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THE ESSENTIAL SPECTRUM
OF A MODEL OPERATOR IN FOCK SPACE

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Abstract

A model operator $H$ associated to a system describing four particles in interaction, without
conservation of the number of particles, is considered. We describe the essential spectrum of $H$
via the spectrum of channel operators and prove the Hunziker-van Winter-Zhislin (HWZ) theorem
for the operator $H$.

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1 INTRODUCTION

Spectral properties of multiparticle Schrödinger operators in Euclidean space are sufficiently well studied in [5]. As is well known, the theorem on the location of the essential spectrum of multiparticle Hamiltonians was named the HWZ theorem in [4, 17] to the honor of Hunziker [7], van Winter [20] and Zhislin [21]. A lattice analogue of this theorem for the four particle Schrödinger operator was proved in [1, 13].

The effective description of the location of the essential spectrum of electromagnetic Schrödinger operators on $\mathbb{R}^N$ is obtained in [14]. The well-known methods for the investigation of the location of essential spectra of Schrödinger operators are Weyl criterion for the one particle problem and the HWZ theorem for multiparticle problems, the modern proof of which is based on the Ruelle-Simon partition of unity. In [15] by means of the limit operators method the essential spectrum of discrete Schrödinger operators on lattice $\mathbb{Z}^N$ is studied. This method has been applied by one of the authors to describe the essential spectrum of continuous electromagnetic Schrödinger operators, square-root Klein-Gordon operators and Dirac operators under quite weak assumptions on the behavior of the magnetic and electric potential at infinity.

The systems considered above have a fixed number of quasi-particles. In statistical physics [10, 11], solid-state physics [12] and the theory of quantum fields [6] some important problems arise in the systems where the number of quasi-particles is not fixed. The study of systems with a non conserved, but bounded number of particles, is reduced to the study of the spectral properties of self-adjoint operators associated to a system describing $n$ particles in interaction without conservation of the number of particles, acting in the cut subspace $\mathcal{H}^{(n)}$ of Fock space, consisting of $r \leq n$ particles [6, 11, 12, 18, 22].

In [18] geometric and commutator techniques have been developed in order to find the location of the spectrum and to prove the absence of singular continuous spectrum for Hamiltonian without conservation of the particle number. The model operators acting in $\mathcal{H}^{(3)}$ were well studied in [2, 3, 8, 9, 16, 19].

In the present paper we consider a model operator $H$ associated to a system describing four particles in interaction, without conservation of the number of particles, acting in the cut subspace $\mathcal{H}^{(4)}$.

For the study of location of the essential spectrum of $H$, we introduce the channel operators and prove that the essential spectrum of $H$ is the union of spectra of channel operators. The channel operators have a more simple structure than $H$. The two-, three- and four-particle branches of the essential spectrum of $H$ are singled out. We also prove the HWZ theorem in the location of the essential spectrum of $H$.

The plan of the present paper is as follows.

In Section 2 the model operator $H$ is described as a bounded self-adjoint operator in $\mathcal{H}^{(4)}$ and the main results are formulated. In Section 3 we study the spectrum of channel operators by the spectrum of corresponding families of operators. In Section 4 we obtain an analogue of the Faddeev-Yakubovskii type system of integral equations for the eigenfunctions of $H$. Section 5 is devoted to the proof of the main results of the paper (Theorems 2.1 and 2.2).

Throughout this paper we adopt the following convention: Denote by $T^\nu$ the $\nu$-dimensional torus, the cube $(-\pi, \pi]^\nu$ with appropriately identified sides. The torus $T^\nu$ will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations of the $\nu$-dimensional space $\mathbb{R}^\nu$ modulo $(2\pi \mathbb{Z})^\nu$. 

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2 THE MODEL OPERATOR AND STATEMENTS OF THE MAIN RESULTS

Let us introduce some notations used in this work. Let \( \mathbb{C} = \mathbb{C}^1 \) be the field of complex numbers and let \( L_2((T^\nu)^n), n = 1, 2, 3 \) be the Hilbert space of square-integrable (complex) functions defined on \( (T^\nu)^n, n = 1, 2, 3 \).

Denote
\[
\mathcal{H}_0 = \mathbb{C}, \quad \mathcal{H}_1 = L_2(T^\nu), \quad \mathcal{H}_2 = L_2((T^\nu)^2), \quad \mathcal{H}_3 = L_2((T^\nu)^3),
\]
\[
\mathcal{H}^{(n,m)} = \bigoplus_{i=n}^m \mathcal{H}_i, \quad 0 \leq n < m \leq 3.
\]

The Hilbert space \( \mathcal{H}^{(n)} \equiv \mathcal{H}^{(0,n-1)}, n = 2, 3, 4 \) is called an \( n \)-particle cut subspace of Fock space.

Let the model operator \( H \) act in the Hilbert space \( \mathcal{H}^{(0,3)} \) as a matrix operator
\[
H = \begin{pmatrix}
H_{00} & H_{01} & 0 & 0 \\
H_{10} & H_{11} & H_{12} & 0 \\
0 & H_{21} & H_{22} & H_{23} \\
0 & 0 & H_{32} & H_{33}
\end{pmatrix}
\]

and let its components \( H_{ij} : \mathcal{H}_j \to \mathcal{H}_i, \ i, j = 0, 1, 2, 3 \) are defined by the rule
\[
(H_{00} f_0)_0 = w_0 f_0, \quad (H_{01} f_1)_0 = \int_{T^\nu} v_1(q') f_1(q') dq', \quad (H_{10} f_0)_1(p) = v_1(p) f_0,
\]
\[
(H_{11} f_1)_1(p) = w_1(p) f_1(p), \quad (H_{12} f_2)_1(p) = \int_{T^\nu} v_2(q') f_2(p, q') dq',
\]
\[
(H_{21} f_1)_2(p, q) = v_2(q) f_1(p), \quad H_{22} = H_{22}^0 - V_{21} - V_{22},
\]
\[
(H_{22}^0 f_2)_2(p, q) = w_2(p, q) f_2(p, q), \quad (V_{21} f_2)_2(p, q) = v_21(p) \int_{T^\nu} v_21(p') f_2(p', q) dq',
\]
\[
(H_{32}^0 f_2)_3(p, q, t) = v_23(p, q, t) f_2(p, q, t), \quad (V_{32} f_2)_3(p, q, t) = v_33(p, q, t) f_3(p, q, t).
\]

Here \( w_0 \) is a real number, \( v_i(\cdot), i = 1, 2, 3 \), \( w_2(\cdot, \cdot) \), \( j = 1, 2 \), \( w_1(\cdot) \) are real-valued continuous functions on \( T^\nu \) and \( w_2(\cdot, \cdot) \) resp. \( w_3(\cdot, \cdot, \cdot) \) is a real-valued continuous function on \( (T^\nu)^2 \) resp. \( (T^\nu)^3 \).

