-th] to charged black hole solutions of four-dimensional $U(1)^4$ gauged supergravity which carry three charges. There are two decoupling near-horizon limits, one over the near-BPS black hole solution and the other over the near-extremal, but non-BPS geometry. Taking the limit over the eleven dimensional uplift of these black hole solutions, for both of these cases we obtain a geometry which has a piece (conformal) to rotating $\text{BTZ} \times S^2$. We study the $4d$, $11d$ and $5d$, $3d$ black hole properties. Moreover, we show that the $\text{BTZ} \times S^2$ geometry obtained after the near-BPS (near-extremal) limit is also a solution to five-dimensional $U(1)^3$ un-gauged (gauged) STU supergravity. Based on these decoupling limits we argue that there should be sectors of $3d$ CFT resulting from low energy limit of theory on $N$ M2-branes ($N \to \infty$), which are decoupled from the rest of the theory and are effectively described by a $2d$ CFT. The central charge of the $2d$ CFT in both near-BPS and near-extremal case scales as $N$. The engineering dimension of the operators in these decoupled sectors scales as $N^{4/3}$ (for near-BPS case) while as $N^{3/2}$ (for the near-extremal case). Moreover, we discuss the description of the decoupled sectors as certain deformations of $6d$ CFT residing on the intersecting M5-brane giants.
1 Introduction and Summary

Understanding the statistical mechanical origin of black hole thermodynamics has been an ever challenging question posed to string theory. In this regard BPS or extremal black holes in AdS background in various dimensions have been of particular interest, as via AdS/CFT [1] we have the possibility of a description for black hole microstates through the dual CFT operators. Here we focus on a specific class of black hole solutions to $\mathcal{N} = 2, d = 4 \ U(1)^4$ gauged supergravity [2, 3]. These solutions can be uplifted to 11d as (black brane) deformations to $AdS_4 \times S^7$ geometry by the addition of various intersecting spherical M5-brane giant gravitons [4, 5] and their excitations. These M5-branes wrap various five-spheres inside the $S^7$ [4].

Since the 11d description of these 4d charged black holes is as deformations of $AdS_4 \times S^7$, these black holes should also have a description in the 3d dual CFT. Although our knowledge of this CFT is very limited, we may still study certain sectors of this theory. For example the BMN sector of this 3d CFT is described by the plane-wave matrix model (also known as BMN matrix model) [6, 7], which in turn is the DLCQ of M-theory on $AdS_4 \times S^7$ [8, 9]. In this work, among other things, we make the first steps in gaining a better understanding of certain sectors of the dual 3d CFT by relating it to better manageable theories like 2d CFT’s.

Here we focus on the three-charge black hole solutions to $U(1)^4$ 4d gauged supergravity and argue that the near-extremal black hole solutions admit near-horizon decoupling limits to a geometry which has $X_{M,J} \times S^2$, $X_{M,J}$ being global $AdS_3$, or $AdS_3$ with conical singularity or a rotating BTZ black hole. Throughout the paper we will discuss the limit from the 4d, 11d and 5d gravity theories as well as the 3d and 2d CFT’s (noting the appearance of $AdS_3$ factor in the decoupled geometry). In this way we support our statement about existence of decoupled sectors. This is an M-theory version of the “two-charge” black holes discussed in [10], see also [11]. Similar to the “two-charge” black hole of [10], there are two decoupling limits. The first is a near-horizon limit on the near-extremal black hole while also going to the near-BPS limit; we will refer to this near-horizon near-extremal limit as simply the “near-BPS” limit. The second is a near-horizon limit on the near-extremal black hole and continuing to be far-from-BPS; we will refer to this near-horizon near-extremal limit as the “far-from-BPS” limit. Throughout this work, we discuss the near-BPS and far-from-BPS cases in parallel.

This paper is organized as follows. In section 2, we first review generic charged black hole solutions to 4d $U(1)^4$ gauged SUGRA, and discuss their uplift to 11d supergravity where they appear as black five-brane deformations of $AdS_4 \times S^7$. In section 3, we focus on the three-charge black holes when they are close to saturating the extremality bound. We perform both the near-horizon limits i.e. the near-BPS and the far-from-BPS ones. In both cases we obtain a 11d geometry containing an $AdS_3 \times S^2$ (or more generally a static BTZx$S^2$) factor. Then we turn on the fourth charge in a perturbative manner, that is we choose the fourth charge to be much smaller than the other three. The effect of the perturbative addition of the fourth charge in the
near-horizon decoupling limits is the addition of angular momentum to the static BTZ black hole, to obtain a rotating BTZ black hole.

In section 4, we show that the BTZ × S^2 geometries obtained in the near-horizon limits are indeed solutions to certain 5d supergravities; that of the near-BPS case is a solution to ungauged 5d STU model [12] while that of the far-from-BPS geometry is a solution to the 5d gauged U(1)^5 supergravity [13].

In section 5, we analyze the Bekenstein-Hawking entropy of the 4d black holes, the near-horizon limits of which was discussed in section 3. We show that for both the near-BPS and far-from-BPS cases the 4d entropy and the entropy of the rotating BTZ black hole obtained after the limit are exactly matching.

In section 6, we present dual CFT pictures of the gravity analysis of the previous sections. The near-horizon limit in the language of 3d CFT (dual to M-theory on AdS_4 with N units of four-form flux) translates into N → ∞ limit, while focusing on specific BMN-type sectors of the 3d CFT. In the near-BPS case this sector is identified with operators which carry three R-charges J_i of the dual CFT with Δ ∼ J_i ∼ N^{4/3} → ∞ (Δ is the engineering dimension of the operators), while Δ − ∑_i J_i ∼ N. In the far-from-BPS limit, however, we are dealing with operators with Δ ∼ J_i ∼ N^{3/2}, while a certain combination of Δ, J_i (see (6.20)) scales as N. Due to the presence of AdS_3 factors in the decoupled geometry, one then expects the BMN-type operators to have a description in terms of 2d CFT’s. We briefly discuss the corresponding 2d CFT’s, and read the central charge of the theory, which in both the near-BPS and far-from-BPS cases scales as N. In addition, we also give a mapping between the 3d CFT charges, Δ, J_i and the L_0 and ¯L_0 of the 2d CFT’s.

The last section is devoted to discussions and concluding remarks. In appendix A, we have gathered computations showing that our decoupled geometries after the limit are indeed solutions to 11d supergravity. In appendix B, we show that running the entropy function machinery [14] for the BTZ × S^2 geometries correctly reproduces the black hole entropy.

2 Charged Black Holes in 4d U(1)^4 SUGRA

The black hole solutions that we are interested in and will be reviewed in this section are the static charged solutions to \( N = 2 \) U(1)^4 gauged supergravity in four dimensions which were first obtained in [15, 16] (see [3] for a review). These solutions can be uplifted to eleven dimensions [2]. They were analyzed in [4] where it was discussed that they correspond to condensates of intersecting smeared (delocalized) M-theory spherical M5-brane giant gravitons and referred to as superstars/black holes. As solutions to 11d supergravity they are specified by their metric and the three-form field:

\[
ds_{11}^2 = \Delta^\frac{2}{3} \left[ -\frac{f}{H} dt^2 + \frac{dr^2}{f} + r^2 d\Omega_2^2 \right] + \Delta^{-\frac{1}{3}} \left[ \sum_{i=1}^{4} L_i H_i \left( d\mu_i^2 + \mu_i^2 \left[ d\phi_i + a_i \frac{dt}{L_i} \right]^2 \right) \right], \tag{2.1}
\]
\[ C^{(3)} = -\frac{r^3}{2} \Delta dt \wedge d^2 \Omega_2 - \frac{L^2}{2} \sum_{i=1}^{4} \tilde{q}_i \mu_i^2 \left( d\phi_i - \frac{q_i dt}{\tilde{q}_i L} \right) \wedge d^2 \Omega_2. \]  

(2.2)

In the above

\[ H = H_1 H_2 H_3 H_4, \quad H_i = 1 + \frac{q_i}{r}, \quad f = 1 - \frac{\mu}{r} + \frac{4 r^2}{L^2} H, \]

\[ \Delta = H \left[ \frac{\mu_1^2}{H_1} + \frac{\mu_2^2}{H_2} + \frac{\mu_3^2}{H_3} + \frac{\mu_4^2}{H_4} \right], \quad a_i = \frac{\tilde{q}_i}{q_i} \left[ \frac{1}{H_i} - 1 \right], \]

(2.3)

\( \mu_1 = \cos \theta_1, \quad \mu_2 = \sin \theta_1 \cos \theta_2, \quad \mu_3 = \sin \theta_1 \sin \theta_2 \cos \theta_3, \quad \mu_4 = \sin \theta_1 \sin \theta_2 \sin \theta_3, \)

and \( d\Omega_2^2 \) and \( d^2 \Omega_2 \) are respectively the metric and the volume form on a unit radius two-sphere.

These geometries asymptote to, i.e. as \( r \to \infty \), \( AdS_4 \times S^7 \) with radii \( L/2 \) and \( L \) respectively, and \( \theta_1, \theta_2, \theta_3, \phi_1, \phi_2, \phi_3, \phi_4 \) parameterize the angles of the seven-sphere. They constitute a five parameter family of solutions, specified with \( \mu, q_i, i = 1, 2, 3, 4 \) and

\[ \tilde{q}_i = \sqrt{q_i (q_i + \mu)}. \]

(2.4)

From the metric and the three-from expressions it is evident that in our conventions the parameters, \( \mu, q_i, \tilde{q}_i \) and \( L \), are all of dimension of length.

Upon a specific reduction of the 11d supergravity on \( S^7 \) [2], the above geometries are mapped onto the electrically charged black hole solutions of 4d \( \mathcal{N} = 2 \ U(1)^4 \) gauged supergravity [15] which are also solutions of \( \mathcal{N} = 8 \ SO(8) \) gauged supergravity [16]:

\[ ds_4^2 = -H^{-1/2} f \ dt^2 + H^{1/2} \left( \frac{dr^2}{f} + r^2 d\Omega_2^2 \right), \]

(2.5a)

\[ A_i = \frac{\tilde{q}_i}{q_i} \left( \frac{1}{H_i} - 1 \right) dt, \quad X_i = \frac{H_i^{1/4}}{H_i}, \]

(2.5b)

where \( f, H_i \) and \( H \) are given in (2.3) and \( A_i \) and \( X_i \) are parameterizing the four gauge fields and three scalars of the four-dimensional theory (note that \( X_1 X_2 X_3 X_4 = 1 \)). The physical observable charges associated with these 4d black holes are the ADM mass [17]

\[ M = \frac{1}{2 G_N^{(4)}} (2 \mu + q_1 + q_2 + q_3 + q_4). \]

(2.6)

and electric charges

\[ J_i = \frac{L}{2 G_N^{(4)} \tilde{q}_i}. \]

(2.7)

In the above expressions \( G_N^{(4)} \) is the four-dimensional Newton constant which is related to the 11d Newton constant as

\[ G_N^{(4)} = \frac{G_N^{(11)}}{L^7} = \frac{3}{8 \sqrt{2}} \frac{L^2}{N^{3/2}}, \quad G_N^{(11)} = 16 \pi^2 \ell_p^9, \quad L^6 = 32 \pi^2 \ell_p^6 N. \]

(2.8)

As argued from the 11d viewpoint these geometries correspond to (at most) four stacks of intersecting spherical five-branes; each stack carries angular momentum \( J_i \) [2] and consists of \( N_i \) number of M5-branes [4]

\[ N_i = \frac{3 J_i}{4 \tilde{N}} = \sqrt{2} N^{1/2} \tilde{q}_i \]

(2.9)
The above number has been computed noting that each M5-brane carries one unit of the flux of the four-form field strength \( F_4 = dC_3 \), explicitly

\[
N_i = \frac{1}{16\pi G_N^{(11)} T_{M_5}} \int F_4, \quad T_{M_5} = \frac{1}{(2\pi)^5 l_p^6},
\]  

(2.10)

and noting the expression for the three-form (2.2).

