TO WHAT EXTENT ARE STOCHASTIC THE ARITHMETICAL PROGRESSIONS OF THE FRACTIONAL PARTS?

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Abstract

For the residues of the division of the $n$ members of an arithmetical progression by a real number $N$ is proved the tending to 0 of the Kolmogorov’s stochasticity parameter $\lambda_n$, when $n$ tends to infinity, providing that the progression step is commensurable with $N$.

On the contrary, when the step is incommensurable with $N$, the paper describes some examples, where the stochasticity parameter $\lambda_n$ does not tend to zero, and even attains (infrequently) some arbitrary large values.

Both the too small and the too large values of the stochasticity parameter show the small probability of the randomness of the sequence, for which they have been counted. Thus, the long arithmetical progressions’ stochasticity degree is much smaller than that of the geometrical progressions (which provide temperate values of the stochasticity parameter, similarly to its value for the genuinely random sequences).
1. The measurement of the randomness degree

A.N. Kolmogorov [1] discovered in 1933 a remarkable statistical property of the sequences of independent values of any random real variable (having a continuous distribution function).

He introduced for this the stochasticity parameter $\lambda_n$, whose value indicates the probability of the conjecture that the given $n$ real numbers are $n$ independent observations of the same random variable.

To define his stochasticity parameter $\lambda_n$, Kolmogorov starts from the empirical counting function $C_n$, which describes the $n$ observed values

$$x_1 \leq x_2 \leq \cdots \leq x_n$$

(ordered here in the order of growing numbers).

The value of the counting function $C_n$ at any point $X \in \mathbb{R}$ is the number $C_n(X)$ of those observed values that do not exceed the number $X$.

In other terms, $C_n(X) = 0$ for $X < x_1$, $C_n(X) = m$ for $x_m \leq X < x_{m+1}$ and $C_n(X) = n$ for $X \geq x_n$.

Next Kolmogorov compared this empirical counting function $C_n$, describing the $n$ observed values, with the theoretical counting function $C_0$, whose values are defined the following way:

$$C_0(X) = (\text{the mathematical expectation of the number of the } n \text{ observed values, which do not exceed the value of } X) = (n \cdot \text{probability of the event } x \leq X).$$

Then Kolmogorov normalized the deviation of the empirical counting function from the theoretical one, measuring it in the uniform convergence norm:

$$F_n = \sup_X |C_n(X) - C_0(X)|$$

(differing here from the preceding work of Von Mises, who used the quadratic mean norm of the deviation).

To get the value of the stochasticity parameter $\lambda_n$, Kolmogorov normalized the deviation norm, dividing it by $\sqrt{n}$ (that is by the deviation expected for $n$ independent observations of a genuinely random variable)

$$\lambda_n = \frac{F_n}{\sqrt{n}}.$$

This Kolmogorov stochasticity parameter $\lambda_n$ of the set of $n$ observed values $\{x_1, \ldots, x_n\}$ is itself a random quantity. The astonishing Kolmogorov’s theorem [1] describes the asymptotical behaviour of the distribution of this random quantity $\lambda_n$ for $n \to \infty$.

Namely, he proved that this random quantity $\lambda_n$ is distributed (in the limit $n \to \infty$, in terms of the uniform convergence of the distribution functions of $\lambda_n$ to the limiting Kolmogorov’s distribution $\Phi$) some universal, standard way, which does not depend on the distribution of the initial random variable $x$. 

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The Kolmogorov’s theorem states that the limiting distribution of the stochasticity parameter (on the axis of real variable Λ) has the following form: define

\[ \Phi(\Lambda) = \text{probability of the event } \lambda \leq \Lambda, \]

then (for any real Λ > 0)

\[ \Phi(\Lambda) = \sum_{k \in \mathbb{Z}} (-1)^k e^{-2k^2 \Lambda^2}. \]

**Example.** The approximate values of this (growing) Kolmogorov’s distribution function are

\[ \Phi(0, 2) \approx 0.0000, \quad \Phi(2, 2) \approx 1.0000, \]

while for the intermediate values of the stochasticity parameter Λ Kolmogorov provided the following approximate values of function Φ (with 4 decimal accuracy):

<table>
<thead>
<tr>
<th>Λ</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Φ</td>
<td>0.028</td>
<td>1.57</td>
<td>4.56</td>
<td>7.3</td>
<td>8.87</td>
<td>9.6</td>
<td>9.88</td>
<td>9.96</td>
<td>9.99</td>
</tr>
</tbody>
</table>

In this Kolmogorov’s distribution both the too small (say, smaller than 0.4) and the too large (say, greater than 1.8) values of the stochasticity parameter Λ have small probabilities (smaller than one third of a percent).