Under these assumptions the operator \( H \) is bounded and self-adjoint in \( \mathcal{H}^{(0,3)} \).

We remark that the operators \( H_{01}, H_{12} \) and \( H_{23} \) resp. \( H_{10}, H_{21} \) and \( H_{32} \) defined in the Fock space are called creation resp. annihilation operators.

Let us introduce the channel operators \( H_n, n = 1, 3 \) resp. \( H_2 \) acting in \( \mathcal{H}^{(2,3)} \) resp. \( \mathcal{H}^{(1,3)} \) by the following rule
\[
H_1 = \begin{pmatrix}
H_{22}^0 - V_{21} & H_{23} \\
H_{32} & H_{33}
\end{pmatrix}, \quad H_3 = \begin{pmatrix}
H_{22}^0 & H_{23} \\
H_{32} & H_{33}
\end{pmatrix},
\]

resp.
\[
H_2 = \begin{pmatrix}
H_{11} & H_{12} & 0 \\
H_{21} & H_{22}^0 - V_{22} & H_{23} \\
0 & H_{32} & H_{33}
\end{pmatrix}.
\]

The essential spectrum of the operator \( H \) can be precisely described as well, as in the following
Theorem 2.1 The essential spectrum $\sigma_{\text{ess}}(H)$ of the operator $H$ is the union of spectra of channel operators $H_1$, $H_2$ and $H_3$, i.e. the equality

$$\sigma_{\text{ess}}(H) = \bigcup_{n=1}^{3} \sigma(H_n)$$

holds, where $\sigma(H_n)$, $n = 1, 2, 3$ stands the spectrum of the operator $H_n$, $n = 1, 2, 3$.

The following theorem shows that the least element of the essential spectrum of $H$ can belong to the spectrum of channel operators $H_1$ or $H_2$.

Theorem 2.2 (a HWZ theorem). The formula

$$\min \sigma_{\text{ess}}(H) = \min\{\min \sigma(H_1), \min \sigma(H_2)\}$$

holds for the essential spectrum of the operator $H$.

3 THE SPECTRUM OF THE CHANNEL OPERATORS

In this section we study some spectral properties of the operator $H_n$, $n = 1, 2, 3$, which plays a crucial role in the study of the essential spectrum of $H$. Using decomposition into direct integrals (see [17]) we reduce to study the spectral properties of the operator $H_n$, $n = 1, 2$ resp. $H_3$ to the investigation of the spectral properties of the family of operators $h_n(p)$, $p \in T^\prime$, $n = 1, 2$ resp. $h_3(p,q)$, $p,q \in T^\prime$ defined below.

First we consider the operator $H_3$, which commutes with any multiplication operator $U_{\alpha}^{(3)}$ by the bounded function $\alpha_3$ on $(T^\prime)^2$

$$U_{\alpha}^{(3)} \left( \begin{array}{c} g_2(p,q) \\ g_3(p,q,t) \end{array} \right) = \left( \begin{array}{c} \alpha_3(p,q)g_2(p,q) \\ \alpha_3(p,q)g_3(p,q,t) \end{array} \right), \left( \begin{array}{c} g_2 \\ g_3 \end{array} \right) \in \mathcal{H}^{(2,3)}.$$

Therefore the decomposition of the space $\mathcal{H}^{(2,3)}$ into the direct integral

$$\mathcal{H}^{(0,1)} = \int_{(T^\prime)^2} \oplus(\mathcal{H}^{(2,3)}) dp dq$$

yields the decomposition into the direct integral

$$H_3 = \int_{(T^\prime)^2} \oplus h_3(p,q) dp dq,$$

where a family of the Friedrichs models $h_3(p,q)$, $p,q \in T^\prime$ acts in $\mathcal{H}^{(0,1)}$ as

$$h_3(p,q) = \left( \begin{array}{cc} h_{00}^{(3)}(p,q) & h_{01}^{(3)}(p,q) \\ h_{10}^{(3)} & h_{11}^{(3)}(p,q) \end{array} \right).$$

Here

$$(h_{00}^{(3)}(p,q)f_0)_0 = w_2(p,q)f_0, \quad (h_{01}^{(3)}f_1)_0 = \int_{T^\prime} v_3(q')f_1(q')dq',$$

$$(h_{10}^{(3)}f_0)_1(t) = v_3(t)f_0, \quad (h_{11}^{(3)}(p,q)f_1)_1(t) = w_3(p,q,t)f_1(t).$$
In analogy with the operator $H_3$ one can give the decomposition

$$H_n = \int T_v \oplus h_n(p)dp, \quad n = 1, 2,$$  \hspace{1cm} (3.2)

where a family of the operators $h_1(p), p \in T_v$ resp. $h_2(p), p \in T_v$ acts in $H^{(1,2)}$ resp. $H^{(0,2)}$ as

$$h_1(p) = \left( \begin{array}{cc}
  h_{11}^{(1)}(p) & h_{12}^{(1)}(p) \\
  h_{21}^{(1)} & h_{22}^{(1)}(p)
\end{array} \right) \quad \text{resp.} \quad h_2(p) = \left( \begin{array}{ccc}
  h_{00}^{(2)}(p) & h_{01}^{(2)}(p) & 0 \\
  h_{10}^{(2)} & h_{11}^{(1)}(p) & h_{12}^{(1)}(p) \\
  0 & 0 & h_{22}^{(1)}(p)
\end{array} \right)$$