For generic \( \mu > 0 \) the above solutions are non-BPS breaking all the supersymmetries, for \( \mu = 0 \), however, the solution is BPS. For a solution with \( n \) number of non-zero charges \( q_i \), the BPS solutions, as solutions to 11d supergravity, preserve \( 32/2^n \) number of supersymmetries. In the four-dimensional setting black holes with regular horizons can occur only when \( \mu \neq 0 \) and hence are all non-supersymmetric. The supersymmetric solution corresponds to a naked singularity and hence were termed superstars in [4]. Although in this paper we will mainly be interested in the three-charge and four-charge M-theory superstar/black holes, for completeness we present a brief account of each of one, two, three and four charge black holes separately.

• **One-charge black hole:** For \( \mu = 0 \) we have a space with naked, null singularity which preserves 16 supercharges. The eleven-dimensional LLM solutions [18] are de-singularized counter-parts of these solutions. As soon as we turn on \( \mu \) we obtain a space-like singularity at \( r = 0 \) which is hidden behind a regular horizon.

From the 11d viewpoint these solutions correspond to M5-branes wrapping \( S^5 \) inside the \( S^7 \), while rotating on a circle on \( S^7 \). The spherical M5-branes, the giant five-branes, are smeared on the two directions transverse to their worldvolume. The non-extremality parameter \( \mu \) corresponds to turning on (membrane) excitations on the five-brane. These excitations are such that they do not change the \( S^5 \) shape of the five-brane.

• **Two-charge black hole:** For \( \mu = 0 \) we again have a null, naked singularity and the background is \( 1/4 \) BPS. For any \( \mu > 0 \) we have a space-like singularity at \( r = 0 \) and a horizon sitting at the larger root of function \( f \).

In the 11d picture we have two stacks of spherical five-branes which are intersecting on an \( S^3 \) inside \( S^7 \), while each moving on a circle and are smeared on directions transverse to their worldvolume.

• **Three-charge M-theory superstar/black hole:** When only one of the four charges, say the \( q_1 \), is vanishing, the 4d black hole has a different causal and singularity structure than the one and two charge cases. For \( 0 \leq \mu < \mu_c \), with

\[
\mu_c = \frac{4}{L^4 q_2 q_3 q_4},
\]

(2.11)

we have a solution with a time-like singularity sitting at \( r = 0 \) and no regular horizon. This case resembles a standard Reissner-Nordstrom black hole which violates the extremality bound. For the \( \mu = 0 \) case the solution becomes BPS, preserving 4 supercharges. For
$\mu = \mu_c$ the solution is “extremal”, but non-supersymmetric (non-BPS), with a vanishing horizon size. Similar to the extremal Reissner-Nordstrom solution, the geometry has a null singularity. When $\mu > \mu_c$ we have a regular Reissner-Nordstrom-type black hole with a finite size horizon and a space-like singularity sitting behind the horizon.

In the eleven-dimensional viewpoint we have three stacks of smeared spherical M5-branes on $S^7$ while interesting on an $S^1$. As we will argue one may interpret the stack of intersecting giants at $\mu = \mu_c$ as a (non-marginal) bound state of M5-branes which are wrapping (holomorphic) 4-cycles $\Sigma_4$ on a $CP^3$, and hence with worldvolume $R \times S^1 \times \Sigma_4$. Turning on the non-extremality parameter $\mu > \mu_c$ is then like turning on the five-brane type excitations on the system of intersecting giants.

- **Four-charge M-theory superstar/black hole:** As far as the causal and singularity structures are concerned, this case is very similar to the three charge case, with two differences. First, $\mu_c$ has now a different complicated expression in terms of $q_i$’s than (2.11), but in any case $\mu_c \geq \frac{4}{L^2}(q_1 q_2 q_3 + q_1 q_2 q_4 + q_1 q_3 q_4 + q_2 q_3 q_4)$. Second, in this case at $\mu = \mu_c$, unlike the three-charge case, we have an extremal solution with non-vanishing horizon size. Note that in this case one can extend the geometry past $r = 0$ to $r \geq -q_1$, where $q_1$ is the smallest of the four charges. For $0 \leq \mu < \mu_c$ the time-like naked singularity is sitting behind $r = 0$, at $r = -q_1$.

In terms of M5-branes, the 11d geometry corresponds to four stacks of smeared M5-brane giants their worldvolume intersecting only on the time direction.

3 Near-Horizon Limits of Three-Charge Black Holes

Among the black hole solutions reviewed in the previous section, the three-charge case has certain unique features. In particular, for a given set of three non-zero charges, there are two possible near-extremal limits. The first is what we have already referred to as near-BPS case; the near-extremal black holes thus obtained have $\mu \sim 0$ with $\mu_c/L \ll 1$. The second is what we have already referred to as the far-from-BPS limit. The near-extremal black holes thus obtained have $\mu \sim \mu_c$ (2.11) with with $\mu_c \sim L$. Both these limits result in vanishing horizon area. The M-theory three-charge case that we consider here is analogous to the IIB two-charge case discussed in [10].

In taking the near-horizon limit of the three-charge extremal black hole in four dimensions, we face the same problems as we did for the five-dimensional two-charge extremal black hole [10], namely we do not obtain a product geometry with $AdS$ and sphere factors. However, working with the uplifted eleven dimensional solution, we do obtain a product geometry: $AdS_3 \times S^2$.

3.1 The near-horizon near-BPS limit

We require $\mu_c \rightarrow 0$ together with $r \rightarrow 0$. More precisely, we consider $\epsilon \rightarrow 0$ with the following scalings
\bullet \mu_1 \sim 1 \text{ case}

\begin{align}
q_i = \epsilon q_i \Rightarrow \mu_c = \epsilon^3 \hat{\mu}_c, \quad \hat{\mu}_c \equiv \frac{4q_2q_3q_4}{L^2}, \quad \mu - \mu_c = \epsilon^3 M^2, \\
r = \epsilon^3 \hat{\mu}_c \rho^2, \quad \theta_i = \epsilon^{1/2} x_i, \quad \psi_i = \phi_i - t/L, \quad (i = 2, 3, 4)
\end{align}

(3.1)

while keeping \(\rho, \rho_c, \hat{q}_i, M, x_i, \psi_i, L\) fixed. Note also that, as \(\mu_1^2 = 1 - \mu_2^2 - \mu_3^2 - \mu_4^2\), in this limit \(\mu_1 = 1 + \mathcal{O}(\epsilon^2)\). This limit corresponds to \(\theta_1 \sim \epsilon^{1/2}, \theta_2, \theta_3 = \text{fixed} \ cf. \ (2.3)\).

\begin{align}
q_i = \epsilon q_i \Rightarrow \mu_c = \epsilon^3 \hat{\mu}_c, \quad \hat{\mu}_c \equiv \frac{4q_2q_3q_4}{L^2}, \quad \mu - \mu_c = \epsilon^3 M^2, \\
r = \epsilon^3 \hat{\mu}_c \rho^2, \quad \theta_i = \theta_i - \epsilon \hat{\theta}_i \Rightarrow d\mu_i = \epsilon d\hat{\mu}_i, \quad \psi_i = \frac{1}{\epsilon}(\phi_i - t/L), \quad (i = 2, 3, 4),
\end{align}

(3.2)

where \(0 \leq \theta_i^0 \leq \pi/2\) are fixed values for \(\theta_i\) and \(\rho, \hat{\theta}_i, \psi_i, \hat{q}_i, \hat{\mu}_c\) are kept fixed.

Note that in both of the above limits the physical charges \(\hat{q}_i \sim q_i \sim \epsilon\) and the function \(f\) of (2.1) becomes

\[ f = 1 - \frac{\gamma^2}{\rho^2}. \]

(3.3)

where

\[ \gamma^2 \equiv \frac{\hat{\mu} - \mu_c}{\mu_c} = \frac{\mu - \mu_c}{\mu_c}. \]

(3.4)

Performing this limit on (2.1), we end up with the following BTZ \(\times S^2 \times T^6\) metric

\[ \frac{ds^2}{\epsilon^2} = (R_A^2 + R_S^2 d\Omega_2^2) + \frac{L^2}{R_S} d\Omega_6^2 \\
\]

(3.5)

where

\[ d\Omega_6^2 = -(\rho^2 - \gamma^2) d\tau^2 + \frac{d\rho^2}{\rho^2 - \gamma^2} + \rho^2 d\phi_1^2 \]

(3.6)

with \(\tau = t/L\) and

\[ R_A = 2R_S. \]

(3.7)

The radius of the two-sphere \(R_S\) and the six-dimensional part \(d\Omega_6^2\) have different expressions for the two cases:

\begin{itemize}
\item \(\mu_1 \sim 1 \text{ case}\)

\[ R_S^3 = \hat{q}_2\hat{q}_3\hat{q}_4, \quad d\Omega_6^2 = \sum_{i=2,3,4} \hat{q}_i(dx_i^2 + x_i^2 d\psi_i^2). \]

(3.8)

\item \(\mu_1 \sim \mu_1^0 \neq 1 \text{ case}\)

\[ R_S^3 = (\mu_1^0)^2 \hat{q}_2\hat{q}_3\hat{q}_4, \quad d\Omega_6^2 = \sum_{i=2,3,4} \hat{q}_i(\hat{\mu}_i^2 + (\mu_1^0)^2 d\psi_i^2). \]

(3.9)
\end{itemize}

Upon appropriate periodic identifications, the \(M_6\) part of the metric, in both of the above cases describes a \(T^6\). For \(\gamma^2 = -1\), the three dimensional part (3.6) describes a global \(AdS_3\) space. Note that this corresponds to \(\mu = 0\) i.e. the BPS point. For \(-1 < \gamma^2 < 0\), the space
becomes conical whereas for \( \gamma = 0 \) we have a massless BTZ black hole. Finally, for \( \gamma^2 > 0 \) the space is a massive BTZ black hole with mass \( \gamma R_A \). (For a concise review of our terminology see Appendix A of [10].)

Starting from the geometry caused by three stacks of flat M5-branes, intersecting on a \( R^4 \) and taking the near-horizon limit, results in a BTZ \( \times S^2 \times T^6 \) geometry \([19]\), which is essentially the geometry we obtained here. We will return to this point later in section 6.

### 3.2 The near-horizon far-from-BPS limit

When \( \mu_c \) has a finite value, we are far from the BPS point. A near-extremal black hole is described by \( \mu - \mu_c \ll \mu_c \sim L \). To take the near-horizon limit of such a solution we consider the following scalings

\[
\begin{align*}
& r = \frac{\epsilon^2 \mu_c}{f_0} \rho^2, \quad t = \frac{1}{\epsilon \sqrt{f_0}} \tau, \quad \phi_1 = \frac{\varphi}{\epsilon}, \\
& \mu - \mu_c = \epsilon^2 \mu_c M, \quad d\psi_i = d\phi_i - \frac{q_i}{\epsilon} \frac{\tau}{\sqrt{f_0}} (i = 2, 3, 4),
\end{align*}
\]

where

\[
f_0 = 1 + \frac{4(q_2 q_3 + q_2 q_4 + q_3 q_4)}{L^2}
\]

and \( q_i, \mu_c, M; \rho, \tau, \varphi, \psi_i \) are fixed in the \( \epsilon \to 0 \) limit. Note also that in this limit

\[
f = f_0 (1 - \frac{M}{\rho^2}).
\]

Performing the above limit on (2.1) will result in the following metric

\[
ds_2 = \mu_1^{4/3} \left( R_A^2 ds_{BTZ}^2 + R_S^2 d\Omega_2^2 \right) + \frac{1}{\mu_1^{2/3}} ds_{\mathcal{M}_6}^2
\]

where

\[
ds_{BTZ}^2 = -(\rho^2 - M) d\tau^2 + \frac{d\rho^2}{\rho^2 - M} + \rho^2 d\varphi^2,
\]

and

\[
ds_{\mathcal{M}_6}^2 = \frac{L^2}{R_S} \sum_{i=2,3,4} q_i (d\mu_i^2 + \mu_i^2 d\psi_i^2).
\]

with

\[
R_S^2 = q_2 q_3 q_4 = \frac{\mu_c L^2}{4}, \quad R_A^2 = \frac{4}{f_0} R_S^2.
\]

For \( M > 0 \), the metric (3.14) locally describes a stationary BTZ black hole with mass \( M \). For \( M < 0 \) we have a conical space with a deficit angle \( 2\pi(1 - \epsilon^2 M) \). Also note that the range of the angle \( \varphi \) in \( AdS_3 \) is \([0, 2\pi \epsilon]\). Nonetheless, the causal boundary of this locally \( AdS_3 \) space is still \( R \times S^1 \) [10].