Therefore, the observation of the value \( \lambda_n \) provides some information on the randomness of the observed series of \( n \) values: this randomness is improbable, if the value of \( \lambda_n \) is too small or too large.

The mean value of the stochasticity parameter Λ, distributed according to the Kolmogorov distribution Φ, is

\[ \Lambda = \sqrt{\pi/2 \ln 2} \approx 0.869. \]

The median value (such that both the higher as well as the lower values of the stochasticity parameter have probability 1/2 for the Kolmogorov’s distribution Φ) is approximately Λ ≈ 0.83.

In the present paper I used the method, described above, to study the “stochasticity degrees” of the arithmetical progressions of the residues for the division by some real number \( N \).

**Example.** The sequence of \( n \) two-digits numbers

\[ 37, 74, 11, 48, 85, 22, 59, 96, 33, 70, 07, 44, \ldots \]

(whose numbers are the mod 100 residues of the numbers 37\( x \), where \( x = 1, 2, 3, \ldots, n \)) looks as a (pseudo)random sequence.

To obtain an objective measure of the randomness degree of a given set of \( n \) numbers, we compute the value of the Kolmogorov’s randomness parameter \( \lambda_n \) for this set.

Comparing the observed value with the Kolmogorov’s distribution Φ, we interpret the improbable values of the stochasticity parameter \( \lambda_n \) as a sign of the nonrandomness of the given set, thinking that the probable values, (being not far from the mean value 0.87) provide a confirmation of the (pseudo)randomness of this set.

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1we denote below by 0, 1 the number 1/10 and so on.
To make this, I used as the “theoretical distribution” $C_0$ (of the arithmetical progressions of the residues modulo $N$) the uniform distribution, whose distribution function is linear on the segment of the values:

$$C_0(X) = (n/N)X \quad \text{for} \quad 0 \leq X < N.$$  

The uniform distribution of the sequence of the residues mod $N$ of the members of arithmetical progression, whose step is incommensurable with $N$, is the “fractional parts equipartition theorem” of H. Weyl.

The measurement of the stochasticity degree, in terms of the Kolmogorov’s stochasticity parameter $\lambda_n$, described above, can be applied to other (pseudorandom) sequences.

The numerical experiments of paper [2] have confirmed in this sense the “empirical randomness” of the geometrical progressions of the residues

$$\{a^x \pmod{N}\}, \quad x = 1, 2, \ldots, n,$$

and also the empirical randomness of the residues of the Fibonacci numbers (and of the members of other exponentially growing recurrent sequences).

For the “theoretical distribution” $C_0$ in these experiments I have used still the uniform distribution (on the segment from 0 to $N$), while the old conjecture of the equipartition of the fractional parts of generic geometrical progressions is still unproved (while some “adiabatic” arguments for his validity are described in the book [3]).

The values of the Kolmogorov’s stochasticity parameter $\lambda_n$, obtained in these numerical experiments, were approximately $\lambda_n \approx 0.7$, or 0.8, or 0.9. In this sense the fractional parts of the geometrical progressions members behave similarly to the independent values of a random variable (equidistributed along the segment $[0, 1]$).

On the contrary, for the arithmetical progressions of the integer residues (mod $N$) of $n$ subsequent prime numbers the observed values of the Kolmogorov’s parameters $\lambda_n$ are so small, that the probability to obtain such a value of the stochasticity parameter for $n$ independent observations of a genuinely random variable is a small part of one percent.

In the case of the prime numbers statistics the “theoretical distribution” $C_0$ was taken in these numerical experiments to be the Legendre-Chebychev’s distribution, whose distribution function is the integral logarithm.

The logarithm of a large number is slowly changing, when this number is growing. Therefore the sequence of the (large) consecutive primes behaves similarly to an arithmetical progression (whose step is slightly growing along the sequence).