with the entries

$$(h_{11}^{(1)}(p)f_1(q))_1 = w_2(p,q)f_1(q) - v_21(q) \int_{T_v} v_21(q')f_1(q')dq', \quad (h_{12}^{(1)}f_2)_1(q) = \int_{T_v} v_3(q')f_2(q,q')dq',$$

$$(h_{21}^{(1)}f_1)_2(q,t) = v_3(t)f_1(q), \quad (h_{22}^{(1)}f_2)_2(q,t) = w_3(p,q,t)f_2(q,t),$$

$$(h_{00}^{(2)}(p)f_0)_0 = w_1(p)f_0, \quad (h_{01}^{(2)}f_1)_0 = \int_{T_v} v_2(q')f_1(q')dq', \quad (h_{10}^{(2)}f_0)_1(q) = v_2(q)f_0, \quad v_2(q), \quad (h_{11}^{(2)}f_1)_1(q) = w_2(p,q)f_1(q) - v_22(q) \int_{T_v} v_22(q')f_1(q')dq'.$$

Let us introduce the notations

$$m = \min_{p,q,t \in T_v} w_3(p,q,t), \quad M = \max_{p,q,t \in T_v} w_3(p,q,t),$$

$$\sigma_{four}(H_n) = [m; M], \quad n = 1, 2, 3,$$

$$\sigma_{three}(H_n) = \bigcup_{p,q \in T_v} \sigma_{disc}(h_3(p,q)), \quad n = 1, 2, 3,$$

$$\sigma_{two}(H_n) = \bigcup_{p \in T_v} \sigma_{disc}(h_n(p)), \quad n = 1, 2.$$

The spectrum of the operators $H_n, n = 1, 2, 3$ can be precisely described as well, as in the following

**Theorem 3.1** The following equalities hold:

(i) $\sigma(H_3) = \sigma_{three}(H_3) \cup \sigma_{four}(H_3);$ 
(ii) $\sigma(H_2) = \sigma_{two}(H_2) \cup \sigma_{three}(H_2) \cup \sigma_{four}(H_2);$ 
(iii) $\sigma(H_1) = \sigma_{two}(H_1) \cup \sigma_{three}(H_1) \cup \sigma_{four}(H_1).$

Before proving the Theorem 3.1 we introduce a new subsets of the essential spectrum of $H.$

**Definition 3.2** The sets $\sigma_{four}(H) = \sigma_{four}(H_3), \quad \sigma_{three}(H) = \sigma_{three}(H_3), \quad \sigma_{two}(H) = \sigma_{two}(H_1) \cup \sigma_{two}(H_2)$ are called four-particle, three-particle and two-particle branches of the essential spectrum of $H$, respectively.
We start the proof of Theorem 3.1 with the following auxiliary statements.

Let the operator \( h_3^0(p, q), p, q \in T' \) act in \( \mathcal{H}^{(0,1)} \) as

\[
h_3^0(p, q) = \begin{pmatrix} \begin{array}{c} 0 \\ 0 \\ h_1^3(p, q) \end{array} \end{pmatrix}, p, q \in T'.
\]

The perturbation \( h_3(p, q) - h_3^0(p, q), p, q \in T' \) of the operator \( h_3^0(p, q), p, q \in T' \) is a self-adjoint operator of rank 2. Therefore in accordance with the invariance of the essential spectrum under finite rank perturbations the essential spectrum \( \sigma_{ess}(h_3(p, q)) \) of \( h_3(p, q), p, q \in T' \) fills the following interval on the real axis:

\[
\sigma_{ess}(h_3(p, q)) = [m_3(p, q); M_3(p, q)],
\]

where the numbers \( m_3(p, q) \) and \( M_3(p, q) \) are defined by

\[
m_3(p, q) = \min_{t \in T'} w_3(p, q, t), \quad M_3(p, q) = \max_{t \in T'} w_3(p, q, t).
\]

For any fixing \( p, q \in T' \) we define an analytic function \( \Delta_3(p, q; \cdot) \) (the Fredholm determinant associated with the operator \( h_3(p, q), p, q \in T' \)) in \( C \setminus \sigma_{ess}(h_3(p, q)) \) by

\[
\Delta_3(p, q; z) = w_2(p, q) - z - \int_{T'} \frac{v_3^2(q')dq'}{w_3(p, q, q') - z}.
\]

The following lemma establishes a connection between eigenvalues of \( h_3(p, q), p, q \in T' \) and the zeroes of the function \( \Delta_3(p, q; \cdot), p, q \in T' \).

**Lemma 3.3** For any fixing \( p, q \in T' \) the number \( z \in C \setminus \sigma_{ess}(h_3(p, q)) \) is an eigenvalue of the operator \( h_3(p, q), p, q \in T' \) if and only if \( \Delta_3(p, q; z) = 0 \).

**Proof.** "Only If Part." Let for any fixing \( p, q \in T' \) the number \( z \in C \setminus \sigma_{ess}(h_3(p, q)) \) be an eigenvalue of the operator \( h_3(p, q), p, q \in T' \) and let \( f = (f_0, f_1) \in \mathcal{H}^{(0,1)} \) be the corresponding eigenfunction, i.e. the equation \( h_3(p, q)f = zf \) or the system of equations

\[
\begin{align*}
( w(p, q) - z ) f_0 + \int_{T'} v_3(q')f_1(q')dq' &= 0 \\
v_3(t)f_0 + ( w_3(p, q, t) - z ) f_1(t) &= 0
\end{align*}
\]

has a nonzero solution \( f = (f_0, f_1) \in \mathcal{H}^{(0,1)} \).