The metric (3.13) is a solution to eleven-dimensional supergravity with the three-form

\[
C_3 = -\frac{L^2}{2} \sum_{i=2,3,4} \tilde{q}_i \mu_i^2 d\psi_i \wedge d^2 \Omega_2
\]

where in the near-horizon far-from-BPS limit

\[
\tilde{q}_i^2 = q_i (q_i + \mu_c), \quad i = 2, 3, 4.
\]
3.3 Perturbative addition of the fourth charge

The black hole solution we have discussed so far has been a three charge one. When the non-extremality parameter is small, the near-horizon limit of these black holes resulted in decoupled geometries having an $AdS_3 \times S^2$ factor as a subspace. In this section we extend our analysis of these black holes by turning on the fourth charge. We require, for both of the near-BPS and far-from-BPS cases, that this charge to be much smaller than the other three, $q_1 \ll q_2, q_3, q_4$, so that it can be considered as a perturbation to the previous case. One expects that the near-horizon limit of such four-charge near extremal black holes results in similar decoupled geometries as above but with an additional angular momentum in the $AdS_3$ factor to obtain a rotating BTZ black hole. In the following we will show that this is in fact the case for both the near and far-from-BPS cases.

3.3.1 The near-horizon, near-BPS case

To extend the near-horizon limit to include the fourth charge $q_1$, we supplement the limit (3.1) or (3.2) with

$$ q_1 = \epsilon^3 \hat{q}_1, \quad r = \epsilon^3 (\hat{\mu}_c \rho^2 - \hat{q}_1), $$

(3.19)

while keeping $\hat{q}_1$ fixed. As discussed at the end of section 2, when the fourth charge is also turned on one can extend the geometry past $r = 0$, to $r \geq -q_1$. The shift in the scaling of $r$ in (3.19) is a reflection of this fact. In the limit $\epsilon \to 0$

$$ f = 1 - \frac{\hat{\mu}_c \gamma}{\hat{\mu}_c \rho^2 - \hat{q}_1} + \frac{\hat{\mu}_c \hat{q}_1}{(\hat{\mu}_c \rho^2 - \hat{q}_1)^2}, \quad a_1 = -\frac{\mathcal{J}}{2\rho^2}, $$

(3.20)

where $\gamma$ is defined in (3.3) and

$$ \mathcal{J} = \frac{2\sqrt{q_1 (\hat{q}_1 + \hat{\mu})}}{\hat{\mu}_c}. $$

(3.21)

Performing the limit on (2.1) we find the metric in (3.5) with the same expressions for $R_A$ and $R_S$, but the $ds^2_{BTZ}$ is now replaced with

$$ ds^2_{AdS} = -\frac{F(\rho)}{\rho^2} dt^2 + \frac{\rho^2}{F(\rho)} d\rho^2 + \rho^2 (d\phi - \frac{\mathcal{J}}{2\rho^2} dt)^2. $$

(3.22)

where

$$ F(\rho) = \rho^4 - (\gamma + 2 \frac{\hat{q}_1}{\hat{\mu}_c}) \rho^2 + \frac{\mathcal{J}^2}{4} $$

(3.23)

The metric (3.22) describes a rotating BTZ with (see Appendix A of [10] for our conventions)

$$ M_{BTZ} = \gamma + 2 \frac{\hat{q}_1}{\hat{\mu}_c} = \frac{\hat{\mu} + 2 \hat{q}_1}{\hat{\mu}_c} - 1, \quad J_{BTZ} = \mathcal{J}. $$

(3.24)

---

1One can study near-horizon limit of a generic extremal four-charge black hole (when all four charges are of the same order). In the near-horizon limit a generic four-dimensional four-charge black hole leads to $AdS_3 \times S^2$ [20]. This near-horizon limit and application of entropy function to study the entropy of these four-charge extremal black holes has been carried out in [21].
3.3.2 The near-horizon, far-from-BPS case

Again the only ingredient we need to add to (3.10) is the scaling of the fourth charge,

\[ q_1 = \epsilon q_1, \quad (3.25) \]

while keeping the rest of the scalings unchanged. In the limit

\[ f = f_0(1 - \frac{M}{\rho^2} + \frac{J^2}{4\rho^2}), \quad a_1 = -\sqrt{f_0} \frac{J}{2\rho^2}, \quad (3.26) \]

where \( M \), as defined in (3.10), is \( M = \frac{\mu - \mu_c}{\epsilon^2 \mu_c} \) and

\[ J \equiv 2 \left( \frac{q_1 f_0}{\mu_c} \right)^{1/2}. \quad (3.27) \]

Performing the limit on (2.1), we end up with the metric (3.13) but \( ds_{BTZ}^2 \) replaced with

\[ ds_{AdS}^2 = -\frac{G(\rho)}{\rho^2} d\tau^2 + \frac{\rho^2}{G(\rho)} d\rho^2 + \rho^2 (d\varphi^2 - \frac{J}{2\rho^2} d\tau)^2, \quad (3.28) \]

where

\[ G(\rho) = \rho^4 - M\rho^2 + \frac{J^2}{4}. \quad (3.29) \]

The metric (3.28) describes a rotating BTZ with

\[ M_{BTZ} = M, \quad J_{BTZ} = J. \quad (3.30) \]

Similar to the non-rotating case of section 3.2, the metric obtained in this case is also a solution to eleven-dimensional supergravity with the three-form field given in (3.17). The explicit verification of this point will be given in the Appendix.

4 The 5d Description of BTZ×S^2

In this section we show that the BTZ×S^2 geometries of previous section can be realized as solutions to five-dimensional supergravities. The near-BPS case is a solution to ungauged STU model and the far-from-BPS case is a solution to gauged U(1)^3 supergravity. We also show that the former can be obtained as near-horizon limit of magnetically charged string solutions of the the STU model, first constructed in [22].

4.1 The near-BPS case

The action of 5d ungauged STU model [23, 24] is given by

\[ S_{\text{ungauged}} = \frac{1}{16\pi G_N^{(5)}} \int dx^5 \sqrt{-g^{(5)}} \left( R^{(5)} - \sum_{i=1,2,3} \left( \frac{1}{2} (X^i)^{-2} \partial_\mu X^i \partial^{\mu} X^i \right) + \frac{1}{4} (X^i)^{-2} F_{\mu\nu}^i F^{i\mu\nu} + \frac{1}{4} \epsilon^{\mu\nu\rho\lambda} F_{\mu\nu}^i F_{\rho\sigma}^i A_\lambda^3 \right), \quad (4.1) \]
where scalars $X^i$ obey the constraint

$$X^1 X^2 X^3 = 1.$$  \hspace{1cm} (4.2)

This action is a truncation of the general $\mathcal{N} = 2$, $d = 5$ supergravity [12] obtained from reduction of 11$d$ supergravity on Calabi-Yau threefold [25, 26]. Moreover, (4.1) can be obtained by compactification of heterotic string theory on $K_3 \times S^1$ [27].

We seek a string solution of this theory which has $\text{BTZ} \times S^2$ geometry. Our ansatz for the field configuration of this solution is

$$ds^2 = R_A^2 \left( - f(\rho) d\tau^2 + \frac{d\rho^2}{f(\rho)} + \rho^2 d\phi^2 \right) + R_S^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

$$X^i = u^i$$

$$F^i_{\theta \phi} = p^i \sin \theta$$ \hspace{1cm} (4.3a, 4.3b, 4.3c)

where the function $f(\rho)$ and constants $R_A$, $R_S$, $u_i$ are determined in terms of the magnetic charges $p_i$ using the equations of motion.

It is straightforward to check that the above ansatz is a solution to the STU model (4.1) if

$$f(\rho) = \rho^2 - M + \frac{J^2}{4\rho^2}$$ \hspace{1cm} (4.4)

and when

$$R_S = \frac{R_A}{2} = \frac{p^i}{u^i}, \quad i = 1, 2, 3.$$ \hspace{1cm} (4.5)

The constraint (4.2) implies $u^1 u^2 u^3 = 1$ and hence

$$R_S^3 = p^1 p^2 p^3.$$ \hspace{1cm} (4.6)

Note that $M$ and $J$ in (4.4) are independent of the $p^i$’s. That is, the $\text{BTZ} \times S^2$ geometry we obtained as a result of taking the near-horizon near-BPS limit is a solution to STU model once we rename magnetic charges as

$$p^i = \hat{q}_i \text{ for } \mu_1 \sim \mu_1^0, \quad p^i = \hat{q}_i \text{ for } \mu_1 \sim 1.$$ \hspace{1cm} (4.7)

In the conventions of [22, 28], the above $\text{BTZ} \times S^2$ solutions are characterized by the “magnetic central charge” $Z = \frac{1}{3} u^i p_i = R_S^3$.

As discussed in [29] it is possible to obtain the same $\text{BTZ} \times S^2$ solution as the near-horizon limit of “near-BPS” magnetically charged string solutions, which are charged under all three $U(1)$ fields of the STU model.

From the 11$d$ viewpoint these black strings can be obtained as geometries corresponding to three stacks of intersecting M5-branes wrapping holomorphic four-cycles of a CY threefold [30, 19] whose near-horizon limit coincides with the 11$d$ solutions we obtained after the near-horizon,
near-BPS limit (3.5). In our case too the $AdS_3 \times S^2$ is coming as near-horizon limit of specific intersecting M5-branes, but spherical (rather than flat) M5-branes in the $AdS_4 \times S^7$ background. Note, however, that the BTZ and the $AdS_3$ factor we obtain in our limit is already in global coordinates, while those obtained in [22, 30, 19] have $AdS_3$ in the Poincare patch. From the $AdS_3 \times S^2 \times C_6$ geometry (3.5) one can read off the 5d Newton constant in terms of the eleven-dimensional one,

$$G_N^{(5)} = \frac{G_N^{(11)}}{V_6} \cdot \epsilon^3 = \frac{16\pi^7 g_\text{gN}}{(2\pi)^3 L^6 \frac{\mu_0^4 \mu_1^4}{|\mu_1|^2}} \cdot \epsilon^{-3} = \frac{\pi}{64\sqrt{2}} L^3 (N\epsilon^2)^{-3/2} \frac{(\mu_1^0)^2}{\mu_0^2 \mu_2^2 \mu_4^2}. \quad (4.8)$$

Note that, in the above, after the reduction on the $C_6$, to remove the $\epsilon$ factor appearing in front of the metric, we have rescaled the resulting 5d metric by a factor of $\epsilon$.

As discussed in [22, 31, 28] the rotating BTZ$\times S^2$ solutions for $M = J \geq 0$, where we have $AdS_3 \times S^2$ geometry, are half BPS, meaning that they preserve 4 out of 8 supersymmetries of the $\mathcal{N} = 2$, 5d theory. The supersymmetry of the magnetically charged black strings is half of supersymmetry of the rotating BTZ$\times S^2$ obtained after the near-horizon limit.

A few comments about the near-horizon near-BPS limit that we have presented above are in order here. While we started from a 4d electrically charged black hole solution, after the limit we ended up with a 5d magnetically charged solution. In the literature [29], there exists a duality between electrically and magnetically charged solutions to 5d STU model. This duality relates electric $AdS_2 \times S^3$ solutions to the magnetic $AdS_3 \times S^2$ and was established in [29] by reducing the solutions to 4d, more precisely to solutions of 4d ungauged STU model, where the duality is an electric-magnetic S-duality.