One of the most known applications of the Kolmogorov’s stochasticity parameter is provided by his study [4] of the statistical data of the students of T.D. Lysenko, who had tried to refute the genetical Mendel’s laws, counting the deviations of the empirical data from the Mendel’s law prediction.
Kolmogorov proved that, rather than to refute the Mendel’s law, the published observations of the Lysenko’s students were confirming it. Namely, if the observed deviations were smaller than they reported, it would show that the data were falsified, while the published numbers provide just a reasonably temperate value of the Kolmogorov’s parameter $\lambda_n$ (not far from the mean value $\lambda_n \approx 0.87$ along the Kolmogorov distribution).

In the present article the theoretical studies of the arithmetical progressions of residues

$$\{ux \pmod{v}\}, \quad x = 1, 2, \ldots, n,$$

where $u$ and $v$ are integers are described.

It is proved that the Kolmogorov’s stochasticity parameter $\lambda_n$ tends to 0 for $n \to \infty$ in this case, showing that the “randomness degree” of the long part of such an arithmetical progression declines to 0 when the length of the progression grows.

Choosing the lengths unit to be $v$, one reduces this problem to that of the study of the arithmetical progressions of the fractional parts,

$$\{kx \pmod{1}\}, \quad x = 1, 2, \ldots, n,$$

where $k = u/v$.

For the arithmetical progressions of the fractional parts, whose step $k$ is a rational number, the Kolmogorov’s stochasticity parameter $\lambda_n$ tends to zero for $n \to \infty$ (indicating the asymptotical loss of randomness for such a long progression).

**Remark.** The Von Mises stochasticity parameter also tends to 0 in this case (the quadratic mean value being smaller than the uniform convergence metric’s norm).

On the contrary to the rational $k$ case, for the irrational step $k$ case the article contains some examples, where the Kolmogorov’s parameter $\lambda_n$ of an arithmetical progression of the $n$ fractional parts does not tend to 0 for $n \to \infty$. It even attains, while infrequently, some arbitrarily large values (which can’t, however, exceed $\sqrt{n}$) for some sufficiently large lengths $n$ of the progressions.

A similar growth of the Von Mises stochasticity parameter also occurs for similar examples.

For the majority of the irrational values of the step $k$ of the arithmetical progression of the fractional parts the behaviour of the stochasticity parameter $\lambda_n$ for large $n$ is not clear: all the observed phenomena are only proved for a Lebesgue measure zero set of the values of step $k$.

The large values of the Kolmogorov stochasticity parameter $\lambda_n$ are observed for some infrequent values of the length $n$ of the arithmetical progression. Therefore, it might be interesting to study the (Cesaro) averaged values (along the progression’s length $n$).

$$\hat{\lambda}_n = (\lambda_1 + \cdots + \lambda_n)/n.$$

But I have not made the corresponding numerical experiments and formulated no conjectures on the behaviour of $\hat{\lambda}_n$ for typical $k$.

These averaged values $\hat{\lambda}_n$ might depend on the progressions’s step $k$ in a less chaotic way than the values of the Kolmogorov’s parameter $\lambda_n$. 
To study this dependence of the behaviour of the “randomness degrees” on the parameter $k$ one might also use the local averaging along $k$ (in a neighbourhood of a fixed value $k$), that is the weak asymptotics (of $\lambda_n$ and of $\hat{\lambda}_n$), defined in the article [5].

This averaging, defining the “weak asymptotics”, might be performed both in the continuous case ($k \in \mathbb{R}$) and in the discrete version ($((u, v) \in \mathbb{Z}^2$).

In the continuous case one might average either along the Lebesgue measure on the axis of $k \in \mathbb{R}$, or along the Gauss-Kuz'min invariant measure, usual in the continued fractions theory (this measure is invariant under the transformation $z \mapsto \{1/z\}$ of the interval $(0, 1)$).

While these two measures (on the axis of the variable $k$) are different, one might suppose that the “weak asymptotics”, provided by both local averagings near the same point $k$, would be the same: the “averaged” behaviour of the numbers $\lambda_n$ (or at least of the numbers $\hat{\lambda}_n$) for typical central point $k$ would be the same for both measures.