Since \( z \in C \setminus \sigma_{ess}(h_3(p, q)) \) from the second equation of the system (3.3) we find

\[
f_1(t) = -\frac{v_3(t)f_0}{w_3(p, q, t) - z}.
\]

Substituting expression (3.4) for \( f_1 \) into the first equation of the system (3.3), we get \( f_0\Delta_3(p, q; z) = 0 \). If \( f_0 = 0 \), then \( f_1(q) = 0 \). This contradicts the fact that \( f = (f_0, f_1) \) is an eigenfunction of the operator \( h(p, q) \). Thus, \( \Delta_3(p, q; z) = 0 \).

"If Part." For some \( z \in C \setminus \sigma_{ess}(h_3(p, q)) \) let the equality \( \Delta_3(p, q; z) = 0 \) hold. It is easy to show that for any \( p, q \in T' \) the vector-function \( f = (f_0, f_1) \in \mathcal{H}^{(0,1)} \) is an eigenvector of the operator \( h(p, q), p, q \in T' \) corresponding to the eigenvalue \( z \in C \setminus \sigma_{ess}(h_3(p, q)) \), where \( f_0 = const \neq 0 \) and the \( f_1 \) is defined by (3.4). ☐

From Lemma 3.3 it immediately follows the following equality

\[
\sigma_{disc}(h_3(p, q)) = \{ z \in C \setminus \sigma_{ess}(h_3(p, q)) : \Delta_3(p, q; z) = 0 \}, p, q \in T'.
\]
For any fixed $p \in T^\nu$ we define an analytic function $\Delta_2(p; \cdot)$ resp. $\Delta_1(p; z)$ (the Fredholm determinant associated with the operator $h_2(p), p \in T^\nu$ resp. $h_1(p), p \in T^\nu$) in $C \setminus \sigma_{ess}(h_2(p))$ by

$$\Delta_2(p; z) = \left(1 - \int_{T^\nu} \frac{v_{22}^2(q')dq'}{\Delta_3(p, q'; z)} \right) \left( w_1(p) - z - \int_{T^\nu} \frac{v_{2}^2(q')dq'}{\Delta_3(p, q'; z)} \right) - \left( \int_{T^\nu} \frac{v_2(q')v_{22}^2(q')dq'}{\Delta_3(p, q'; z)} \right)^2.$$

resp.

$$\Delta_1(p; z) = 1 - \int_{T^\nu} \frac{v_{21}^2(q')dq'}{\Delta_3(p, q'; z)}.$$

Analogously to (3.5) one can derive the equalities

$$\sigma_{disc}(h_2(p)) = \{z \in C \setminus \sigma_{ess}(h_2(p)) : \Delta_2(p; z) = 0\}, p \in T^\nu$$

(3.6)

and

$$\sigma_{disc}(h_1(p)) = \{z \in C \setminus \sigma_{ess}(h_1(p)) : \Delta_1(p; z) = 0\}, p \in T^\nu.$$  

(3.7)

**Proof of Theorem 3.1.** The assertions of Theorem 3.1 follows from the representations (3.1), (3.2) and the theorem on decomposable operators (see [17]) and the equalities (3.5)-(3.7). □

**Corollary 3.4** The following inclusion

$$\sigma(H_3) \subset \sigma(H_1) \cup \sigma(H_2)$$

holds.

The proof of Corollary 3.4 immediately follows from Theorem 3.1.

**4 THE FADDEEV-YAKUBOVSKII TYPE SYSTEM OF INTEGRAL EQUATIONS AND THE OPERATOR $T(z)$**

In this section we derive an analogue of the Faddeev-Yakubovskii type system of integral equations for the eigenvectors, corresponding to the eigenvalues lying outside of the essential spectrum of the operator $H$.

Let us introduce the notation

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3,$$

where

$$\mathcal{H}_0 = \mathcal{H}_0, \mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \mathcal{H}_1.$$

For each $z \in C \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3))$ let the operator matrices $A(z)$ and $K(z)$ act in the Hilbert space $\mathcal{H}$ as

$$A(z) = \begin{pmatrix} A_{00}(z) & 0 & 0 & 0 \\ 0 & A_{11}(z) & 0 & A_{13}(z) \\ 0 & 0 & A_{22}(z) & 0 \\ 0 & A_{31}(z) & 0 & A_{33}(z) \end{pmatrix},$$

$$K(z) = \begin{pmatrix} K_{00}(z) & K_{01}(z) & 0 & 0 \\ K_{10}(z) & 0 & K_{12}(z) & 0 \\ 0 & K_{21}(z) & 0 & K_{23}(z) \\ 0 & 0 & K_{32}(z) & 0 \end{pmatrix}.$$
where \( A_{ij}(z) : \overline{H}_j \rightarrow \overline{H}_i, \ i, j = 0, 1, 2, 3 \) is the multiplication operator by the function \( a_{ij}(p; z) \):

\[
a_{00}(p; z) \equiv 1, \quad a_{11}(p; z) = w_1(p) - z - \int_{T^p} \frac{v_2(q')dq'}{\Delta_3(p, q' ; z)},
\]

\[
a_{13}(p; z) \equiv a_{31}(p; z) = \int_{T^p} \frac{v_2(q')v_{22}(q')dq'}{\Delta_3(p, q' ; z)},
\]

\[
a_{22}(p; z) = 1 - \int_{T^p} \frac{v_2(q')dq'}{\Delta_3(q', p; z)}, \quad a_{33}(p; z) = 1 - \int_{T^p} \frac{v_2(q')dq'}{\Delta_3(p, q' ; z)},
\]

and the operators \( K_{ij}(z) : \overline{H}_j \rightarrow \overline{H}_i, \ i, j = 0, 1, 2, 3 \) are defined as

\[
(K_{00}(z)\psi_0) = (w_0 - z + 1)\psi_0, \quad K_{01}(z) \equiv H_{01}, \quad K_{10}(z) \equiv -H_{10},
\]

\[
(K_{12}(z)\psi_2)_1(p) = -v_{21}(p) \int_{T^p} \frac{v_2(q')v_2(q')dq'}{\Delta_3(p, q' ; z)},
\]

\[
(K_{21}(z)\psi_1)_2(p) = -v_2(p) \int_{T^p} \frac{v_2(q')v_1(q')dq'}{\Delta_3(q', p; z)},
\]

\[
(K_{23}(z)\psi_3)_2(p) = v_{22}(p) \int_{T^p} \frac{v_2(q')v_3(q')dq'}{\Delta_3(q', p; z)},
\]

\[
(K_{32}(z)\psi_2)_3(p) = v_2(p) \int_{T^p} \frac{v_2(q')v_2(q')dq'}{\Delta_3(p, q' ; z)}.
\]

We note that for each \( z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3)) \) the operators \( K_{ij}(z), \ i, j = 0, 1, 2 \) belong to the Hilbert-Schmidt class and therefore \( K(z) \) is a compact operator.