This duality between electrically and magnetically charged solutions may be relevant for us. Although the electrically charged 4d black hole is a solution to the gauged SUGRA, in the near-horizon, near-BPS limit that we take, the $X_i$ (cf. (2.5)) are scaled in such a way that the potential term in the gauged SUGRA vanishes and hence it reduces to an ungauged theory action.

### 4.2 The far-from-BPS case

The action for the 5d gauged $U(1)^3$ supergravity is [13] (for a review e.g. see [3])

$$S_{\text{gauged}} = \frac{1}{16\pi G_N^{(5)}} \int dx^5 \sqrt{-g^{(5)}} \left( R^{(5)} - \sum_{i=1,2,3} \left( \frac{1}{2} (X^i)_{-2} \partial_\mu X^i \partial^\mu X^i - \frac{4}{L^2} (X^i)^{-1} \right.ight.$$

$$\left. + \frac{1}{4} (X^i)^{-2} F_{\mu \nu}^i F^{i \mu \nu} \right) + \frac{1}{4} \epsilon^{\mu \nu \rho \sigma \lambda} F_{\mu \nu}^1 F_{\rho \sigma}^2 A_3^\lambda,$$

where $X$'s are subject to the constraint (4.2). The above action is the same as (4.1) but with the extra potential term for $X^i$'s. This potential has appeared as a result of the gauging of $U(1)^3$ in the supergravity.

To check that the BTZ$\times S^2$ geometry we obtained from the near-horizon far-from-BPS limit of section 3.2 is a solution to the above theory, we plug the ansatz (4.3) into the action and solve...
for $R_A$ and $R_S$ in terms of the magnetic charges $p^i$’s. We then get

$$R_A^2 = \frac{4L^2(Q_1Q_2Q_3)^{\frac{2}{3}}}{L^2 + 4(Q_1Q_2 + Q_1Q_3 + Q_2Q_3)} \quad (4.10a)$$

$$R_S^2 = \frac{Q_1}{(Q_1Q_2Q_3)^{\frac{1}{3}}} \quad (4.10b)$$

$$u^i = \frac{Q_i}{(Q_1Q_2Q_3)^{\frac{1}{3}}} \quad (4.10c)$$

where $Q^i$’s are given in terms of $p^i$’s through the relations

$$p^i = \sqrt{Q_i \left( Q_i + \frac{4Q_1Q_2Q_3}{L^2} \right)} \quad (i = 1, 2, 3). \quad (4.11)$$

It is readily seen that the above is the same $AdS_3 \times S^2$ geometry of section 3.2 once we identify $Q^i$’s with the $q^i$’s there.

To read the 5d Newton constant in this case, we recall the decoupled metric (3.13) and do the reduction of the 11d SUGRA over $M_6$ (3.15):

$$G_N^{(5)} = \frac{G_N^{(11)}}{V_{M_6}} = \frac{16\pi^7\eta^9}{\eta L^6} = \frac{3\pi}{4\sqrt{2}}L^3N^{-3/2}, \quad (4.12)$$

where the $V_{M_6}$ is the volume of the six-dimensional manifold $M_6$ (3.15). Note that the range of $\mu^i$’s in $M_6$ is such that $\mu_2^2 + \mu_3^2 + \mu_4^2 = 1$ and all of them are non-negative.\(^2\)

It is straightforward to examine whether the above $AdS_3 \times S^2$ geometry is a supersymmetric solution to the 5d gauged SUGRA by checking the Killing spinor equations. It turns out that the above solution does not preserve any supersymmetry. This is consistent with the results of [28, 20, 32] and [33] where the classification of all BPS solutions of this 5d gauged SUGRA has been carried out. We would also like to comment that, although we expect it to exist, the magnetic string solution to the gauged SUGRA which leads to the above $AdS_3 \times S^2$ geometry in the near-horizon limit has not yet been constructed.

5  Entropy Analysis of the 4d and 3d Black Holes

In this section, we first compute the entropy of the four-dimensional near extremal black hole for both the near-BPS and far-from-BPS cases. We do this for a four-charge black hole, one of the charges being much smaller than the rest. We will then compute the entropy of the rotating BTZ black hole which is a subspace of the decoupled geometry after the near-horizon limit. We find that the results coincide i.e. the four and three dimensional black holes, in the limits of our interest, produce the same value for the entropy. This provides supportive evidence for the fact that our limits are indeed decoupling limits.

\(^2\)This is in contrast with the original eleven-dimensional metric in which $\mu_2^2 + \mu_3^2 + \mu_4^2 = 1 - \mu_1^2$.\(\)
5.1 4d black hole entropy

As mentioned earlier, the four-dimensional black holes discussed above are solutions to four-dimensional \( U(1)^4 \) gauged SUGRA with the metric given in (2.5). The position of the horizon \( r_h \) is determined by the zeroes of \( f/H_{1/2} \) and

\[
S_{B.H.}^{(4)} = \frac{A_h}{4G_N^{(4)}}, \quad A_h = 4\pi r_h^2(H_1H_2H_3H_4)^{1/2}|_{r=r_h}.
\]  

(5.1)

Using (2.8) that is

\[
S_{BH} = \frac{2\sqrt{2}}{3} N^{3/2} \cdot \frac{A_h}{L^2}.
\]  

(5.2)

As we see for generic horizon areas of order \( L^2 \) the entropy scales as \( N^{3/2} \). The three-charge extremal black holes have vanishing horizon size. However the near extremal solutions have a non-zero horizon size, where their horizon areas scale to zero with some power of \( \epsilon \). Moreover, as discussed in section 2, the \( r = 0 \) is the curvature singularity of the three-charge black holes. Therefore, it is necessary to make sure that in our limit we can still trust (classical) gravity description. In order this we should make sure that

\[
S_{BH} \gg 1,
\]  

(5.3)

and that all the curvature invariants of the “decoupled” geometries remain small (in the relevant Planck units). The condition (5.3) can only be fulfilled when together with taking \( \epsilon \to 0 \) we also take the large \( N \) limit in an appropriate rate.

5.1.1 The near-BPS limit

The horizon radius is the larger root of function \( f/H_{1/2} \), which turns out to be same as the larger root of \( F(\rho) \), (3.23)

\[
r_h + q_1 = \frac{\mu_c}{4} \left( \sqrt{\frac{\mu + 2q_1}{\mu_c}} - \mathcal{J} - 1 + \sqrt{\frac{\mu + 2q_1}{\mu_c}} + \mathcal{J} - 1 \right)^2 \epsilon^3,
\]  

(5.4)

and the horizon area is

\[
A_h = 2\pi L \frac{\mu_c^{1/2}}{L} (r_h + q_1)^{1/2} \epsilon^{3/2},
\]  

(5.5)

where \( \mathcal{J} \) is defined in (3.21). The entropy is found to be

\[
S_{Near-BPS}^{BH} = \frac{2\sqrt{2}}{3} \frac{\mu_c}{L} \left( \sqrt{\frac{\mu + 2q_1}{\mu_c}} - \mathcal{J} - 1 + \sqrt{\frac{\mu + 2q_1}{\mu_c}} + \mathcal{J} - 1 \right) (N\epsilon^2)^{3/2}.
\]  

(5.6)

As the above Bekenstein-Hawking entropy only makes sense for classical black holes, we should make sure that (5.3) is fulfilled. This is only possible if together with \( \epsilon \) we scale \( N \sim \epsilon^{-\alpha} \), \( \alpha \geq 2 \). The components of the curvature tensor for the decoupled geometry all scale as \( \epsilon^{-2} \) (in units of \( L \)) which in the limit remain large. Noting the factor of \( \epsilon^2 \) in front of the decoupled metric components (3.5) and that eleven-dimensional Planck length should be the shortest length,

\[
\epsilon \sim l_p/L \Rightarrow N \sim \epsilon^{-6} \rightarrow \infty,
\]  

(5.7)
where we have used (2.8) and that \( L \) is fixed. This choice of (5.7) will become more clear in the next section. With the above scaling \( S_{BH}^{N_{\text{Near-BPS}}} \sim N \to \infty \).

### 5.1.2 The far-from-BPS case

In the far-from-BPS scaling the horizon radius is the larger root of \( f/H^1/2 \), which turns out to be the same as the larger root of \( G(\rho) \) (3.29),

\[
    r_h = \frac{\mu c}{4 f_0} \left( \sqrt{M - J} + \sqrt{M + J} \right)^2 \epsilon^2, \tag{5.8}
\]

and the horizon area is

\[
    A_h = 2\pi \mu \epsilon^{1/2} r_h^{1/2} = \frac{\pi \mu c L}{\sqrt{f_0}} \left( \sqrt{M - J} + \sqrt{M + J} \right) \epsilon, \tag{5.9}
\]

where \( M \) and \( J \) are defined in (3.10) and (3.27) respectively. The Bekenstein-Hawking entropy is found to be

\[
    S_{BH}^{\text{far-from-BPS}} = \frac{2\sqrt{2\pi}}{3} \frac{\mu c}{\sqrt{f_0} L} \left( N^{3/2} \epsilon \right) \left( \sqrt{M - J} + \sqrt{M + J} \right). \tag{5.10}
\]

To ensure validity of our classical treatments and (5.3) we need to scale \( N \sim \epsilon^{-\beta} \to \infty \), \( \beta \geq 3/2 \) as we take \( \epsilon \to 0 \). Arguments of next section and the 3d CFT analysis specifies \( \beta = 2 \), that is

\[
    N \sim \epsilon^{-2} , \quad \epsilon \sim \left( \frac{L_p}{L} \right)^{1/3}. \tag{5.11}
\]

With this choice \( S_{BH}^{\text{far-from-BPS}} \sim N \). It is notable that the scaling of entropy with \( N \) in both of the the near-BPS and far-from-BPS cases are the same. This is in accord with similar results for five-dimensional two-charge black holes [10].

### 5.2 3d rotating BTZ entropy

As discussed in the near-horizon limit for both of the near-BPS and far-from-BPS cases we obtain a rotating BTZ black hole, cf. (3.22) and (3.28). In this sub-section, we compute the Bekenstein-Hawking entropy of these three-dimensional black holes and compare it to the entropy of four-dimensional black holes computed in the previous subsections. As we will see they are equal. But first, we need to compute the corresponding three-dimensional Newton constant \( G_N^{(3)} \). As discussed in section 4, both the near-BPS and far-from-BPS rotating \( \text{BTZ} \times S^2 \) geometries are solutions to five-dimensional supergravities both of which have the same Newton constant (4.8), (4.12). The rotating BTZ black hole is then the solution to the three-dimensional gravity obtained from the reduction of (either of) the five-dimensional gravity theories on the \( S^2 \) of radius \( R_S \), therefore

\[
    G_N^{(3)} = \frac{G_N^{(5)}}{4\pi R_S^2}. \tag{5.12}
\]
The Bekenstein-Hawking area-law for a rotating BTZ of angular momentum $J$ and mass $M$ is (e.g. see [34])

$$S_{BTZ} = \frac{\pi}{2G_N^{(3)}} R_A \left( \sqrt{M_{BTZ} - J_{BTZ}} + \sqrt{M_{BTZ} + J_{BTZ}} \right),$$

(5.13)

where $R_A$ is the $AdS$ radius. Next we consider the near-BPS and far-from-BPS cases separately.