In the discrete version case one considers the averages (of the quantities $\lambda_n$ and $\hat{\lambda}_n$) along the neighbourhood of the integral point $(mu, mv) \in \mathbb{Z}^2$, which neighbourhood has a fixed radius (or a radius, growing slowly with $m$). Next one considers the asymptotical behaviours of these averages along the neighbourhoods for the growing values of the parameter $m \to \infty$, fixing the central direction $(u : v \in \mathbb{R}P^1$).

One might expect some similarities between the weak asymptotics in the continuous and in the discrete versions of this problem (like the similarity of the Gauss-Kuz’min statistics of the continuous fractions’ elements of random real numbers to its discrete version, where it reappears as the asymptotics of the averages along the balls in the discrete plane $\mathbb{Z}^2$ of the quadratic equations $x^2 + px + q = 0$, defining the quadratic irrational numbers, whose periodic continued fractions statistics is averaged).

But I haven’t made even the experimental verification of the above conjectures on the weak asymptotics of the Kolmogorov’s parameter behaviour for the arithmetical progressions of fractional parts, all my numerical experiments being performed by hand-made calculation, using no computers.

Of course, Kolmogorov proved his theorem for the real random variables, whose distribution functions are continuous. We use below his distribution for the discrete case of the variables, whose values belong to $\mathbb{Z}$, $\mathbb{Z}_N$ or even $\mathbb{Q}$. I hope the Kolmogorov’s distribution provides here a good approximation (in spite of the fact, that this statement is neither proved nor formulated in [1]).

2. ARITHMETICAL PROGRESSIONS OF FRACTIONAL PARTS WITH RATIONAL DIFFERENCES

Below is proved the convergence to zero for $n \to \infty$ of the values of the Kolmogorov’s stochasticity parameter $\lambda_n$ of the first $n$ terms of the arithmetical progressions of the residues modulo $N$ of the $n$ numbers $kx$ ($x = 1, 2, \ldots, n$), provided that the progression’s step, $k$, is commensurable with the number $N$. 
Choosing \( N \) to be the unity of the measurements of the lengths on the real axis, we reduce this to the study of the arithmetical progressions of fractional parts \((\mod 1)\), whose step \( k \) is a rational number.

Let \( k = u/v, \) where \((u,v) = 1,\) be a rational number. We shall suppose that \( u > 0, \ v > 0. \) Consider the arithmetical progression of the fractional parts, that is of the residues modulo 1, consisting of the \( n \) numbers

\[ r(x) = \{kx\}, \text{ where } x = 1, 2, \ldots, n. \]

We define here the fractional part \( \{t\} \) of an integer to be rather 1 than 0:

\[ t = [t] + \{t\}, \quad 0 < \{t\} \leq 1. \]

Fix a particular value \( X \in \{1, 2, \ldots, n\}. \) Consider the fractional part

\[ R = \{kX\} = \rho/v, \]

with its integer numerator \( \rho,\ 1 \leq \rho \leq v. \)

For this integer \( \rho \) we have the identity

\[ Xu = mv + \rho \]

for some integer \( m = M, \) such that

\[ \rho = vR, \quad M = [kX] = kX - R. \]

Consider (Fig. 1) on the plane with cartesian coordinates \((x, m)\) the following parallelogram \( \Pi \) (depending on the parameters \( k, n \) and \( R \)):

\[ \Pi = \{ \begin{array}{l} 0 < x \leq n, \\
        kx - R \leq m < kx. \end{array} \]

Denote by \( S(X) \) the area of the parallelogram \( \Pi \): this rational number is equal to

\[ S(X) = nR = n\rho/v \in \mathbb{Q}. \]

Denote by \( \#(X) \) the number of the integer points, belonging to \( \Pi \) (including the part of the boundary denoted by the double line in Fig. 1, but not including the complementary part of the boundary of parallelogram \( \Pi \), where either \( x = 0 \) or \( m = kx \)).
Fig. 1. Parallelogram $\Pi$ (counting the predecessors of the value $R = \{kX\}$ among the first $n$ terms of the arithmetical progression of the fractional parts of the multiples of the step $k$).

The point $(X, M = \lfloor kX \rfloor = kX - R)$ belongs to the parallelogram $\Pi$ (and to its boundary, included in it).