**Lemma 4.1** The operator \( A(z), \ z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3)) \) is bounded and invertible and the inverse operator \( A^{-1}(z) \) is given by

\[
A^{-1}(z) = \begin{pmatrix}
B_{00}(z) & 0 & 0 & 0 \\
0 & B_{11}(z) & 0 & B_{13}(z) \\
0 & 0 & B_{22}(z) & 0 \\
0 & B_{31}(z) & 0 & B_{33}(z)
\end{pmatrix},
\]

where \( B_{ij}(z) : \overline{H}_j \rightarrow \overline{H}_i, \ i, j = 0, 1, 2, 3 \) is the multiplication operator by the function \( b_{ij}(p; z) \):

\[
b_{00}(p; z) \equiv 1, \quad b_{11}(p; z) = \frac{a_{33}(p; z)}{\Delta_2(p; z)}, \quad b_{13}(p; z) = b_{31}(p; z) = -\frac{a_{13}(p; z)}{\Delta_2(p; z)},
\]

\[
b_{22}(p; z) = \frac{1}{\Delta_1(p; z)}, \quad b_{33}(p; z) = \frac{a_{11}(p; z)}{\Delta_2(p; z)}.
\]

**Proof.** By definition \( A(z) \) is the multiplication operator by the matrix \( A(p; z) \), where

\[
A(p; z) = \begin{pmatrix}
a_{00}(p; z) & 0 & 0 & 0 \\
0 & a_{11}(p; z) & 0 & a_{13}(p; z) \\
0 & 0 & a_{22}(p; z) & 0 \\
0 & a_{31}(p; z) & 0 & a_{33}(p; z)
\end{pmatrix}.
\]
Obviously, \( A(\cdot; z) \) is a matrix-valued continuous function on \( T' \). This implies that \( A(z) \) is bounded. Since \( \det A(p; z) = \Delta_1(p; z)\Delta_2(p; z) \) and \( z \not\in \sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3) \), we have \( \det A(p; z) \neq 0 \). Therefore for each \( p \in T' \) the matrix \( A(p; z) \) is invertible and its inverse matrix has the form

\[
A^{-1}(p; z) = \begin{pmatrix}
  a_{00}(p; z) & 0 & 0 & -a_{13}(p; z) \\
  0 & \frac{a_{03}(p; z)}{\Delta_2(p; z)} & 0 & \frac{-1}{\Delta_2(p; z)} \\
  0 & 0 & \frac{a_{13}(p; z)}{\Delta_1(p; z)} & 0 \\
  0 & \frac{-a_{13}(p; z)}{\Delta_2(p; z)} & 0 & -\frac{a_{13}(p; z)}{\Delta_2(p; z)}
\end{pmatrix}.
\]

Then for each \( z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3)) \) the matrix-valued function \( A^{-1}(\cdot; z) \) is continuous on \( T' \). Let \( A^{-1}(z) \) be the multiplication operator by the matrix \( A^{-1}(p; z) \) acting in \( H \). It is easy to show that \( A^{-1}(z) \) is the inverse of \( A(z) \). □

The following lemma established a connection between of eigenvalues of \( H \) and \( T(z) = A^{-1}(z)K(z) \).

**Lemma 4.2** The number \( z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3)) \) is an eigenvalue of the operator \( H \) if and only if the operator \( T(z) \) has eigenvalue 1.

**Proof.** Let \( z \in \mathbb{C} \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3)) \) be an eigenvalue of the operator \( H \) and \( f = (f_0, f_1, f_2, f_3) \in \mathcal{H}^{(0,3)} \) be the corresponding eigenvector, that is, the equation \( Hf = zf \) or the system of equations

\[
((H_{00} - z)f_0)_0 + (H_{01}f_1)_0 = 0;
\]

\[
(H_{10}f_0)_1(p) + ((H_{11} - z)f_1)_1(p) + (H_{12}f_2)_1(p) = 0;
\]

\[
(H_{21}f_1)_2(p, q) + ((H_{22} - z)f_2)_2(p, q) + (H_{23}f_3)_2(p, q) = 0;
\]

\[
(H_{32}f_2)_3(p, q, t) + ((H_{33} - z)f_3)_3(p, q, t) = 0
\]

has a nontrivial solution \( f = (f_0, f_1, f_2, f_3) \in \mathcal{H}^{(0,3)} \). Since \( z \not\in \sigma_{four}(H_3) \), from the fourth equation of the system (4.1) for \( f_3 \) we have

\[
f_3(p, q, t) = \frac{-v_3(t)f_2(p, q)}{w_3(p, q, t) - z}.
\]

Substituting the expression (4.2) for \( f_3 \) into the third equation of the system (4.1) we obtain that the system of equations

\[
((H_{00} - z)f_0)_0 + (H_{01}f_1)_0 = 0;
\]

\[
(H_{10}f_0)_1(p) + ((H_{11} - z)f_1)_1(p) + (H_{12}f_2)_1(p) = 0;
\]

\[
(H_{21}f_1)_2(p, q) + ((H_{22} - z)f_2)_2(p, q) + (H_{23}f_3)_2(p, q) = 0
\]

has a nontrivial solution if and only if the system of equations (4.1) has a nontrivial solution, where \( R_{33}(z) \) is the resolvent of \( H_{33} - zI \) and \( I \) is an identical operator in \( \mathcal{H}_3 \).