### 5.2.1 The near-BPS case

In taking the near-BPS limit we are focusing on fixed values for $\theta_i$ and hence each near-horizon geometry describes a “strip” of the original four-dimensional black hole, similarly to the five-dimensional case discussed in [11, 10]. The entropy of the four-dimensional black hole is hence expected to be distributed among these strips each corresponding to a rotating BTZ$\times S^2$. The mass and angular momentum of all of these BTZ’s, in units of the corresponding $AdS_3$ radius, are equal. However, these BTZ’s have different $AdS_3$ radii (cf. (3.9)). Moreover, the corresponding three-dimensional Newton constant is different for each of them, depending on the value of $\mu_i^0$. The entropy of each strip is then

$$dS_{\text{strip}} = 2\pi R_A \cdot \frac{(2\pi)^3 L^6 \cdot 4 R_S^2}{4G_N^{(4)}} \left( \sqrt{M_{BTZ} + J_{BTZ}} + \sqrt{M_{BTZ} - J_{BTZ}} \right) \epsilon^3 \frac{1}{8} d\hat{\mu}_2^2 \ d\hat{\mu}_3^2 \ d\hat{\mu}_4^2,$$

(5.14)

where $R_A$ and $R_S$ are respectively given in (3.7) and (3.8), (3.9). Noting that

$$\int_{\mu_2^2 + \mu_3^2 + \mu_4^2 = 1} d\hat{\mu}_2^2 \ d\hat{\mu}_3^2 \ d\hat{\mu}_4^2 = \frac{1}{6},$$

(5.15)

the total entropy becomes

$$S_{BTZ} = \frac{2\sqrt{2}\pi}{3} \left( N\epsilon^2 \right)^{3/2} \frac{\mu_c}{L} \left( \sqrt{M_{BTZ} - J_{BTZ}} + \sqrt{M_{BTZ} + J_{BTZ}} \right),$$

(5.16)

where $M_{BTZ}$ and $J_{BTZ}$ are given by (3.24). As we see there is a perfect matching between the four-dimensional entropy (5.6) and the collection of the “strip-wise” three-dimensional rotating BTZ black holes (5.16).

### 5.2.2 The far-from-BPS case

The three-dimensional Newton constant (5.12) is

$$G_N^{(3)} = \frac{3}{16\sqrt{2}} \frac{L^3}{R_S^2} N^{-\frac{3}{2}},$$

(5.17)

where $R_S$ is given by (3.16). Noting that the angular variable in BTZ, $\varphi$ ranges over $[0, 2\pi\epsilon]$, the entropy is then

$$S_{BTZ} = \frac{2\sqrt{2}\pi}{3} \left( N^{3/2}\epsilon \right) \frac{\mu_c}{L} \left( \sqrt{M_{BTZ} - J_{BTZ}} + \sqrt{M_{BTZ} + J_{BTZ}} \right),$$

(5.18)

where $M_{BTZ}$ and $J_{BTZ}$ are given in (3.30). Again we see a perfect matching between the three and four dimensional entropies.
6 Dual Field Theory Descriptions

In this section we study the near-horizon decoupling limits from the dual (conformal) field theory viewpoints. First we give the 3d description, motivated by the fact that the original geometry is a four-dimensional black hole in the \( AdS_4 \) background. Next, we focus on the 2d dual field theory description arising from the appearance of \( AdS_3 \) factors in the decoupled geometries.

6.1 3d CFT description

According to the \( AdS_4/CFT_3 \) duality [1] there is a one-to-one correspondence between the four-dimensional black holes, as deformations about the \( AdS_4 \times S^7 \) geometry and certain sectors of the 3d CFT. We choose the \( AdS_4 \times S^7 \) background to correspond to the near-horizon limit of \( N \) coincident M2-branes. The operators of this 3d CFT are specified by \( SO(3,2) \times SO(8) \) quantum numbers. In our case the five parameters of the black hole geometry, \( q_i \) and \( \mu \), are mapped to the four R-charges \( J_i \) and the engineering dimension of the operators \( \Delta \) as

\[
\Delta = \frac{4\sqrt{2}}{3} N^{3/2} \frac{2\mu + q_1 + q_2 + q_3 + q_4}{L}, \\
J_i = \frac{L}{2G_N^{(4)} q_i} = \frac{4\sqrt{2}}{3} N^{3/2} \frac{\hat{q}_i}{L},
\]

where \( M_{ADM} \) is given in (2.6). Operators of interest to us are singlets of \( SO(3) \in SO(3,2) \).

As discussed in the previous section, in both of the near-BPS and far-from-BPS limits we are taking \( N \to \infty \), and \( \Delta \) and \( J_i \) are hence becoming large. Similarly to the case of two-charge five-dimensional black holes of [10], we search for a “BMN-type” sector in the 3d CFT whose dynamics is decoupled from the rest of the theory.

6.1.1 The near-horizon near-BPS limit, \( N = 8 \) 3d CFT description

In the near-BPS limit together with some of the coordinates we also scale \( \mu \sim \epsilon^3 \) and \( q_i \sim \epsilon \). As discussed in section 5.1.1 we need to also scale \( N \sim \epsilon^{-6} \), which we choose

\[
\epsilon = \frac{1}{\sqrt{2}} N^{-1/6}.
\]

Therefore, in this limit, for the three-charge case \( \Delta \) and \( J_i \) of the operators scale as:

\[
\Delta = \frac{4\sqrt{2}}{3} N^{3/2} \epsilon^{3/4} \frac{(\hat{q}_2 + \hat{q}_3 + \hat{q}_4 + \mathcal{O}(\epsilon^2))/L}{L} \sim N^{4/3} \to \infty \\
J_i = \frac{4\sqrt{2}}{3} N^{3/2} \epsilon^{1/4} (\hat{q}_i + \mathcal{O}(\epsilon))/L \sim N^{4/3} \quad i = 2, 3, 4.
\]

That is, the sector of the \( N = 8 \) 3d CFT corresponding to M-theory on the geometries in question have large scaling dimension and R-charge, \( \Delta \sim J_i \sim N^{4/3} \). In the same spirit as the BMN limit [6, 10], one can find certain combinations of \( \Delta \) and \( J_i \) which are finite and describe
physics of the operators after the limit. To find these combinations we recall the way the limit was taken (3.1),

\[ iL \frac{\partial}{\partial \tau} = iL \frac{\partial}{\partial t} + i \sum_{i=2,3,4} \frac{\partial}{\partial \phi_i} = \Delta - \sum_{i=2,3,4} J_i \quad (6.4a) \]

\[ -i \frac{\partial}{\partial \psi_i} = -i \frac{\partial}{\partial \phi_i} = J_i . \quad (6.4b) \]

For the limit (3.2), (6.4b) should be replaced with

\[ -i \frac{\partial}{\partial \psi_i} = -i \varepsilon \frac{\partial}{\partial \phi_i} = \varepsilon J_i . \]

Up to leading order we have

\[ \Delta - \sum_{i=2,3,4} J_i = 2 \sqrt{2} N^{3/2} \varepsilon^{3/2} \frac{\hat{\mu}}{L} , \]

\[ J_i = 4 \sqrt{2} \frac{N^{3/2}}{3} \varepsilon \frac{\hat{q}_i}{L} , \quad i = 2, 3, 4 . \]

As we see \( \Delta - \sum J_i \) scales as \( N^{3/2} \cdot N^{-1/2} = N \rightarrow \infty \), while \( J_i \sim N^{4/3} \) and therefore the “BPS-deviation-parameter” \[ \eta \equiv \frac{\Delta - \sum J_i}{J_i} \sim \varepsilon^2 \sim N^{-1/3} \rightarrow 0 , \]

and hence we are dealing with an “almost-BPS” sector.\(^3\) Moreover, \( \Delta - \sum J_i \) is linearly proportional to the non-extremality parameter \( \hat{\mu} \). It is also notable that \( S_{BH} \) (5.6) scales the same as \( \Delta - \sum J_i \).

In sum, the sector we are dealing with is composed of “almost 1/8 BPS” operators of the \( \mathcal{N} = 8 \) 3d CFT, with

\[ \Delta \sim J_i \sim N^{4/3} , \]

\[ \frac{J_i}{N^{4/3}} = \frac{4 \hat{q}_i}{3L} = fixed, \quad (\Delta - \sum_{i=2,3,4} J_i) \cdot \frac{1}{N} = \frac{\hat{\mu}}{3L} = fixed . \]

The dimensionless physical quantities that describe this sector are therefore \( \hat{q}_i/L, \hat{\mu}/L \).

Here we are dealing with a system of intersecting multi M5-brane giants. The “number of giants” in each stack (2.9) in the near-BPS, near-horizon limit is

\[ N_i = \sqrt{2} N \varepsilon \cdot \frac{\hat{q}_i}{L} = N^{1/3} \frac{\hat{q}_i}{L} , \]

and therefore,

\[ \Delta - \sum_{i} J_i = \frac{4 N_2 N_3 N_4}{3} \frac{\hat{\mu}}{\mu_e} . \]

\(^{3}\)It is instructive to make parallels with the BMN sector [6]. In the BMN sector of the 3d CFT we are dealing with operators with

\[ \Delta \sim J \sim N^{1/3} , \quad \text{while} \quad \Delta - J = finite , \]

implying that, similar to our case, \( \eta_{BMN} \sim N^{-1/3} \rightarrow 0 \). As we see the \( \eta \) parameter for our case and the case of BMN scale in the same way.
Finally, let us consider the rotating case of section 3.3.1, where besides $J_2$, $J_3$ and $J_4$ we have also turned on the fourth $R$-charge $J_1$,

$$J_1 = \frac{4\sqrt{2}}{3} N^{3/2} \epsilon^3 \cdot \frac{1}{L} \sqrt{\hat{q}_1 (\hat{q}_1 + \hat{\mu})} = \frac{2}{3} N \cdot \frac{1}{L} \sqrt{\hat{q}_1 (\hat{q}_1 + \hat{\mu})}. \quad (6.10)$$

As we see $\Delta - \sum_{i=2,3,4} J_i \sim J_1 \sim N^{3/2} \epsilon^3 \sim N \rightarrow \infty$. Instead of $\Delta - \sum_{i=2,3,4} J_i$ it is more appropriate to define another positive definite quantity:

$$\Delta - \sum_{i=1}^{4} J_i = N \cdot \left( \frac{\hat{\mu} + 2 \hat{q}_1 - \sqrt{(\hat{\mu} + 2 \hat{q}_1)^2 - \hat{\mu}^2}}{3L} \right) \geq 0. \quad (6.11)$$

It is remarkable that the above BPS bound is exactly the bound on the rotating BTZ parameters, $M_{BTZ} - J_{BTZ} \geq -1$, in which it becomes a sensible geometry in string theory [10]. This bound is more general than just the extremality bound of the rotating BTZ black hole $M_{BTZ} \geq J_{BTZ} \geq 0$. This bound besides the rotating black hole cases also includes the case in which we have a conical singularity which could be resolved in string theory (cf. Appendix B and section 5 of [10]). We will comment on this point further in section 6.2.1.

6.1.2 The near-horizon far-from-BPS limit, $\mathcal{N} = 8$ 3d CFT description

Since in the near-horizon, far-from-BPS limit of (3.10) we do not scale $\mu$ and $q_i$'s, we expect to deal with a sector of the 3d CFT in which $\Delta \sim J_i \sim N^{3/2}$. As mentioned in 5.1.2, $N \sim \epsilon^{-2}$ which for convenience we choose

$$\epsilon^2 = \frac{9}{32N}. \quad (6.12)$$

To deduce the correct “BMN-type” combination of $\Delta$ and $J_i$ which correspond to physical observables, we recall the way the limit has been taken, and in particular

$$\tau = \epsilon \frac{R_S}{R_{AdS}} \frac{t}{L}, \quad \phi_i = \psi_1 + \frac{\hat{q}_i R_{AdS} \tau}{q_i R_S} \frac{\epsilon}{\epsilon}, \quad i = 2, 3, 4. \quad (6.13)$$

Therefore,

$$-i \frac{\partial}{\partial \tau} = -i \frac{\partial}{\partial \phi_i} = J_i$$

and

$$\mathcal{E} \equiv -i \frac{\partial}{\partial \tau} = - \frac{R_{AdS}}{\epsilon R_S} \left( i \frac{\partial}{\partial t} + i \sum_{i=2,3,4} \frac{\hat{q}_i}{q_i} \frac{\partial}{\partial \phi_i} \right) = - \frac{R_{AdS}}{\epsilon R_S} \left( \Delta - \frac{3L}{4\sqrt{2} N^{3/2}} \sum_{i=2,3,4} \frac{J_i^2}{q_i} \right) \quad (6.14)$$

The last equality can be understood in an intuitive way. In this case we are dealing with massive giant gravitons which are far from being BPS and hence are behaving like non-relativistic objects. They are rotating with angular momentum $J_i$ over circles with radii $R_i$, $R_i^2 = \frac{L^2}{R_S} q_i$ (3.15).