**Theorem 1.** If $n = cv + w$, where $w$ is the residue of the division of the integer $n$ by the integer $v$, then

$$|S(X) - \#(X)| \leq w.$$ 

**Example.** In the case where $n$ is divisible by $v$, the area of the parallelogram $\Pi$ equals the number of its integer points:

$$S(X) = \#(X).$$

**Corollary.** For $n \to \infty$ the Kolmogorov’s stochasticity parameter value $\lambda_n$ tends to zero:

$$\lim_{n \to \infty} \max_{1 \leq X \leq n} \frac{|S(X) - \#(X)|}{\sqrt{n}} = 0$$

Indeed, the numerator is between 0 and $v$ (by Theorem 1), while the denominator tends to infinity.

The main point of the proof of Theorem 1 is

**Lemma 1.** In the case $n = v$ the area of the parallelogram $\Pi$ equals the number of its integer points:

$$S(X) = \#(X).$$

**Proof.** Consider a larger (containing $\Pi$) parallelogram $P$, for which the side length $R$ be 1 (Fig. 2):

$$P = \begin{cases} 0 < x \leq n, \\ kx - 1 \leq m < kx. \end{cases}$$
Fig. 2. The decomposition of the integer parallelogram $P$ into $n = v$ layers (containing one integer point in each layer).

The area of parallelogram $P$ equals $n$. Subdivide it into $v$ layers by the $v$ oblique parallel lines

\[
(*) \quad m = kx - \xi/v, \quad \xi = 1, \ldots, v.
\]

Each layer has the area equal to $n/v$ (which is 1 in the case of Lemma 1).

Parallelogram $P$ contains exactly $n$ integer points \( \{x, m = [kx]\} \) (one at each vertical line $x =$ const). Each of these $n$ points belongs to one of the $n$ oblique straight lines $(*).

No such line contains two integer points of parallelogram $P$. Indeed, if there were two such points $(x, m)$ and $(x', m')$ on the same line, that is with the same value $\xi$, then one would have

\[
m - kx = m' - kx',
\]

and therefore one would have

\[
v(m - m') = u(x - x').
\]

The integers $u$ and $v$ being relatively prime, the difference $x - x'$ ought be divisible by $v$. But $|x - x'| < v$, therefore $x = x'$ (and hence also $m = m'$).

Thus, each oblique line $(*),$ corresponding to any of the $n$ integers $\xi \in (0, 1, \ldots, v-1),$ contains at most one of the integer points of $P$. Therefore it contains exactly one integer point of $P$, and parallelogram $P$ contains no other integer points.

Among these oblique lines those ones for which $0 < \xi \leq R$ do belong to parallelogram II. Therefore, parallelogram II contains exactly $\rho$ integer points (namely, the points

\[
(x, m = [kx] = kx - \xi/v), \quad \text{where } \xi = 1, 2, \ldots, \rho.
\]
The area of the parallelogram $\Pi$ is also equal to $\rho$. Therefore, for $n = v$ we obtain
\[ S(X) = \#(\pi) \]
(for every point $X$, defining parallelogram $\Pi$ and its area $\rho$).

Lemma 1 is thus proved.

**Lemma 2.** If the number $n = cv$ is divisible by $v$, then the area of parallelogram $\Pi$ equals the number of the integer points in it:

\[ S(X) = \#(\pi) \]

**Proof.** Subdivide parallelogram $\Pi$ into $c$ parts (Fig. 3) ($\Pi_1, \Pi_2, \ldots, \Pi_c$) by the vertical lines $x = sv$ ($s = 1, 2, \ldots, c - 1$):

\[ \Pi_s = \begin{cases} (sv - 1)v < x \leq sv, \\ kx - R \leq m < kx, \end{cases} \]

where $R = \{kX\}$.

![Fig. 3. The decomposition of parallelogram $\Pi$ of height $cv$ into $c$ parallelograms $\Pi_s$ of height $v$.](image)

Parallellogram $\Pi_s$ is obtained from $\Pi_1$ by an integer shift
\[ (x, m) \rightarrow (x + (s - 1)v, m + (s - 1)u). \]

Therefore each of these parallelograms $\Pi_s$ contains the same number $\rho$ of integer points as parallelogram $\Pi_1$, and has the same area, which is also $\rho$ (by Lemma 1). Thus, we see that
\[ S(X) = cp, \quad \#(\pi) = cp, \]
proving Lemma 2.