Since \( z \not\in \sigma_{three}(H_3) \), from the third equation of system (4.3) for \( f_2 \), we have

\[
f_2(p, q) = -\frac{v_2(q)f_1(p)}{\Delta_3(p, q; z)} + \frac{v_21(p)c_1(q) + v_22(q)c_2(p)}{\Delta_3(p, q; z)},
\]

where

\[
c_1(q) = \int_{T'} v_21(q')f_2(q', q)dq',
\]

\[
c_2(p) = \int_{T'} v_22(q')f_2(p, q')dq'.
\]
Next we transform the system (4.3) using $f_0, f_1, c_1, c_2$. Substituting the expression (4.4) for $f_2$ into the second equation of the system (4.3) and the equalities (4.5), (4.6) we obtain that the system of equations
\[ f_0 = (w_0 - z + 1)f_0 + \int_{T^v} v_1(q')f_1(q')dq'; \]
\[ \left( w_1(p) - z - \int_{T^v} \frac{v_2^2(q')dq'}{\Delta_3(p, q'; z)} \right) f_1(p) + \int_{T^v} \frac{v_2(q')v_2(q')dq'}{\Delta_3(p, q'; z)} c_2(p) = -v_1(p)f_0 - v_21(p) \int_{T^v} \frac{v_2(q')c_1(q')dq'}{\Delta_3(p, q'; z)}. \]
\[ (4.7) \]
\[ \left( 1 - \int_{T^v} \frac{v_2^2(q')dq'}{\Delta_3(q', q'; z)} \right) c_1(q) = -v_2(q) \int_{T^v} \frac{v_21(q')f_1(q')dq'}{\Delta_3(q', q'; z)} + v_22(q) \int_{T^v} \frac{v_2(q')c_2(q')dq'}{\Delta_3(q', q'; z)}; \]
\[ \int_{T^v} \frac{v_2(q')v_22(q')dq'}{\Delta_3(p, q'; z)} f_1(p) + \left( 1 - \int_{T^v} \frac{v_2^2(q')dq'}{\Delta_3(p, q'; z)} \right) c_2(p) = v_21(p) \int_{T^v} \frac{v_2(q')c_1(q')dq'}{\Delta_3(p, q'; z)} \]

or the equation
\[ A(z)\psi = K(z)\psi, \quad \psi = (f_0, f_1, c_1, c_2) \in \overline{H} \]
has a nontrivial solution if and only if the system of equations (4.3) has a nontrivial solution.

By Lemma 4.1 the operator $A(z)$ is invertible and hence the following equation
\[ \psi = A^{-1}(z)K(z)\psi \]
or
\[ \psi = T(z)\psi \]
has a nontrivial solution if and only if the system of equations (4.7) has a nontrivial solution. □

**Remark 4.3** We point out that the equation $T(z)\psi = g$ is an analogue of the Faddeev-Yakubovskii type system of integral equations for eigenvectors of the operator $H$.

## 5 THE PROOF OF THE MAIN RESULTS

In this section we prove Theorems 2.1 and 2.2.

**Proof of Theorem 2.1.** The inclusion $\sigma(H_3) \subset \sigma_{ess}(H)$ can be proven quite similarly to the corresponding inclusion of [8].

We prove that $\sigma(H_1) \cup \sigma(H_2) \subset \sigma_{ess}(H)$. Let $z_0 \in \sigma(H_1) \cup \sigma(H_2)$ be an arbitrary point.

Two cases are possible:

1) $z_0 \in \sigma(H_3)$,
2) $z_0 \notin \sigma(H_3)$.

If $z_0 \in \sigma(H_3)$, then $z_0 \in \sigma_{ess}(H)$. Let $z_0 \notin \sigma(H_3)$. By the definition of $\sigma(H_1) \cup \sigma(H_2) \setminus \sigma(H_3)$, there exists a point $p_0 \in T^v$ such that $\Delta_1(p_0; z_0)\Delta_2(p_0; z_0) = 0$. Then the system of homogeneous linear equations
\[ l_0 = 0; \]
\[ \left( w_1(p_0) - z_0 - \int_{T^v} \frac{v_2^2(q')dq'}{\Delta_3(p_0, q'; z_0)} \right) l_1 + \int_{T^v} \frac{v_2(q')v_22(q')dq'}{\Delta_3(p_0, q'; z_0)} l_3 = 0; \]

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Proposition 5.1

There exists an orthonormal system \( \{k_n\} \subset L_2(\mathbb{T}^\nu) \), satisfying the conditions

1. \( \text{supp } k_n \subset V_n(p_0) \),
2. \( k_n \) is a system of orthogonal functions.

Proof.

The existence of \( k_n \) follows from the following proposition.

Proposition 5.1

There exists an orthonormal system \( \{k_n\} \subset L_2(\mathbb{T}^\nu) \), satisfying the conditions

\[
\left( 1 - \int_{\mathbb{T}^\nu} \frac{v_{22}(q')dq'}{\Delta_3(p_0, q'; z_0)} \right) l_2 = 0; \\
\int_{\mathbb{T}^\nu} \frac{v_2(q')v_{22}(q')dq'}{\Delta_3(p_0, q'; z_0)} l_1 + \left( 1 - \int_{\mathbb{T}^\nu} \frac{v_{22}(q')dq'}{\Delta_3(p_0, q'; z_0)} \right) l_3 = 0
\]

has an infinite number of solutions. It is easy to verify that there exists a nontrivial solution \( l = (0, l_1, l_2, l_3) \) of system of equations (5.1) satisfying one of the following conditions:

1. If \( \Delta_2(p_0; z_0) = 0 \), then either \( l_1 \neq 0 \) and \( l_2 = 0 \) or \( l_1 = 0 \), \( l_2 = 0 \) and \( l_3 \neq 0 \).
2. If \( \Delta_1(p_0; z_0) = 0 \), then \( l_2 \neq 0 \) and \( l_1 = l_3 = 0 \).