Therefore, the kinetic energy of these rotating branes is proportional to $\sum J_i^2/q_i$.

Recalling that $\Delta$ is measuring the “total” energy of the system, $\mathcal{E}$ should have two parts: the rest-mass of the system of giants and the energy corresponding to the “internal” excitations of the branes. We can see this explicitly from (6.1) and (2.6),

$$\mathcal{E} = \frac{4\sqrt{2}}{3} \frac{R_{AdS}}{R_S} \cdot \frac{N^{3/2}}{\epsilon} \cdot \frac{\mu}{L} \cdot \frac{\epsilon}{L} \cdot \frac{M}{L}$$

$$= \mathcal{E}_0 + \frac{R_{AdS}}{R_S} \cdot N \frac{\mu L}{\epsilon} \cdot M \quad (6.15)$$
where have used \( \mu = \mu_3 + \epsilon^2 \mu_3 M \) (\( M \) is related to the mass of BTZ black hole (3.14)), and
\[
\mathcal{E}_0 = \frac{128 R_{AdS} R^2_S}{9 L^3} \cdot N^2.
\]
\( \mathcal{E}_0 \) which is basically \( \mathcal{E} \) evaluated at \( \mu = \mu_3 \), is the rest-mass of the brane system.\(^4\)

\( \mathcal{E} - \mathcal{E}_0 \), which is proportional to \( M \), corresponds to the fluctuations of the intersecting, deformed M5-brane giants about the extremal point. From the 11d viewpoint we start with the geometries corresponding to M5-brane giants intersecting on a string, the string which lives in five dimensions, it is very suggestive to associate \( \mathcal{E} - \mathcal{E}_0 \) to the mass of these 5d strings. These strings hence correspond to five-brane-type fluctuations of the original “intersecting M5-brane giants”.

In sum, from the 3d CFT viewpoint the sector describing the near-horizon, far-from-BPS limit consists of operators specified with

\[
\Delta \sim J_1 \sim N^{3/2}, \quad N \rightarrow \infty,
\]
\[
\frac{J_1}{N^{3/2}} = \frac{4\sqrt{2} \hat{q}_1}{3L} = \text{fixed}, \quad \frac{\mathcal{E} - \mathcal{E}_0}{N} = \text{fixed},
\]

where \( \mathcal{E}, \mathcal{E}_0 \) in equations (6.14),(6.15) and (6.16) are defined in terms of \( \Delta, J_1 \).

As discussed in section 3.3.2 one may obtain a rotating BTZ if we turn on the fourth \( R \)-charge in a perturbative manner. In the 3d CFT language this means considering the operators which besides being in the sector specified by (6.17) carry the fourth \( R \)-charge \( J_1 \), \( J_1 \sim N^{3/2} \epsilon^2 \sim N^{1/2} \). Explicitly,
\[
J_1 = \frac{4\sqrt{2}}{3} N^{3/2} L^2 \epsilon^2 \sqrt{\hat{q}_1 \mu_3}.
\]

One should note that in terms of the \( AdS_3 \) parameters, since \( \varphi = \epsilon \phi \),
\[
\mathcal{J} \equiv -i \frac{\partial}{\partial \varphi} = -i \epsilon \frac{\partial}{\partial \phi} = \frac{\hat{q}_1}{\epsilon} \frac{J_1}{\epsilon} = \frac{4\sqrt{2}}{3} N^{3/2} \epsilon^2 \mu_3 \sqrt{\hat{q}_1 \mu_3} = N \frac{\mu_3}{L} \mu_3 \sqrt{\hat{q}_1 \mu_3}.
\]

As we see \( \mathcal{J} \), like \( \mathcal{E} - \mathcal{E}_0 \), is also scaling like \( N^2 \epsilon \sim N \) in our decoupling limit. When \( J_1 \) is turned on the expressions for the \( \Delta \) and hence \( \mathcal{E} \) are modified, receiving contributions from \( q_1 \). These corrections, recalling (6.1) and that \( q_1 \) scales as \( \epsilon^4 \) (3.25), vanish in the leading order. However,\(^4\)

---

\(^4\)One should keep in mind that at the extremal point the system is not BPS and hence the “rest-mass” of the system is not simply the sum of the masses of individual stacks of giants but also includes their “binding energy” (stored in the non-spherical shape of the giants). Nonetheless, it should still be proportional to the number of giants times mass of a single giant. Eq.(6.16), however, seems to suggest a simpler interpretation in terms of dual M2-brane giants [4]. Inspired by the expression for the 11d three-form flux, the system of giant M5-branes we start with, e.g. through SUGRA solution (2.1), may also be interpreted as spherical membranes wrapping \( S^2 \subset AdS_4 \) while rotating on \( S^7 \). In terms of dual membrane giants, after the limit, we are dealing with a system of M2-branes wrapping \( S^2 \subset AdS_3 \times S^2 \) which has radius \( R_S \). The mass of a single such dual giant \( m_0 \) (as measured in \( R_{AdS_3} \) units and also noting the scaling of \( AdS_3 \) time with respect to \( AdS_3 \) time) is then
\[
\frac{m_0}{R_{AdS_3}/\epsilon} = T_{M2}(4\pi R_S^2) = \frac{4\sqrt{2} R_S^2}{L^3} \cdot N^{1/2}.
\]

The number of dual membrane giants is proportional to \( N^{3/2} \) (this could be seen from a relation like (2.10)) and hence one expects the total “rest-mass” of the system \( m_0 \) to be proportional to \( N^2 R_S^2 \).
one may still define physically interesting combinations like $\mathcal{E} - \mathcal{E}_0 \pm J$. We will elaborate further on this point in section 6.2.2.

Before closing this subsection some comments are in order:

- A remarkable point which is seen directly from (6.14) is that $-\mathcal{E}$ is negative definite, i.e. there is an extremality bound:
  \[ \Delta - \sum_i f_i(J_i) \leq 0. \]  
  (6.20)

  where
  \[ f_i(J_i) = \frac{3L}{4\sqrt{2}N^{3/2}} \frac{J_i^2}{q_i}. \]

  (Note that one can express $q_i$’s in terms of the $J_i$’s but since the explicit expressions are not illuminating we do not present them here.) This could be thought of as a complement to the usual BPS bound, $\Delta - \sum_i J_i \geq 0$.

- We note that both $\mathcal{E} - \mathcal{E}_0$ and $J$ scale as $N^{3/2} \sim N$ which is the same scaling as the black hole entropy (5.10).

- Finally, the system of original intersecting giants is composed of three stacks of M5-brane giants each containing $N_i = \sqrt{2}N^{1/2}\frac{\tilde{q}_i}{L}$ branes and $N_i \sim N^{1/2} \rightarrow \infty$.

6.2 2d CFT description

As we showed in either of the near-BPS or far-from-BPS near-horizon limits we obtain a spacetime which has an $AdS_3 \times S^2$ factor. This, within the AdS/CFT ideology, is suggesting that M-theory on the corresponding geometries should have a 2d dual CFT description. In this section we elaborate on this 2d description.

6.2.1 The near-BPS case

To find the possible dual field theory which describes our decoupled geometries of (3.5), and their rotating generalizations (3.22), we recall that the original 11d background is a deformation of $AdS_4 \times S^7$ by the addition of three stacks of intersecting M5-brane giants which intersect on a circle while also wrapping four cycles of the $S^7$. In the process of taking the decoupling limit, we take the volume of these four cycles to be very large while keeping the radius of the circle fixed. Therefore, the situation becomes essentially the same as the near-horizon limit of three stacks of intersecting flat M5-branes in a flat eleven-dimensional background [19].

A closely related system is coming from M-theory on a Calabi-Yau ($CY_3$) and three stacks of M5-branes wrapping holomorphic four cycles of the $CY_3$ [35]. The intersection of the M5-branes from the 5d supergravity viewpoint is then a (5d black) string. The near-horizon limit of the geometry corresponding to the above intersecting M5-branes is $AdS_3 \times S^2 \times CY_3$. M-theory on this decoupled geometry has been conjectured to be described by the $\mathcal{N} = (0,4)$ 2d CFT, the
MSW CFT [35] (see also [36] and [37] for a more recent study and a complete list of references on the topic).

In our case we have a very similar situation with the difference that the $AdS_3$ geometry we obtain is in the global coordinates (rather than the Poincare patch as in the MSW case [19, 35]). We hence conjecture that M-theory on the rotating $BTZ \times S^2 \times C_6$ of section 3.3.1 is dual to an $N = (0,4)$ 2d CFT, the degrees of freedom of which are coming from the low energy fluctuations of the intersecting M5-branes. Here are some further comments regarding the conjecture:

- The geometry (3.5) or (3.22) were obtained as the near-horizon, near-BPS limit of a 4d black hole or a deformation of $AdS_4 \times S^7$ and in the process of the limit we focus on a narrow strip on $\mu_2, \mu_3, \mu_4$ directions. The $BTZ \times S^2 \times C_6$ geometry and hence the corresponding 2d CFT description is only describing the narrow strips on the original 4d black hole. Therefore, our 4d black hole is described in terms of not a single 2d CFT, but a collection of (infinitely many of) them. The only property which is different among these 2d CFT’s is their central charge and as is seen from the decoupled metric (3.22) they all have the same $L_0$ and $\bar{L}_0$.

The “metric” on the space of these 2d CFT’s is exactly the same as the metric on $C_6$. As far as the entropy and the overall (total) number of degrees of freedom are concerned, one can define an effective central charge of the theory which is the integral over the central charge of the theory corresponding to each strip. To compute the central charge we use the Brown-Henneaux central charge formula [38],

$$c = \frac{3R_{AdS}}{2G_N^{(3)}}. $$

The effective total central charge is obtained by integrating strip-wise $c$ over the $C_6$. Noting that

$$\int_{\mu_2^2 + \mu_3^2 + \mu_4^2 \leq 1} d\mu_2^2 d\mu_3^2 d\mu_4^2 = \frac{1}{6},$$

the effective central charge of the system is

$$c_L = c_R = c = 8N_2N_3N_4 = 2N \cdot \frac{\hat{\mu}_c}{L} \quad (6.21)$$

It is notable that the central charge $c \sim N \to \infty$.

- The 2d CFT is described by $L_0, \bar{L}_0$ which are related to the BTZ black hole mass and angular momentum [39] as

$$L_0 = \frac{6}{c}N_L = \frac{1}{4}(M_{BTZ} - J_{BTZ}), \quad \bar{L}_0 = \frac{6}{c}N_R = \frac{1}{4}(M_{BTZ} + J_{BTZ}). \quad (6.22)$$

The above expressions for $L_0, \bar{L}_0$ are given for $M_{BTZ} - J_{BTZ} \geq 0$ when we have a black hole description. When $-1 \leq M_{BTZ} - J_{BTZ} < 0$, we need to replace them with $L_0 = -\frac{c}{24}a_+^2$, $\bar{L}_0 = -\frac{c}{24}a_-^2$ (in the conventions introduced in the Appendix B of [10]) [39, 34]. In the special case of global $AdS_3$ background, where $a_+ = a_- = 1/2$ formally corresponding to
$M_{BTZ} = -1, J_{BTZ} = 0$, the ground state is describing an NSNS vacuum of the 2d CFT [40]. The expressions for $M_{BTZ}$ and $J_{BTZ}$ in terms of the system of giants are given in (3.24) and (3.21).

- With the above identification, it is readily seen that the Cardy formula for the entropy of a 2d CFT

$$S_{2d \ CFT} = 2\pi \left( \sqrt{cN_L/6} + \sqrt{cN_R/6} \right)$$

exactly reproduces the expressions for the entropy we got in the previous section, (5.6), once we substitute for the central charge from (6.21) and $M_{BTZ}, J_{BTZ}$ from (3.24) and (3.21).