**Lemma 3.** Subdivide parallelogram $\Pi$ of height $n = cv + w$ into two parts (Fig. 4)

\[ \Pi' = \begin{cases} 0 < x \leq cv, \\ kx - R \leq m < kx; \end{cases} \quad \Pi'' = \begin{cases} cv < x \leq n, \\ kx - R \leq m < kx. \end{cases} \]
Then the integer points numbers \( \#' \) and area \( S' \) coincide for the part \( \Pi' \), while for the part \( \Pi'' \) both these numbers do not exceed the residue \( w \) of the division of the integer \( n \) by the integer \( v \):

\[
\#'' \leq w, \quad S'' \leq w.
\]

**Proof.** The statement \( \#' = S' \) is Lemma 2. The number of the integer points in the part \( \Pi'' \) does not exceed \( w \), since on each of its \( n \) vertical lines \( x = cv + r \ (r = 1, 2, \ldots, w) \) there is at most one integer point in \( \Pi'' \) (since \( R \leq 1 \)).

The area of parallelogram \( \Pi'' \) equals \( wR \leq w \) (since \( R \leq 1 \)).

Thus, we have proved that

\[
|S'' - \#''| \leq w.
\]

For the total parallelogram \( \Pi \) characteristics

\[
S = S' + S'', \quad \# = \#' + \#'',
\]

we get the inequality

\[
|S - \#| \leq w,
\]

proving Theorem 1 (and also its Corollary).

For the integer base \( a \) geometrical progression of the residues modulo an integer \( N \ \{a^x \pmod{N}\} \), \( x = 1, \ldots, n \), the Kolmogorov parameter \( \lambda_n \) tends to 0 for \( n \to \infty \) (the theoretical distribution being uniform if \( N > a \) is prime). It follows from the Fermat’s and Euler’s periodicity theorem for these residues. But the conjecture that for almost all non integer bases \( a \) case \( \lambda_n \nrightarrow 0 \) is unproved.
3. Arithmetical progressions of fractional parts in the cases of irrational differences

To study the irrational step \( k \) arithmetical progressions of fractional parts, I start from some general statements, for which validity this irrationality is possible, but not required.

The point is that if the convergence statement (1) were accompanied by some uniformity of this convergence for the variable rational parameter \( k = u/v \) (for instance, if the deviation \( |S - \#| \) were uniformly bounded from above), then one would be able to deduce the calculation of the limit \( \lambda_n \to 0 \) for the irrational \( k \) from its validity for the rational values of parameter \( k \), approximating an irrational value by the rational neighbours.

But the next results show that such estimations, uniform with respect to the values \( k \in \mathbb{Q} \), are impossible: the convergence rate for \( n \to \infty \) is different for different rational values of \( k \), and there is no uniform convergence.

**Theorem 2.** The differences \( S - \# \) between the areas and the integer points numbers of parallelograms \( \Pi \), calculated for all the rational numbers \( k = u/v \) for all \( n \) and \( X \), are unbounded: they reach arbitrarily large values for some convenient quadruples of integers \((u, v, n, X)\).

**Proof.** Take \( u = 2n + 1 \), \( v = 2n \) (that is, take \( k = \frac{2n - 1}{2n} = \frac{1}{1 + \frac{1}{2n}} \)).

Calculating the values of the areas \( S(X) = R(X)n \) and the numbers \( \#(X) \) of the integer points for the corresponding parallelograms

\[
\Pi = \{ 0 < x \leq n, \quad kx - R \leq m < kx \}, \quad \text{where} \quad R = \{kX\},
\]
we start from \( R = \rho/v, \rho = uX - vM \), \( S = \rho/2 \) and obtain the following table of values

<table>
<thead>
<tr>
<th>( X )</th>
<th>( uX )</th>
<th>( M )</th>
<th>( vM )</th>
<th>( \rho )</th>
<th>( S )</th>
<th>( # )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2n - 1</td>
<td>0</td>
<td>2n</td>
<td>2n - 1</td>
<td>n - 1/2</td>
<td>n - 1/2</td>
</tr>
<tr>
<td>2</td>
<td>4n - 2</td>
<td>1</td>
<td>2n - 2</td>
<td>2n - 3</td>
<td>n - 1</td>
<td>n - 1</td>
</tr>
<tr>
<td>3</td>
<td>6n - 3</td>
<td>2</td>
<td>4n</td>
<td>(s - 1)2n</td>
<td>n - s/2</td>
<td>n - (s - 1)</td>
</tr>
<tr>
<td></td>
<td>2sn - s</td>
<td>s - 1</td>
<td></td>
<td>2n - s</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>2n^2 - n</td>
<td>n - 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We choose now the value \( X = n \), and in this case we obtain the difference

\[
S(X) - \#(X) = n/2 - 1.
\]

It is an arbitrarily large number for a sufficiently large value of \( n \).