The system of equations (5.1) can be written in the form

\[
A(p_0; z_0)l = 0, \quad l = (0, l_1, l_2, l_3) \in \mathbb{C}^4.
\]

Let \( \chi_{V_n}(\cdot) \) be the characteristic function of the set

\[
V_n(p_0) = \left\{ p \in \mathbb{T}^\nu : \frac{1}{n+1} < |p-p_0| < \frac{1}{n} \right\}, \quad n = 1, 2, \ldots
\]

and \( \mu(V_n(p_0)) \) be the Lebesgue measure of the set \( V_n(p_0) \).

We choose a sequence of orthogonal functions \( \{f^{(n)}\} \) as

\[
f^{(n)} = \begin{pmatrix} 0 \\ f_1^{(n)}(p) \\ f_2^{(n)}(p, q) \\ f_3^{(n)}(p, q, t) \end{pmatrix},
\]

where

\[
f_1^{(n)}(p) = \psi_1^{(n)}(p), \quad p \in \mathbb{T}^\nu,
\]

\[
f_2^{(n)}(p, q) = -v_2(q)\psi_1^{(n)}(p) + v_{21}(p)\psi_2^{(n)}(q) + v_{22}(q)\psi_3^{(n)}(p) \quad \Delta_3(p, q; z_0), \quad p, q \in \mathbb{T}^\nu,
\]

\[
f_3^{(n)}(p, q, t) = -v_3(t)f_2^{(n)}(p, q) \quad w_3(p, q, t) - z_0, \quad p, q, t \in \mathbb{T}^\nu,
\]

\[
\psi_i^{(n)}(p) = l_i k_n(p) \chi_{V_n}(p) (\mu(V_n(p_0)))^{-1/2}, \quad i = 1, 2, 3.
\]

Here \( \{k_n\} \subset L_2(\mathbb{T}^\nu) \) is to be found from the orthogonality condition for \( \{f^{(n)}\} \), i.e.

\[
(f^{(n)}, f^{(m)}) = \frac{l_2}{\sqrt{\mu(V_n(p_0))\mu(V_m(p_0))}} \int_{V_n(p_0)} \int_{V_m(p_0)} \left( 1 + \int_{\mathbb{T}^\nu} \frac{v_3(t)dt}{w_3(p, q, t) - z_0} \right) \times
\]

\[
\left[ \frac{v_{21}(p)(l_3v_{22}(q) - l_4v_{21}(q))}{\Delta_2^2(p, q; z_0)} + \frac{v_{21}(q)(l_3v_{22}(p) - l_4v_{21}(p))}{\Delta_2^2(p, q; z_0)} \right] k_n(p)k_m(q)dq = 0, \quad n \neq m.
\]

The existence of \( k_n(p) \) follows from the following proposition.
satisfying the conditions of the proposition. Proposition 5.1 is proved.

We choose the function \( \tilde{\varepsilon} \), inequality holds. Therefore

\[
[\frac{v_{21}(p)(l_3v_{22}(q) - l_1v_2(q))}{\Delta_3^2(p, q; z_0)} + \frac{v_{21}(q)(l_3v_{22}(p) - l_1v_2(p))}{\Delta_3^2(p, q; z_0)}]k_n(p)k_m(q)dqd = 0, n \neq m.
\]

**Proof.** We construct the sequence \( \{k_n\} \) by induction. Let \( k_1(p) = \chi_{V_1}(p) \left( \sqrt{\mu(V_1(p_0))} \right)^{-1} \).

We choose the function \( \tilde{k}_2 \in L_2(V_2(p_0)) \), such that \( \|\tilde{k}_2\| = 1 \) and \( (\tilde{k}_2, \varepsilon_1^{(2)}) = 0 \), where

\[
\varepsilon_1^{(2)}(p) = \int_{T^\nu} \left( 1 + \frac{v_3^2(t)dt}{\|w_3(p, q, t) - z_0\|^2} \right) \times
\[
\left[ \frac{v_{21}(p)(l_3v_{22}(q) - l_1v_2(q))}{\Delta_3^2(p, q; z_0)} + \frac{v_{21}(q)(l_3v_{22}(p) - l_1v_2(p))}{\Delta_3^2(p, q; z_0)} \right]k_1(q)\chi_{V_2}(p)dq.
\]

We set \( k_2(p) = \tilde{k}_2(p)\chi_{V_2}(p) \) and continue the process. Assuming that the functions \( k_1(\cdot), \ldots, k_n(\cdot) \)
are constructed, we choose the function \( \tilde{k}_{n+1} \in L_2(V_{n+1}(p_0)) \), such that \( \|\tilde{k}_{n+1}\| = 1 \) and it is orthogonal to the functions

\[
\varepsilon_1^{(n+1)}(p) = \int_{T^\nu} \left( 1 + \frac{v_3^2(t)dt}{\|w_3(p, q, t) - z_0\|^2} \right) \times
\[
\left[ \frac{v_{21}(p)(l_3v_{22}(q) - l_1v_2(q))}{\Delta_3^2(p, q; z_0)} + \frac{v_{21}(q)(l_3v_{22}(p) - l_1v_2(p))}{\Delta_3^2(p, q; z_0)} \right]k_i(q)\chi_{V_{n+1}}(p)dq, \ i = 1, n.
\]

We set \( k_{n+1}(p) = \tilde{k}_{n+1}(p)\chi_{V_{n+1}}(p) \). We have thus constructed an orthonormalized system \( \{k_n\} \)
satisfying the conditions of the proposition. Proposition 5.1 is proved. \( \square \)

**We resume the proof of Theorem 2.1.**

We assume that \( \Delta_2(p_0; z_0) = 0 \) and \( l_1 \neq 0, l_2 = 0 \). Then

\[
\|f^{(n)}\|^2 \geq \|f_1^{(n)}\|^2 = \frac{d_1}{\mu(V_1(p_0))}, \ d_1 = l_1^2 > 0.
\]