- It is also instructive to directly compare the 3d description discussed in 6.1.1 and the 2d field theory descriptions. Comparing the expressions for $M_{BTZ}, J_{BTZ}$ and $\Delta - \sum_{i=2,3,4} J_i, J_1$, we see that they match; explicitly

$$\Delta - \sum_{i=2,3,4} J_i = \frac{c}{6} (M_{BTZ} + 1), \quad J_1 = \frac{c}{6} J_{BTZ}.$$  \hspace{1cm} (6.24)

This is very remarkable because it makes a direct contact between the 2d and 3d CFT descriptions. The 3d CFT BPS bound, i.e. $\Delta - \sum_{i=1,2,3,4} J_i \geq 0$ now translates into the bound $M_{BTZ} - J_{BTZ} \geq -1$. This means that the 3d CFT, besides being able to describe the rotating BTZ black holes, can describe the conical spaces too. In other words, $\Delta - \sum_{i=1}^{4} J_i = 0$ and $N \hat{\mu}$ respectively correspond to global $AdS_3$ and massless BTZ cases and when

$$0 < \Delta - \sum_{i=1}^{4} J_i < \frac{c}{6} = N \hat{\mu} \frac{c}{3L},$$

the 3d CFT is describing a conical space, provided that $\gamma$,

$$\gamma^2 \equiv \frac{6}{c} \left( \Delta - \sum_{i=1}^{4} J_i \right) - 1,$$

is a rational number. One should also keep in mind that entropy and temperature are sensible only when $\Delta - \sum_{i=1}^{4} J_i \geq \frac{c}{6}$; for smaller values the degeneracy of the operators in the 3d CFT is not large enough to form a horizon of finite size (in 3d Planck units).

### 6.2.2 The far-from-BPS case

In the far-from-BPS case, the near-horizon limit results in a background which again contained $AdS_3 \times S^2$ as a subspace. This background, however is quite different from what we obtained in the near-BPS case, as there is a different relation between the radii of the $AdS$ and $S$, and the $S^1 \subset AdS_3$ ranges over $[0, 2\pi \epsilon] = [0, 3/(4\sqrt{2N})]$. In addition, the 11d metric is now a warped
product of five and six dimensional subspaces. According to the usual AdS/CFT ideology we expect M-theory on \(\text{AdS}_3 \times S^2 \times \mathcal{M}_6\) to have a dual 2d CFT description. Noting that this background is a non-supersymmetric one, and also the points stressed above, this 2d CFT cannot be the \(\mathcal{N} = (0, 4)\) expanded about its maximally supersymmetric ground state. In the following we just make some general remarks on the conjectured 2d CFT:

- Like the 10d example [10], we expect this 2d CFT to reside on the \(R \times S^1\) causal boundary of the \(\text{AdS}_3 \times S^2\) geometry.

- One may use the Brown-Henneaux analysis [38] to compute the central charge of this 2d CFT:
  \[
  c = \frac{3R_{\text{AdS}_3} \epsilon}{2G^{(3)}_N} = 3 \frac{\mu_c}{L \sqrt{f_0}} N.
  \]  
  As in the near-BPS case (6.21) the central charge scales as \(N \to \infty\). Its expression in terms of the number of five-branes in the stack is, however, more complicated (cf. (3.18) and (2.9)).

- The 4d or 3d black hole entropies given in (5.10) and (5.18) take exactly the same form obtained from counting the number of microstates of a 2d CFT, i.e. the Cardy formula (6.23), with the central charge (6.25) and \(M_{\text{BTZ}}\) and \(J_{\text{BTZ}}\) given in (3.30).

- As discussed in section 6.1.2, there is a sector of \(\mathcal{N} = 8, d = 3\) CFT which describes M-theory on the background found in section 3.3.2. This sector is characterized by \(\mathcal{E} - \mathcal{E}_0\) and \(\mathcal{J}\). From (6.15) and (6.19) one can readily express the 3d parameters in terms of 2d parameters, namely:
  \[
  \mathcal{E} - \mathcal{E}_0 = \frac{c}{6} M_{\text{BTZ}} , \quad \mathcal{J} = \frac{c}{6} J_{\text{BTZ}} ,
  \]  
  where \(c\) is given in (6.25) and \(M_{\text{BTZ}}, J_{\text{BTZ}}\) are given in (3.30). The above relations have the form of the near-BPS case discussed in section 6.2.1. Note, however, that in this case \(\mathcal{E} - \mathcal{E}_0\) is measuring the mass of the BTZ with the zero point energy set at the massless BTZ case (rather than global \(\text{AdS}_3\)).

### 7 Discussion

In this paper we studied near-horizon decoupling limits of near-extremal three-charge black holes of \(U(1)^4\) 4d gauged SUGRA; a parallel analysis for the two-charge black holes of \(U(1)^3\) 5d gauged SUGRA was carried out in [10], see also [11]. We showed that there are two such near-extremal limits, the near-BPS and the far-from-BPS limits. In both cases the eleven-dimensional uplift of the 4d black holes lead to space-times with \(X_{M,J} \times S^2\) factors, \(X_{M,J}\) being a rotating BTZ black hole. As the eleven-dimensional uplift of the 4d black holes are deformations around \(\text{AdS}_4 \times S^7\) geometry, the M-theory on the geometries obtained after the near-horizon limit and the process of taking the decoupling limit should have a description in terms of the 3d \(\mathcal{N} = 8\) CFT. As
we argued taking the near-horizon limit corresponds to working in BMN-type sectors of large R-charges in this 3d CFT. Moreover, appearance of the $AdS_3$ (or rotating BTZ) factors indicates that there should be a description in terms of 2d CFT’s, the central charge and $L_0$ and $\tilde{L}_0$ of which we identified in terms of the rotating BTZ parameters. In the near-BPS case this 2d CFT description is closely related to the three-charge black holes coming in the near horizon limits of three stacks of intersecting M5-branes [35, 37]. In our case, however, the M5-brane picture originates not from flat M5-branes wrapping holomorphic four cycles of $CY_3$ over which the M-theory is compactified, but from spherical M5-branes, M5-brane giants, wrapping five-spheres in $S^7$ of the original $AdS_4 \times S^7$ geometry. In other words, the 2d CFT in the near-BPS case is a specific sector of the 6d $\mathcal{N} = (0, 2)$ CFT on $R \times S^1 \times \Sigma_4$, where $\Sigma_4$ is a four-dimensional space over which the M5-branes overlap.

Our knowledge of M-theory and the $\mathcal{N} = 8$ 3d CFT are both very limited, so identification of sectors in the 3d CFT which describes the M-theory on the above backgrounds involving $AdS_3 \times S^2$ factors is not very illuminating. Nonetheless, one can use our results to learn more about, at least specific decoupled sectors, of the 3d CFT using the better understood 2d CFT’s. On the other hand, M-theory in the Discrete Light-Cone Quantization (DLCQ) is conjectured to be described by Matrix models [6, 7, 41]. In particular, it has been argued that in the DLCQ description of M-theory on the $AdS_4 \times S^7$ and on the corresponding 11d plane-wave are the same [8, 9]. It is then reasonable to look for a Matrix theory description of the sectors of the 3d CFT or the 2d CFT’s we have identified. Apart from specific sectors in the BMN (or plane-wave) matrix model one may also search for matrix model description of (DLCQ of) M-theory on the geometries obtained after the near-horizon limits. In the same spirit as the BMN matrix model [7, 9] this matrix model is presumably coming from “quantization” of spherical M2-branes on the corresponding background.

As we argued the $X_{M,J} \times S^2$ factors in both the near-BPS and far-from-BPS cases are solutions to five-dimensional supergravities; the near-BPS case is a solution to the ungauged $U(1)^3$ SUGRA, the STU model, while the far-from-BPS case is a solution to the gauged $U(1)^3$ SUGRA. As discussed the $AdS_3 \times S^2$ obtained in the near-BPS case can also be obtained from the near-horizon limit of magnetically charged string solutions of the STU model, with the important difference that in our limit we obtain $AdS_3$ in global coordinates rather than the Poincare patch of [28]. In the far-from-BPS limit, it is an open question to check if our $AdS_3 \times S^2$ can be obtained as the near-horizon limit of a 5d string solution. As the $AdS_3 \times S^2$ in this case is not a supersymmetric solution, this 5d string solution, if it exists, is expected to be a non-supersymmetric solution. Searching for such a 5d string solution is an interesting open question because, this solution, if it exists, should be a circular string solution in the $AdS_3$ background. As such, one then expects to have another description in terms of sectors of $\mathcal{N} = 4$, 4d SYM.

As we showed in section 5, the $AdS_3 \times S^2$ solution coming from the far-from-BPS limit is a solution to 5d $U(1)^3$ gauged SUGRA with three magnetic fluxes over the $S^2$. Recalling that this
AdS$_3 \times S^2$ is a part of a solution to 11d SUGRA with metric and three-form (3.13) and (3.17), it is then plausible to expect that the 5d gauged $U(1)^3$ theory could be obtained from warped reduction of 11d SUGRA over (3.15). Explicitly, we expect the metric reduction ansatz to be

$$ds^2_{(11)} = \Delta^4 g^{(5)}_{\mu\nu}(x) dx^\mu dx^\nu + \Delta^{-\frac{2}{3}} \sum_{i=2,3,4} X_i^{-1} \left( d\mu_i^2 + \mu_i^2 (d\psi_i + L A_i)^2 \right) \quad (7.1)$$

where $x_\mu$ denote the five-dimensional coordinates and $A_i$ are the three $U(1)$ gauge fields. $X_i$, which are constrained by $X_2 X_3 X_4 = 1$, constitute the two scalars of the 5d $U(1)^3$ gauged SUGRA. For the case with vanishing electric charges, like our $AdS_3 \times S^2$ we expect $\Delta = \mu_1 = \sqrt{1 - (\mu_2^2 + \mu_3^2 + \mu_4^2)}$. In [10] we proposed a similar reduction of IIB SUGRA to a six-dimensional $U(1)^2$ gauged SUGRA. Verifying the consistency of this reduction and completing the reduction ansatz for the three-from is postponed to feature works [42]. Although for the cases where $\Delta = \mu_1$, $\mu_1 = 0$ is a curvature singularity we expect this singularity to be resolved once the quantum gravity (M-theory) effects are taken into account. If the above proposal for obtaining the $U(1)^3$ 5d gauged SUGRA is verified, it will be very interesting to see if there is a direct relation between the other reduction which leads to the same 5d theory, i.e. reduction of 10d IIB theory on $S^5$ [2, 3].