**Remark.** The reason of this large deviation value is the large value \( (a_2 = 2n - 1) \) of the second element of continuous fraction for \( k \).

We show below that this reasoning allows one to construct such irrational numbers \( k \), that the values of the Kolmogorov’s parameter \( \lambda_n \) for the arithmetical progression of the fractional parts \( \{kx\} \), \( x = 1, 2, \ldots, n \),

\[
\lambda_n = \sup_X \frac{|S(X) - \#(X)|}{\sqrt{n}},
\]
does not tend to zero for $n \to \infty$ (which would take place for any rational step $k$), but attains, contrarily to the rational step $k$ case, arbitrarily large values $\lambda_n > K$ an infinite number of times (in spite of the boundary $\lambda_n \leq \sqrt{n}$).

**Theorem 3.** Suppose that the continued fraction

$$k = \frac{1}{a_1 + \frac{1}{a_2 + \ldots}}$$

has an even value of the even order element, $a_{s+2} = 2r$.

Then the Kolmogorov’s stochasticity parameter of the arithmetical progression of fractional parts ($\{k\}, \{2k\}, \{3k\}, \ldots$) takes for a convenient length $n = (q_s + q_{s+2})/2$ of the progression a value $\lambda_n$, exceeding any given number $K$, provided that the element $a_{s+2}$ is sufficiently large: it suffices that

$$a_{s+2} \geq \alpha(a_1, \ldots, a_{s+1}; K),$$

for some explicitly defined function $\alpha$.

The example of such function $\alpha$ is written below explicitly (in the inequality (3) at the end of the proof of Theorem 3).

**Remark.** Some similar large lower bounds are also verified in these examples by the Von Mises’ stochasticity parameter (where the uniform norm of the deviation is replaced by its quadratic mean value). The function $\alpha$ ought to be slightly larger to obtain such a minoration of the stochasticity parameter of Von Mises.

**Remark.** These exceedingly large values of the Kolmogorov’s stochasticity parameter $\lambda_n$ make highly improbable the randomness conjecture for the above arithmetical progression, similarly to the unprobability of this conjecture in the case of the observation of some too small values of the Kolmogorov’s parameter $\lambda_n$.

**Proof.** Consider on the plane of Fig. 5 with cartesian coordinates $(x, m)$ the following vectors, representing the continued fraction approximation $k \approx p_s/q_s$:

$$\xi_s := (x = q_s, m = p_s).$$
When \( s \) is even, this vector lies below the line \( m = kx \), and when \( s \) is odd, it lies up this line.

The continued fraction theory provides the classical identity

\[
\xi_{s+2} = \xi_s + a_s + 2 \xi_{s+1}.
\]

Construct the arithmetical mean vector between \( \xi_s \) and \( \xi_{s+2} \) (supposing index \( s \) be even and supposing even the continued fraction element \( a_{s+2} = 2r \)):

\[
\Xi = \xi_s + r \xi_{s+1} = (\xi_s + \xi_{s+2})/2,
\]

whose components are (Fig. 6)

\[
n = X = q_s + r q_{s+1}, \quad M = p_s + r p_{s+1}.
\]
Construct parallelogram II, corresponding to the above choice of the integers

\[ n = X = (q_s + q_{s+2})/2. \]

There are no integer points inside this parallelogram II, except point \( \Xi \) (it is a standard result of the theory of continued fractions). Therefore the number of the integer points in II is \( \#(X) = 1 \) for our choices.

The area of parallelogram II is bounded from below the following way.