Let \( \Delta_2(p_0; z_0) = 0 \) and \( l_3 \neq 0, l_1 = l_2 = 0 \). Then

\[
\|f^{(n)}\|^2 \geq \|f_2^{(n)}\|^2 = \frac{l_2^2}{\mu(V_1(p_0))} \int_{V_1(p_0)} \int_{T^\nu} \left| \frac{v_{22}(q)k_n(p)}{\Delta_3(p, q; z_0)} \right|^2 dqd \geq \frac{d_2}{\mu(V_1(p_0))},
\]

\[
d_2 = \frac{l_2^2\|v_{22}\|^2}{\max_{p, q \in T^\nu} |\Delta_3(p, q; z_0)|^2}.
\]

Similarly, we can prove that in the case where \( \Delta_1(p_0; z_0) = 0 \), \( l_2 \neq 0 \) and \( l_1 = l_3 = 0 \) the inequality

\[
\|f^{(n)}\|^2 \geq \frac{d_3}{\mu(V_1(p_0))}, \ d_3 = \frac{l_2^2\|v_{21}\|^2}{\max_{p, q \in T^\nu} |\Delta_3(p, q; z_0)|^2}
\]

holds. Therefore

\[
\|f^{(n)}\|^2 \geq \frac{\xi_0}{\mu(V_1(p_0))}.
\]

(5.2)
where \( \xi_0 = \min\{d_1, d_2, d_3\} > 0 \).

We set \( \mathbf{f}^{(n)} = f^{(n)}/\|f^{(n)}\| \). It is clear that the system \( \{\mathbf{f}^{(n)}\} \) is orthonormal.

We consider the operator \((H - z_0)\mathbf{f}^{(n)}\) and estimate its norm as
\[
\| (H - z_0)\mathbf{f}^{(n)} \|_{\mathcal{H}(0,3)} \leq \| A(z_0)\tilde{\psi}^{(n)} \|_{\mathcal{H}} + \| K(z_0)\tilde{\psi}^{(n)} \|_{\mathcal{H}},
\]
where
\[
\tilde{\psi}^{(n)} = \left( 0, \frac{\psi_1^{(n)}}{\|f^{(n)}\|}, \frac{\psi_2^{(n)}}{\|f^{(n)}\|}, \frac{\psi_3^{(n)}}{\|f^{(n)}\|} \right).
\]

We note that \( \{\tilde{\psi}^{(n)}\} \subset \overline{\mathcal{H}} \) is a bounded orthonormal system. Indeed, the orthogonality follows since the support of the functions \( \tilde{\psi}^{(n)} \) and \( \tilde{\psi}^{(m)} \) is nonintersecting. The equality
\[
\| \tilde{\psi}^{(n)} \|^2 = \frac{1}{\|f^{(n)}\|} \frac{1}{\mu(V_n(p_0))} (l_1^2 + l_2^2 + l_3^2)
\]
and inequality (5.2) imply that the system \( \{\tilde{\psi}^{(n)}\} \) is uniformly bounded, i.e.
\[
\| \tilde{\psi}^{(n)} \| \leq \frac{1}{\xi_0} \|l\|_{C^4}^2
\]
for all \( n \in \mathbb{N} \), where \( \mathbb{N} \) is the set of positive integers.

Since the operator \( K(z_0) \) is compact, it follows that \( \| K(z_0)\tilde{\psi}^{(n)} \| \to 0 \) as \( n \to \infty \).

We next estimate \( \| A(z_0)\tilde{\psi}^{(n)} \| \). Applying the Schwarz inequality we have
\[
\| A(z_0)\tilde{\psi}^{(n)} \|^2 \leq C^2 \sup_{p \in V_n(p_0)} \| A(p; z_0)l \|_{C^4}^2, \quad C^2 = \max \left\{ \frac{2}{\xi_0} \frac{\|v_{22}\|^2}{\xi_0}, \frac{\|v_{21}\|^2}{\xi_0} \right\}.
\]

The continuity of the matrix-valued function \( A(\cdot; z_0) \) implies that
\[
\sup_{p \in V_n(p_0)} \| A(p; z_0)l \|_{C^4} \to 0 \quad \text{as} \quad n \to \infty.
\]

Therefore, for the sequence of orthonormal vector functions \( \{\mathbf{f}^{(n)}\} \) it follows that \( \|(H - z_0)\mathbf{f}^{(n)}\| \to 0 \) as \( n \to \infty \) and hence \( z_0 \in \sigma_{\text{ess}}(H) \). Since the point \( z_0 \) is an arbitrary, it follows that \( \sigma(H_1) \cup \sigma(H_2) \subset \sigma_{\text{ess}}(H) \). Thus we have proved that \( \sigma(H_1) \cup \sigma(H_2) \subset \sigma_{\text{ess}}(H) \).

We now prove the converse inclusion, that is, \( \sigma_{\text{ess}}(H) \subset \sigma(H_1) \cup \sigma(H_2) \subset \sigma_{\text{ess}}(H) \). Since the operator \( K(z) \) is compact and \( A^{-1}(z) \) is bounded, \( f(z) = A^{-1}(z)K(z) \) is a compact-valued analytic function in \( C(\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3)) \). From the self-adjointness of \( H \) and Lemma 4.2 it follows that the operator \( (I - f(z))^{-1} \) exists for all \( \text{Im} z \neq 0 \), where \( I \) is an identical operator in \( \overline{\mathcal{H}} \).

In accordance with the analytic Fredholm theorem, we conclude that the operator-valued function \( (I - f(z))^{-1} \) exists on \( C(\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3)) \) everywhere except at a discrete set \( S \), where it has finite-rank residues. Hence, with \( \sigma_{\text{disc}}(H) \) denoting the discrete spectrum of \( H \), we have \( \sigma(H) \setminus (\sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3)) \subset \sigma_{\text{disc}}(H) = \sigma(H) \setminus \sigma_{\text{ess}}(H) \), i.e. \( \sigma_{\text{ess}}(H) \subset \sigma(H_1) \cup \sigma(H_2) \cup \sigma(H_3) \). Theorem 2.1 is completely proved. \( \square \)

Proof of Theorem 2.2 follows from Theorems 2.1 and 3.1.
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