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**A Near-Horizon Geometry as a Solution to 11d SUGRA**

Here, we present the details which establish that the near-horizon far-from-BPS limit of the three-charge black hole obtained in section 3.2 and the perturbatively added four-charge black hole obtained in section 3.3.2 are solutions to the eleven-dimensional supergravity equations of motion:

$$R_{MN} - \frac{1}{2} R g_{MN} = T_{MN}, \quad (A.1)$$
$$d* F + \frac{1}{2} F \wedge F = 0, \quad (A.2)$$

where $T_{MN}$ is the energy-momentum tensor of the flux

$$T_{MN} = \frac{1}{12} \left( F_{MPQR} F^P_N F^QR - \frac{1}{8} F_{PQRS} F^{PQRS} \right). \quad (A.3)$$

The metric for the three-charge case is given in equations (3.13), (3.14) and (3.15) and that for the four-charge case is given in (3.13), (3.22) and (3.15). The three-form flux for both the three-charge case and the four-charge case is the same, given in equation (3.17); its four-form
field strength is given by,

\[ F^{(4)} = -L^2 q_2 \left( 1 + \frac{4q_3 q_4}{L^2} \right)^{\frac{1}{2}} \mu_2 d\mu_2 \wedge d\psi_2 \wedge d^3\Omega_2 - L^2 q_3 \left( 1 + \frac{4q_4 q_2}{L^2} \right)^{\frac{1}{2}} \mu_3 d\mu_3 \wedge d\psi_3 \wedge d^3\Omega_2 - L^2 q_4 \left( 1 + \frac{4q_2 q_3}{L^2} \right)^{\frac{1}{2}} \mu_4 d\mu_4 \wedge d\psi_4 \wedge d^3\Omega_2 \]

(A.4)

and its Hodge dual,

\[ \star F^{(4)} = -L^4 q_2 \left( 1 + \frac{4q_3 q_4}{L^2} \right)^{\frac{1}{2}} \rho \mu_3 \mu_4 \rho \mu_4 \wedge d\tau \wedge d\varphi \wedge d\mu_3 \wedge d\mu_4 \wedge d\psi_4 \]

\[ -L^4 q_3 \left( 1 + \frac{4q_4 q_2}{L^2} \right)^{\frac{1}{2}} \rho \mu_4 \mu_2 \rho \mu_2 \wedge d\tau \wedge d\varphi \wedge d\mu_4 \wedge d\mu_2 \wedge d\psi_2 \]

\[ -L^4 q_4 \left( 1 + \frac{4q_2 q_3}{L^2} \right)^{\frac{1}{2}} \rho \mu_2 \mu_3 \rho \mu_3 \wedge d\tau \wedge d\varphi \wedge d\mu_2 \wedge d\mu_3 \wedge d\psi_3. \]

(A.5)

It is clear from (A.5) and (A.4) that the two terms in (A.2) separately vanish and thus the flux equation of motion is satisfied. The non-vanishing components of the energy-momentum tensor (A.3) for the three-charge case are:

\[ T_{\tau\tau}/g_{\tau\tau} = T_{\rho\rho}/g_{\rho\rho} = T_{\varphi\varphi}/g_{\varphi\varphi} = -\frac{f_0 + 2}{4(q_2 q_3 q_4)^{\frac{2}{3}} \mu_1^{\frac{4}{3}}}, \]

\[ T_{ij}/g_{ij} = \frac{f_0 + 2}{4(q_2 q_3 q_4)^{\frac{2}{3}} \mu_1^{\frac{4}{3}}}, \]

\[ T_{\mu_2 \mu_3}/g_{\mu_2 \mu_3} = T_{\psi_3 \psi_4}/g_{\psi_3 \psi_4} = -\frac{f_0 - \frac{8q_3 q_4}{L^2}}{4(q_2 q_3 q_4)^{\frac{2}{3}} \mu_1^{\frac{4}{3}}}, \]

\[ T_{\mu_4 \mu_3}/g_{\mu_4 \mu_3} = T_{\psi_4 \psi_3}/g_{\psi_4 \psi_3} = \frac{f_0 - \frac{8q_2 q_3}{L^2}}{4(q_2 q_3 q_4)^{\frac{2}{3}} \mu_1^{\frac{4}{3}}}, \]

(A.6)

where \( g_{ij} \) is the metric on the two-sphere. Although the flux for the four-charge case is the same as for the three-charge case, since the metric has one extra component, the energy-momentum tensor for the four-charge case has one more component, apart from (A.6)

\[ T_{\varphi\varphi}/g_{\varphi\varphi} = -\frac{f_0 + 2}{4(q_2 q_3 q_4)^{\frac{2}{3}} \mu_1^{\frac{4}{3}}}. \]

(A.7)

The Ricci tensor of the three-charge metric (3.13), (3.14) and (3.15) has the following non-vanishing components:

\[ R_{\tau\tau}/g_{\tau\tau} = R_{\rho\rho}/g_{\rho\rho} = R_{\varphi\varphi}/g_{\varphi\varphi} = -\frac{f_0 + 2}{6(q_2 q_3 q_4)^{\frac{2}{3}} \mu_1^{\frac{4}{3}}}, \]

\[ R_{ij}/g_{ij} = \frac{f_0 + 2}{3(q_2 q_3 q_4)^{\frac{2}{3}} \mu_1^{\frac{4}{3}}}, \]

\[ R_{\mu_2 \mu_3}/g_{\mu_2 \mu_3} = R_{\psi_2 \psi_3}/g_{\psi_2 \psi_3} = -\frac{f_0 - \frac{12q_3 q_4}{L^2}}{6(q_2 q_3 q_4)^{\frac{2}{3}} \mu_1^{\frac{4}{3}}}, \]

\[ R_{\mu_4 \mu_3}/g_{\mu_4 \mu_3} = -\frac{f_0 - \frac{12q_2 q_3}{L^2}}{6(q_2 q_3 q_4)^{\frac{2}{3}} \mu_1^{\frac{4}{3}}}. \]

(A.8)
The Ricci tensor for the four-charge metric (3.13), (3.22) and (3.15) is (A.8) plus one more component:

\[
R_{\tau \varphi} = -\frac{f_0 + 2}{6(q_2 q_3 q_4)^{\frac{2}{3}} \mu_1^4}.
\]

(A.9)

Using the Ricci scalar

\[
R = \frac{f_0 + 2}{6(q_2 q_3 q_4)^{\frac{2}{3}} \mu_1^4}
\]

(A.10)

and plugging (A.8) and (A.6) into (A.1), it is clear that the near-horizon far-from-BPS limit of the three-charge black hole is indeed a solution to eleven-dimensional supergravity. Similarly plugging (A.8), (A.9) and (A.6), (A.7) into (A.1) proves that the small-charge near-horizon far-from-BPS limit of the four-charge black hole is a solution the eleven-dimensional supergravity.

B The Entropy Function Analysis

To study thermodynamic property of both of the near-BPS and far-from-BPS black holes one may use Sen’s entropy function [14], see also [21, 43]. The procedure and computations are essentially the same as the one presented in [10] for the BTZ\(\times S^3\) geometries. To run the entropy function machinery (for a review see [14]) we plug the ansatz (4.3) into the entropy function \(F\):

\[
F(R_A, R_S, u^i; p^i) = \frac{1}{16G_N^{(5)}} \int d^Hx \sqrt{-g^{(5)}} \frac{\partial L}{\partial F_{\tau \mu}} - L = \frac{-1}{16G_N^{(5)}} \int d^Hx \sqrt{-g^{(5)}} L.
\]

(B.1)

In writing the second equality we have used the fact that for our ansatz (4.3) we do not have electric charges/flux \((F_{\tau \mu} = 0)\). We take \(L\) to be the ungauged 5\(d\) supergravity Lagrangian (4.1) for the near-BPS case and the gauged 5\(d\) supergravity Lagrangian (4.9) for the far-from-BPS case. \(\{x^H\}\) denotes the three-dimensional horizon of the 5\(d\) black string solution which for our case it is \(S^1 \times S^2\), where \(S^1\) is a circle of radius \(\rho_h\).

According to the entropy function procedure [14], the minimum value of the above entropy function is equal to the entropy of the corresponding 5\(d\) near-extremal black hole. The entropy function is minimized on the solutions of the “field” equations

\[
\frac{\partial F(R_A, R_S, u^i; p^i)}{\partial R_A} = 0, \quad \frac{\partial F(R_A, R_S, u^i; p^i)}{\partial R_S} = 0 \quad \text{(B.2a)}
\]

\[
\frac{\partial f(R_A, R_S, u^i; p^i)}{\partial u^i} = 0, \quad i = 1, 2. \quad \text{(B.2b)}
\]

Note that \(u^3 = (u^1 u^2)^{-1}\).

B.1 The near-BPS case

Evaluating the entropy function for the near-BPS case using the ansatz (4.3) we obtain

\[
F(R_A, R_S, u^i; p^i) = \frac{-1}{16\pi G_N^{(5)}} \int \sin \theta d\theta d\phi_d \phi_1 R_A^2 R_S^2 \rho_h \left[ \frac{2}{R_S^2} - \frac{6}{2R_A^2} \left( \frac{p^1}{u^1} \right)^2 + \left( \frac{p^2}{u^2} \right)^2 + \left( \frac{p^3}{u^3} \right)^2 \right]
\]

(B.3)
where \(u^1 u^2 u^3 = 1\) and \(\phi_1, \phi, \theta\) parameterize the \(S^1 \times S^2\) horizon. The field equations (B.2) take the form

\[
\frac{1}{R_S} - \frac{1}{R_A} - \frac{1}{4R_S} \left( \frac{p^1}{u_1} \right)^2 + \left( \frac{p^2}{u_2} \right)^2 + (u^1 u^2 p^3)^2 \right) = 0 \quad (B.4a)
\]

\[
-\frac{6}{R_A} + \frac{1}{2R_S} \left( \frac{p^1}{u_1} \right)^2 + \left( \frac{p^2}{u_2} \right)^2 + (u^1 u^2 p^3)^2 \right) = 0 \quad (B.4b)
\]

\[
\frac{p^1}{u_1} = u^1 u^2 p^3 \quad (B.4c)
\]

\[
\frac{p^2}{u_2} = u^1 u^2 p^3. \quad (B.4d)
\]

Solutions to these equations are given by (4.5), (4.6) and (4.7).

The minimum value of the entropy function for given charges \(q^i\) and \(\rho_h\) is then,

\[
F_{\text{min}}^{\text{Near-BPS}} = \frac{4\pi}{G^{(5)}_N} R^3_S \rho_h = 32\sqrt{2} (N\epsilon)^{3/2} \left( \frac{\hat{\mu}_c}{L^3} \cdot \rho_h \mu^2 \right) \mu^3 \mu^4 \rho^4 \quad (B.5)
\]

where in the second equality we have used (4.8) and (4.7). Upon integration over the angles \(\mu_2, \mu_3\) and \(\mu_4\), we obtain the “total” entropy of the system which has exactly the same value as the 3d rotating BTZ black hole (5.16), or the original 4d black hole (5.6).

**B.2 The far-from-BPS case**

The entropy function for the far-from-BPS case (when we have 5d gauged SUGRA) is

\[
F(R_A, R_S, u^i; p^i) = -\frac{1}{16\pi G^{(5)}_N} \int \sin \theta d\theta d\phi d\phi_1 R^3_A R^2_S \rho_h \left[ \frac{2}{R_S} - \frac{6}{R_A} + \frac{4}{L^2} \left( \frac{1}{u^1} + \frac{1}{u^2} + \frac{1}{u^3} \right) \right.
\]

\[
- \frac{1}{2R_S} \left( \frac{p^1}{u_1} \right)^2 + \left( \frac{p^2}{u_2} \right)^2 + \left( \frac{p^3}{u_3} \right)^2 \right] \quad (B.6)
\]

The field equations after some simplification take the form

\[
-\frac{2}{R_A} + \frac{2}{R_S} + \frac{4}{L^2} \left( \frac{1}{u^1} + \frac{1}{u^2} + \frac{1}{u^3} \right) - \frac{1}{2R_S} \left( \frac{p^1}{u_1} \right)^2 + \left( \frac{p^2}{u_2} \right)^2 + \left( \frac{p^3}{u_3} \right)^2 \right) = 0 \quad (B.7a)
\]

\[
- \frac{4}{R_A} - \frac{2}{R_S} + \frac{1}{R_S} \left( \frac{p^1}{u_1} \right)^2 + \left( \frac{p^2}{u_2} \right)^2 + \left( \frac{p^3}{u_3} \right)^2 \right) = 0 \quad (B.7b)
\]

\[
\frac{4}{L^2} \left( -\frac{1}{u^1} + u^1 u^2 \right) - \frac{1}{R_S} \left( -\left( \frac{p^1}{u_1} \right)^2 + (u^1 u^2 p^3)^2 \right) = 0 \quad (B.7c)
\]

\[
\frac{4}{L^2} \left( -\frac{1}{u^2} + u^1 u^2 \right) - \frac{1}{R_S} \left( -\left( \frac{p^2}{u_2} \right)^2 + (u^1 u^2 p^3)^2 \right) = 0. \quad (B.7d)
\]

The solution to the above set of equations is given in (4.10) and (4.11).

The minimum value of the entropy function for a given set of \(p^i\) and \(\rho_h\), after substituting for \(G^{(5)}_N\) from (4.12), is

\[
F_{\text{min}}^{\text{far-from-BPS}} = \frac{8\sqrt{2}}{3} \cdot N^{3/2} \frac{R^2_S R_A}{L^3} \rho_h \quad (B.8)
\]

which is the same expression as (5.10) once we replace for the \(R_S\) and \(R_A\) and \(\rho_h\) in terms of the charges and \(M, J\).
References


33