This area equals the product of the height \( n \) of parallelogram II by the length \( R \) of the side \( A\Xi \):

\[ S(X) = nR = n|A\Xi|. \]

The segment \( A\Xi \) is the middle line of the trapeze \( CD\xi_{s+2}\xi_s \):

\[ |A\Xi| = (|C\xi_s| + |D\xi_{s+2}|)/2 \geq |C\xi_s|/2. \]

From the homotetical triangles \((OC\xi_s)\) and \((ODE)\) one obtains the length

\[ |C\xi_s| = |DE|q_s/q_{s+2} \geq |\xi_{s+2}E|q_s/q_{s+2}. \]

The ordinate of point \( \xi_{s+2} \) is \( p_s+2 \), the ordinate of point \( E \) being \( p_sq_{s+2}/q_s \). Therefore, the product in the worthest part of the preceding inequality equals the following area \([,]\) of a parallelogram:

\[ q_s|\xi_{s+2}E| = |p_s+2q_s - p_sq_{s+2}| = ||\xi_{s+2},\xi_s||. \]

The identity (2) provides the value of this parallelogram area:

\[ [\xi_{s+2},\xi_s] = a_{s+2}[\xi_{s+1},\xi_s], \]
and thus we find
\[ q_s |\xi_{s+2}E| = a_{s+2} \]
(the parallelogram with sides \( \xi_s \) and \( \xi_{s+1} \) having area 1, according to the continued fractions theory).

Recollecting the above inequalities, we obtain the following lower bound for the length of the side of the parallelogram \( \Pi \):
\[ |A\Xi| \geq |C\xi_s|/2 \geq (q_s |\xi_{s+2}E|)/(2q_{s+2}) = a_{s+2}/(2q_{s+2}). \]

For the area of this parallelogram one obtains therefore the lower bound
\[ S(X) = n |A\Xi| \geq \frac{q_s + q_{s+2}}{2} \cdot \frac{a_{s+2}}{2q_{s+2}} \geq \frac{a_{s+2}}{4}. \]

Theorem 3 is an easy corollary of this inequality, but I describe below an explicit lower bound \( \alpha \) for the continued fraction’s element \( a_{s+2} \), implying inequality \( \lambda_n \geq K \).

For the case \( S \geq 2 \) we have \( S - 1 \geq S/2 \), and therefore for parallelogram \( \Pi \) holds the inequality
\[ |S(X) - \#(X)| = S(X) - 1 \geq S/2 \geq a_{s+2}/8. \]

For the value of the Kolmogorov’s parameter
\[ \lambda_n = \frac{\max_X |S(X) - \#(X)|}{\sqrt{n}} \]
one gets therefore the lower bound
\[ \lambda_n \geq K, \]
provided that \( a_{s+2}/8 \geq K \sqrt{n} \). This inequality is satisfied if \( a_{s+2} \geq 64K^2n \), that is if
\[ a_{s+2}^2 \geq 32K^2(q_s + q_{s+2}). \]

According to formula (2), we have the inequality
\[ q_{s+2} = q_s + a_{s+2}q_{s+1} \leq (1 + a_{s+2})q_{s+1}, \]
and we conclude that for the validity of the inequality \( \lambda_n \geq K \) it is sufficient that holds the inequality
\[ a_{s+2}^2 \geq 32K^2(2 + a_{s+2})q_{s+1}, \]
which might be written in the form
\[ (a_{s+2} - 16K^2q_{s+1})^2 \geq 256K^4q_{s+1}^2 + 64K^2q_{s+1}, \]
which proves Theorem 3 (the explicit form of the function \( \alpha \) being the right hand side of formula (3)).

**Remark.** I do not know whether the value of the Kolmogorov’s stochasticity parameter \( \lambda_n \) of an arithmetical progression of the fractional parts of the \( n \) numbers \( kx \) tends to zero for almost
all real numbers $k$, or whether it is unbounded equally frequently (it might also be “generically” bounded from 0 and from infinity).

The ergodicity of the Gauss-Kuz’min dynamical system, $z \mapsto \{1/z\}$, suggests that any such asymptotic behaviour of the stochasticity parameter $\lambda_n$ should have probability either 0 or 1 (in the space of the values of parameter $k$), provided that it depends only on the asymptotics of the behaviour of the elements $a_s$ of the continued fraction of $k$ for $s \to \infty$. But I do not know, whether the probability is 0 or 1 for the types of the stochasticity parameter behaviour described above.

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**References**


